

An analog of a modular functor from quantized Teichmüller theory

J. Teschner

Institut für theoretische Physik, Freie Universität Berlin,
Arnimallee 14, 14195 Berlin, Germany, and:

Max Planck Institut für Gravitationsphysik (Albert Einstein Institut)
Am Mühlenberg 1, D-14476 Potsdam, Germany

Contents

I	Introduction	3
1	Motivation	4
2	Aims	5
3	Overview	6
4	Outlook	8
II	Coordinates for the Teichmüller spaces	9
5	The Penner coordinates	9
5.1	Triangulations and fat graphs	9
5.2	Penner coordinates	10
5.3	Fock coordinates	11
6	The Ptolemy groupoids	13
6.1	Groupoids vs. complexes	13
6.2	Change of the triangulation	13
6.3	The representation of the mapping class group	15
7	Teichmüller space as the phase space of a constrained system	15
7.1	Kashaev's coordinates	15
	The constraints	16
	Change of fat graph	16

	2
7.2 The structure of the Kashaev space W_φ	17
Fock variables vs. Kashaev's variables	17
Splitting of W_φ	17
8 The Fenchel-Nielsen coordinates	18
8.1 Definition of the Fenchel-Nielsen coordinates	19
8.2 Symplectic structure	20
8.3 Geodesic lengths from the Penner coordinates	20
9 Coordinates for surfaces with holes of finite size	21
9.1 Useful fat graphs on surfaces with holes of finite size	22
9.2 Kashaev type coordinates	22
III Quantization of the Teichmüller spaces	24
10 Quantization of the Teichmüller spaces	24
10.1 Canonical quantization	24
10.2 Quantization of the Kashaev space W_φ	25
10.3 Reduction to the quantized Teichmüller spaces	26
11 Representations of the mapping class groups	28
11.1 Projective unitary representations of groupoids	29
11.2 Representations of symmetries of a groupoid	29
11.3 The projective representation of the Ptolemy groupoid on $\mathcal{K}(\varphi)$	31
11.4 Reduction to the quantized Teichmüller spaces	31
IV A stable modular functor from the quantum Teichmüller spaces	35
12 The notion of a stable unitary modular functor	35
12.1 Rigged Riemann surfaces	36
12.2 Stable unitary modular functors	36
12.3 Representations of the modular groupoid versus modular functors	38
13 The modular groupoid	40
13.1 Notations and conventions	41
13.2 Generators	41
13.3 Relations	41
14 From markings to fat graphs	44
14.1 The complex $\mathcal{M}^{\text{ad}}(\Sigma)$	45
14.2 Separated variables	47
14.3 Quantized Teichmüller spaces for surfaces with holes	48

	3
15 Geodesic length operators	49
15.1 Overview	49
15.2 Construction of the length operators	50
15.3 Spectrum	52
15.4 Relation with the Dehn twist generator	53
16 Passage to the length representation	54
16.1 The length representation	54
16.2 Construction of the operator $D(\sigma)$	55
17 Realization of $M(\Sigma)$	55
17.1 Two constructions for the generators	55
17.2 The main result	56
17.3 Concluding remarks	58
A The special functions $e_b(x)$ and $s_b(x)$	60
B Operator-theoretical background	61
C On the proof of Theorem 2	62
D Proof of Proposition 10	63
E On the proof of Theorem 6	65
E.1 A technical preliminary	65
E.2 Invariance of length operators	66

Part I

Introduction

The program of the quantization of the Teichmüller spaces $\mathcal{T}(\Sigma)$ of Riemann surfaces Σ which was started in [Fo, CF1] and independently in [Ka1] ¹ is motivated by certain problems and conjectures from mathematical physics. One of the main aims of this program is to construct a one-parameter family of maps

$$\Sigma \longrightarrow \left(\mathcal{H}_b^{\mathcal{T}}(\Sigma), \mathbf{O}_b^{\mathcal{T}}(\Sigma), M_b^{\mathcal{T}}(\Sigma) \right), \quad \text{where} \quad (0.1)$$

- (i) b is a deformation parameter, related to the traditional \hbar via $b = \sqrt{\hbar}$,
- (ii) Σ is a two-dimensional topological surface possibly with boundary,
- (iii) $\mathcal{H}_b^{\mathcal{T}}(\Sigma)$ is a Hilbert-space (possibly infinite-dimensional),
- (iv) $\mathbf{O}_b^{\mathcal{T}}(\Sigma)$ is an algebra of bounded operators on $\mathcal{H}_b^{\mathcal{T}}(\Sigma)$ and,
- (v) $M_b^{\mathcal{T}}(\Sigma)$ is a unitary projective representation of the mapping class group of Σ on $\mathcal{H}_b^{\mathcal{T}}(\Sigma)$.

¹See [FG, CP] for recent generalizations.

The data $(\mathbf{O}_b^\tau(\Sigma), \mathbf{M}_b^\tau(\Sigma))$ are restricted by the requirement that a suitably defined limit of $\mathbf{O}_b^\tau(\Sigma)$ for $b \rightarrow 0$ should reproduce the commutative algebra of functions on the Teichmüller space $\mathcal{T}(\Sigma)$, whereas a natural limit $b \rightarrow 0$ of the automorphisms of $\mathbf{O}_b^\tau(\Sigma)$ which are induced by the representation $\mathbf{M}_b^\tau(\Sigma)$ should correspond to the natural action of the mapping class group $\text{MC}(\Sigma)$ on $\mathcal{T}(\Sigma)$.

1. Motivation

Motivation for studying this problem comes from mathematical physics. A conjecture of H. Verlinde [V] can be formulated very schematically as the statement that

$$(\mathcal{H}_b^\tau(\Sigma), \mathbf{M}_b^\tau(\Sigma)) \simeq (\mathcal{H}_c^L(\Sigma), \mathbf{M}_c^L(\Sigma)), \quad \text{where} \quad (1.2)$$

- (i) $\mathcal{H}_c^L(\Sigma)$ has a definition in terms of the representation theory of the Virasoro algebra with central charge c as the so-called space of conformal blocks associated to Σ , and
- (ii) $\mathbf{M}_c^L(\Sigma)$ is an action of the mapping class group $\text{MC}(\Sigma)$ on $\mathcal{H}_c^L(\Sigma)$ which is canonically associated to the representation-theoretic definition of $\mathcal{H}_c^L(\Sigma)$.

Part of the interest in the space $\mathcal{H}_c^L(\Sigma)$ from the side of mathematical physics is due to the fact that the elements of $\mathcal{H}_c^L(\Sigma)$ represent the basic building blocks in the so-called Liouville conformal field theory [TL1]. Deep connections between the perturbative approach to quantum Liouville theory on the one hand and Teichmüller theory on the other hand have been exhibited by Takhtajan, Zograf and Teo see e.g. [TT] and references therein.

This conjecture may be seen as a non-compact analog of similar relations between the quantization of moduli spaces of flat connections on Riemann surfaces on the one hand, and rational conformal field theories on the other hand. For K being a compact group, the geometric quantization of the moduli space $\mathcal{M}_K(\Sigma)$ of flat K -connections on a Riemann surface was performed in [Hi2, ADPW]. Alternative approaches were based on more explicit descriptions of the symplectic structure on $\mathcal{M}_K(\Sigma)$ [FR, AGS, BR, AMR].² In either case one may schematically describe one of the main results of these constructions as an assignment

$$\Sigma \longrightarrow (\mathcal{H}_k^\mathcal{M}(\Sigma), \mathbf{M}_k^\mathcal{M}(\Sigma)), \quad \text{where} \quad (1.3)$$

- (i) $\mathcal{H}_k^\mathcal{M}(\Sigma)$ is a finite-dimensional vector space,
- (ii) $\mathbf{M}_k^\mathcal{M}(\Sigma)$ is a projective representation of the mapping class group of Σ on $\mathcal{H}_k^\mathcal{M}(\Sigma)$.

Part of the interest in these results was due to the close relations between the representation $\mathbf{M}_k^\mathcal{M}(\Sigma)$ and the Reshetikhin-Turaev invariants of three manifolds [RT]. Another source of interest were the relations to rational conformal field theory, which were predicted in [Wi], see

²The equivalence between the different quantization schemes has not been discussed in detail so far. It boils down to the verification that the monodromy representation of the KZ-connection constructed within the geometric quantization framework of [Hi2, ADPW] is equivalent to the mapping class group representation defined in [AGS, BR]. It seems that e.g. combining the results of [La] and [BK2] does the job.

[So] for a review of mathematical approaches to the problem and further references. These relations may, again schematically, be summarized as the existence of canonical isomorphisms

$$\begin{array}{ccc}
 & (\mathcal{H}_k^{\mathcal{M}}(\Sigma), M_k^{\mathcal{M}}(\Sigma)) & \\
 \nearrow \cong & & \nwarrow \cong \\
 (\mathcal{H}_k^{\mathcal{W}}(\Sigma), M_k^{\mathcal{W}}(\Sigma)) & \xleftarrow{\cong} & (\mathcal{H}_k^{\text{RT}}(\Sigma), M_k^{\text{RT}}(\Sigma))
 \end{array} \tag{1.4}$$

where

- (i) $\mathcal{H}_k^{\mathcal{W}}(\Sigma)$ is the space of conformal blocks in the WZNW-model associated to the compact group K , which can be defined in terms of the representation theory of the affine Lie algebra $\hat{\mathfrak{g}}_k$ with level k associated to the Lie algebra \mathfrak{g} of K ,
- (ii) $M_k^{\mathcal{W}}(\Sigma)$ is the natural action of the mapping class group on $\mathcal{H}_k^{\mathcal{W}}(\Sigma)$, which can be defined by means of the monodromy representation of the Knizhnik-Zamolodchikov connection,
- (iii) $\mathcal{H}_k^{\text{RT}}(\Sigma)$ is the space of invariants in certain tensor products of representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$,
- (iv) $M_k^{\text{RT}}(\Sigma)$ is the mapping class group representation on $\mathcal{H}_k^{\text{RT}}(\Sigma)$ defined by the construction of Reshetikhin-Turaev.

The quantization program (0.1) can be seen as a non-compact analog of (1.3) in the following sense. In (0.1) the role of the moduli space of flat connections $\mathcal{M}_K(\Sigma)$ is taken by the Teichmüller space $\mathcal{T}(\Sigma)$, which can be identified with the component in the moduli space $\mathcal{M}_G(\Sigma)$ of flat $G = SL(2, \mathbb{R})$ -connections that has maximal Euler class [Hi1, Go2]. Moreover, the natural symplectic structure on the moduli space of flat $SL(2, \mathbb{R})$ -connections restricts to the Weil-Petersson symplectic form on $\mathcal{T}(\Sigma)$ [Go1]. A quantization of the Teichmüller space $\mathcal{T}(\Sigma)$ may therefore be regarded as providing a quantization of a topological component in the moduli space $\mathcal{M}_G(\Sigma)$.

We expect that any non-compact counterpart of the developments mentioned above will be mathematically at least as rich as the already known results associated to compact groups K . In particular we expect that certain analogs of the constructions of Reshetikhin-Turaev and/or Turaev-Viro will capture information on the geometry of hyperbolic three manifolds, similar and probably related to the appearance of hyperbolic volumes in the asymptotic behavior of certain link invariants [KaV].

2. Aims

A major step towards establishing H. Verlinde's conjecture (1.2) is to show that the quantization of the Teichmüller spaces (0.1) as initiated in [Fo, CF1], [Ka1] produces an analog of a *modular functor*. The basic data of a modular functor are assignments such as (1.3), which are required to satisfy a natural set of axioms as discussed in Section 12. One of the most important implications of the axioms of a modular functor are simple relations between the representations of the mapping class groups associated to Σ and $\Sigma^{\dagger c}$ respectively, where $\Sigma^{\dagger c}$ is the surface that is obtained from Σ by cutting along a simple closed curve c . These relations imply that the

representation $M_k^{\mathcal{M}}(\Sigma)$ restricts to - and is generated by - the representations $M_k^{\mathcal{M}}(\Sigma')$ which are associated to those subsurfaces Σ' that can be obtained from Σ by cutting along a set of non-intersecting simple closed curves. This crucial *locality* property can be seen as the hard core of the notion of a modular functor.

Within the formalisms introduced in [Fo, CF1], [Ka1]-[Ka3] it is far from obvious that the quantization of Teichmüller spaces constructed there has such properties. To show that this is indeed the case is the main problem solved in this paper. The representation $M_b^{\mathcal{T}}(\Sigma)$ constructed and investigated in [Ka1]-[Ka3] is obtained by exploiting the fact that the mapping class group can be embedded into the so-called Ptolemy groupoid associated to the transformations between different triangulations of a Riemann surface Σ . A representation of the Ptolemy groupoid is constructed in [Ka1]-[Ka3], which then canonically induces a projective unitary representation of the mapping class group $MC(\Sigma)$. The simplicity of the Ptolemy groupoid, which underlies the elegance of the constructions in [Ka1]-[Ka3] now turns out to cause a major problem from the point of view of our aims, since the above-mentioned locality properties implied in the notion of a modular functor are not transparently realized by the Ptolemy groupoid.

Essentially our task is therefore to go from triangulations to pants decompositions, which is the type of decomposition of a Riemann surface Σ that is naturally associated to the concept of a modular functor. This requires to construct a change of representation for $\mathcal{H}_b^{\mathcal{T}}(\Sigma)$ from the one naturally associated to triangulations of Σ [Ka1]-[Ka3] to another one which is associated to pants decompositions. The main tool for doing this are the geodesic length operators introduced and studied in [Fo, CF2] and [Ka3, Ka4], which are the observables on the quantized Teichmüller spaces that are associated with the geodesic length functions³ on the classical Teichmüller spaces. The length operators associated to a maximal set of non-intersecting simple closed curves turn out to furnish a set of commuting self-adjoint operators, and the simultaneous diagonalization of these operators defines the sought-for change of representation.

There is a natural groupoid associated with the transformations between different pants decompositions. Of particular importance for us will be a certain refinement of this groupoid which will be called the modular groupoid $M(\Sigma)$. The modular groupoid $M(\Sigma)$ has been introduced for the study of rational conformal field theories by Moore and Seiberg in [MS], and it was further studied in particular in [BK1]. Constructing a modular functor is essentially equivalent to constructing a tower of representations of the modular groupoid. Our main aim in the present paper will be to show that the quantization of Teichmüller spaces allows one to construct a tower of representations of the modular groupoid by unitary operators in a natural way.

3. Overview

This paper has three main parts. The first of these parts collects the necessary results from the “classical” theory of Riemann surfaces. This includes a review of two types of coordinate systems for the Teichmüller spaces $\mathcal{T}(\Sigma)$, one of which is associated to triangulations of Σ , the other to pants decompositions. The coordinates associated to triangulations were first in-

³See [Wo1, Wo2, Wo3] for some classical work on the symplectic nature of the Fenchel-Nielsen coordinates which represents important background for our results.

roduced by Penner in [P1]. We will also need to discuss variants of these coordinates due to Fock [Fo] and Kashaev [Ka1] respectively. The changes of the underlying triangulation of Σ generate a groupoid, the Ptolemy groupoid $\text{Pt}(\Sigma)$, which has a useful representation in terms of generators and relations (Theorem 2).

The coordinates associated to pants decompositions are the classical Fenchel-Nielsen coordinates, which we review briefly in §8.1. We furthermore explain how the coordinates of Penner [P1] and Kashaev [Ka1], which were originally introduced to parameterize the Teichmüller spaces of surfaces Σ with punctures only, can be used to provide coordinates also for the case where the surface Σ has holes represented by geodesics of finite length.

The material in this part is mostly known, but it is scattered over many places in the literature, and some basic results were stated in the original references without a proof. We have therefore tried to give a reasonably self-contained and complete presentation of the relevant material, providing proofs where these are not available elsewhere.

The second part gives a largely self-contained presentation of the foundations of the quantization of Teichmüller spaces. Our presentation is heavily inspired by [Ka1]-[Ka3], but we deviate from these references in some important points. The treatment presented in this paper seems to be the first complete and mathematically rigorous formulation of the quantum theory of the Teichmüller spaces.

The main aims of this paper are finally achieved in the third part. We begin in §12 by introducing the notion of a stable unitary modular functor, and by explaining why having a stable unitary modular functor is equivalent to having a tower of unitary projective representations of the modular groupoid.

In §13 we will reformulate the main result of [MS, BK1] concerning the description of $M(\Sigma)$ in terms of generators and relations in a way that is convenient for us.

Of particular importance for us will be §14, where important first relations between certain subgroupoids of $M(\Sigma)$ and $\text{Pt}(\Sigma)$ are observed.

In §15 we define the geodesic length operators and establish their main properties. These results are of independent interest since some important properties of the geodesic length operators had not been proven in full generality before.

A key step in our constructions is taken in §16 by constructing a change of representation from the original one to a representation in which the length operators associated to a pants decomposition are simultaneously diagonalized. An important feature of this construction is the fact that the unitary operator which describes the change of representation factorizes into operators associated to the individual three holed spheres (trinions) which appear in a pants decomposition.

In §17 we construct the corresponding representation of the modular groupoid $M(\Sigma)$. The operators which represent $M(\Sigma)$ are constructed out of compositions of the representatives for the transformations in $\text{Pt}(\Sigma)$. This makes it relatively easy to verify the relations of $M(\Sigma)$, but the price to pay is that some crucial locality properties are more difficult to prove.

4. Outlook

In a sequel [TT2] to this paper we will calculate the matrix coefficients of the operators which generate the representation of the modular groupoid explicitly. A close relation to the modular double $\mathcal{DU}_q(sl(2, \mathbb{R}))$ of $\mathcal{U}_q(sl(2, \mathbb{R}))$ as defined and studied in [Fa, PT, BT] will be found.

It should be noted that $(\mathcal{H}_b^T(\Sigma), M_b^T(\Sigma))$ will not satisfy all the usual axioms of a modular functor, which require, in particular, that the vector space $\mathcal{V}(\Sigma)$ assigned to each Riemann surface should be finite-dimensional. Most importantly, however, the assignment $\Sigma \rightarrow (\mathcal{H}_b^T(\Sigma), M_b^T(\Sigma))$ was up to now only constructed for surfaces Σ which have at least one boundary component.

