Monte-Carlo-Simulations of Stochastic Differential Equations at the Example of the Forced Burgers' Equation

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We investigate the behaviour of stochastic differential equations, especially Burgers' eq., by means of Monte-Carlo-techniques. By analysis of the produced configurations, we show that direct and often intuitive insight into the fundamentals of the solutions to the underlying equation, like shock wave formation, intermittency and chaotic dynamics, can be obtained. We also demonstrate that very natural constraints for the lattice parameters are sufficient to ensure stable calculations for unlimited numbers of Monte-Carlo-steps.

Keywords: Monte-Carlo-Methods; Burgers' Equation; Shock Waves; Intermittency; Turbulence.

1. Introduction and Motivation

Hydrodynamic turbulence remains a basically unsolved problem of modern physics. This is especially noticeable as the fundamentals seem to be fairly easy — the Navier-Stokes-equations for the velocity and pressure fields v and p,

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha + \frac{1}{\rho}\partial_\alpha p = 0, \tag{1}$$

with the additional constraint

$$\partial_\alpha v_\alpha = 0, \tag{2}$$

express the conservation of momentum in a classical, newtonian, incompressible fluid of viscosity ν and density ρ. Laminar flows are reproduced very accurately; in the turbulent regime, it still is an open question how the universal characteristics of a flow, the scaling exponents ξ_p of the structure functions S_p of order p, defined by

$$S_p(x) := \langle [|v(r+x) - v(r))|^p \rangle_r \propto |x|^{\xi_p}, \tag{3}$$

can be extracted from the basic equations. Intermittency is reflected by exponents ξ_p that differ from those expected by dimensional analysis.
Monte-Carlo-simulations enable us to analyze turbulent flow patterns in detail, to gain direct insight into the formation of localized structures and their behaviour, and to measure observables like the scaling exponents.
Instead of working with the full Navier-Stokes-eqs., we decided to elaborate the methods using Burgers' eq.

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha = 0, \tag{4}$$

which can be interpreted as the flow equation for a fully compressible fluid. A finite viscosity ν and energy dissipation ϵ provide a dissipation length scale λ corresponding to the Kolmogorov-scale in Navier-Stokes-turbulence:

$$\lambda = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}. \tag{5}$$

Besides being of interest of its own (e.g. in cosmology), working with eq. (4) has a number of technical advantages:[a]

- Burgers' eq. is local, while the incompressibility condition acts as a nonlocal interaction in Navier-Stokes-turbulence.
- The fundamental solutions to Burgers' eq. are well-known; in the limit of vanishing viscosity (Hopf-eq.), these form singular shocks as seen in fig. (1). The dissipation-scale provides an UV-regularization of the shock structures.
- A huge variety of analytical methods have been applied to Burgers eq., giving results that can directly be compared to numerical measurements; and the origin of intermittency is well understood.

2. Path Integral

For the moment, we restrict our work to 1-dimensional Burgulence which already shows intermittent statistics. A random force f has to be introduced

[a]For a more complete overview, see[BK 07] .

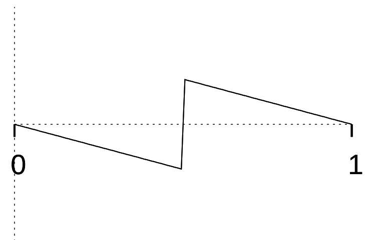

Fig. 1. Typical solution of Burgers' eq. in the limit of vanishing viscosity. The graph shows $v(x,t)$ as a function of x in periodic boundary conditions at constant t.

into eq. (4) so that solutions are statistically homogenous in time:

$$\mathbf{B}[v] \doteq \partial_t v + v\frac{\partial}{\partial x}v - \nu\frac{\partial^2}{\partial x^2}v = f, \tag{6}$$

where we model the stochastic force to be gaussian distributed with zero mean, energy injection rate ϵ and correlation length Λ:

$$\langle f(x,t)\rangle = 0, \tag{7}$$

$$\chi^{-1}(x,x';t,t') \doteq \langle f(x,t)f(x',t')\rangle = \epsilon\delta(t-t')\exp\left(-\frac{|x-x'|}{\Lambda}\right). \tag{8}$$

We then expect intermittent statistics to be found within the inertial subrange $\lambda \ll x \ll \Lambda$. We can further identify a characteristic velocity at the injection scale, $u_0 = (\epsilon\Lambda)^{1/3}$, and a Reynolds-number

$$Re = (\epsilon\Lambda^4/\nu^3)^{1/3}. \tag{9}$$

The path integral is introduced via the standard Martin-Siggia-Rose-formalism, giving for the generating functional[b]

$$Z[J] = \int \mathcal{D}v\,\mathcal{D}f\,\delta(\mathbf{B}[v]-f)\exp\left(-\frac{1}{2}\int f\chi f + \int Jv\right) \tag{10}$$

$$= \int \mathcal{D}v\, e^{-S[v;J]}, \tag{11}$$

with the action

$$S[v;J] = \int dxdtdx'dt'\ \left(\mathbf{B}[v(x,t)]\chi(x,x';t,t')\mathbf{B}[v(x',t')] + J(x,t)v(x,t)\right). \tag{12}$$

[b]The functional determinant can be shown not to contribute for local theories, see e.g.[HMPV 99] .

