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# NONPERTURBATIVE HQET AT ORDER $1/m$

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We summarize the first results for masses and decay constants of bottom-strange (pseudo-scalar and vector) mesons from nonperturbatively renormalized heavy-quark effective theory (HQET), using lattice-QCD simulations in the quenched approximation.

*Keywords*: B physics; lattice QCD; heavy-quark effective theory.

## 1. Introduction

The study of B physics is essential to determine the flavor structure of the standard model, through knowledge of the Cabibbo–Kobayashi–Maskawa (CKM) matrix describing quark mixing and CP violation, which may be associated with the lack of symmetry between matter and antimatter in the universe. In fact, since the amount of baryons in the universe predicted using the CKM mechanism is several orders of magnitude smaller than what is observed by astronomers, extensions of the

standard model propose additional sources of CP violation, which must be tested against standard model predictions. B mesons provide the ideal environment for such tests.[1] In particular, high-precision theoretical inputs are needed for hadronic matrix elements, which may be computed starting from the gauge theory itself using numerical simulations of lattice QCD. At present, however, it is not yet feasible to perform simulations on lattices that can simultaneously represent the two relevant scales of B physics: the low energy scale $\Lambda_{\text{QCD}}$, requiring large physical lattice size, and the high energy scale of the b-quark mass $m_b$, requiring very small lattice spacing $a$. An approximate framework is therefore needed, but one should strive to achieve sufficiently precise results, otherwise the task of over-constraining the parameters of the standard model is compromised.

One such promising framework is to consider (lattice) heavy-quark effective theory (HQET), which allows for an elegant theoretical treatment, with the possibility of fully nonperturbative renormalization.[2,3] The approach is briefly described as follows. HQET provides a valid low-momentum description for systems with one heavy quark, with heavy-quark symmetry manifesed in the limit $m_b \to \infty$. The heavy-quark flavor and spin symmetries are broken at finite values of $m_b$ respectively by kinetic and spin terms, with first-order corrections to the static Lagrangian parametrized by $\omega_{\text{kin}}$ and $\omega_{\text{spin}}$

$$\mathcal{L}^{\text{HQET}} = \overline{\psi}_h(x) \, D_0 \, \psi_h(x) - \omega_{\text{kin}} \mathcal{O}_{\text{kin}} - \omega_{\text{spin}} \mathcal{O}_{\text{spin}}, \tag{1}$$

where

$$\mathcal{O}_{\text{kin}} = \overline{\psi}_h(x)\mathbf{D}^2\psi_h(x), \quad \mathcal{O}_{\text{spin}} = \overline{\psi}_h(x)\sigma \cdot \mathbf{B}\psi_h(x). \tag{2}$$

These $O(1/m_b)$ corrections are incorporated by an expansion of the statistical weight in $1/m_b$ such that $\mathcal{O}_{\text{kin}}$ and $\mathcal{O}_{\text{spin}}$ are treated as insertions into static correlation functions. This guarantees the existence of a continuum limit, with results that are independent of the regularization, provided that the renormalization is done nonperturbatively.

As a consequence, expansions for masses and decay constants are given respectively by

$$m_B = m_{\text{bare}} + E^{\text{stat}} + \omega_{\text{kin}}E^{\text{kin}} + \omega_{\text{spin}}E^{\text{spin}} \tag{3}$$

and

$$f_B\sqrt{\frac{m_B}{2}} = Z_A^{\text{HQET}}p^{\text{stat}}(1 + c_A^{\text{HQET}}p^{\delta A} + \omega_{\text{kin}}p^{\text{kin}} + \omega_{\text{spin}}p^{\text{spin}}), \tag{4}$$

where the parameters $m_{\text{bare}}$ and $Z_A^{\text{HQET}}$ are written as sums of a static and an $O(1/m_b)$ term (denoted respectively with the superscripts "stat" and "$1/m_b$" below), and $c_A^{\text{HQET}}$ is of order $1/m_b$. Bare energies ($E^{\text{stat}}$ *etc.*) and matrix elements ($p^{\text{stat}}$ *etc.*) are computed in the numerical simulation.

