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Fractal dimension of the topological charge density distribution in SU(2) lattice gluodynamics

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We study the effect of cooling on the spatial distribution of the topological charge density in quenched SU(2) lattice gauge theory with overlap fermions. We show that as the gauge field configurations are cooled, the Hausdorff dimension of regions where the topological charge is localized gradually changes from $d = 2 \div 3$ towards the total space dimension. Hence the cooling procedure destroys some of the essential properties of the topological charge distribution.

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Introduction

Topological charge density is an important characteristic of the QCD vacuum, recently involved in phenomenological studies of a plenty of new hypothetical effects [1, 2]. However, the spatial structure of the topological density distribution seems to be not well-defined since the relevant properties of the underlying vacuum structure depend on the measuring procedure [3]. The classical instanton approach [4] assumes that the non-perturbative physics is governed by the scale of Λ_{QCD}, which means that the dimensionful quantities like volumes occupied by topological fermion modes should depend on Λ_{QCD} but not on the lattice spacing. On the contrary, the lattice measurements demonstrate that these volumes do depend on the spacing (i.e. on the measurement resolution) and shrink to zero in the continuum limit [5–8].

It turns out that the continuum definition of the topological charge density

$$q(x) = \frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(G^a_{\mu\nu}\,G^a_{\alpha\beta}\right) \qquad (1)$$

cannot be directly applied to the lattice gauge theory, since the discretized version of (1) is no longer a full derivative. There are two widely used methods to study the topology of gauge fields on the lattice. First, one can apply a smearing procedure, which makes the gauge fields smoother and thus closer to the continuum fields. Second, one can rely on the lattice version of the Atyah-Singer theorem and define the total topological charge of a gauge field configuration as the number of zero modes of the overlap Dirac operator [9] on this configuration. The corresponding local density of topological charge can be defined, for example, as follows [9–11]:

$$q(x) = -\mathrm{Tr}\left[\gamma_5\left(1 - \frac{a}{2}D(x,x)\right)\right], \qquad (2)$$

where $D(x,x)$ is the zero-mass Neuberger operator and the trace is taken over spinor and color indices. Another attractive property of this definition is that it allows us to measure a *local* imbalance in number of left- and right-handed quarks (chirality), which is important for lattice studies of the *local* CP-violation in strong interactions [13]. A typical result of the lattice simulation for this quantity (without cooling) is shown in Fig. 1.

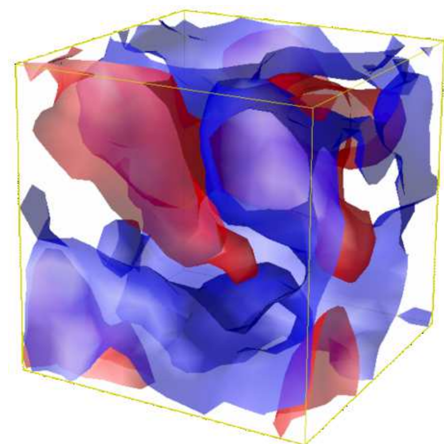

FIG. 1: Isosurfaces of the topological charge density $q(x) = \pm 0.0001$ for a fixed time slice. Red and blue colors represent positive and negative values respectively. For the full animation see [12].

At the moment there are many investigations related to the spatial structure of the topological charge distribution [7, 8, 14–18], which use both of the alternative definitions. The measurements which rely on the cooling procedure mostly suggest an instanton-like picture of the QCD vacuum [19], while the definition (2) typically shows that the topological charge is localized at low-dimensional objects (defects) [7, 8, 17, 18] and has a very-long-range structure of the distribution [18]. At the qualitative level it is known that both definitions yield the topological charge densities which are strongly correlated [14, 20].

The aim of this note is to fill the existing gap in the literature and to demonstrate in what way the cooling procedure affects the dimensionality of regions where the topological charge is localized. We use the definition (2) based on zero modes of overlap Dirac operator

β	a [fm]	$L_s^3 \times L_t$	V [fm^4]	# conf
3.200	0.117	$12^3 \times 12$	3.93	50×11
3.295	0.100	$14^3 \times 14$	3.90	50×11
3.332	0.094	$15^3 \times 15$	3.89	50×11
3.365	0.088	$16^3 \times 16$	3.88	50×11
3.425	0.078	$18^3 \times 18$	3.87	50×11

TABLE I: Lattice parameters used in the calculation: couplings β, lattice spacings a, lattice sizes $L_s^3 \times L_t$, physical volume V and number of gauge field configurations.

and show that as the gauge field configurations are cooled the dimension of these regions gradually tends to 4, which is the total space dimension. The procedure makes the effective resolution of the measurement lower and thus provides a result close to the instanton picture. We verify our result using several measures of the localization [5, 6, 21].