Beginning from an equivalent path integral, it has been shown that intermittent statistics of Burgers' eq. can be understood in terms of instanton solutions ([BFKL 97]).

3. Monte-Carlo-Simulations

We discretized the above action onto a rectangular lattice with L sites in space-, and T sites in time-direction. Derivatives have been written in a symmetric (Stratanovich-) prescription. Lattice spacings will be denoted Δx and Δt, respectively. We mainly used a heat bath algorithm on single nodes.

3.1. *Lattice parameters*

To indentify the lattice parameters with the constants of the continuum theory, we first notice that the viscosity has to be defined as[c]

$$\nu = \alpha \frac{(\Delta x)^2}{\Delta t}, \tag{13}$$

in which we define the arbitrary constant $\alpha = 1$. The so-defined ν gives us Re according to eq. (9). We further find that the dissipation length λ is related to the correlation length Λ simply by

$$\lambda = \frac{\Lambda}{Re^{3/4}}. \tag{14}$$

In any practical application, it is sufficient to define the η and Re, to chose L and Λ, and to calculate from that T and ϵ. Stability considerations lead to further constraints for the lattice size, as will be explained in the following subsection.

3.2. *Stability*

As would be expected, the stability of the simulations over a large number of Monte-Carlo-steps is a big issue, due to the shock-like solutions of eq. (4). Indeed, if certain restrictions to the lattice parameters are not taken care of rigorously, the simulation terminates sooner or later due to divergencies. It is interesting to notice that the occurence of instabilities in our MC-simulations is related to the (non-trivial) existence of the dissipation scale λ. We found that to obtain stable simulations, λ has to be resolved on the

[c] For example, this can be shown via the continuum limit of the symmetric random walk, leading to the diffusion eq..

lattice: $\lambda > \Delta x$. Unstable simultations occur otherwise — we observed that the overall energy of the configurations accumulate in the smallest scale Δx, causing the configuration to separate into two sub-configurations, of which one looks as the expected solution to the Hopf-eq., while the other grows beyond any limit, eventually breaking the simulation.
As long as the dissipation length is resolved, the simulations are stable. Having performed several millions of MC-steps, no further instabilities occured.
If length scales are measured in units of the system size, this translates into a constraint involving the Reynolds-number:

$$Re < \Lambda L, \tag{15}$$

which enforces huge lattices for high Re as $\Lambda \leq 1$.

3.3. *Configurations*

We simulated systems of different sizes, from $(L = 4) \times (T = 16)$ to larger lattices of the same viscosity, as 8×64 and 16×256, and also different viscosities, like 16×16 or 64×32. The Reynolds-number ranged from $Re = 0.01$ to $Re = 100$, not respecting the above conditions for stability. This may seem surprising, but we could see that interesting information could also be extracted from the physical sub-configurations. Stable runs have been obtained for $Re = 1$.
Long runs of several millions of MC-steps, on a single node, are realistic for small lattices like 4×16 or 8×64 only. For larger lattices, a parallelized version of the code will be needed.
We obtained the following results:

- Thermalization and autocorrelation times are very long, up to the order of 10^5 MC-steps.
- In the stable runs, after thermalization, the typical shock solutions of the Burgers' eq. form and can be observed moving and interacting with each other, see fig. (2).
- In the unstable runs before occurence of the instability, configurations resemble the kink-solutions of the Hopf-eq..
- The disctinction between stable and unstable simulations can directly be related to the existence of a dissipation length scale which is either bigger (stable) or smaller (unstable simulations) than the lattice spacing.

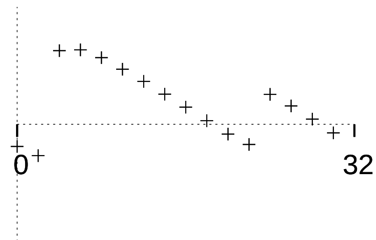

Fig. 2. Calculated configuration $v(x,t)$ of a $(L = 32) \times (T = 8)$-lattice; the plot shows a slize of constant t, periodic boundary conditions in x. Two shock-like structures are clearly visible-

4. Summary and Outlook

We have shown how to perform stable MC-simulations of stochastic partial differential eq., like Burgers' or the Hopf-eq.. The lattice versions of the theories can directly be identified with their continuum counterparts, and, as long as certain constraints on the lattice size are respected, unlimited numbers of configurations produced. Direct insight into the structures leading to intermittency and, thus, multiscaling, can be obtained. Especially, we want to point out that the existence of a dissipation length scale can be observed.
As next steps, complete statistics will be made; especially the scaling exponents of the structure functions have to be compared with the analytic results. Later, we will proceed by analyzing the incompressible Navier-Stokes-eqs..

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