The divergences (with inverse powers of $a$) in the above parameters are canceled through the nonperturbative renormalization, which is based on a matching of

HQET parameters to QCD on lattices of small physical volume — where fine lattice spacings can be considered — and an extrapolation to a large volume by the step-scaling method. Such an analysis has been recently completed for the quenched case.[4] In particular, there are nonperturbative (quenched) determinations of the static coefficients $m_{\text{bare}}^{\text{stat}}$ and $Z_A^{\text{stat}}$ for HYP1 and HYP2 static-quark actions[5] at the physical b-quark mass, and similarly for the $O(1/m_b)$ parameters $\omega_{\text{kin}}$, $\omega_{\text{spin}}$, $m_{\text{bare}}^{1/m_b}$, $Z_A^{1/m_b}$ and $c_A^{\text{HQET}}$.

The newly determined HQET parameters are very precise (with errors of a couple of percentages in the static case) and show the expected behavior with $a$. They are used in our calculations reported here, to perform the nonperturbative renormalization of the (bare) observables computed in the simulation. Of course, in order to keep a high precision, these bare quantities also have to be accurately determined. This is accomplished by an efficient use of the generalized eigenvalue problem (GEVP) for extracting energy levels $E_n$ and matrix elements, as described below.

A significant source of systematic errors in the determination of energy levels in lattice simulations is the contamination from excited states in the time correlators

$$C(t) = \langle O(t)O(0)\rangle = \sum_{n=1}^{\infty} |\langle n|\hat{O}|0\rangle|^2 e^{-E_n t} \tag{5}$$

of fields $O(t)$ with the quantum numbers of a given bound state.

Instead of starting from simple local fields $O$ and getting the (ground-state) energy from an effective-mass plateau in $C(t)$ as defined above, it is then advantageous to consider all-to-all propagators[6] and to solve, instead, the GEVP

$$C(t)v_n(t,t_0) = \lambda_n(t,t_0)C(t_0)v_n(t,t_0), \tag{6}$$

where $t > t_0$ and $C(t)$ is now a matrix of correlators, given by

$$C_{ij}(t) = \langle O_i(t)O_j(0)\rangle = \sum_{n=1}^{\infty} e^{-E_n t}\Psi_{ni}\Psi_{nj}, \quad i,j = 1,\ldots,N. \tag{7}$$

The chosen interpolators $O_i$ are taken (hopefully) as linearly independent, e.g. they may be built from the smeared quark fields using $N$ different smearing levels. The matrix elements $\Psi_{ni}$ are defined by

$$\Psi_{ni} \equiv (\Psi_n)_i = \langle n|\hat{O}_i|0\rangle, \quad \langle m|n\rangle = \delta_{mn}. \tag{8}$$

One thus computes $C_{ij}$ for the interpolator basis $O_i$ from the numerical simulation, and then gets effective energy levels $E_n^{\text{eff}}$ and estimates for the matrix elements $\Psi_{ni}$ from the solution $\lambda_n(t,t_0)$ of the GEVP at large $t$. For the energies

$$E_n^{\text{eff}}(t,t_0) \equiv \frac{1}{a}\log\frac{\lambda_n(t,t_0)}{\lambda_n(t+a,t_0)} \tag{9}$$

it is shown[7] that $E_n^{\text{eff}}(t,t_0)$ converges exponentially as $t \to \infty$ (and fixed $t_0$) to the true energy $E_n$. However, since the exponential falloff of higher contributions may

be slow, it is also essential to study the convergence as a function of $t_0$ in order to achieve the required efficiency for the method. This has been recently done,[8,9] by explicit application of (ordinary) perturbation theory to a hypothetical truncated problem where only $N$ levels contribute. The solution in this case is exactly given by the true energies, and corrections due to the higher states are treated perturbatively. We get