Technical details

We work in quenched $SU(2)$ lattice gauge theory with the tadpole-improved Wilson-Symanzik action [22]. Lattices we used are listed in Table I. We also implement the cooling procedure described in [15] with coefficient $c = 0.5$ for the APE-smearing. For each lattice spacing we consider eleven different stages of the cooling procedure: 0, 1, 2, 5, 7, 9 - 12, 20 and 50 iterations of the algorithm. For valence quarks we use the Neuberger's overlap Dirac operator [9]. Its eigenvalues and eigenfunctions are given by the following relation

$$D\psi_\lambda = \lambda\, \psi_\lambda \,. \tag{3}$$

The quantities we measure in the present work are functions of two basic ingredients - the "chiral condensate" computed on a mode with eigenvalue λ:

$$\rho_\lambda(x) = \psi_\lambda^{*\alpha}(x)\; \psi_{\lambda\alpha}(x) \tag{4}$$

and "chirality" computed on a mode with eigenvalue λ (in agreement with the definition (2)):

$$\rho_\lambda^5(x) = \left(1 - \frac{\lambda}{2}\right) \psi_\lambda^{*\alpha}(x) \gamma^5_{\alpha\beta}\, \psi_\lambda^\beta(x) \,. \tag{5}$$

Here we sum over spinor and (omitted) color indices. The total values of both chiral condensate and chirality are given by the infinite sum over all eigenvalues. Lattice studies [23, 24] suggest that the long distance properties of QCD can be treated with a finite cut-off of the fermionic spectrum. We hereby restrict our consideration to the IR part of the Dirac spectrum consisting of zero modes ($\lambda = 0$) and few low-lying modes ($\lambda \neq 0$).

Inverse participation ratio (IPR) for an arbitrary normalized distribution $\alpha(x)$ is usually defined in the following way

$$\mathrm{IPR} = \left\{ N \sum_x \alpha^2(x) \,\middle|\, \sum_x \alpha(x) = 1 \right\} , \tag{6}$$

where N is the total number of lattice sites x. From this definition one can clearly see that IPR $= N$ if $\alpha(x)$ is localized on a single site and IPR $= 1$ if $\alpha(x) = const$, i.e. the distribution is unlocalized. In general IPR is equal to the inverse fraction of sites occupied by the support of $\alpha(x)$. Since this fraction of sites can be thought of as a number of four-dimensional lattice hypercubes covering the support, the Hausdorff dimension d of these regions can be extracted from the asymptotic behavior of IPR at small lattice spacings a

$$\mathrm{IPR}(a) = \frac{c}{a^d} \,, \tag{7}$$

where c is a constant. It is also useful to mention, that in physical units IPR^{-1} is equal to the part of the total volume occupied by the distribution.

In the following sections we will modify the standard definition (6) to adapt it to our particular cases (i.e. unnormalized or non-normalizable distributions, e.t.c.). The final result will show an equivalence of the chosen definitions.

Ordinary IPR for zero modes.

In this section we compute the inverse participation ratio for fermionic zero modes according to one defined in [6]:

$$\mathrm{IPR}_0 = N \left[\frac{\sum_x \left(\rho_0(x)\right)^2}{\left(\sum_x \rho_0(x)\right)^2} \right]_{\lambda=0} , \tag{8}$$

where the brackets $[...]_{\lambda=0}$ denote an averaging over all zero modes and further averaging over all gauge field configurations. Results are presented in Fig. 2.

The left figure shows how the localization depends on the lattice spacing a - the finer the lattice, the larger IPR. This fits very well to the idea of vanishing total volume occupied by fermionic zero modes in the continuum limit $a \to 0$ (see [3] for a review). Using the fit (7) we recover the fractal (Hausdorff) dimension d of the volume. Results for the fits with fixed numbers of cooling steps are presented the Table II.

Chiral IPR for low-lying modes. First definition.