What will allow us to overcome this unsatisfactory feature are the remarkable analytic properties that the matrix coefficients of the operators which represent the modular groupoid will be shown to have. It turns out that the mapping class group representation $M_b^T(\Sigma)$ assigned to a Riemann surface with a boundary represented by geodesics of a fixed length depends *analytically* on the values of these lengths. The analytic properties of the matrix coefficients will furthermore allow us to “close a hole” by taking a limit where the length parameter assigned to this boundary component approaches a certain *imaginary* value. It will be shown in [TT2] that the resulting mapping class group representation is equivalent to the one on the surface which is obtained by gluing a disc into the relevant boundary component.

Concerning the representation theoretic side of H. Verlinde’s conjecture (1.2) it should be mentioned that a complete mathematical construction of $(\mathcal{H}_c^L(\Sigma), M_c^L(\Sigma))$ is not available so far, but nontrivial steps in the direction of constructing and describing $(\mathcal{H}_c^L(\Sigma), M_c^L(\Sigma))$ precisely have been taken in [TL2] in the case of surfaces Σ of genus zero. This includes in particular the derivation of explicit formulae for a set of basic data which characterize the resulting representation of the braid group uniquely.

The explicit computation of the matrix coefficients of the operators which generate the representation of the modular groupoid carried out in [TT2] will therefore allow us to verify H. Verlinde’s conjecture (1.2) in the case of Riemann surfaces of genus zero.

We furthermore expect that it should be possible to construct non-compact analogs of $(\mathcal{H}_k^{\text{RT}}(\Sigma), M_k^{\text{RT}}(\Sigma))$ based the non-compact quantum group $\mathcal{DU}_q(sl(2, \mathbb{R}))$, and thereby complete a non-compact analog of the triangle (1.4).

Acknowledgements

The author would like to thank L. Chekhov, V. Fock and especially R. Kashaev for useful discussions.

We furthermore gratefully acknowledge financial support from the DFG by a Heisenberg fellowship, as well as the kind hospitality of the Humboldt University Berlin, the School of Natural Sciences of the IAS Princeton, the Physics Department of Chicago University and the Caltech, Pasadena.

Part II

Coordinates for the Teichmüller spaces

We will consider two-dimensional surfaces Σ with genus $g \geq 0$ and $s \geq 1$ boundary components such that

$$M \equiv 2g - 2 + s > 0.$$

On Σ we will consider metrics of constant negative curvature -1 . Our main interest will be the case where the boundary components can be represented by geodesics of finite length. Such boundary components will also be called holes in the following. However, to begin with we will focus on the case where the boundary components are *punctures*, i.e. holes of vanishing geodesic circumference.

The space of deformations of the metrics of constant negative curvature is called the Teichmüller space $\mathcal{T}(\Sigma)$. It will be of basic importance for us to have useful systems of coordinates for $\mathcal{T}(\Sigma)$.

We will consider two classes of coordinate systems which are associated to two types of graphs drawn on the Riemann surfaces respectively. The first class of coordinates goes back to Penner and is associated to triangulations of the Riemann surface or the corresponding dual graphs, the so-called fat graphs. We will also describe two useful variants of the Penner coordinates due to Kashaev and Fock respectively.

The second class of coordinates are the classical Fenchel-Nielsen length-twist coordinates. One may view them as being associated to a second type of graph on a surface Σ called *marking* that determines in particular a decomposition of the surface into three-holed spheres (trinions).

In the following first part of this paper we shall describe these coordinate systems in some detail, discuss the graphs on Σ that these coordinates are associated to, as well as the groupoids generated by the transformations between different choices of these graphs.

5. The Penner coordinates

5.1 Triangulations and fat graphs

Consider a fixed oriented topological surface Σ of genus g with $s \geq 1$ punctures. An ideal triangulation τ of Σ is the isotopy class of a collection of disjointly embedded arcs in Σ running between the punctures such that τ decomposes Σ into triangles. There are $2M$ triangles and $3M$ edges for any ideal triangulation. As an example we have drawn a triangulation of the once-punctured torus in Figure 1.

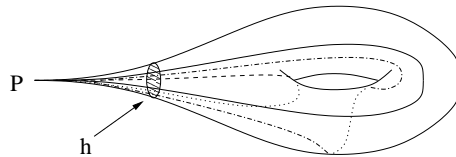


Figure 1: Triangulation of the once-punctured torus.

The graph dual to a triangulation is a trivalent fat graph i.e. a trivalent graph embedded in the surface with fixed cyclic order of the edges incident to each vertex. An example for a fat graph on the once-punctured torus is depicted in Figure 2. The sets of vertices and edges of a

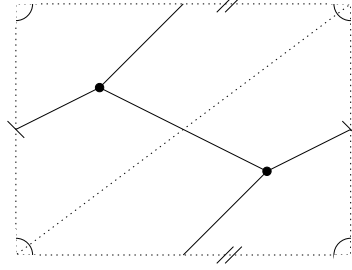


Figure 2: Another representation of the triangulation from Figure 1 and the dual fat graph.

fat graph φ will be denoted φ_o and φ_1 respectively.

An ideal triangulation is called *decorated* if the triangles are numbered and if a corner is marked for each triangle [Ka1]. The decoration of the triangle t_v dual to a vertex $v \in \varphi_o$ can be used to fix a numbering convention for the edges e_i^v , $i = 1, 2, 3$ which emanate from v as defined in Figure 3.

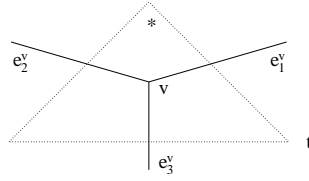


Figure 3: Graphical representation of the vertex v dual to a triangle t . The marked corner defines a corresponding numbering of the edges that emanate at v .

Remark 1. Decorated ideal triangulations are dual to decorated fat graphs, which means that the vertices are numbered, and for each vertex $v \in \varphi_o$ one has chosen a distinguished edge $e_v \in \varphi_1$. As a convention we will assume that fat graphs always carry such a decoration unless otherwise stated.

5.2 Penner coordinates

It turns out to be useful to consider a somewhat enlarged object which keeps track of the choices of horocycles around each of the punctures, the so-called decorated Teichmüller space $\tilde{\mathcal{T}}(\Sigma)$. $\tilde{\mathcal{T}}(\Sigma)$ is defined as a principal \mathbb{R}_+^s -bundle over $\mathcal{T}(\Sigma)$ by taking the s -tuple of horocycles around each of the punctures as the fiber over a point of $\mathcal{T}(\Sigma)$. The ordered s -tuple of hyperbolic lengths of the horocycles gives coordinates for the fibers.

Given any point P in the decorated Teichmüller space $\tilde{\mathcal{T}}(\Sigma)$ and an ideal triangulation τ of Σ , Penner assigns a coordinate value to each of the edges in τ by means of the following construction. By means of Fuchsian uniformization one may equip the surface Σ with a unique

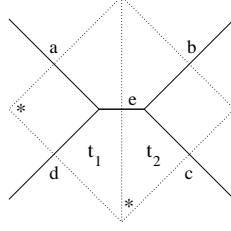


Figure 4: Two adjacent triangles and the dual fat graph.

hyperbolic metric g associated to our chosen point $P \in \tilde{\mathcal{T}}(\Sigma)$. Let τ_1 be the set of edges of a triangulation τ . Each edge e in τ_1 may be straightened to a geodesic for the hyperbolic metric g . The coordinate $l_e(P)$ is defined as the hyperbolic length of the segment of e that lies between the two horocycles surrounding the punctures that e connects, taken with positive sign if the two horocycles are disjoint, with negative sign otherwise. We are going to consider the tuple $(l_e)_{e \in \tau_1}$ as a vector in the vector space \mathbb{R}^{τ_1} of dimension $3M$.

Theorem 1. Penner [P1][P2]

(a) For any fixed ideal triangulation τ of Σ , the function

$$l : \tilde{\mathcal{T}}(\Sigma) \rightarrow \mathbb{R}^{\tau_1}, \quad P \rightarrow (l_e(P))_{e \in \tau_1}$$

is a homeomorphism.

(b) The pull-back of the Weil-Petersson two-form ω on $\mathcal{T}(\Sigma)$ is given by the expression

$$\omega = - \sum_{t \in \tau_2} (dl_{e_1(t)} \wedge dl_{e_2(t)} + dl_{e_2(t)} \wedge dl_{e_3(t)} + dl_{e_3(t)} \wedge dl_{e_1(t)}),$$

where the summation is extended over the set τ_2 of triangles of τ , and $e_i(t)$, $i = 1, 2, 3$ are the edges bounding the triangle t , labelled in the counter-clockwise sense.

The Teichmüller space $\mathcal{T}(\Sigma)$ itself can finally be described as the space of orbits in $\tilde{\mathcal{T}}(\Sigma)$ under the following symmetry. Choose a number $d(\mathfrak{p})$ for each puncture \mathfrak{p} . Let the action of the symmetry be defined by

$$l'_e \equiv l_e + d(\mathfrak{p}) + d(\mathfrak{p}') \quad (5.1)$$

if the edge e connects the punctures \mathfrak{p} and \mathfrak{p}' .

5.3 Fock coordinates

There is a useful variant of the Penner coordinates which was introduced by V. Fock in [Fo]. In terms of the Penner coordinates one may define the Fock coordinates in terms of certain cross-ratios. Given a quadrilateral formed by two adjacent triangles we are going to keep the labelling of the edges introduced in Figure 4. Let

$$z_e = l_a + l_c - l_b - l_d. \quad (5.2)$$

The dependence of the Penner coordinates on the choice of horocycles drops out in the Fock coordinates. However, the variables z_e assigned to the $3M$ edges in a triangulation are not all

independent. To describe the relations that they satisfy it is convenient to think of the Fock coordinates as being assigned to the edges of the fat graph dual to the given triangulation. Each closed curve c on Σ is homotopic to a unique path g_c on the fat graph which has minimal length w.r.t. the metric defined by assigning each edge of φ the length one. Such paths will also be called graph geodesics in the following. The path g_c may be described by a sequence of edges $e_1^c, \dots, e_{n_c}^c \in \varphi_1$. To a closed curve c let us associate

$$f_{\varphi, c} \equiv \sum_{i=1}^{n_c} z_{e_i^c}. \quad (5.3)$$

The definition (5.2) then implies the relations

$$f_{\varphi, c} = 0 \quad (5.4)$$

for any closed curve c that encircles one puncture only. If one uses the equations (5.4) to express s of the variables z_e in terms of the others one obtains a set of coordinates for $\mathcal{T}(\Sigma)$.

On the $3M$ -dimensional space F_φ that is spanned by the coordinate functions $z_e(P)$ one may define a Poisson bracket Ω_{WP} which is such that

- (i) the elements f_a , $a = 1, \dots, s$ span the set C_φ of all $\mathfrak{c} \in F_\varphi$ such that

$$\Omega_{\text{WP}}(\mathfrak{c}, \mathfrak{w}) = 0 \quad \text{for all } \mathfrak{w} \in F_\varphi, \quad (5.5)$$

- (ii) the Poisson bracket which is induced by Ω_{WP} on the quotient F_φ/C_φ coincides with the Poisson bracket which corresponds to the Weil-Petersson symplectic form on $\mathcal{T}(\Sigma)$.

There exists a rather simple description for this Poisson-bracket [Fo]:

$$\Omega_{\text{WP}}(z_e, z_{e'}) = n_{e, e'}, \quad \text{where } n_{e, e'} \in \{-2, -1, 0, 1, 2\}. \quad (5.6)$$

The value of $n_{e, e'}$ depends on how edges e and e' are embedded into a given fat graph. If e and e' don't have a common vertex at their ends, or if one of e, e' starts and ends at the same vertex then $n_{e, e'} = 0$. In the case that e and e' meet at two vertices one has $n_{e, e'} = 2$ (resp. $n_{e, e'} = -2$) if e' is the first edge to the right⁴ (resp. left) of e at both vertices, and $n_{e, e'} = 0$ otherwise. In all the remaining cases $n_{e, e'} = 1$ (resp. $n_{e, e'} = -1$) if e' is the first edge to the right (resp. left) of e at the common vertex.

The coordinates z_e also have a nice geometrical meaning [Fo]. In the Fuchsian uniformization the two triangles that share the common edge e will be mapped into ideal hyperbolic triangles in the upper half plane. The edges are then represented by half-circles, and the corners will be at points x_1, \dots, x_4 on the real line, see Figure 5.

We then have

$$\exp(z_e) = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_3)(x_2 - x_1)}. \quad (5.7)$$

By means of M'obius-transformations $x_i \rightarrow \frac{ax_i + b}{cx_i + d}$ one may map the corners of one of the two triangles to $-1, 0$ and ∞ respectively. The variable z_e , being expressed in terms of the M'obius-invariant cross-ratio therefore parameterizes the different ways of gluing two ideal hyperbolic

⁴w.r.t. to the orientation induced by the embedding of the fat-graph into the surface

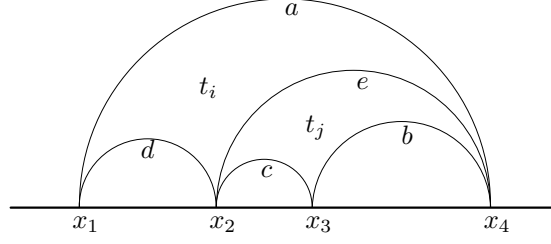


Figure 5: Representation of the triangles t_i and t_j in the upper half plane.

triangles along a common edge modulo Möbius-transformations. Given the variables z_e one may reconstruct the Riemann surface as represented in the Fuchsian uniformization by successively mapping ideal hyperbolic triangles into the upper half-plane, glued along the edges e in the way prescribed by the given value z_e [Fo].

6. The Ptolemy groupoids

In the previous section we had associated coordinate systems to fat graphs on a surface Σ : A change of graph will of course induce a change of coordinates. The groupoid generated by the moves between different fat graphs will be the subject of the present section.

6.1 Groupoids vs. complexes

The groupoids that we will be interested in can be conveniently described as 2-dimensional connected CW complexes \mathcal{G} . The set of vertices \mathcal{G}_0 of \mathcal{G} will be represented by certain sets of graphs, within this section called fat graphs. The (directed) edges $E \in \mathcal{G}_1$ that connect these vertices correspond to the generators (“elementary moves”) of the groupoid, while the faces $F \in \mathcal{G}_2$ of \mathcal{G} yield the relations.

The groupoid G associated to the 2-dimensional connected CW complexes \mathcal{G} will then simply be the path groupoid of \mathcal{G} , which has the vertices in \mathcal{G}_0 as objects and the homotopy classes of edge paths between two vertices as morphisms. The homotopy class of paths leading from vertex $V_1 \in \mathcal{G}_0$ to vertex $V_2 \in \mathcal{G}_0$ will be denoted by $[V_2, V_1]$. Similarly we will sometimes use the notation $[W_E, V_E]$ for the element of G which corresponds to an edge $E \in \mathcal{G}_1$.

A path π which represents an element in the homotopy class $[W, V]$ may be represented by a *chain* of edges $E \in \mathcal{G}_1$, i.e. an ordered sequence $(E_{\pi, n(\pi)}, \dots, E_{\pi, 1})$, $E_{\pi, j} \in \mathcal{G}_1$ for $j = 1, \dots, n(\pi)$ such that $E_{\pi, j} \in [V_{\pi, j+1}, V_{\pi, j}]$ for $j = 1, \dots, n(\pi) - 1$, and $V_{\pi, 1} = V$, $V_{\pi, n(\pi)} = W$. We will also use the suggestive notation $E_n \circ E_{n-1} \circ \dots \circ E_1$ to denote a chain.

6.2 Change of the triangulation

In the case of the Ptolemy groupoids $\text{Pt}(\Sigma)$ we will consider a complex $\mathcal{Pt}(\Sigma)$, where the set $\mathcal{G}_0 = \mathcal{Pt}_0(\Sigma)$ is defined to be the set of fat graphs on Σ . Let us furthermore define $\mathcal{Pt}_1(\Sigma)$ to consist of the following elementary moves.

- (i) *Permutation* (vw) : Exchanges the labels of the vertices v and w .
- (ii) *Rotation* ρ_v : See Figure 6.

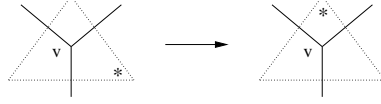


Figure 6: Transformation ρ_v changes the marked corner of the triangle dual to a vertex $v \in \varphi_\circ$.

(iii) *Flip* ω_{vw} : See Figure 7.

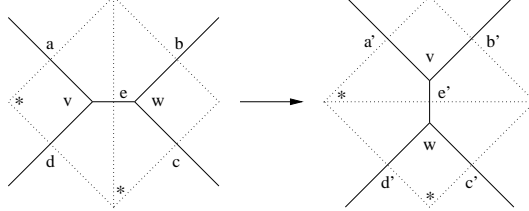


Figure 7: The flip transformation ω_{vw} changes the diagonal in the quadrilateral formed by the two adjacent triangles t_v and t_w .

Proposition 1. — [P1, Proposition 7.1][Ka3] —

The complex $\mathcal{Pt}(\Sigma)$ is connected, i.e. for a given surface Σ , any two fat graphs φ and φ' can be connected by a chain of elementary transformations.

Validity of the following relations in $\mathcal{Pt}(\Sigma)$ can easily be verified pictorially [Ka3].

$$\begin{aligned}
 \rho_v \circ \rho_v \circ \rho_v &= id, \\
 \omega_{u_1 u_2} \circ \omega_{u_3 u_4} &= \omega_{u_3 u_4} \circ \omega_{u_1 u_2}, \quad u_r \neq u_s \text{ for } r \neq s, \\
 \omega_{vw} \circ \omega_{uw} \circ \omega_{uv} &= \omega_{uv} \circ \omega_{vw}, \\
 (\rho_v^{-1} \times \rho_w) \circ \omega_{vw} &= \omega_{vw} \circ (\rho_v^{-1} \times \rho_w), \\
 \omega_{wv} \circ \rho_v \circ \omega_{vw} &= (vw) \circ (\rho_v \times \rho_w).
 \end{aligned} \tag{6.1}$$

Theorem 2. The complex $\mathcal{Pt}(\Sigma)$ is simply connected, i.e. any relation between the generators (vw) , ρ_v and ω_{vw} of the Ptolemy groupoid is a consequence of the relations (6.1) together with the relations of the permutation group.

The proof of this theorem is explained in Appendix C.

One of the main virtues of the Penner coordinates is that the corresponding change of coordinates can be described rather simply.