$$E_n^{\text{eff}}(t, t_0) = E_n + \varepsilon_n(t, t_0) \tag{10}$$

for the energies and

$$e^{-\hat{H}t}(\hat{\mathcal{Q}}_n^{\text{eff}}(t, t_0))^\dagger|0\rangle = |n\rangle + \sum_{n'=1}^{\infty} \pi_{nn'}(t, t_0)|n'\rangle \tag{11}$$

for the eigenstates of the Hamiltonian, which may be estimated through

$$\hat{\mathcal{Q}}_n^{\text{eff}}(t, t_0) = R_n(\hat{O}, v_n(t, t_0)), \tag{12}$$

$$R_n = (v_n(t, t_0), \quad C(t)v_n(t, t_0))^{-1/2} \left[\frac{\lambda_n(t_0 + a, t_0)}{\lambda_n(t_0 + 2a, t_0)}\right]^{t/2}. \tag{13}$$

In our analysis we see that, due to cancellations of $t$-independent terms in the effective energy, the first-order corrections in $\varepsilon_n(t, t_0)$ are independent of $t_0$ and very strongly suppressed at large $t$. We identify two regimes: (1) for $t_0 < t/2$, the second-order corrections dominate and their exponential suppression is given by the smallest energy gap $|E_m - E_n| \equiv \Delta E_{m,n}$ between level $n$ and its neighboring levels $m$; and (2) for $t_0 \geq t/2$, the first-order corrections dominate and the suppression is given by the large gap $\Delta E_{N+1,n}$. Amplitudes $\pi_{nn'}(t, t_0)$ get main contributions from the first-order corrections. For fixed $t - t_0$ these are also suppressed with $\Delta E_{N+1,n}$. Clearly, the appearance of large energy gaps in the second regime improves convergence significantly. We therefore work with $t$, $t_0$ combinations in this regime.

A very important step of our approach is to realize that the same perturbative analysis may be applied to get the $1/m_b$ corrections in the HQET correlation functions mentioned previously

$$C_{ij}(t) = C_{ij}^{\text{stat}}(t) + \omega C_{ij}^{1/m_b}(t) + \text{O}(\omega^2), \tag{14}$$

where the combined $\text{O}(1/m_b)$ corrections are symbolized by the expansion parameter $\omega$. Following the same procedure as above, we get similar exponential suppressions (with the static energy gaps) for static and $\text{O}(1/m_b)$ terms in the effective theory. We arrive at

$$E_n^{\text{eff}}(t, t_0) = E_n^{\text{eff,stat}}(t, t_0) + \omega E_n^{\text{eff},1/m_b}(t, t_0) + \text{O}(\omega^2) \tag{15}$$

with

$$E_n^{\text{eff,stat}}(t, t_0) = E_n^{\text{stat}} + \beta_n^{\text{stat}}e^{-\Delta E_{N+1,n}^{\text{stat}}t} + \cdots, \tag{16}$$

$$E_n^{\text{eff},1/m_b}(t, t_0) = E_n^{1/m_b} + [\beta_n^{1/m_b} - \beta_n^{\text{stat}}t\Delta E_{N+1,n}^{1/m_b}]e^{-\Delta E_{N+1,n}^{\text{stat}}t} + \cdots \tag{17}$$

and similarly for matrix elements. Preliminary results of our application of the methods described in this section were presented recently[10] and are summarized in the next section. A more detailed version of this study will be presented elsewhere.[11]

## 2. Results

We carried out a study of static-light $B_s$-mesons in quenched HQET with the non-perturbative parameters described in the previous section, employing the HYP1 and HYP2 lattice actions for the static quark and an $O(a)$-improved Wilson action for the strange quark in the simulations. The lattices considered were of the form $L^3 \times 2L$ with periodic boundary conditions. We took $L \approx 1.5$ fm and lattice spacings 0.1 fm, 0.07 fm and 0.05 fm, corresponding respectively to $\beta = 6.0219, 6.2885$ and 6.4956. We used all-to-all strange-quark propagators constructed from approximate low modes, with 100 configurations. Gauge links in interpolating fields were smeared with three iterations of (spatial) APE smearing, whereas Gaussian smearing (eight levels) was used for the strange-quark field. A simple $\gamma_0\gamma_5$ structure in Dirac space was taken for all eight interpolating fields. Also, the local field (no smearing) was included in order to compute the decay constant.