In this section we modify the IPR to measure localization properties of the topological charge distribution. The average chirality $\left[\sum_x \rho_\lambda^5(x)\right]_\lambda$ is zero, therefore we have to use either the absolute value $|\rho_\lambda^5(x)|$ or the square $\left(\rho_\lambda^5(x)\right)^2$. Here we stick to the definition from [21], which

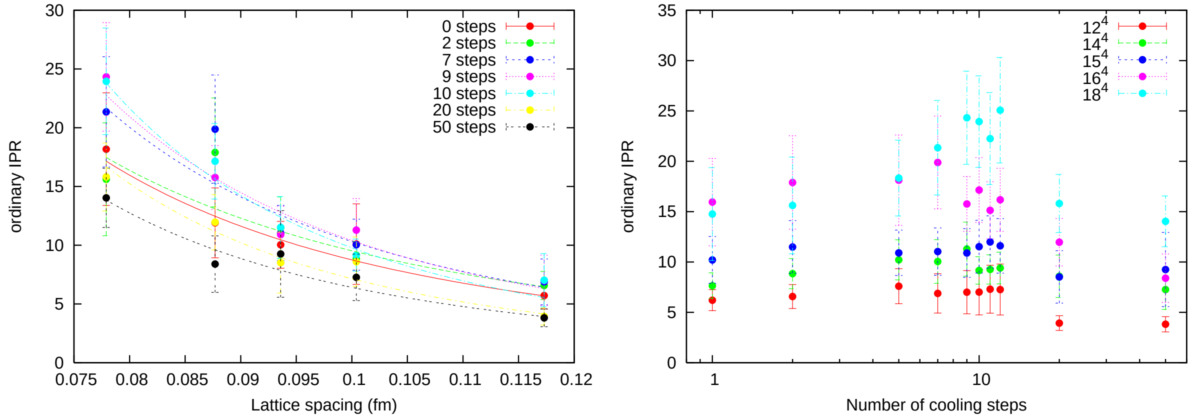

FIG. 2: Ordinary IPR for zero modes.

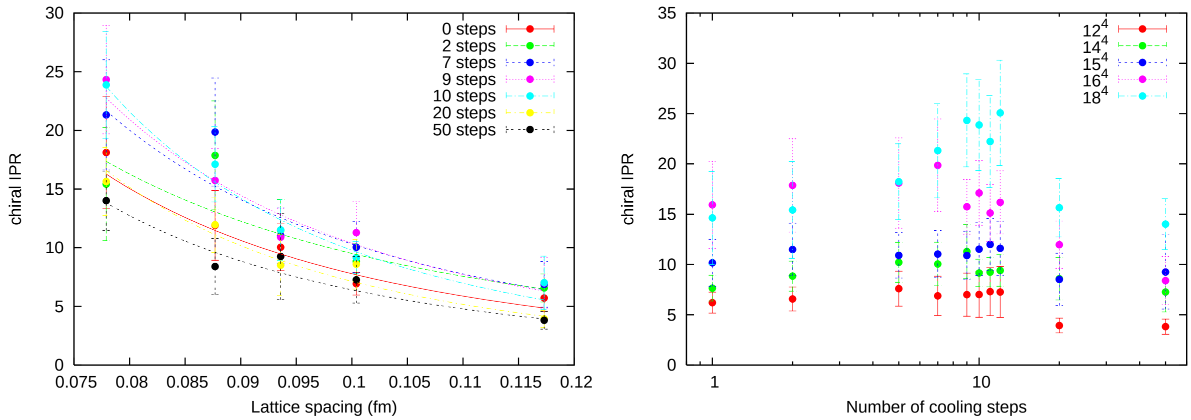

FIG. 3: Chiral IPR for zero modes. Definition 1.