Lemma 1. — Lemma A.1a of [P2] —

Let τ' be the triangulation obtained by applying the flip of Figure 7 to a pair of adjacent triangles in a given initial triangulation τ , and denote e and e' the diagonal edge before and after the flip. The coordinates associated to τ and τ' will then agree for each edge that the two triangulations have in common, and

$$\lambda_{e'} = \frac{1}{\lambda_e} (\lambda_a \lambda_c + \lambda_b \lambda_d), \quad \lambda_f \equiv \sqrt{2} \exp(\frac{1}{2} l_f), \quad \forall f \in \varphi_1, \tag{6.2}$$

where we have labelled the edges according to Figure 7.

The corresponding transformation of the Fock variables is also easy to describe:

$$\begin{aligned} e^{-z_{a'}} &= e^{-z_a}(1 + e^{-z_e}) & z_{e'} &= -z_e & e^{-z_{b'}} &= e^{+z_b}(1 + e^{+z_e}) \\ e^{+z_{d'}} &= e^{+z_d}(1 + e^{+z_e}) & & & e^{+z_{c'}} &= e^{-z_c}(1 + e^{-z_e}) \end{aligned} \quad (6.3)$$

6.3 The representation of the mapping class group

The mapping class group $\text{MC}(\Sigma)$ of the topological surface Σ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ . An element $\mu \in \text{MC}(\Sigma)$ of the mapping class group will map a given (decorated) triangulation τ into another one, $\mu.\tau$. The fact that any two triangulations can be mapped into each other by a composition of the elementary transformations introduced in subsection 6.2 therefore leads to an embedding of the mapping class group into the Ptolemy groupoid:

$$\Phi_\tau : \text{MC}(\Sigma) \rightarrow \text{Pt}(\Sigma), \quad \Phi_\tau(f) \equiv [\mu.\tau, \tau]. \quad (6.4)$$

More precisely, Φ_τ induces a homomorphism $\text{MC}(\Sigma) \rightarrow \text{Pt}(\Sigma)$ in the sense that

$$\Phi_\tau(\mu_2 \circ \mu_1) = \Phi_{\mu_1.\tau}(\mu_2) \circ \Phi_\tau(\mu_1) \text{ for any } \mu_2, \mu_1 \in \text{MC}(\Sigma),$$

which embeds $\text{MC}(\Sigma)$ *injectively* into $\text{Pt}(\Sigma)$ [P3, Theorem 1.3].

7. Teichmüller space as the phase space of a constrained system

As a preparation for the description of the quantum Teichmüller spaces it will be useful to parameterize the Teichmüller spaces by means of variables assigned to the *triangles* instead of the *edges* of a triangulation [Ka1]. In the following section we shall elaborate upon the results and constructions in [Ka1], strengthening them somewhat.

7.1 Kashaev's coordinates

Assume given a fat graph φ with set of vertices φ_\circ . For each vertex $v \in \varphi_\circ$ one may introduce a pair of variables (q_v, p_v) according to the following rule. Let us label the edges that emanate from the vertex v by $e_i^v, i = 1, 2, 3$ according to Figure 3. We will denote the Penner coordinates associated to the edges e_i^v by $l_i^v, i = 1, 2, 3$. Let us then define the pair of variables (q_v, p_v) as

$$(q_v, p_v) = (l_3^v - l_2^v, l_1^v - l_2^v). \quad (7.1)$$

Following Kashaev [Ka1] we will consider the vector space $V_\varphi \simeq \mathbb{R}^{4M}$ obtained by regarding the variables q_v, p_v as the components $q_v(\mathfrak{v}), p_v(\mathfrak{v})$ of vectors $\mathfrak{v} \in V_\varphi$. The space of linear coordinate functions on V_φ will be called the Kashaev space W_φ . On W_φ we will consider the Poisson bracket Ω_φ defined by

$$\Omega_\varphi(p_v, q_w) = \delta_{vw}, \quad \Omega_\varphi(q_v, q_w) = 0, \quad \Omega_\varphi(p_v, p_w) = 0. \quad (7.2)$$

The assignment (7.1) associates a vector $\mathfrak{v}(P)$ in a subspace $T_\varphi \subset V_\varphi$ to each point $P \in \tilde{\mathcal{T}}(\Sigma)$. Kashaev has observed that the subspace T_φ can be characterized by a suitable set of linear forms $h_c \in W_\varphi$ (“constraints”).

The constraints

To define the linear forms h_c let us introduce an embedding of the first homology $H_1(\Sigma, \mathbb{R})$ into W_φ as follows. Each graph geodesic g_c which represents an element $[c] \in H_1(\Sigma, \mathbb{R})$ may be described by an ordered sequence of vertices $v_i \in \varphi_\circ$, and edges $e_i \in \varphi_\bullet$, $i = 0, \dots, n$, where $v_0 = v_n$, $e_0 = e_n$, and we assume that v_{i-1} , v_i are connected by the single edge e_i . We will define $\omega_i = 1$ if the arcs connecting e_i and e_{i+1} turn around the vertex v_i in the counterclockwise sense, $\omega_i = -1$ otherwise. The edges emanating from v_i will be numbered e_j^i , $j = 1, 2, 3$ according to the convention introduced in Figure 3. To each $[c] \in H_1(\Sigma, \mathbb{R})$ we will assign

$$h_c \equiv \sum_{i=1}^n u_i, \quad u_i := \omega_i \begin{cases} -q_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_3^i, e_1^i\}, \\ p_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_2^i, e_3^i\}, \\ q_{v_i} - p_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_1^i, e_2^i\}. \end{cases} \quad (7.3)$$

h_c is independent of the choice of representative c within the class $[c]$. Let C_φ be the subspace in W_φ that is spanned by the h_c , $[c] \in H_1(\Sigma, \mathbb{R})$.

Lemma 2. — [Ka1] —

(i) The mapping $H_1(\Sigma, \mathbb{R}) \ni [c] \mapsto h_c \in C_\varphi$ is an isomorphism of vector spaces.

(ii) The restriction of Ω_φ to C_φ coincides with the intersection form I on $H_1(\Sigma, \mathbb{R})$,

$$\Omega_\varphi(h_{c_1}, h_{c_2}) = I(c_1, c_2).$$

(iii) The linear forms h_c , $[c] \in H_1(\Sigma, \mathbb{R})$ vanish identically on the subspace T_φ .

The equations $h_c(\mathbf{v}) = 0$, $[c] \in H_1(\Sigma, \mathbb{R})$ characterize the image of $\tilde{\mathcal{T}}(\Sigma)$ within V_φ . It is useful to recall that $H_1(\Sigma, \mathbb{R})$ splits as $H_1(\Sigma, \mathbb{R}) = H_1(\Sigma_{\text{cl}}, \mathbb{R}) \oplus B(\Sigma)$, where $B(\Sigma)$ is the $s - 1$ -dimensional subspace spanned by the homology classes associated to the punctures of Σ , and Σ_{cl} is the compact Riemann surface which is obtained by “filling” the punctures of Σ . The corresponding splitting of C_φ will be written as $C_\varphi = H_\varphi \oplus B_\varphi$.

Change of fat graph

In order to describe the change of Kashaev variables induced by a change of fat graph let us, following [Ka1], define the following two transformations associated to the elementary moves ω_{vw} and ρ_v respectively.

$$A_v : (q_v, p_v) \mapsto (p_v - q_v, -q_v), \quad (7.4)$$

$$T_{vw} : \begin{cases} (U_v, V_v) \mapsto (U_v U_w, U_v V_w + V_v), \\ (U_w, V_w) \mapsto (U_w V_v (U_v V_w + V_v)^{-1}, V_w (U_v V_w + V_v)^{-1}), \end{cases} \quad (7.5)$$

where we have set $U_v \equiv e^{q_v}$ and $V_v \equiv e^{p_v}$ for all $v \in \varphi_\circ$.

Lemma 3. The maps $A_v : W_\varphi \rightarrow W_{\rho_v \circ \varphi}$ and $T_{vw} : W_\varphi \rightarrow W_{\omega_{vw} \circ \varphi}$ defined in (7.4) and (7.5) respectively are canonical, i.e. they preserve the Poisson structure Ω_φ .

The proof is again straightforward. Lemma 3 implies in particular that the mapping class group acts on W_φ by *canonical* transformations.

7.2 The structure of the Kashaev space W_φ

Fock variables vs. Kashaev's variables

There is a canonical way to reconstruct the Fock-variables in terms of Kashaev's variables which is found by combining equations (5.2) and (7.1). The result may be formulated as follows. Let $v, w \in \varphi_\circ$ be the vertices that are connected by the edge $e \in \varphi_1$, and let e_i^v , $i = 1, 2, 3$ be the edges introduced in Figure 3.

$$\hat{z}_e = \hat{z}_{e,v} + \hat{z}_{e,w}, \quad \hat{z}_{e,v} = \begin{cases} p_v & \text{if } e = e_1^v, \\ -q_v & \text{if } e = e_2^v, \\ q_v - p_v & \text{if } e = e_3^v. \end{cases} \quad (7.6)$$

The definition (7.6) defines a linear map $I_\varphi : F_\varphi \rightarrow \hat{F}_\varphi \subset W_\varphi$. It will be useful to describe the properties of this map a bit more precisely.

Lemma 4.

- (i) $\hat{z}_e(\mathbf{v}(P)) = z_e(P) \quad \forall e \in \varphi_1, \quad \forall P \in \tilde{T}(\Sigma),$
- (ii) $\Omega_\varphi(\hat{z}_e, \hat{z}_f) = \Omega_{\text{WP}}(z_e, z_f) \quad \forall e, f \in \varphi_1,$
- (iii) $\Omega_\varphi(\hat{z}_e, h_c) = 0 \quad \forall e \in \varphi_1, \quad \forall c \in H_1(\Sigma, \mathbb{R}),$
- (iv) $\hat{f}_c \equiv I_\varphi(f_c) = h_c, \quad \forall [c] \in B(\Sigma).$

Proof. Straightforward verifications. □

It is also useful to remark that the transformation of the Fock variables \hat{z}_e , $e \in \varphi_1$ that is induced by (7.4), (7.5) coincides with (6.3).

Splitting of W_φ

The linear forms $h_c \in B_\varphi$ turn out to be the Hamiltonian generators for the symmetry (5.1) [Ka1]. It is therefore natural to consider the subspace $M_\varphi \subset W_\varphi$ which is spanned by the Hamiltonian vector fields that are generated by the linear forms $h_c \in B_\varphi$, as well as $N_\varphi \equiv M_\varphi \oplus B_\varphi$.

Proposition 2. *There exists a canonical transformation establishing the isomorphism of Poisson vector spaces $W_\varphi \simeq T_\varphi \oplus N_\varphi \oplus H_\varphi$, such that*

- (i) $T_\varphi \simeq T'(\Sigma)$ is the space of linear functions on the Teichmüller space $T(\Sigma)$,
- (ii) The restriction of Ω_φ to T_φ coincides with the Poisson bracket induced by the Weil-Petersson symplectic form.

Proof. As a warmup it may be instructive to count dimensions: We have $\dim(W_\varphi) = 8g - 8 + 4s$ and $\dim(C_\varphi) = \dim(H_1(\Sigma, \mathbb{R})) = 2g + s - 1$. In order to determine $\dim(N_\varphi)$ let us choose a canonical basis for $H_1(\Sigma, \mathbb{R})$, represented by curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_{s-1}$ such that the only nontrivial intersection pairings are $I(\alpha_i, \beta_j) = \delta_{ij}$. N_φ is spanned by the images

of the classes $[\gamma_1], \dots, [\gamma_{s-1}]$, together with the Hamiltonian vector fields that they generate. It follows that $\dim(N_\varphi) = 2s - 2$.

The main point that remains to be demonstrated is the existence of a decomposition of \widehat{F}_φ as the direct sum

$$\widehat{F}_\varphi = T_\varphi \oplus B_\varphi \quad \text{such that} \quad \Omega_\varphi(t, h) = 0 \quad \forall t \in T_\varphi, \quad \forall h \in N_\varphi. \quad (7.7)$$

Thanks to Lemma 3 we may choose a convenient fat graph to this aim. Let us pick a basis $\mathfrak{B}(\Sigma)$ for $B(\Sigma)$ represented by the curves which encircle $s - 1$ of the s punctures. It is easy to see that we can always construct a fat graph φ' such that the elements of $\mathfrak{B}(\Sigma)$ are represented by single edges in φ'_1 . These edges end in a unique vertex $v(c)$ for each $c \in \mathfrak{B}(\Sigma)$. It is clear that the expression for h_c only involves the variables $(q_{v(c)}, p_{v(c)})$ for all $c \in \mathfrak{B}(\Sigma)$. It follows from (7.2) that the Hamiltonian vector field generated by h_c can likewise be expressed in terms of $(q_{v(c)}, p_{v(c)})$ only. The existence of the sought-for decomposition (7.7) is obvious in this case.

The result is carried over to the general case with the help of Lemma 3. It is clear that the subspace $T_\varphi \subset \widehat{F}_\varphi$ is defined uniquely by the condition (7.7). T_φ may then also be described as the quotient of \widehat{F}_φ by the conditions $h_c = 0$ for all $[c] \in B(\Sigma)$. It therefore follows from our discussion in §5.3 that T_φ is canonically isomorphic to $\mathcal{T}'(\Sigma)$, the vector space of linear functions on $\mathcal{T}(\Sigma)$.

To complete the proof it remains to observe that we have

$$\begin{aligned} \text{(i)} \quad & \Omega_\varphi(t, h) = 0 \quad \forall t \in T_\varphi, \quad \forall h \in H_\varphi, \\ \text{(ii)} \quad & \Omega_\varphi(h, n) = 0 \quad \forall h \in H_\varphi, \quad \forall n \in N_\varphi. \end{aligned} \quad (7.8)$$

(i) follows directly from Lemma 4, whereas part (ii) can easily be verified by considering the fat graph φ' above. \square

8. The Fenchel-Nielsen coordinates

We will now be interested in the case of Riemann surfaces with a boundary that is represented by a collection of $s > 0$ geodesics. Another standard set of coordinates for the Teichmüller spaces is associated to the decomposition of a Riemann surface into trinions (three-holed spheres). We are now going to review the definition of these coordinates. Different sets of Fenchel-Nielsen coordinates will be associated to different *markings* of the Riemann surface in a way which is analogous to the relation between the Penner coordinates and fat graphs.

Let us denote by S_3 the sphere with three holes (trinion). As a concrete model we may e.g. choose

$$S_3 \equiv \{z \in \mathbb{C}; |z| \geq \epsilon, |1 - z| \geq \epsilon, |z| < 1/\epsilon\}. \quad (8.1)$$

Any trinion with a smooth boundary is diffeomorphic to S_3 .

Definition 1. A marking σ of a surface Σ consists of the following data.

- (i) A cut system \mathcal{C}_σ , which is a set $\mathcal{C}_\sigma = \{c_1, \dots, c_{3g-3+s}\}$ of simple non-intersecting oriented closed curves c_i on Σ . Cutting Σ along the curves in \mathcal{C} decomposes the surface into a collection \mathcal{P}_σ of trinions.

(ii) A choice of a trivalent graph Γ_T with a single vertex v_T in each trinion $T \in \mathcal{P}_\sigma$ such that the graphs on the different trinions glue to a connected graph Γ_σ on Σ .

(iii) A choice of a distinguished boundary component for each trinion $T \in \mathcal{P}_\sigma$.

These data will be considered up to isotopy.

An example for the graphs Γ_T is depicted in Figure 10.

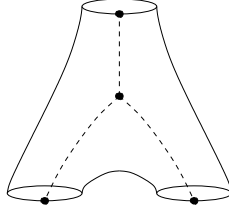


Figure 8: A trinion equipped with a marking graph

8.1 Definition of the Fenchel-Nielsen coordinates

The basic observation underlying the definition of the Fenchel-Nielsen twist coordinates is the fact that for each triple (l_1, l_2, l_3) of positive real numbers there is a *unique* metric of constant curvature -1 on the three-holed sphere (trinion) such that the boundary components are geodesics with lengths l_i , $i = 1, 2, 3$. A trinion with its metric of constant curvature -1 will be called hyperbolic trinion. There furthermore exist three distinguished geodesics on each hyperbolic trinion that connect the boundary components pairwise.

Let us assume that the geodesic c separates two trinions T_a and T_b . Pick boundary components c_a and c_b of T_a and T_b respectively by starting at c , following the marking graphs, and turning left at the vertices. As mentioned above, there exist distinguished geodesics on T_a and T_b that connect c with c_a and c_b respectively. Let δ_c be the signed geodesic distance between the end-points of these geodesics on c , and let

$$\theta_c = 2\pi \frac{\delta_c}{l_c} \quad (8.2)$$

be the corresponding twist-angle. In a similar way one may define θ in the case that cutting along c opens a handle.

Given a cut system $\mathcal{C} = \{c_1, \dots, c_\kappa\}$, $\kappa = 3g - 3 + s$, we thereby associate to each Riemann surface Σ a tuple $(l_1, \dots, l_\kappa; \theta_1, \dots, \theta_\kappa)$ of real numbers. It can be shown (see e.g. [IT]) that the Riemann surface Σ is characterized uniquely by the tuple $(l_1, \dots, l_\kappa; e^{i\theta_1}, \dots, e^{i\theta_\kappa}) \in (\mathbb{R}^+)^{\kappa} \times (S^1)^{\kappa}$. In order to describe the Teichmüller space $\mathcal{T}(\Sigma)$ of deformations of Σ it suffices to allow for arbitrary *real* values of the twist angles θ_i . Points in $\mathcal{T}(\Sigma)$ are then parametrized by tuples $(l_1, \dots, l_\kappa; \theta_1, \dots, \theta_\kappa) \in (\mathbb{R}^+)^{\kappa} \times \mathbb{R}^{\kappa}$.

Remark 2. The marking graph Γ_σ allows one to distinguish systems of Fenchel-Nielsen coordinates which are related to each other by Dehn twists, $\theta'_c = \theta_c + 2\pi k_c$, $k \in \mathbb{Z}$, $c \in \mathcal{C}$. To use the

markings for the parametrization of different systems of Fenchel-Nielsen coordinates is then closely analogous to using fat graphs for the specification of systems of Penner coordinates.

The definition of the Fenchel-Nielsen coordinates does not use the choice of a distinguished boundary component for each trinion. The latter has been included into the definition 1 for later convenience only.

8.2 Symplectic structure

Let us furthermore notice that the Weil-Petersson symplectic form becomes particularly simple in terms of the Fenchel-Nielsen coordinates:

Theorem 3. [Wo2] —

$$\omega_{\text{WP}} = \sum_{i=1}^{\kappa} d\tau_i \wedge dl_i, \quad \tau_i = \frac{1}{2\pi} l_i \theta_i. \quad (8.3)$$

The content of the theorem may also be paraphrased as follows:

- (i) The geodesic length functions associated to non-intersecting closed curves Poisson-commute.
- (ii) The Hamiltonian flows generated by the geodesic length functions coincide with the Fenchel-Nielsen twist flows.