The resulting $(8 \times 8)$ correlation matrix may be conveniently truncated to an $N \times N$ one and the GEVP solved for each $N$, so that results can be studied as a function of $N$. We then pick a basis from unprojected interpolators, by sampling the different smearing levels (from 1 to 7) as $\{1, 7\}, \{1, 4, 7\}$, *etc.* We perform fits of the various energy levels and values of $N$ to the behavior in Eq. (16) and extract our results from the predicted plateaus. Next, we take the continuum limit, extrapolating our results to $a \to 0$. We see that the correction to the ground-state energy
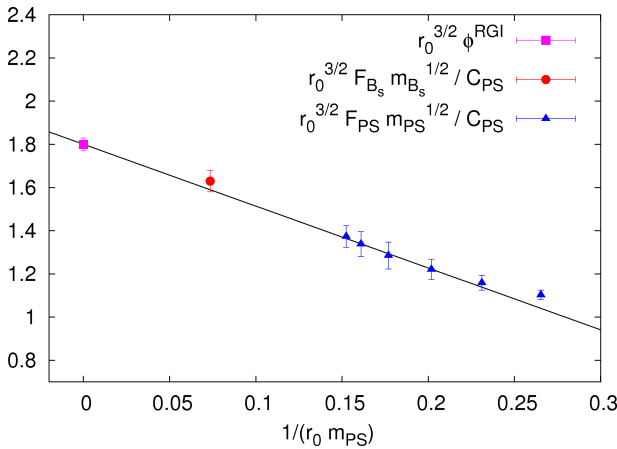


Fig. 1.   Comparison of the continuum values for the pseudoscalar meson decay constant from Fig. 4 in Ref. 12 to fully relativistic data in the charm region. The solid line is a linear interpolation between the static limit and the points around the charm-quark mass, which corresponds to $1/r_0 \, m_{\rm PS} \approx 0.2$.

due to terms of order $1/m_b$, which is positive for finite $a$, is quite small (consistent with zero) in the continuum limit. Our results for the pseudoscalar meson decay constant, both in the static limit and including $O(1/m_b)$ corrections, are shown in terms of the combination $\Phi^{\mathrm{HQET}} \equiv F_{\mathrm{PS}}\sqrt{m_{\mathrm{PS}}}/C_{\mathrm{PS}}$, where $C_{\mathrm{PS}}(M/\Lambda_{\mathrm{QCD}})$ is a known matching function and $\Phi^{\mathrm{RGI}}$ denotes the renormalization-group-invariant matrix element of the static axial current.[12] These two continuum extrapolations are shown in comparison with fully relativistic heavy-light (around charm-strange) data[12] in Fig. 1. Note that, up to perturbative corrections of order $\alpha^3$ in $C_{\mathrm{PS}}$, HQET predicts a behavior $\mathrm{const} + O(1/r_0 m_{\mathrm{PS}})$ in this graph. Surprisingly no $1/(r_0 m_{\mathrm{PS}})^2$ terms are visible, even with our rather small errors.

## 3. Conclusions

The combined use of nonperturbatively determined HQET parameters (in action and currents) and efficient GEVP allows us to reach precisions of a few percent in matrix elements and of a few MeV in energy levels, even with only a moderate number of configurations. The method is robust with respect to the choice of interpolator basis. All parameters have been determined nonperturbatively and in particular power divergences are completely subtracted. We see that HQET plus $O(1/m_b)$ corrections at the b-quark mass agrees well with an interpolation between the static point and the charm region, indicating that linearity in $1/m$ extends even to the charm point. A corresponding study for $N_f = 2$ is in progress.

## Acknowledgments

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