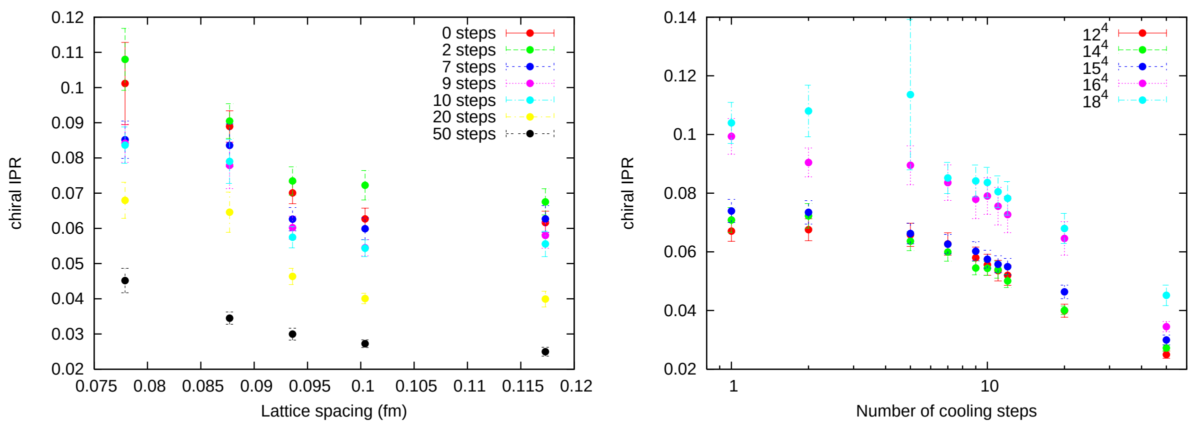

FIG. 4: Chiral IPR for the lowest non-zero modes. Definition 1.

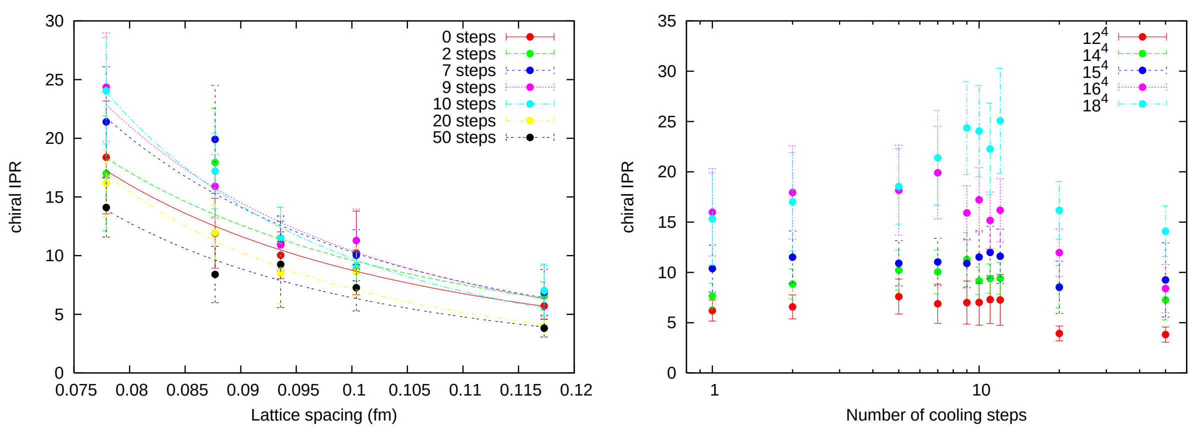

FIG. 5: Chiral IPR for zero modes. Definition 2.

in our terms has the following form

$$\mathrm{IPR}_0^5 = N \left[\frac{\sum_x \left(\rho_0^5(x)\right)^2}{\left(\sum_x \rho_0(x)\right)^2} \right]_{\lambda=0} . \qquad (9)$$

Results are presented in Fig. 3 and in Table III. From the plots we conclude that the topological charge distribution behaves similar to the zero modes, tending to occupy a vanishing volume in the continuum limit. We can also compute the chiral IPR for small but non-zero eigenvalues

$$\mathrm{IPR}_{\lambda\neq 0}^5 = N \left[\frac{\sum_x \left(\rho_\lambda^5(x)\right)^2}{\left(\sum_x \rho_\lambda(x)\right)^2} \right]_{\lambda\neq 0} . \qquad (10)$$

Chiral IPR for these modes is small (Fig. 4) and thus the topological charge distribution at this part of the spectrum is delocalized.

Chiral IPR for zero modes. Second definition.

Finally we consider a second definition of the chiral IPR according to [5]:

$$\mathrm{IPR}_0^5 = N \left[\frac{\sum_x \left|\rho_0^5(x)\right|^2}{\left(\sum_x \left|\rho_0^5(x)\right|\right)^2} \right]_{\lambda=0} , \qquad (11)$$

where, as before, $\rho_0^5(x)$ denotes the chirality on a zero mode (2). Results are presented in Fig. 5 and in Table IV.

Fractal dimension. Results and conclusions.

To conclude, we demonstrate that the topological charge is localized on low-dimensional fractal structures, whose fractal (Hausdorff) dimension depends on the number of cooling steps. The obtained dimension is about $d = 2 \div 3$ for a few ($n < 5$) steps of the cooling, while it grows to $d = 3 \div 4$ for many iterations of the procedure (see Fig. 6). For a very long cooling ($n > 50$) the result becomes insignificant, because the dependence of IPR on the lattice spacing is no longer a power function.