8.3 Geodesic lengths from the Penner coordinates

A nice feature of the Fock coordinates is that they lead to a particularly simple way to reconstruct the Fuchsian group corresponding to the point P in Teichmüller space that is parametrized by the variables $z_e(P)$, $e \in \varphi_1$. Assume given a graph geodesic g_c on the fat graph homotopic to a simple closed curve c on Σ . Let the edges be labelled e_i , $i = 1, \dots, r$ according to the order in which they appear on g_c , and define σ_i to be 1 if the path turns left⁵ at the vertex that connects edges e_i and e_{i+1} , and to be equal to -1 otherwise. The generator $X(c)$ of the Fuchsian group that corresponds to c is then constructed as follows [Fo].

$$X_c = V^{\sigma_r} E(z_{e_r}) \dots V^{\sigma_1} E(z_{e_1}), \quad (8.4)$$

where the matrices $E(z)$ and V are defined respectively by

$$E(z) = \begin{pmatrix} 0 & +e^{+\frac{z}{2}} \\ -e^{-\frac{z}{2}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \quad (8.5)$$

Given the generator X_c of the Fuchsian group one may then calculate the hyperbolic length of the closed geodesic isotopic to c via

$$2 \cosh\left(\frac{1}{2}l_c\right) = |\text{tr}(X_c)|. \quad (8.6)$$

The proof of (8.4) was omitted in [Fo]. We are therefore now going to explain how to verify the validity of equation (8.4).

⁵w.r.t. to the orientation induced by the embedding of the fat-graph into the surface

It was remarked in the Subsection §5.3 that for given values of the coordinates z_e one may construct a uniformized representation of the corresponding Riemann surface by successively mapping ideal hyperbolic triangles into the upper half plane which are glued according to the values z_e . Iterating this procedure ad infinitum one generates a tessellation of the upper half plane by ideal hyperbolic triangles. Let us now consider a generator $X(c)$ of the Fuchsian group, represented on the upper half plane by a Möbius transformation $M_{X(c)}$, where $M_X(u) \equiv \frac{au+b}{cu+d}$ if $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The element $c \in \pi_1(\Sigma)$ may then be represented by an open path on the upper half plane which leads from a chosen base point u to its image under M_X .

But one may equivalently represent the motion along the path by standing still at the base point and moving the tessellation around by means of Möbius-transformations. More precisely, let us assume that our base point u is located within the ideal hyperbolic triangle t_0 with corners at $-1, 0, \infty$, and that the path \mathcal{P}_c representing our chosen element $c \in \pi_1(\Sigma)$ crosses the edges e_i $i = 1, \dots, r$ in the order of the labelling. We may assume that the edge e_1 connects the points 0 and $e^{z_{e_1}}$. After having crossed edge e_1 one would have left the triangle t_0 into the triangle t_1 with corners at $0, e^{z_{e_1}}, \infty$. The Möbius-transformation M_1 corresponding to $E(z_{e_1})$ brings one back into t_0 : It can be checked that it leaves the set of corners $\{-1, 0, e^{z_{e_1}}, \infty\}$ on the two triangles glued along e_1 invariant, but exchanges the two triangles. To continue along the path \mathcal{P}_c in this fashion we now need to map the next edge e_2 to be crossed to the edge going from 0 to ∞ before we can apply the Möbius-transformation corresponding to $E(z_{e_2})$ in the same manner as before. This is precisely what the Möbius-transformation $M_{V\sigma_1}$ does: It simply rotates the edges of our fundamental triangle t_0 . Moreover, the triangle t_2 that would be reached when leaving t_0 through e_2 will be mapped by $M_{V\sigma_1}$ into the ideal hyperbolic triangle with corners $0, e^{z_{e_2}}, \infty$. The fact that $e^{z_{e_2}}$ is indeed the position that the corner of t_2 is mapped into by $M_{V\sigma_1}$ follows from the prescription for gluing t_1 and t_2 along e_2 in terms of z_{e_2} and the fact that $M_{V\sigma_1}$ preserves cross-ratios.

By continuing in this fashion one generates the Möbius transformation $M_{X(c)}$ that evidently maps the original tessellation representing the chosen point P in $\mathcal{T}(\Sigma)$ into another one that is equally good as a representation for P . By assumption, $M_{X(c)}$ represents a closed path on the considered fat graph. This means that the points that are mapped into each other by $M_{X(c)}$ are to be identified as different representatives for the same points on the surface corresponding to our point $P \in \mathcal{T}(\Sigma)$.

9. Coordinates for surfaces with holes of finite size

In the present paper we are mainly interested in the case of Riemann surfaces which have a boundary $\partial\Sigma$ represented by s geodesics of finite length. We therefore need to discuss how to introduce analogs of the previously described coordinate systems for $\mathcal{T}(\Sigma)$ for the cases of interest here. When considering surfaces with holes of finite size one has to choose if one wants to keep the geodesic lengths of the boundary components variable, or if one wants to consider surfaces Σ_Λ which have fixed boundary length given by the tuple $\Lambda = (l_1, \dots, l_s) \in \mathbb{R}_+^s$. We shall find the first option often more convenient to work with. Passing to a representation in which the boundary lengths are fixed will then be almost trivial.

9.1 Useful fat graphs on surfaces with holes of finite size

Riemann surfaces Σ with s holes can always be represented by considering a Riemann surface Σ^e with s pairs of punctures, from which Σ is obtained by cutting Σ^e along the geodesics b_1, \dots, b_s that encircle the pairs of punctures. This simple observation allows us to use the coordinates discussed previously in order to define coordinates for the Teichmüller spaces of surfaces with s holes. In order to spell out more precisely how to do this, let us first introduce a convenient class of fat graphs.

Let us consider a pair of punctures (P_1, P_2) . Let c be a geodesic such that cutting Σ along c produces two connected components one of which is a two-punctured disc D with punctures P_1 and P_2 . A given fat graph φ^e will be said to have standard form near D if there exists a neighborhood of the disc D in which φ^e is homotopic to the fat graph depicted on the left half of Figure 9.

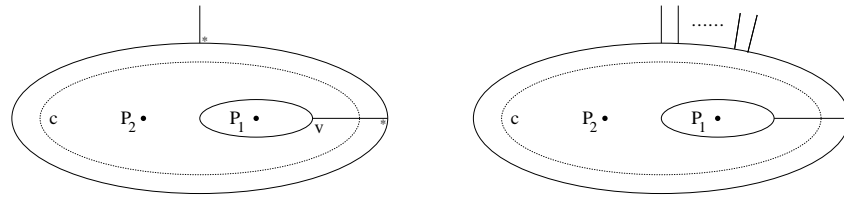


Figure 9: Simple fat graphs in a neighborhood of the disc defined by the geodesic c which encircles two punctures P_1 and P_2 .

For surfaces Σ^e with s pairs of punctures there exist fat graphs φ^e which are of standard form in the neighborhood of $s - 1$ discs D_i . The simplest possible form of a fat graph around the remaining two punctures is indicated on the right of Figure 9. A fat graph φ^e on a surface with $2s$ punctures will be said to have standard form if it has standard form near $s - 1$ discs D_i , and if it has the form depicted on the right of Figure 9 in a neighborhood of the remaining disc.

We will finally say that a fat graph φ on a Riemann surface Σ with s geodesic boundaries has standard form if (Σ, φ) can be obtained from a pair (Σ^e, φ^e) consisting of a $2s$ -punctured surface Σ^e and a fat graph φ^e of standard form by cutting Σ^e along s geodesics b_1, \dots, b_s , each of which encircles a pair of punctures. The embedding $\Sigma \hookrightarrow \Sigma^e$ furthermore induces an embedding $\text{MC}(\Sigma) \hookrightarrow \text{MC}(\Sigma^e)$ of the respective mapping class groups. The subgroup of $\text{MC}(\Sigma^e)$ which is generated by the diffeomorphisms that are supported on $\Sigma \subset \Sigma^e$ preserves the set of fat graphs which have standard form.

9.2 Kashaev type coordinates

If we only use fat graphs of standard form, it becomes easy to adapt the previously discussed systems of coordinates to the case of interest in the rest of this paper. We may in particular consider the space W_φ of Kashaev variables associated to the fat graph φ . A subspace T_φ of W_φ can again be defined by means of the decomposition (7.7). It is furthermore convenient to introduce the set φ'_1 which only contains the edges of φ that do not end in boundary components of Σ .

Lemma 5. (i) We have $T_\varphi \simeq T_{\varphi^e}$.

(ii) The Fock coordinates $\{z_e; e \in \varphi'_1\}$ form a set of coordinates for T_φ .

Proof. Let us first note that the linear form h_1 which is via (7.3) associated to puncture P_1 can be expressed in terms of the variables (p_v, q_v) associated to the vertex v in Figure 9 *only*. This means that both p_v and q_v are contained in N_{φ^e} . Instead of the linear form h_2 associated to puncture P_2 we may consider $h_c = h_1 + h_2$, which can be expressed exclusively in terms of the variables associated to the vertices contained in Σ . Part (i) of the lemma follows easily from these observations.

In order to verify part (ii) one may again consider φ^e . When writing the relations $f_c = 0$, $c \in B(\Sigma)$ in terms of the Fock variables z_e , $e \in \varphi_1^e$ one will always find contributions containing the z_e , $e \in \varphi_1^e \setminus \varphi'_1$. It is then easy to convince oneself that the relations $\hat{f}_c = 0$ may be used to express the z_e , $e \in \varphi_1^e \setminus \varphi'_1$ in terms of the z_e , $e \in \varphi'_1$. After this is done, all relations $f_c = 0$, $c \in B(\Sigma)$ are satisfied, the variables z_e , $e \in \varphi'_1$ are therefore unconstrained. \square

However, in this case the relation between the Teichmüller space $\mathcal{T}(\Sigma)$ and T_φ is slightly more complicated. In order to describe this relation let us consider the spaces $\text{Fun}(\mathcal{T}(\Sigma))$ and $\text{Fun}(T_\varphi)$ of smooth functions on $\mathcal{T}(\Sigma)$ and T_φ respectively. These spaces carry canonical Poisson brackets $\{ \cdot, \cdot \}_{\text{WP}}$ and $\{ \cdot, \cdot \}_\varphi$ uniquely defined by the bilinear forms Ω_{WP} and Ω_φ respectively.

Proposition 3. *We may represent $\text{Fun}(\mathcal{T}(\Sigma))$ as the subspace of $\text{Fun}(T_\varphi)$ which is defined by the conditions*

$$\{ F, l_i \}_\varphi = 0, \quad i = 1, \dots, s, \quad F \in \text{Fun}(T_\varphi), \quad (9.1)$$

where l_i is the length function which is associated to the i -th boundary component via equations (8.4) and (8.6).

It should be noted that the length functions associated to the boundary components are contained in $\text{Fun}(\mathcal{T}(\Sigma))$.

Part III

Quantization of the Teichmüller spaces

Our aim is to construct certain classes of infinite-dimensional representations of the mapping class groups $\text{MC}(\Sigma)$. One possible approach to this problem is to “quantize” Poisson manifolds like the Teichmüller spaces on which $\text{MC}(\Sigma)$ acts as a group of symmetries.

Our construction will proceed in two main steps. Quantization of the Kashaev spaces W_φ leads to a rather elegant construction of projective unitary representations of the mapping class groups [Ka1]. However, these representations turn out to be reducible. The second step will therefore be to identify distinguished subrepresentations within the representations coming from the quantization of the Kashaev spaces as the mapping class group representations which are naturally associated to the quantization of the Teichmüller spaces. A direct construction of the latter is not known, which is why this somewhat indirect construction seems to be most efficient at the moment.

10. Quantization of the Teichmüller spaces

10.1 Canonical quantization

Quantization of a Poisson manifold \mathcal{P} means “deforming” the space of functions on \mathcal{P} into a one-parameter (\hbar) family of noncommutative algebras \mathcal{P}_\hbar in such a way that the deformed product $f *_\hbar g$ satisfies

$$f *_\hbar g = fg + \hbar \Omega(f, g) + \mathcal{O}(\hbar^2), \quad (10.2)$$

where fg is the ordinary commutative product of functions on \mathcal{P} and $\Omega(f, g)$ is the Poisson bracket on \mathcal{P} . If the Poisson manifold \mathcal{P} has a group G of symmetries it is natural to demand that these symmetries are preserved by quantization in the sense that any $g \in G$ is realized as an automorphism $f \rightarrow a_g(f)$ of \mathcal{P}_\hbar .

Representations of the group G can be constructed by studying representations of the algebra \mathcal{P}_\hbar by operators $\mathcal{O}(f)$ on a Hilbert space \mathcal{H} ,

$$\mathcal{O}(f) \cdot \mathcal{O}(g) = \mathcal{O}(f *_\hbar g).$$

The Hilbert space \mathcal{H} will then typically come equipped with a unitary projective representation of the group G of symmetries by operators U_g such that the automorphisms $A_g(\mathcal{O}(f)) \equiv \mathcal{O}(a_g(f))$ are realized as

$$A_g(\mathcal{O}) = U_g \cdot \mathcal{O} \cdot U_g^{-1}.$$

Quantization is particularly simple if there exist coordinate functions q_1, \dots, q_N and p_1, \dots, p_N defined globally on \mathcal{P} such that the Poisson bracket takes the form

$$\Omega(p_v, q_w) = \delta_{vw}, \quad \Omega(q_v, q_w) = 0, \quad \Omega(p_v, p_w) = 0, \quad (10.3)$$

for $v, w \in \{1, \dots, N\}$. One may then define \mathcal{P}_\hbar in such a way that the relations

$$i[p_v, q_w] = \hbar \delta_{vw}, \quad [q_v, q_w] = 0, \quad [p_v, p_w] = 0, \quad (10.4)$$

hold for $[f, g] \equiv f *_{\hbar} g - g *_{\hbar} f$. There exists a standard representation O of these commutation relations on (dense subspaces of) the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^N)$ of square-integrable functions $\Psi(\mathbf{q})$, $\mathbf{q} = (q_1, \dots, q_N)$, which is generated by pairs of operators

$$\mathbf{p}_v \equiv \frac{1}{2\pi\hbar} O(p_v), \quad \mathbf{q}_v \equiv O(q_v), \quad v = 1, \dots, N,$$

that are defined respectively by

$$\mathbf{q}_v \Psi(\mathbf{q}) \equiv q_v \Psi(\mathbf{q}), \quad \mathbf{p}_v \Psi(\mathbf{q}) \equiv \frac{1}{2\pi i} \frac{\partial}{\partial q_v} \Psi(\mathbf{q}). \quad (10.5)$$

This simple example for the quantization of Poisson manifolds is often referred to as “canonical quantization”.

Remark 3. The representation that is constructed in this way is irreducible in the following sense. If O is a bounded operator on $L^2(\mathbb{R}^N)$ which commutes⁶ with the operators \mathbf{p}_v and \mathbf{q}_v for all $v = 1, \dots, N$ then $O = \chi$, the operator of multiplication with the complex number χ .

Following the discussion in Sections 5, it seems natural to define the quantized Teichmüller spaces as the noncommutative algebra $\mathcal{T}_{\hbar}(\Sigma)$ with generators z_e and relations

$$(i) \quad i [z_{e_2}, z_{e_1}] = \hbar \Omega_{\text{WP}}(z_{e_2}, z_{e_1}) \quad (10.6)$$

$$(ii) \quad f_{\varphi, c} = 0, \quad \forall c \in B(\Sigma). \quad (10.7)$$

The space $\mathcal{H}(\Sigma)$ will be defined as an irreducible representation of the commutation relations (10.6) which satisfies the additional conditions (10.7).

10.2 Quantization of the Kashaev space W_{φ}

For each given fat graph φ let us define the Hilbert space $\mathcal{K}(\varphi)$ as the space of square integrable functions $\Psi(\mathbf{q})$ of the Kashaev-variables $\mathbf{q} = (q_1, \dots, q_{2M})$. On $\mathcal{K}(\varphi)$ we shall consider the basic operators $\mathbf{p}_v, \mathbf{q}_v$ defined in (10.5) for $v = 1, \dots, 2M$. The noncommutative algebras of operators which are generated by the operators $\mathbf{q}_v, \mathbf{p}_w$ with the commutation relations

$$[\mathbf{p}_v, \mathbf{q}_w] = (2\pi i)^{-1} \delta_{v,w} \quad (10.8)$$

may be considered as representing quantized algebras of functions on the Kashaev space W_{φ} .

Remark 4. It is worth noting that the decoration of the triangles is used to define the concrete realization of the space $\mathcal{K}(\varphi)$ as a space of square integrable functions.

When quantizing the Kashaev spaces W_{φ} we get more than we ultimately want. In order to see this, let us introduce quantum analogs of the coordinate functions h_c and \hat{z}_e respectively, i.e.

⁶Commutativity $[O, A] = 0$ with a self-adjoint unbounded operator A is, by convention, understood in the sense of commutativity with the spectral projections of A . For the reader's convenience, we have collected the relevant operator-theoretical results in Appendix B.

self-adjoint operators \hat{h}_c and \hat{z}_e on $\mathcal{H}(\varphi)$ which are defined by formulae very similar to (7.3) and (7.6) respectively, normalized in such a way that the following commutation relation hold:

$$(i) \quad i[\hat{z}_{e_1}, \hat{z}_{e_2}] = \hbar \Omega_{WP}(z_{e_1}, z_{e_2}), \quad (10.9)$$

$$(ii) \quad i[\hat{h}_{c_1}, \hat{h}_{c_2}] = \hbar I(c_1, c_2). \quad (10.10)$$

$$(iii) \quad [\hat{z}_e, \hat{h}_c] = 0 \quad \forall c \in H_1(\Sigma, \mathbb{R}). \quad (10.11)$$

We observe that we *do* have a representation, henceforth denoted Z_φ , of the algebra (10.6), but this representation is neither irreducible, nor does it fulfill the additional relations (10.7). The latter point becomes most clear if one introduces the operators \hat{f}_c , $c \in B(\Sigma)$ associated to the relations (10.7) which are defined by replacing $z_e \rightarrow \hat{z}_e$ in (5.3). We may then observe that

$$\hat{f}_c = h_c \quad \text{for } c \in B(\Sigma), \quad (10.12)$$

which is verified in the same way as statement (iv) in Lemma 4.

10.3 Reduction to the quantized Teichmüller spaces

In order to see that the representations Z_φ “contain” irreducible representations of the quantized Teichmüller spaces $\mathcal{T}_h(\Sigma)$, let us consider the noncommutative algebra $\mathcal{Z}_h(\Sigma)$ with generators z_e and the only relations (10.6). One should observe that the f_c generate the center of the algebra \mathcal{Z}_h . Irreducible unitary representations Z_f of this algebra are parametrized by linear functions $f : B(\Sigma) \rightarrow \mathbb{R}$. Such representations are such that the operators $f_c = Z_f(f_c)$ are realized as the operators of multiplication with the real numbers $f(c)$, $c \in B(\Sigma)$.

The representations can be constructed concretely by forming linear combinations t_k and $t_{k'}^\vee$, $k = 1, \dots, 3g - 3 + s$ of the z_e which

(i) are mutually linearly independent, and linearly independent of the f_c ,

(ii) and which satisfy the commutation relations

$$2\pi [t_k, t_{k'}^\vee] = i \delta_{kk'}. \quad (10.13)$$

Canonical quantization realizes Z_f on the Hilbert space $\mathcal{H}_f(\varphi) \simeq L^2(\mathbb{R}^{3g-3+s})$ which consists of square-integrable functions $\Phi(t)$, $t = (t_1, \dots, t_{3g-3+s})$. The operators $z_{f,e} \equiv Z_f(z_e)$ are then realized as linear combination of the operators

$$t_k \Phi_f(t) = t_k \Phi_f(t), \quad t_{k'}^\vee \Phi_f(t) = \frac{1}{2\pi i} \frac{\partial}{\partial t_{k'}} \Phi_f(t), \quad f_c \Phi_f(t) = f(c) \Phi_f(t),$$

where $k = 1, \dots, 3g - 3 + s$ and $c \in B(\Sigma)$.

Our aim is to describe how the representation Z_φ decomposes into the representations Z_f . In order to do this, let us introduce the representation

$$Z'_\varphi \equiv \int_{B'(\Sigma)}^\oplus df Z_f \quad \text{on the space} \quad \mathcal{H}_z(\varphi) \equiv \int_{B'(\Sigma)}^\oplus df \mathcal{H}_f(\varphi).$$

The space $\mathcal{H}_z(\varphi)$ is spanned by square-integrable families $\Phi \equiv (\Phi_f)_{f \in B'(\Sigma)}$ of functions $\Phi_f \in \mathcal{H}_f(\varphi)$ which are associated to the linear functions $f : B(\Sigma) \rightarrow \mathbb{R}$ in the dual $B'(\Sigma) \simeq \mathbb{R}^{s-1}$ up to a set of measure zero. The representatives $z_e \equiv Z_\varphi(z_e)$ are defined as follows

$$z_e \Phi \equiv (z_{f,e} \Phi_f)_{f \in B'(\Sigma)}. \quad (10.14)$$

Proposition 4. *The decomposition of the representation Z_φ into irreducible representations of \mathcal{Z}_h may be written as follows:*

$$Z_\varphi \simeq \left(\int_{B'(\Sigma)}^\oplus df Z_f \right) \otimes 1_{\mathcal{H}_h(\varphi)}, \quad (10.15)$$

where the space $\mathcal{H}_h(\varphi)$ is isomorphic to $L^2(\mathbb{R}^g)$. There exists a unitary operator $l_\varphi, l_\varphi : \mathcal{K}(\varphi) \rightarrow \mathcal{H}_z(\varphi) \otimes \mathcal{H}_h(\varphi)$ such that

$$l_\varphi \cdot \hat{z}_e \cdot l_\varphi^{-1} = z_e \otimes 1 \quad \text{and} \quad l_\varphi \cdot \hat{h}_c \cdot l_\varphi^{-1} = 1 \otimes h_c, \quad (10.16)$$

for any $e \in \varphi_1$ and $c \in H_1(\Sigma_{\text{cl}}, \mathbb{R})$, respectively.

Proof. Let us recall the direct sum decomposition

$$W_\varphi \simeq T_\varphi \oplus N_\varphi \oplus H_\varphi. \quad (10.17)$$

To each of the three spaces $T_\varphi, N_\varphi, H_\varphi$ one may choose coordinates which bring the Poisson bracket to the canonical form (10.3). The corresponding operators

$$\begin{aligned} (\hat{t}_k, \hat{t}_k^\vee), \quad k &= 1, \dots, 3g - 3 + s, \\ (\hat{f}_l, \hat{f}_l^\vee), \quad l &= 1, \dots, s - 1, \\ (\hat{h}_m, \hat{h}_m^\vee), \quad m &= 1, \dots, g, \end{aligned}$$

can be constructed as linear combinations of the (p_v, q_v) , $v \in \varphi_1$, in such a way that the only nontrivial commutation relations are

$$\begin{aligned} 2\pi [\hat{t}_k, \hat{t}_{k'}^\vee] &= ib^2 \delta_{kk'}, \quad k, k' = 1, \dots, 3g - 3 + s, \\ 2\pi [\hat{f}_l, \hat{f}_{l'}^\vee] &= ib^2 \delta_{ll'}, \quad l, l' = 1, \dots, s - 1, \\ 2\pi [\hat{h}_m, \hat{h}_{m'}^\vee] &= ib^2 \delta_{mm'}, \quad m, m' = 1, \dots, g. \end{aligned} \quad (10.18)$$

It will be convenient to form the following vectors with $4M$ operator-valued components:

$$\begin{aligned} \mathbf{v} &= (\dots, q_v, \dots, p_w, \dots), \\ \hat{\mathbf{v}} &= (\dots, \hat{t}_k, \dots, \hat{f}_l, \dots, \hat{h}_m, \dots, \hat{t}_k^\vee, \dots, \hat{f}_l^\vee, \dots, \hat{h}_m^\vee, \dots). \end{aligned}$$

The linear change of variables $\hat{\mathbf{v}} = \hat{\mathbf{v}}(\mathbf{v})$ can then be represented by a symplectic $(4M \times 4M)$ -matrix J_φ ,

$$\hat{\mathbf{v}} = J_\varphi \mathbf{v}, \quad J_\varphi \in Sp(2M, \mathbb{R}). \quad (10.19)$$

On the other hand let us note that $\mathcal{H}_z(\varphi) \otimes \mathcal{H}_h(\varphi)$ is canonically isomorphic to $L^2(\mathbb{R}^{2M})$ via

$$\mathcal{K} : \Phi \otimes \psi \rightarrow \Psi, \quad \Psi(t, f, h) \equiv \Phi_f(t) \psi(h). \quad (10.20)$$

The corresponding representation of the commutation relations (10.18) on the Hilbert space $L^2(\mathbb{R}^{2M})$ is obtained by renaming the operators q_v, p_v as follows

$$\begin{aligned} (\dots, q_v, \dots, p_w, \dots) \\ \equiv (\dots, t_k, \dots, f_l, \dots, h_m, \dots, t_k^\vee, \dots, f_l^\vee, \dots, h_m^\vee, \dots). \end{aligned}$$

It follows from the Stone - von Neumann uniqueness theorem for the representation of the commutation relations (10.18) that these two representation must be related by a unitary transformation. This transformation may be characterized more precisely as follows.

Lemma 6.

a) To each $\gamma \in Sp(2M, \mathbb{R})$ there exists a unitary operator J_γ on $L^2(\mathbb{R}^{2M})$ such that

$$J_\gamma \cdot v \cdot J_\gamma^{-1} = \gamma v. \quad (10.21)$$

The operators J_γ generate a projective unitary representation of $Sp(2M, \mathbb{R})$.

b) The operators J_γ can be represented in the form

$$J_\gamma = \exp(iJ_\gamma(v)), \quad (10.22)$$

where $J_\gamma(v)$ is a quadratic expression in the operators v .

Proof. Part a) is a classical result of I.E. Segal, [Se]. Part b) follows easily from the observation that the quadratic functions of the operators v generate a representation of the Lie algebra of $Sp(2M, \mathbb{R})$ which satisfies the infinitesimal version of (10.21), see e.g. [GS] for more details. \square

One may therefore find an operator J_φ on $L^2(\mathbb{R}^{2M})$ which represents the transformation (10.19) in the sense that

$$J_\varphi \cdot v \cdot J_\varphi^{-1} = \hat{v}(v) = J_\varphi v. \quad (10.23)$$

The sought-for isomorphism l_φ can finally be constructed as $l_\varphi = K^{-1} \cdot J_\varphi^{-1}$. \square

Remark 5. It is worth noting that the definition of $\mathcal{H}_f(\varphi)$ depends only on the combinatorial structure of the fat graph φ , not on the way it is embedded into the Riemann surface Σ . It follows that the isomorphism $\mathcal{H}_f(\mu.\varphi) \simeq \mathcal{H}_f(\varphi)$, $\mu \in MC(\Sigma)$ is *canonical*.

11. Representations of the mapping class groups

The representations of the mapping class group associated to the quantized Teichmüller spaces will be obtained by means of a very general construction which produces representations of the group G of symmetries of a two-dimensional CW complex \mathcal{G} out of representations of the edge path groupoid of \mathcal{G} . We will first describe this construction, before we discuss how to construct representations of the Ptolemy groupoid on the quantized Teichmüller spaces. The latter will then induce the sought-for representation of $MC(\Sigma)$.

11.1 Projective unitary representations of groupoids

Let us recall that a groupoid G is a category such that all morphisms are invertible. The objects of G will here be denoted by letters U, V, W, \dots . Anticipating that the groupoids G we will be interested in are path groupoids of some topological space we will use the notation $[W, V] \equiv \text{Hom}_G(V, W)$. The elements of $[W, V]$ will also be called "paths".

Definition 2. A unitary projective representation of the groupoid G consists of the following data:

- (i) A Hilbert space $\mathcal{H}(V)$ associated to each object $V \in \text{Ob}(G)$,
- (ii) a map u which associates to each path $\pi \in [W, V]$ in G a unitary operator

$$u(\pi) : \mathcal{H}(V) \rightarrow \mathcal{H}(W),$$

- (iii) a family of maps $\zeta_V, V \in \text{Ob}(G)$ which associate to each closed path $\pi \in [V, V]$ a number $\zeta_V(\pi) \in \mathbb{C}$ with $|\zeta(\pi)| = 1$.

These data are required to satisfy the relations

$$\begin{aligned} \text{a)} \quad u(\pi_2 \circ \pi_1) &= u(\pi_2) \cdot u(\pi_1), & \text{b)} \quad u(\pi^{-1}) &= u^\dagger(\pi), \\ \text{c)} \quad \zeta_V(\pi_2 \circ \pi_1) &= \zeta_V(\pi_2)\zeta_V(\pi_1), & \text{d)} \quad \zeta_V(\pi^{-1}) &= (\zeta_V(\pi))^*, \\ \text{e)} \quad u(\pi) &= \zeta_V(\pi) \text{ if } \pi \in [V, V], & \text{f)} \quad u(\text{id}) &= 1, \end{aligned} \quad (11.24)$$

where we use the notation $\zeta_V(\pi)$ also to denote the operator which multiplies each vector of \mathcal{H}_V by the number $\zeta_V(\pi)$.

The groupoids of interest will be the path groupoids G of two-dimensional CW complexes \mathcal{G} . The set of objects is given by the set of vertices \mathcal{G}_0 , whereas the set of morphisms coincides with the set of paths in the complex \mathcal{G} . Since each path π may be represented as a chain $E_{\pi,n(\pi)} \circ \dots \circ E_{\pi,2} \circ E_{\pi,1}$ of edges in \mathcal{G}_1 it is clear that a projective unitary representation of the path groupoid G of a two-dimensional CW complex \mathcal{G} is characterized completely by specifying the images $u(E)$ for $E \in \mathcal{G}_1$. Existence of the family of maps ζ_V such that relation e) is fulfilled represents a rather nontrivial constraint that the operators $u(E)$, $E \in \mathcal{G}_1$ have to satisfy. Of course it suffices to satisfy these constraints for the 2-cells $\pi \in \mathcal{G}_2$.

11.2 Representations of symmetries of a groupoid

The group of symmetries G of a two-dimensional CW complex \mathcal{G} is the group of all invertible mappings

$$\mu : \begin{cases} \mathcal{G}_0 \ni V \longrightarrow \mu.V \in \mathcal{G}_0, \\ \mathcal{G}_1 \ni E \longrightarrow \mu.E \in \mathcal{G}_1. \end{cases}$$

There is an associated action on the edge paths in the complex \mathcal{G} ,

$$[W, V] \ni \pi \longrightarrow \mu.\pi \in [\mu.W, \mu.V],$$

which is such that

$$\mu(\pi_2 \circ \pi_1) = \mu(\pi_2) \circ \mu(\pi_1), \quad \mu(\pi^{-1}) = (\mu(\pi))^{-1}. \quad (11.25)$$

We will assume that we are given a unitary projective representation of G which is compatible with the symmetry G in the sense that $\mathcal{H}(V)$ is canonically isomorphic with $\mathcal{H}(\mu.V)$, $\mathcal{H}(V) \simeq \mathcal{H}(\mu.V)$. We are going to show that the given representation of the groupoid G canonically induces a representation of its group G of symmetries.

Let us fix a base point $V \in \text{Ob}(G)$ and assume having chosen a path $\pi_V(\mu) \in [\mu.V, V]$ for each $\mu \in G$. Let then $R(\mu) : \mathcal{H}(V) \rightarrow \mathcal{H}(V) \simeq \mathcal{H}(\mu.V)$ be defined by

$$R_V(\mu) = u(\pi_V(\mu)). \quad (11.26)$$

We are going to assume that the paths $\pi_{\mu_1.V}(\mu_2)$ are the translates of $\pi_V(\mu_2)$ under μ_1 , i.e. that $\pi_{\mu_1.V}(\mu_2) = \mu_1.\pi_V(\mu_2)$. It follows that

$$R_{\mu_1.V}(\mu_2) = R_V(\mu_2).$$

The operators $R_V(\mu)$ satisfy the relations

$$R_V(\mu_2) \cdot R_V(\mu_1) = \vartheta_V(\mu_2, \mu_1) R_V(\mu_2 \circ \mu_1), \quad (11.27)$$

$$\vartheta_V(\mu_2, \mu_1) = \zeta_V(\pi_V^{-1}(\mu_2 \circ \mu_1) \circ \pi_{\mu_1.V}(\mu_2) \circ \pi_V(\mu_1)). \quad (11.28)$$

We may next observe that the apparent dependence on the base point $V \in \mathcal{G}_o$ is inessential. Let $V, W \in \mathcal{G}_o$, and let us pick a path $\pi_{W,V} \in [W, V]$. For an operator $O_V : \mathcal{H}(V) \rightarrow \mathcal{H}(V)$ we will define

$$A_{[W,V]}(O_V) = u(\pi_{W,V}) \cdot O_V(\mu) \cdot u^\dagger(\pi_{W,V}). \quad (11.29)$$

It is easy to convince oneself that $A_{W,V}(O_V(\mu))$ does not depend on the choice of a path $\pi_{W,V} \in [W, V]$. We furthermore have

$$A_{W,V}(R_V(\mu)) = R_W(\mu). \quad (11.30)$$

It easily follows that $\vartheta_V(\mu_2, \mu_1)$ does not depend on V , i.e. $\vartheta_V(\mu_2, \mu_1) \equiv \vartheta(\mu_2, \mu_1)$.

To summarize: The operators $R_V(\mu)$ generate a projective unitary representation $R_V(G)$ of G on \mathcal{H} ,

$$R_V(\mu_2) \cdot R_V(\mu_1) = \vartheta(\mu_2, \mu_1) R_V(\mu_2 \circ \mu_1). \quad (11.31)$$

The operators $A_{[W,V]}$ express the unitary equivalence of the representations R_V associated to the different $V \in \mathcal{G}_o$, which allows us to regard

$$R \equiv \left[(R_V)_{V \in \mathcal{G}_o}, (A_E)_{E \in \mathcal{G}_1} \right]$$

as the representation of G canonically associated to the given representation of the groupoid G .

Remark 6. There is of course some ambiguity in the construction, coming from the choice of a representative $\pi_V(\mu) \in [\mu.V, V]$. However, it is clearly natural to consider two representations r, r' as equivalent if the generators $r_V(\mu)$ and $r'_V(\mu)$ differ from each other just by multiplication with a (possibly μ -dependent) central element. The cocycle ϑ of the representation r will differ from the cocycle ϑ' of r' by a coboundary.

11.3 The projective representation of the Ptolemy groupoid on $\mathcal{K}(\varphi)$

Following [Ka3] closely we shall define a projective representation of the Ptolemy groupoid in terms of the following set of unitary operators on $\mathcal{K}(\varphi)$

$$\begin{aligned} A_v &\equiv e^{\frac{\pi i}{3}} e^{-\pi i(\mathbf{p}_v + \mathbf{q}_v)^2} e^{-3\pi i \mathbf{q}_v^2} \\ T_{vw} &\equiv e_b(\mathbf{q}_v + \mathbf{p}_w - \mathbf{q}_w) e^{-2\pi i \mathbf{p}_v \mathbf{q}_w}, \end{aligned} \quad \text{where } v, w \in \varphi_\circ. \quad (11.32)$$

The special function $e_b(U)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$\log e_b(z) \equiv \frac{1}{4} \int_{i0-\infty}^{i0+\infty} \frac{dw}{w} \frac{e^{-2izw}}{\sinh(bw) \sinh(b^{-1}w)}. \quad (11.33)$$

We refer to Appendix A for more details on this remarkable special function. These operators are unitary for $(1 - |b|)\Im b = 0$. They satisfy the following relations [Ka3]

$$(i) \quad T_{vw} T_{uw} T_{uv} = T_{uv} T_{vw}, \quad (11.34)$$

$$(ii) \quad A_v T_{uv} A_u = A_u T_{vu} A_v, \quad (11.35)$$

$$(iii) \quad T_{vu} A_u T_{uv} = \zeta A_u A_v P_{uv}, \quad (11.36)$$

$$(iv) \quad A_u^3 = \text{id}, \quad (11.37)$$

where $\zeta = e^{\pi i c_b^2/3}$, $c_b \equiv \frac{i}{2}(b + b^{-1})$. The relations (11.34) to (11.37) allow us to define a projective representation of the Ptolemy groupoid as follows.