The main conclusions of the paper are the following:

number of cooling steps	fractal dimension	standard error	$\chi^2/d.o.f.$
0	2.69 ± 0.24	9%	0.10
1	2.34 ± 0.54	23%	0.51
2	2.43 ± 0.51	21%	0.47
5	2.28 ± 0.45	20%	0.38
7	2.98 ± 0.59	20%	0.50
9	3.13 ± 0.49	15%	0.33
10	3.56 ± 0.43	12%	0.30
11	3.13 ± 0.32	10%	0.15
12	3.54 ± 0.41	12%	0.22
20	3.40 ± 0.31	9%	0.30
50	3.08 ± 0.29	9%	0.21

TABLE II: Fractal dimension of the fermionic zero modes

number of cooling steps	fractal dimension	standard error	$\chi^2/d.o.f.$
0	2.95 ± 0.57	19%	0.49
1	2.33 ± 0.54	23%	0.51
2	2.41 ± 0.51	21%	0.47
5	2.26 ± 0.45	20%	0.38
7	2.97 ± 0.59	20%	0.50
9	3.13 ± 0.49	16%	0.33
10	3.55 ± 0.43	12%	0.29
11	3.13 ± 0.33	10%	0.15
12	3.54 ± 0.41	12%	0.22
20	3.38 ± 0.32	9%	0.30
50	3.08 ± 0.29	9%	0.21

TABLE III: Fractal dimension of the topological charge distribution. Definition 1.

number of cooling steps	fractal dimension	standard error	$\chi^2/d.o.f.$
0	2.71 ± 0.25	9%	0.11
1	2.43 ± 0.34	22%	0.51
2	2.58 ± 0.47	18%	0.41
5	2.31 ± 0.45	20%	0.39
7	2.98 ± 0.59	20%	0.50
8	2.78 ± 0.65	23%	0.59
10	3.58 ± 0.44	12%	0.30
11	3.1 ± 0.30	10%	0.13
12	3.54 ± 0.41	12%	0.22
20	3.44 ± 0.30	9%	0.26
50	3.09 ± 0.29	9%	0.22

TABLE IV: Fractal dimension of the topological charge distribution. Definition 2.

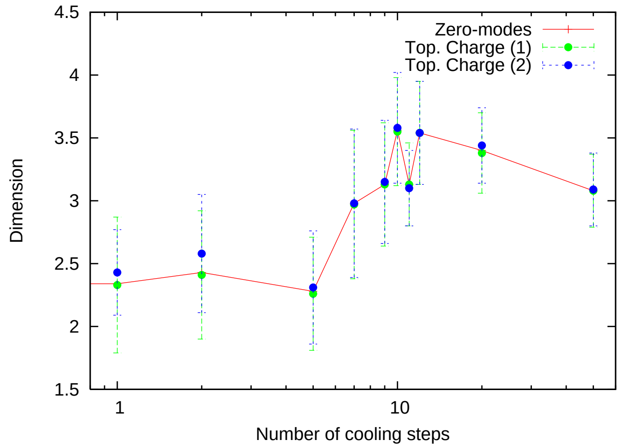

FIG. 6: Fractal dimensions at various cooling stages. The solid line is shown to guide the eye.

(1) Fermionic zero modes and chirality are localized on structures with fractal dimension $d = 2 \div 3$ which is an argument in favor of the vortex/domain-wall nature of the localization [25, 26].

(2) A long sequence of iterations of the cooling procedure provides a result close to the instanton picture, i.e. destroys the low-dimensional structure of the QCD vacuum.

Finally, let us briefly mention a possible phenomenological consequence of our study. One of the most promising effects appearing due to the nontrivial topology of the QCD vacuum is the Chiral Magnetic Effect [1], which states the generation of an electric current in parallel to an external magnetic field. Topological charge density in this case can be understood as an imbalance in the number of left- and right-handed light quarks induced by the nontrivial gluonic background. This effect is expected to explain charge asymmetries observed at RHIC [13, 27]. Some evidences of the CME on the lattice as well as numerical estimates for the values of the local topological charge were also obtained in [28–32]. At the current level of analytic studies CME is considered as an effect on the background of spatially homogeneous axial fields [33], while the lattice simulations predict an irregular structure of the would-be axial field (see Fig. 1). This spatial inhomogeneity can be treated within a chiral-superfluid model [34], where the topological charge density is carried by an effective axion-like field. Knowledge of the nature of the topological charge localization can help us to translate lattice Euclidean properties of the chirality to the language of an effective Minkowski field theory [35].

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