- Assume that $\omega_{uv} \in [\varphi', \varphi]$. To ω_{uv} let us associate the operator

$$u(\omega_{uv}) \equiv T_{uv} : \mathcal{K}(\varphi) \ni \mathbf{v} \rightarrow T_{uv} \mathbf{v} \in \mathcal{K}(\varphi').$$

- For each fat graph φ and vertices $u, v \in \varphi_\circ$ let us define the following operators

$$A_u^\varphi : \mathcal{K}(\varphi) \ni \mathbf{v} \rightarrow A_u \mathbf{v} \in \mathcal{K}(\rho_u \circ \varphi).$$

$$P_{uv}^\varphi : \mathcal{K}(\varphi) \ni \mathbf{v} \rightarrow P_{uv} \mathbf{v} \in \mathcal{K}((uv) \circ \varphi).$$

It follows immediately from (11.34)-(11.37) that the operators T_{uv} , A_u and P_{uv} can be used to generate a unitary projective representation of the Ptolemy groupoid in $\mathcal{K}(\varphi) \simeq L^2(\mathbb{R}^{2M})$.

11.4 Reduction to the quantized Teichmüller spaces

Theorem 4. *The isomorphism \mathfrak{l}_φ maps the operators $u(\pi)$ which represent the Ptolemy groupoid on $\mathcal{K}(\varphi)$ to operators of the form $u'(\pi) = V_z(\pi) \otimes V_h(\pi)$, where $V_z(\pi) \equiv (V_{\mathfrak{f}}(\pi))_{\mathfrak{f} \in B'(\Sigma)}$ is a family of unitary operators $V_{\mathfrak{f}}(\pi)$ on $\mathcal{H}_{\mathfrak{f}}(\varphi)$.*

For each fixed $\mathfrak{f} \in B'(\Sigma)$ one may use the operators $V_{\mathfrak{f}}(\pi)$ to generate a unitary projective representation of the Ptolemy groupoid.

Proof. To begin with, let us note that each path $\pi \in \rho \equiv [\varphi', \varphi]$ canonically defines a map $c \mapsto c' \equiv \mathfrak{A}(c)$ for each $c \in H_1(\Sigma, \mathbb{R})$. This map is defined in an obvious way for the elementary moves depicted in Figures 6, 7 if we require that c' coincides with c outside the triangles depicted in these figures.

Lemma 7.

- a) The map $c \mapsto \mathfrak{A}(c)$ preserves the symplectic (intersection) form on $H_1(\Sigma, \mathbb{R})$.
- b) The operator $u(\pi)$ maps $u(\pi) \cdot h_{\varphi, c} \cdot (u(\pi))^{-1} = h_{\varphi', \mathfrak{A}(c)}$.

Proof. Direct verifications. □

Proposition 5. For each path $\pi \in [\varphi', \varphi]$ there exists an operator $H(\pi)$ on $\mathcal{K}(\varphi)$ such that the operators $V(\pi)$ on $L^2(\mathbb{R}^{2M})$ defined by

$$V(\pi) = H(\pi) \cdot U(\pi), \quad U(\pi) \equiv J_{\varphi'}^{-1} \cdot u(\pi) \cdot J_{\varphi},$$

- (i) commute with all operators $h_m, h_m^\vee, m = 1, \dots, g$ and $f_l, l = 1, \dots, s-1$,
- (ii) generate a unitary projective representation of $\text{Pt}(\Sigma)$ on $L^2(\mathbb{R}^{2M})$.

The operators $H(\pi)$ can be represented in the form

$$H(\pi) \equiv \exp(iH_\pi(h_{\varphi'})), \quad (11.38)$$

where $H_\pi(h_{\varphi'})$ is a quadratic function of the $2g$ operators $h'_1, \dots, h'_g, h_1^\vee, \dots, h_g^\vee$ on $\mathcal{K}(\varphi')$.

Proof. The existence of operators $H(\pi)$ of the form (11.38) which are such that statement (i) of the Proposition is verified follows directly from Lemma 6 if one takes into account that the transformation $h_{\varphi', c} \rightarrow h_{\varphi', c'}$ is represented by an element of $Sp(g, \mathbb{R})$ according to part a) of Lemma 7.

In order to prove statement (ii) of the proposition, we mainly need to check that the operators $V(\pi)$ satisfy the relations of the Ptolemy groupoid. Let us consider a closed path $\pi \in [\varphi, \varphi]$ which decomposes into a chain of edges as $\pi = \pi_n \circ \dots \circ \pi_1$.

$$V(\pi) \equiv V(\pi_n) \cdots V(\pi_1)$$

On the one hand one may observe that $V(\pi)$ can be factorized as

$$V(\pi) = H(\pi) \cdot U(\pi_n) \cdots U(\pi_1) = H(\pi) \zeta_\varphi(\pi). \quad (11.39)$$

The operator $H(\pi)$ in (11.39) can be represented as follows:

$$H(\pi) = H(\pi_n) \cdot [U(\pi'_n) \cdot H(\pi_{n-1}) \cdot U(\pi'_n)^\dagger] \cdots [U(\pi'_1) \cdot H(\pi_1) \cdot U(\pi'_1)^\dagger],$$

where $\pi'_{n-k} \equiv \pi_n \circ \dots \circ \pi_{n-k}$. It follows from Lemma 7 together with (11.38) that

$$U(\pi'_{j+1}) \cdot H(\pi_j) \cdot U(\pi'_{j+1})^\dagger = \exp(iH_{\pi_j}(h_\varphi)).$$

Taking into account equation (10.16) we conclude that

$$K^{-1} \cdot H(\pi) \cdot K = 1 \otimes H_h(\pi). \quad (11.40)$$

On the other hand let us note that $V(\pi)$ commutes with all operators $h_m, h_m^\vee, m = 1, \dots, g$. Equation (11.39) implies that the same is true for $H(\pi)$. However, the representation of the operators $h_m, h_m^\vee, m = 1, \dots, g$ on $\mathcal{H}_h(\varphi)$ is irreducible (see Remark 3). This allows us to conclude that $H(\pi) = \eta_\varphi(\pi) \in \mathbb{C}, |\eta_\varphi(\pi)| = 1$. Inserting this into (11.39) proves our claim. \square

It follows from statement (i) in Proposition 5 that

$$K^{-1} \cdot V(\pi) \cdot K = V_z(\pi) \otimes 1. \quad (11.41)$$

The task remains to describe the operators $V_z(\pi)$ more precisely.

Proposition 6. *There exists a family of unitary operators $V_f(\pi) : \mathcal{H}_f(\varphi) \rightarrow \mathcal{H}_f(\varphi'), f \in B'(\Sigma)$ which represent the action of the operator $V_z(\pi) : \mathcal{H}_z(\varphi) \rightarrow \mathcal{H}_z(\varphi')$.*

Proof. To begin with, let us observe that it follows from (11.32) that the operators $u(E)$ associated to the edges $E = \rho_v, E = \omega_{vw}$ in $\text{Pt}_1(\Sigma)$ can all be factorized as $u(E) = Q(E) \cdot G(\hat{z}_e)$, where $Q(E)$ is of the form $Q(E) = \exp(iQ_E(v))$ for a quadratic function Q_E , and $G \equiv 1$ if $E = \rho_v$ and $G(z) = e_b(z)$ if $E = \omega_{vw}$. It follows that the corresponding operator $V(E)$ defined in Proposition 5 can be factorized as

$$V(E) = Q'(E) \cdot G(z_e), \text{ where } Q'(E) \equiv H(E) \cdot J_{\varphi'}^{-1} \cdot Q(E) \cdot J_\varphi. \quad (11.42)$$

The operator $Q'(E)$ is a product of four operators $J_{\gamma_k}, \gamma_k \in Sp(2M, \mathbb{R})$ for $k = 1, 2, 3, 4$. It follows from Lemma 6 that it can be represented in the form

$$Q'(E) = \exp(iJ_E(v)),$$

for some expression $J_E(v)$ which is quadratic in v . Note that the operators $G(z_e)$ and $V(E)$ commute with all operators $h_m, h_m^\vee, m = 1, \dots, g$ and $f_l, l = 1, \dots, s-1$. It follows that the same is true for $Q'(E)$, which implies that $J_E(v) \equiv J_E(z)$ depends only on the vector $z \equiv (z_e)_{e \in \varphi_1}$. $V(E)$ is therefore of the form $V(E) = \exp(iJ_E(z)) \cdot G(z_e)$. Our claim follows easily, $W_f(E) = \exp(iJ_E(z_{f,e})) \cdot G(z_{f,e})$ does the job. \square

Theorem 4 follows by combining Propositions 5 and 6. \square

We are finally in the position to define more precisely what we will regard as the quantized Teichmüller spaces. To this aim let us note that the Hilbert spaces $\mathcal{H}_o(\varphi)$ associated to the origin o in $B'(\Sigma)$ form irreducible representations of the relations (10.6), (10.7). Functions of the operators $z_{o,e}$ generate the algebras $B(\mathcal{H}_o(\varphi))$ of all bounded operators on $\mathcal{H}_o(\varphi)$, which suggests to interpret $B(\mathcal{H}_o(\varphi))$ as particular representations of the quantized algebras $\mathcal{T}_h(\Sigma)$ of functions on the Teichmüller spaces.

The operators $V_o(\pi)$ generate a unitary projective representation of the Ptolemy groupoid which allows us to regard two operators $O_{\varphi_2} \in B(\mathcal{H}_o(\varphi_2))$ and $O_{\varphi_1} \in B(\mathcal{H}_o(\varphi_1))$ as equivalent, $O_{\varphi_2} \sim O_{\varphi_1}$, iff

$$O_2 = V_o(\pi) \cdot O_1 \cdot (V_o(\pi))^{-1}, \quad \pi \in [\varphi_2, \varphi_1]. \quad (11.43)$$

Let furthermore MC_φ be the unitary projective representation of the mapping class group $\text{MC}(\Sigma)$ which is generated from the operators $V_\circ(\pi)$ by means of the construction in Subsection 11.2.

Definition 3.

- (i) We define the algebra $\mathcal{T}_\hbar(\Sigma)$ as the algebra generated by the families $\mathcal{O} \equiv (\mathcal{O}_\varphi)_{\varphi \in \mathcal{P}t_\circ(\Sigma)}$ of bounded operators \mathcal{O}_φ on $\mathcal{H}_\circ(\varphi)$ such that $\mathcal{O}_{\varphi_2} \sim \mathcal{O}_{\varphi_1}$ for all $\varphi_2, \varphi_1 \in \mathcal{P}t_\circ(\Sigma)$. The algebra $\mathcal{T}_\hbar(\Sigma)$ will be called the quantized algebra of functions on the Teichmüller spaces.
- (ii) Let $\mathcal{MC}_\hbar(\Sigma)$ be the subalgebra of $\mathcal{T}_\hbar(\Sigma)$ generated by the families $\mathcal{MC}(\mu) \equiv (\text{MC}_\varphi(\mu))_{\varphi \in \mathcal{P}t_\circ(\Sigma)}$ for all $\mu \in \text{MC}(\Sigma)$.

Part IV

A stable modular functor from the quantum Teichmüller spaces

Let us recall that systems of Fenchel-Nielsen coordinates are naturally associated to markings of the surfaces Σ . The transformations between the different markings of a Riemann surface Σ generate yet another groupoid, called the stable modular groupoid. Given that the quantum version of the changes between the Penner coordinates associated to different fat graphs was represented by a unitary projective representation of the Ptolemy groupoid it is natural to expect that the quantization of the Fenchel-Nielsen coordinates should similarly come with a unitary projective representation of the modular groupoid.

Pants decompositions have one big advantage over ideal triangulations: The gluing operation allows us to build “larger” surfaces from simple pieces of the same type, namely hyperbolic surfaces with holes. It is natural to require that the unitary projective representations of the modular groupoid assigned to surfaces Σ should correspondingly be organized in a “tower-like” fashion: They should allow restriction to, and should be generated by the representations assigned to the surfaces Σ' which are obtained from Σ by cutting along simple closed curves on Σ .

Our aim in the rest of this paper will be to show that such a structure can be constructed from the quantized Teichmüller spaces considered in the previous part of this paper. This is of great importance since

having a tower of projective unitary representations of the stable modular groupoid is equivalent to having a stable unitary modular functor.

The notion of a stable unitary modular functor will be introduced in the next section. One main difference to the more conventional (two-dimensional) modular functors as defined e.g. in [Tu, BK2] is that one restricts attention to Riemann surfaces Σ of genus g and with n parametrized boundary components which are stable in the sense that $2g - 2 + n > 0$.

We will then explain why having a stable unitary modular functor is equivalent to having a tower of projective unitary representations of the modular groupoid before we take up the task to actually construct the latter from the quantization of the Teichmüller spaces as described previously.

12. The notion of a stable unitary modular functor

Given that the usual definitions of a modular functor take several pages to fully write them down [Tu, BK2], we shall only briefly explain the most important features. The missing details will not differ much from the definitions discussed in [Tu, BK2].

12.1 Rigged Riemann surfaces

We will consider compact oriented surfaces Σ with boundary $\partial\Sigma = \coprod_{\beta \in A(\Sigma)} b_\beta$, where $A(\Sigma) \equiv \pi_0(\partial\Sigma)$ is the set of connected components of $\partial\Sigma$. A surface Σ is called an extended surface if one has chosen orientation-preserving homeomorphisms $p_\beta : b_\beta \rightarrow S^1$ for each connected component b_β of the boundary. To be concrete, let $S^1 = \{z \in \mathbb{C}; |z| = 1\}$. An e-surface Σ of genus g and with n boundary circles is called stable if $2g - 2 + n > 0$.

We will use the terminology rigged Riemann surface, or r-surface for short, for triples $\widehat{\Sigma} = (\Sigma, y, \mathfrak{c})$, where

- ▷ Σ is a stable extended surface,
- ▷ y is a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$, and
- ▷ $\mathfrak{c} : A(\Sigma) \rightarrow \mathcal{L}$ is a coloring of the boundary of Σ by elements of a set \mathcal{L} .

Given an r-surface $\widehat{\Sigma}$ and given $\beta, \beta' \in A(\Sigma)$ such that $\mathfrak{c}(\beta) = \mathfrak{c}(\beta')$ we can define a new r-surface $\widehat{\Sigma}' \equiv \sqcup_{\beta\beta'} \widehat{\Sigma} = (\Sigma', y', \mathfrak{c}')$ which is obtained from $\widehat{\Sigma}$ by gluing the boundary components b_β and $b_{\beta'}$.

- ▷ The surface $\Sigma' \equiv \sqcup_{\beta\beta'} \Sigma$ is defined by identifying all points $p \in b_\beta$ with $(p_{\beta'}^{-1} \circ p_\beta^o)(p) \in b_{\beta'}$, where $p_\beta^o : b_\beta \rightarrow S^1$ is defined by $p_\beta^o(p) = \overline{-p_\beta(p)}$. There is a corresponding projection $P_{\beta\beta'} : \Sigma \rightarrow \Sigma' \equiv \sqcup_{\beta\beta'} \Sigma$ which maps $b_\beta, b_{\beta'}$ to the same simple closed curve on Σ' .
- ▷ The Lagrangian subspace y' is given by the image of $H_1(\Sigma, \mathbb{R})$ under the projection $P_{\beta\beta'}$,
- ▷ The coloring \mathfrak{c}' is obtained from \mathfrak{c} by putting $\mathfrak{c}'(\alpha') = \mathfrak{c}(\alpha)$ if $P_{\beta\beta'}(b_\alpha) = b_{\alpha'} \in \partial\Sigma'$.

12.2 Stable unitary modular functors

Let \mathcal{L} now be a space with a measure $d\nu$. A stable modular functor with central charge is the following collection of data.

- ▷ Assignment $\widehat{\Sigma} = (\Sigma, y, \mathfrak{c}) \rightarrow \mathcal{H}(\Sigma, y, \mathfrak{c})$, where
 - $\widehat{\Sigma}$ is an r-surface,
 - $\mathcal{H}(\Sigma, y, \mathfrak{c})$ is a separable Hilbert space,
- ▷ *Mapping class group*: Assignment

$$[f] \longrightarrow (U_{[f]} : \mathcal{H}(\Sigma, y, \mathfrak{c}) \rightarrow \mathcal{H}(\Sigma', y'_f, \mathfrak{c}'_f)),$$

- $[f]$ is the isotopy class of a homeomorphism $f : \Sigma \rightarrow \Sigma'$,
- $U_{[f]}$ is a unitary operator,
- y'_f is the Lagrangian subspace of $H_1(\Sigma', \mathbb{R})$ determined from y via f ,
- $\mathfrak{c}'_f : A(\Sigma') \rightarrow \mathcal{L}$: the coloring of boundary components of Σ' induced from \mathfrak{c} via f .

▷ *Disjoint union*: There exist unitary operators

$$G_{21} : \mathcal{H}(\Sigma_2 \sqcup \Sigma_1, y_2 \oplus y_1, \mathbf{c}_2 \sqcup \mathbf{c}_1) \xrightarrow{\sim} \mathcal{H}(\Sigma_2, y_2, \mathbf{c}_2) \otimes \mathcal{H}(\Sigma_1, y_1, \mathbf{c}_1).$$

▷ *Gluing*: Let (Σ', y') be obtained from (Σ, y) by gluing of two boundary components α, β . There then exists a unitary operator

$$G_{\alpha\beta} : \int_{\mathcal{L}} d\nu(s) \mathcal{H}(\Sigma, y, \mathbf{c}_{s|_{\alpha\beta}}) \xrightarrow{\sim} \mathcal{H}(\Sigma', y', \mathbf{c}'),$$

where the coloring $\mathbf{c}_{s|_{\alpha\beta}} : A(\Sigma) \rightarrow \mathcal{L}$ is defined from \mathbf{c}' via

$$\begin{aligned} \mathbf{c}_{s|_{\alpha\beta}}(c) &= \mathbf{c}'(P_{\alpha\beta}(c)) \quad \text{if } c \in A(\Sigma) \setminus \{\alpha, \beta\}, \\ \mathbf{c}_{s|_{\alpha\beta}}(c) &= s \quad \text{if } c \in \{\alpha, \beta\}. \end{aligned}$$

These data are required to satisfy the following “obvious” consistency and compatibility conditions:

Multiplicativity: For all homeomorphisms $f : \Sigma_1 \rightarrow \Sigma_2$, $g : \Sigma_2 \rightarrow \Sigma_3$ there exists $\zeta(f, g) \in S^1$ such that we have

$$U_{[f]} U_{[g]} = \zeta(f, g) U_{[f \circ g]}. \quad (12.1)$$

$\zeta(f, g)$ has to satisfy the condition $\zeta(f, g) \zeta(f \circ g, h) = \zeta(g, h) \zeta(f, g \circ h)$.

Functoriality: The gluing isomorphisms and the disjoint union isomorphisms are functorial in Σ .⁷

Compatibility: The gluing isomorphisms and the disjoint union isomorphisms are mutually compatible.

Symmetry of Gluing: $G_{\alpha\beta} = G_{\beta\alpha}$.

It would take us several pages to write out all conditions in full detail, we therefore refer to [Tu, BK2] for more details. However, it seems that the following two “naturality” requirements represent a key to the understanding of the notion of the modular functor:

Naturality:

a) Let $f_1 : \Sigma_1 \rightarrow \Sigma'_1$, $f_2 : \Sigma_2 \rightarrow \Sigma'_2$ be r-homeomorphisms. We then have

$$G_{21} \cdot U_{[f_2 \sqcup f_1]} \cdot G_{21}^\dagger \equiv U_{[f_2]} \otimes U_{[f_1]}.$$

b) If the r-homeomorphism $f : \Sigma_1 \rightarrow \Sigma_2$ induces an r-homeomorphism $f' : \Sigma'_1 \rightarrow \Sigma'_2$ of the surfaces Σ'_1, Σ'_2 obtained from Σ_1, Σ_2 by the gluing construction, we have

$$G_{\alpha\beta} \cdot \left(\int_{\mathcal{L}} d\nu(s) U_{[f]}(\Sigma, y, \mathbf{c}_{s|_{\alpha\beta}}) \right) \cdot G_{\alpha\beta}^\dagger = U_{[f']}(\Sigma', y', \mathbf{c}').$$

⁷One is considering the category with objects r-surfaces, and morphisms isotopy classes of homeomorphisms of r-surfaces, equipped additionally with the gluing and disjoint union operations.

These requirements make clear how the mapping class group representations on the spaces $\mathcal{H}(\Sigma', y', \mathfrak{c}')$ restrict to and are generated by the representations assigned to the surfaces Σ which are obtained from Σ' by cutting along simple closed curves on Σ' .

Remark 7. The standard definitions of modular functors assume that the Hilbert spaces $\mathcal{H}(\Sigma, y, \mathfrak{c})$ are *finite-dimensional*. They are therefore not suitable for *nonrational* conformal field theories. Our definition should be seen as a first step towards the definition of analogs of modular functors which are associated to nonrational conformal field theories in a way similar to the connections between rational conformal field theories and modular functors mentioned in the introduction.

However, there is one important ingredient of the usual definition that does not have an obvious counterpart in our framework. In the more standard definitions of modular functors it is required that there exists a distinguished element s_\circ in \mathcal{L} which has the property that coloring a boundary component β with s_\circ is equivalent to “closing” this boundary component. More precisely, let $\mathfrak{c}_{s_\circ|\beta}$ be a coloring of the boundary components of an extended surface Σ which assigns $s \in \mathcal{L}$ to the component with label β , and let $\widehat{\Sigma}_{\tilde{\beta}} = (\Sigma_{\tilde{\beta}}, y_{\tilde{\beta}}, \mathfrak{c}_{\tilde{\beta}})$ be the r-surface obtained from $\widehat{\Sigma} = (\Sigma, y, \mathfrak{c})$ by gluing a disc to b_β . The more standard definitions of modular functors assume or imply existence of an element s_\circ in \mathcal{L} such that

$$\mathcal{H}(\Sigma, y, \mathfrak{c}_{s_\circ|\beta}) \simeq \mathcal{H}(\Sigma_{\tilde{\beta}}, y_{\tilde{\beta}}, \mathfrak{c}_{\tilde{\beta}}), \quad \mathcal{U}_{[f]}(\Sigma, y, \mathfrak{c}_{s_\circ|\beta}) \simeq \mathcal{U}_{[f]}(\Sigma_{\tilde{\beta}}, y_{\tilde{\beta}}, \mathfrak{c}_{\tilde{\beta}}).$$

This yields additional relations between the mapping class group representations assigned to surfaces with different numbers of boundary components.

In the case of the quantized Teichmüller spaces it ultimately turns out that an analog of the element s_\circ in \mathcal{L} exists only if one considers the analytic continuation of $\mathcal{H}(\Sigma, y, \mathfrak{c})$ and $\mathcal{U}_{[f]}(\Sigma, y, \mathfrak{c})$ with respect to the boundary labels $\mathfrak{c}(\beta) \in \mathcal{L}$. This fact, and the corresponding improvement of our definition of a stable modular functor will be elaborated upon elsewhere.

12.3 Representations of the modular groupoid versus modular functors

It turns out to be very useful to reformulate the notion of a stable unitary modular functor in terms of generators and relations as follows. Any surface Σ' can be glued from a surface $\Sigma_\circ = \coprod_{p \in \sigma_\circ} T_p$ which is a disjoint union of trinions. The different ways of doing this can be parametrized⁸ by markings σ . The gluing construction determines a canonical Lagrangian subspace y'_σ of $H_1(\Sigma', \mathbb{R})$ from the tautological Lagrangian subspace $y_\circ \equiv H_1(\Sigma_\circ, \mathbb{R})$. By iterating the gluing and disjoint union isomorphisms one defines unitary operators $G(\sigma, \mathfrak{c}') : \mathcal{H}(\Sigma', y'_\sigma, \mathfrak{c}') \rightarrow \mathcal{H}(\sigma, \mathfrak{c}')$, where

$$\mathcal{H}(\sigma, \mathfrak{c}') \equiv \int_{\mathcal{L}} d\nu_\sigma(S) \bigotimes_{p \in \sigma_\circ} \mathcal{H}(T_p, \mathfrak{c}_{\sigma, p}^S). \quad (12.2)$$

We have used the following notation:

⁸Note that markings with the same cut system will yield equivalent representations for Σ' . This redundancy will give a useful book-keeping device when the representation of the mapping class group is considered.

- The integration is extended over the set \mathcal{L} of all colorings $S : \mathcal{C}_\sigma \ni c \rightarrow s_c \in \mathcal{L}$ equipped with the canonical product measure $d\nu_\sigma(S)$ obtained from $d\nu$ by choosing any numbering of the elements of \mathcal{C}_σ .
- $\mathbf{c}_{\sigma,p}^S$ is the coloring of the boundary components of T_p which is defined by assigning

$$\mathbf{c}_{\sigma,p}^S(\beta) = \begin{cases} \mathbf{c}'(\beta') & \text{if } P_{\sigma,p}(\beta) = \beta' \in A(\Sigma'), \\ s_c & \text{if } P_{\sigma,p}(\beta) = c \in \mathcal{C}_\sigma. \end{cases}$$

$P_{\sigma,p}(\beta)$ is the embedding $T_p \hookrightarrow \Sigma'$ defined from σ by the gluing construction.

Unitary mappings between the different spaces $\mathcal{H}(\sigma, \mathbf{c})$ arise in two ways: First, one may have different markings σ_2, σ_1 such that the Lagrangian subspaces defined by the gluing construction coincide, $y_{\sigma_1} = y_{\sigma_2}$. In this case one gets unitary operators $F_{\sigma_2\sigma_1}(\mathbf{c}) : \mathcal{H}(\sigma_1, \mathbf{c}) \rightarrow \mathcal{H}(\sigma_2, \mathbf{c})$ from the composition

$$F_{\sigma_2\sigma_1}(\mathbf{c}) \equiv G(\sigma_2, \mathbf{c}) \cdot (G(\sigma_1, \mathbf{c}))^\dagger. \quad (12.3)$$

Secondly, one has the mappings $U_{[f]}$ which may map between spaces $\mathcal{H}(\Sigma, y_\sigma, \mathbf{c})$ and $\mathcal{H}(\Sigma', (y_\sigma')_f, \mathbf{c}'_f)$. In the case that $\mu : \Sigma \rightarrow \Sigma'$ represents an element of the mapping class group it is natural to define operators $V_{\sigma\mu}(\mathbf{c}) : \mathcal{H}(\sigma, \mathbf{c}) \rightarrow \mathcal{H}(\sigma'_\mu, \mathbf{c}'_\mu)$ by

$$V_{\sigma\mu}(\mathbf{c}) = G(\sigma'_\mu, \mathbf{c}'_\mu) \cdot U_{[\mu]} \cdot (G(\sigma, \mathbf{c}))^\dagger, \quad (12.4)$$

where σ'_μ is the image of the marking σ under μ .

It turns out - as will be reviewed in the next section - that there exists a set $\mathcal{M}_1(\Sigma)$ of elementary moves between the different markings σ such that any two markings σ_2, σ_1 can be connected by paths π which are composed out of the elementary moves. There furthermore exists a set $\mathcal{M}_2(\Sigma)$ of relations which makes the resulting two-dimensional CW complex $\mathcal{M}(\Sigma)$ simply connected. The corresponding path groupoid is called the modular groupoid and denoted by $M(\Sigma)$.

Our construction of the operators $F_{\sigma_2\sigma_1}(\mathbf{c})$, $V_{\sigma\mu}(\mathbf{c})$ assigns unitary operators to each of the elementary moves. This yields unitary operators $U(\pi, \mathbf{c}) : \mathcal{H}(\sigma_1, \mathbf{c}) \rightarrow \mathcal{H}(\sigma_2, \mathbf{c}_\pi)$ for each path $\pi \in [\sigma_2, \sigma_1]$, where \mathbf{c}_π is the coloring of boundary components which is obtained from \mathbf{c} by tracking the relabeling of boundary components defined by π . Due to the fact that multiplicativity holds only projectively, cf. eqn. (12.1), one will find that there exists $\zeta(\pi) \in S^1$ such that $U(\pi) = \zeta(\pi)$ for all *closed* paths, starting and ending at the same marking σ . In other words, a stable unitary modular functor canonically defines projective unitary representations of the stable modular groupoid.

Conversely, the definition of a *tower of projective unitary representations of the modular groupoids* $M(\Sigma)$ involves the following data.

- ▷ Assignment $(\sigma, \mathbf{c}) \rightarrow \mathcal{H}(\sigma, \mathbf{c})$, where $\mathcal{H}(\sigma, \mathbf{c})$ is constructed out of spaces $\mathcal{H}(S_3, \mathbf{c}_3)$ assigned to trinions as in (12.2).

- ▷ Assignment

$$\pi \in [\sigma_2, \sigma_1] \longrightarrow (U(\pi, \mathbf{c}) : \mathcal{H}(\sigma_1, \mathbf{c}) \rightarrow \mathcal{H}(\sigma_2, \mathbf{c}_\pi)).$$

It is required that the operators $U(\pi, \mathbf{c})$ generate unitary projective representations of the modular groupoids associated to the surfaces Σ . In order to formulate the additional requirements which turn this collection of representations of $M(\Sigma)$ into a *tower* let us note that markings can be glued in a natural way. We will use the notation $\sqcup_{\alpha\beta}\sigma$ for the marking obtained by gluing boundary components α and β . Gluing and disjoint union now become realized in a trivial manner,

$$\mathcal{H}(\sigma_2 \sqcup \sigma_1, \mathbf{c}_2 \sqcup \mathbf{c}_1) = \mathcal{H}(\sigma_2, \mathbf{c}_2) \otimes \mathcal{H}(\sigma_1, \mathbf{c}_1), \quad (12.5)$$

$$\mathcal{H}(\sqcup_{\alpha\beta}\sigma, \mathbf{c}') = \int_{\mathcal{L}} d\nu(s) \mathcal{H}(\sigma, \mathbf{c}_{s|\alpha\beta}). \quad (12.6)$$

The data specified above are then required to satisfy the following naturality conditions.

Naturality:

a) Let $\pi = \pi_2 \sqcup \pi_1 \in [\sigma'_2, \sigma_2] \sqcup [\sigma'_1, \sigma_1]$ We then have

$$U(\pi_2 \sqcup \pi_1, \mathbf{c}_2 \sqcup \mathbf{c}_1) \equiv U(\pi_2, \mathbf{c}_2) \otimes U(\pi_1, \mathbf{c}_1). \quad (12.7)$$

b) Given a path $\pi \in [\sigma_2, \sigma_1]$ in $\mathcal{M}(\Sigma)$ let $\sqcup_{\alpha\beta}\pi \in [\sqcup_{\alpha\beta}\sigma_2, \sqcup_{\alpha\beta}\sigma_1]$ be the corresponding path in $\mathcal{M}(\sqcup_{\alpha\beta}\Sigma)$ defined by the gluing construction. We then have

$$U(\sqcup_{\alpha\beta}\pi, \mathbf{c}') = \int_{\mathcal{L}} d\nu(s) U(\pi, \mathbf{c}_{s|\alpha\beta}). \quad (12.8)$$

From a tower of projective unitary representations of the modular groupoids one can reconstruct a stable unitary modular functor as follows. The system of isomorphisms $U(\pi, \mathbf{c}) : \mathcal{H}(\sigma_1, \mathbf{c}) \rightarrow \mathcal{H}(\sigma_2, \mathbf{c})$ allows us to identify the Hilbert spaces associated to different markings σ , thereby defining $\mathcal{H}(\Sigma, \mathbf{c})$. The representation of the modular groupoid defines a representation of the mapping class group on $\mathcal{H}(\Sigma, \mathbf{c})$ via the construction in subsection 11.1. We refer to [BK2] for more details.

It is important to note (see next section for a detailed discussion) that the elements of the set $\mathcal{M}_1(\Sigma)$ of elementary moves only change the markings within subsurfaces of genus zero with three or four holes, or within subsurfaces of genus one with one hole. The corresponding operators will be called Moore-Seiberg data. They characterize a tower of representations of the modular groupoid completely. Let us furthermore note that the faces/relations that one needs to define a two-dimensional CW complex $\mathcal{M}(\Sigma)$ which has as set of edges $\mathcal{M}_1(\Sigma)$ turn out to involve only subsurfaces of genus zero with three to five holes, and subsurfaces of genus one with one or two holes. This means that one only needs to verify a finite number of relations to show that a given set of Moore-Seiberg data defines a stable unitary modular functor.

13. The modular groupoid

The modular groupoid is the groupoid generated from the natural transformations relating the different markings of a Riemann surface Σ [MS][BK1]. A complete set of generators and relations has been determined in [MS, BK1, FuG].

Remark 8. The CW complex $\mathcal{M}(\Sigma)$ defined below will be a subcomplex of the complex denoted $\mathcal{M}^{\max}(\Sigma)$ in [BK1], since that reference allowed for cut systems that yield connected components with less than three boundary components, whereas we will exclusively consider cut systems that yield connected components with exactly three holes. All other deviations from [BK1] are due to slightly different conventions in the definition of the generators.

13.1 Notations and conventions

We have a unique curve $c(\sigma, e) \in \mathcal{C}'$ associated to each edge $e \in \sigma_1$ of the marking graph Γ_σ . The trinions $T_p \in \mathcal{P}_\sigma$ are in one-to-one correspondence with the vertices $p \in \sigma_\circ$ of Γ_σ .

The choice of a distinguished boundary component c_p for each trinion T_p , $p \in \sigma_\circ$ will be called the *decoration* of the marking graph Γ_σ . The distinguished boundary component c_p will be called *outgoing*, the other two boundary components of T_p *incoming*. Two useful graphical representation for the decoration are depicted in Figure 10.

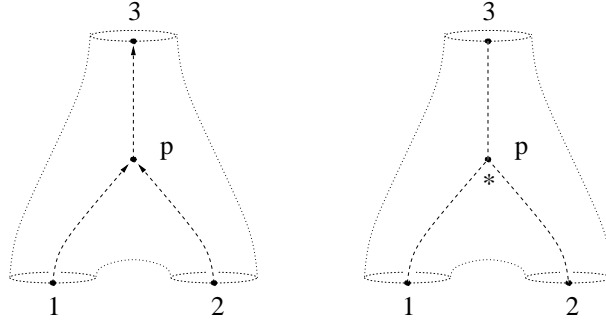


Figure 10: Two representations for the decoration on a marking graph

13.2 Generators

The set of edges $\mathcal{M}_1(\Sigma)$ will be given by elementary moves denoted as (pq) , Z_p , B_p , F_{pq} and S_p . The indices $p, q \in \sigma_\circ$ specify the relevant trinions within the pants decomposition of Σ that is determined by σ . The move (pq) will simply be the operation in which the labels p and q get exchanged. Graphical representations for the elementary moves Z_p , B_p , F_{pq} and S_p are given in Figures 11-14.

13.3 Relations

The relations of $\mathcal{M}(\Sigma)$ correspond to the faces $\varpi \in \mathcal{M}_2(\Sigma)$. In the following we will define a set \mathcal{R} of faces which is large enough to make the complex $\mathcal{M}(\Sigma)$ simply connected. A face ϖ in $\mathcal{M}_2(\Sigma)$ may be characterized by choosing a chain $\Gamma_\varpi = E_{\varpi, n(\varpi)} \circ \cdots \circ E_{\varpi, 1}$, where $E_{\varpi, i} \in \mathcal{M}_1(\Sigma)$ for $i = 1, \dots, n(\varpi)$. In order to simplify notation we will generically factorize Γ_ϖ as $\Gamma_\varpi = \Gamma_\varpi^2 \circ \Gamma_\varpi^1$ and write $(\Gamma_\varpi^2)^{-1} = \Gamma_\varpi^1$ instead of $\Gamma_\varpi = \text{id}$.

Locality. Let us introduce the notation $\text{supp}(m)$ by $\text{supp}(m) = \{p\}$ if $m = Z_p, B_p, S_p$, $\text{supp}(m) = \{p, q\}$ if $m = (pq), F_{pq}$ and $\text{supp}(m_2 \circ m_1) = \text{supp}(m_2) \cup \text{supp}(m_1)$. We then

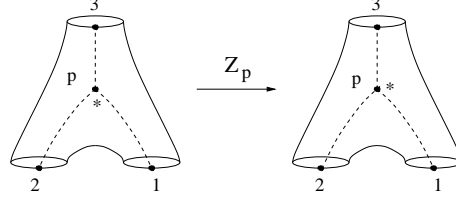


Figure 11: The Z-move

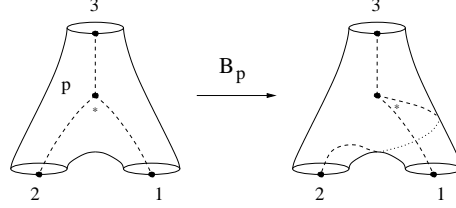


Figure 12: The B-move

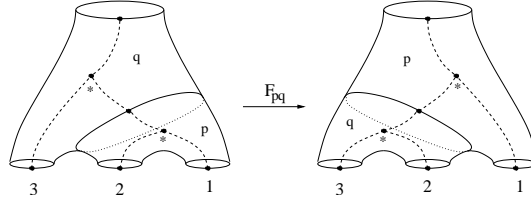


Figure 13: The F-move

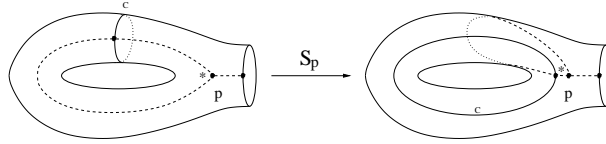


Figure 14: The S-move

have

$$m_2 \circ m_1 = m_1 \circ m_2 \quad \text{whenever} \quad \text{supp}(m_1) \cap \text{supp}(m_2) = \emptyset. \quad (13.1)$$

We will list the remaining relations ordered by the topological type of the surfaces on which the relevant graphs can be drawn.

Relations supported on surfaces of genus zero.

$$g = 0, \quad s = 3 : \quad Z_p \circ Z_p \circ Z_p = \text{id}. \quad (13.2)$$

$$\begin{aligned} & \text{a) } F_{qp} \circ B_p \circ F_{pq} = (pq) \circ B_q \circ F_{pq} \circ B_p, \\ g = 0, \quad s = 4 : & \quad \text{b) } F_{qp} \circ B_p^{-1} \circ F_{pq} = (pq) \circ B_q^{-1} \circ F_{pq} \circ B_p^{-1}, \\ & \quad \text{c) } A_{pq} \circ A_{qp} = (pq). \end{aligned} \quad (13.3)$$

$$g = 0, \quad s = 5 : \quad F_{qr} \circ F_{pr} \circ F_{pq} = F_{pq} \circ F_{qr}. \quad (13.4)$$

We have used the abbreviation

$$A_{pq} \equiv Z_q^{-1} \circ F_{pq} \circ Z_q^{-1} \circ Z_p. \quad (13.5)$$

In Figures 15 and 16 we have given diagrammatic representations for relations (13.3), b) and (13.4) respectively.

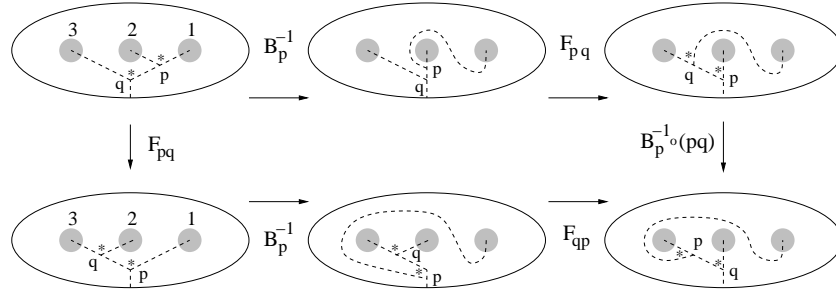


Figure 15: The hexagon relation.

Relations supported on surfaces of genus one. In order to write the relations transparently let us introduce the following composites of the elementary moves.

$$g = 0, \quad s = 3 : \quad \begin{aligned} \text{a)} \quad B'_p &\equiv Z_p^{-1} \circ B_p \circ Z_p^{-1}, \\ \text{b)} \quad T_p &\equiv Z_p^{-1} \circ B_p \circ Z_p \circ B_p, \end{aligned} \quad (13.6)$$

$$g = 0, \quad s = 4 : \quad B_{qp} \equiv Z_q^{-1} \circ F_{qp}^{-1} \circ B'_q \circ F_{pq}^{-1} \circ Z_q^{-1} \circ (pq), \quad (13.7)$$

$$g = 1, \quad s = 2 : \quad S_{qp} \equiv (F_{qp} \circ Z_q)^{-1} \circ S_p \circ (F_{qp} \circ Z_q). \quad (13.8)$$

It is useful to observe that the move T_p , represents the Dehn twist around the boundary component of the trinion t_p numbered by $i = 1$ in Figure 10. With the help of these definitions we may write the relations supported on surfaces of genus one as follows:

$$g = 1, \quad s = 1 : \quad \begin{aligned} \text{a)} \quad S_p^2 &= B'_p, \\ \text{b)} \quad S_p \circ T_p \circ S_p &= T_p^{-1} \circ S_p \circ T_p^{-1}. \end{aligned} \quad (13.9)$$

$$g = 1, \quad s = 2 : \quad B_{qp} = S_{qp}^{-1} \circ T_q^{-1} T_p \circ S_{pq}. \quad (13.10)$$

Relation (13.10) is represented diagrammatically in Figure 17.

Theorem 5. *The complex $\mathcal{M}(\Sigma)$ is connected and simply connected for any e -surface Σ .*

Proof. The theorem follows easily from [BK1], Theorem 5.1. We noted previously that our complex $\mathcal{M}(\Sigma)$ differs from the complex $\mathcal{M}^{\max}(\Sigma)$ of [BK1] in having a set of vertices which corresponds to decompositions of Σ into connected components with *exactly* three holes. The edges of $\mathcal{M}_1^{\max}(\Sigma)$ which correspond to the F-move of [BK1] simply can't appear in $\mathcal{M}_1(\Sigma)$. Otherwise the set of edges of $\mathcal{M}(\Sigma)$ coincides with the relevant subset of $\mathcal{M}_1^{\max}(\Sigma)$, with the exception that our move B_p is a composition of the B - and the Z -move of [BK1]. To complete the proof it remains to check that our set of faces is equivalent to the subset of $\mathcal{M}_2^{\max}(\Sigma)$ which involves only the vertices $\mathcal{M}_0(\Sigma)$ of our smaller complex $\mathcal{M}(\Sigma)$. This is a useful exercise. \square

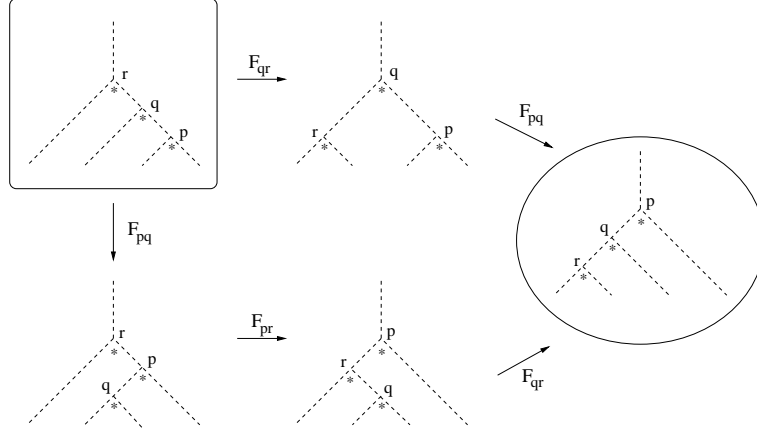


Figure 16: The pentagon identity

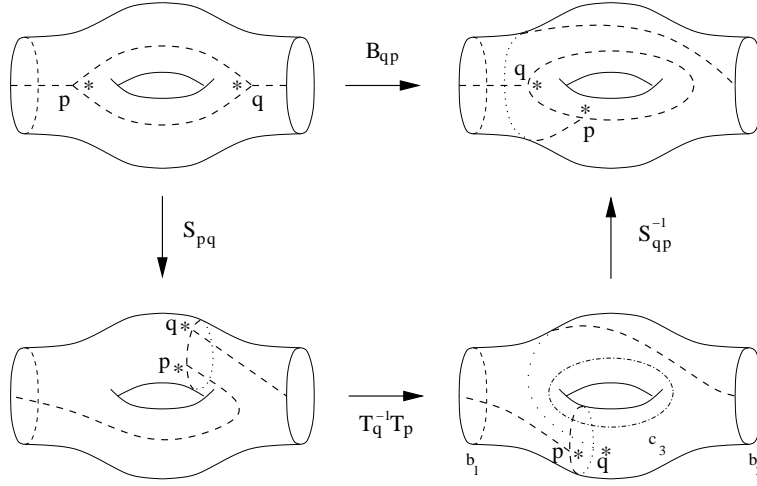


Figure 17: Relation for the two-punctured torus.

Definition 4. Let $\mathcal{M}'(\Sigma)$ be the complex which has the same set of vertices as $\mathcal{M}(\Sigma)$, a set of edges given by the moves

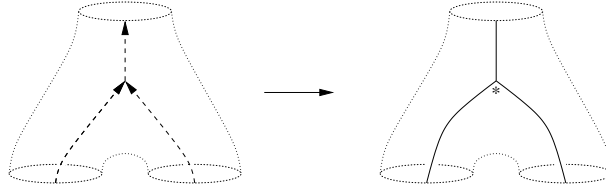
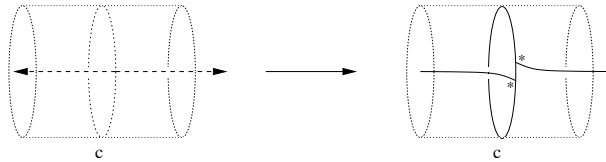
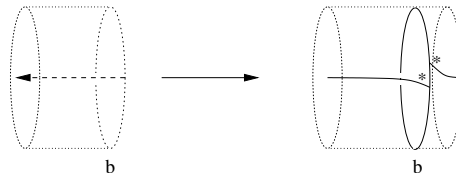
$$(pq), Z_p, B_p, B'_p, S_p, T_p, F_{pq}, A_{pq}, B_{pq}, S_{pq} \quad (13.11)$$

defined above, as well as faces given by equations (13.1)-(13.10).

14. From markings to fat graphs

A key step in our construction of a stable modular functor from the quantized Teichmüller spaces will be the definition of a distinguished class of fat graphs φ_σ which are associated to the elements σ of a certain subset of the set $\mathcal{M}_o(\Sigma)$ of all markings of Σ .

Definition 5. Let $\mathcal{A}_\sigma \subset \mathcal{C}_\sigma$ be the set of all curves c which are incoming for both adjacent trinions. We will say that a marking σ is admissible iff there is no curve $c \in \mathcal{C}_\sigma$ which is outgoing for both adjacent trinions, and if cutting Σ along all curves $c \in \mathcal{A}_\sigma$ yields connected components all of which have genus zero. The set of all admissible markings will be denoted by $\mathcal{M}_o^{\text{ad}}(\Sigma)$.

Figure 18: Substitution for a trinion T Figure 19: Substitution for an annulus A Figure 20: Substitution for a boundary component B

To each admissible marking σ we may naturally associate a fat graph φ_σ on Σ by the following construction. In order to construct the fat graph φ_σ , it will be useful to consider a certain refinement of the pants decomposition associated to σ which is defined as follows. For each curve $c \in \mathcal{A}_\sigma$ let A_c be a small annular neighborhood of c which contains c in its interior, and which is bounded by a pair of curves (c^+, c^-) that are isotopic to c . If $c \in A(\Sigma)$ represents a boundary component of Σ we may similarly consider an annular subset B_c of Σ bounded by $c^+ \equiv c$ and another curve c^- isotopic to c . If $c \in \mathcal{C}_\sigma$ belongs to neither of these two classes we will simply set $c^+ \equiv c \equiv c^-$. By cutting along all such curves c^\pm we obtain a decomposition of Σ into trinions T_p , $p \in \sigma_\circ$ and annuli A_c , $c \in \mathcal{A}_\sigma$, each equipped with a marking graph. We may then replace the markings on each of these connected components by fat graphs according to Figures 18-20. The re-gluing of trinions T_p , $p \in \sigma_\circ$ and annuli A_c , B_c to recover the surface Σ may then be performed in such a way that the fat graphs on the connected components glue to a fat graph φ_σ on Σ .

14.1 The complex $\mathcal{M}^{\text{ad}}(\Sigma)$

It is natural to consider the complex $\mathcal{M}^{\text{ad}}(\Sigma)$ for which the set of vertices $\mathcal{M}_\circ^{\text{ad}}(\Sigma)$ is the subset of $\mathcal{M}_\circ(\Sigma)$ which consists of the admissible markings, and which has a set of edges $\mathcal{M}_1^{\text{ad}}(\Sigma)$ given by the subset of $\mathcal{M}'_1(\Sigma)$ that contains those edges which connect two admissible markings.

Proposition 7. *The complex $\mathcal{M}^{\text{ad}}(\Sigma)$ is connected and simply connected.*

Proof. Let us consider two admissible markings $\sigma, \sigma' \in \mathcal{M}_o(\Sigma)$. There exists a path ϖ in $\mathcal{M}(\Sigma)$ which connects σ and σ' . This path may be represented as a chain $C_\varpi = E_{\varpi, n(\varpi)} \circ \dots \circ E_{\varpi, 1}$, $E_{\varpi, i} \in \mathcal{M}_i(\Sigma)$ composed out of the moves Z_p , F_{pq} , B_p and S_p .

For a given marking $\sigma \in \mathcal{M}_o(\Sigma)$ let $[\sigma]$ be the set of all markings σ' which differ from σ only in the choice of decoration. The moves Z_p act transitively on $[\sigma]$. By inserting Z_p moves if necessary we may therefore modify C_ϖ to a chain D_ϖ which takes the form

$$D_\varpi = Z_{\varpi, n(\varpi)} \circ F_{\varpi, n(\varpi)} \circ Z_{\varpi, n(\varpi)-1} \circ \dots \circ F_{\varpi, 1} \circ Z_{\varpi, 0},$$

where $Z_{\varpi, i}$, $i = 0, \dots, n(\varpi)$ are chains composed out of Z_p -moves only, and the moves $F_{\varpi, n(\varpi)} \in \mathcal{M}_1(\Sigma)$ connect markings $\tilde{\sigma}_{\varpi, i}$ and $\sigma_{\varpi, i}$, $i = 1, \dots, n(\varpi)$ which are *admissible*. We clearly must have $[\sigma_{\varpi, i+1}] = [\tilde{\sigma}_{\varpi, i}]$, $i = 1, \dots, n(\varpi) - 1$ and $[\sigma_{\varpi, 1}] = [\sigma]$, $[\sigma_{\varpi, n(\varpi)}] = [\sigma']$. Connectedness of $\mathcal{M}^{\text{ad}}(\Sigma)$ would follow if the chains $Z_{\varpi, i}$, $i = 0, \dots, n(\varpi)$ are homotopic to chains $Y_{\varpi, i}$ which represent paths in $\mathcal{M}^{\text{ad}}(\Sigma)$. That this indeed the case follows from the following lemma.

Lemma 8. *Assume that $\sigma_a, \sigma_b \in \mathcal{M}_o^{\text{ad}}(\Sigma)$ satisfy $\sigma_b \in [\sigma_a]$. There then exists a path ϖ_{ab} in $\mathcal{M}^{\text{ad}}(\Sigma)$ which connects σ_a and σ_b .*

Sketch of proof. — Let us call a marking σ *irreducible* if it is admissible and if there are no edges e in \mathcal{A}_σ such that cutting Σ along $c(e, \sigma)$ yields two disconnected components. A marking σ which is irreducible has only one outgoing external edge. With the help of Lemma 17 in Appendix D it is easy to show that for an admissible graph σ there always exists a sequence of Z_p -moves in $\mathcal{M}_1^{\text{ad}}(\Sigma)$ which transforms σ to a graph σ' that is irreducible.

We may and will therefore assume that σ_a and σ_b are both irreducible. Using Lemma 17 again allows us to transform σ_b to a marking σ_c which is such that the outgoing external edges of σ_a and σ_c correspond to the same outgoing boundary component of Σ .

The graphs σ_a and σ_c can finally be connected by a chain which is composed out of moves B_p and F_{qp} it only. This follows from the connectedness of $\mathcal{M}(\Sigma_o)$ in the case of a surface Σ_o of genus zero [MS]. In this way one constructs a sequence of moves that connects σ_a to σ_b . \square

It remains to prove that $\mathcal{M}^{\text{ad}}(\Sigma)$ is simply connected as well. Let us consider any closed path ϖ in $\mathcal{M}^{\text{ad}}(\Sigma)$. The path ϖ is contractible in $\mathcal{M}'(\Sigma)$. The deformation of ϖ to a trivial path may be performed recursively, face by face. The crucial observation to be made is the following one.

Lemma 9. *The paths ϖ which represent the boundaries of the faces of $\mathcal{M}'_2(\Sigma)$ are paths in $\mathcal{M}^{\text{ad}}(\Sigma)$.*

Proof. By direct inspection of the relations (13.2)-(13.10). \square

The Lemma 9 implies that deforming a path $\varpi \in \mathcal{M}^{\text{ad}}(\Sigma)$ by contracting a face in $\mathcal{M}'_2(\Sigma)$ will produce a path ϖ' which still represents a path in $\mathcal{M}^{\text{ad}}(\Sigma)$. It follows that ϖ is contractible in $\mathcal{M}^{\text{ad}}(\Sigma)$. \square

14.2 Separated variables

We had observed in Section 9 that the Fock variables z_e , $e \in \varphi'_1$ form a set of coordinates for the Teichmüller spaces of surfaces with holes. When considering fat graphs φ_σ associated to a marking σ it will be useful to replace the Fock variables z_e , $e \in \varphi'_1$ by an alternative set of coordinates (q_c, p_c) , $c \in \mathcal{C}_\sigma$ for T_φ which satisfy

$$\begin{aligned} \Omega_\varphi(p_{c_1}, p_{c_2}) &= 0 = \Omega_\varphi(q_{c_1}, q_{c_2}), & c_1, c_2 \in \mathcal{C}_\sigma. \\ \Omega_\varphi(p_{c_1}, q_{c_2}) &= \delta_{c_1, c_2}, \end{aligned} \quad (14.1)$$

These coordinates are constructed as follows.

For $c \in \mathcal{A}_\sigma$ let A_c be an annular neighborhood of c such that the part of φ_σ which is contained in A_c is isotopic to the model depicted in Figure 21. Let e_1 and e_2 be the two edges that are

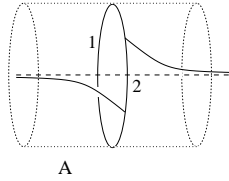


Figure 21: Annulus A_c and fat graph φ on A .

entirely contained in A_c with labelling defined by Figure 21. Out of z_{e_1} and z_{e_2} we may then define

$$q_c \equiv \frac{1}{2}(z_{e_1} - z_{e_2}), \quad p_c \equiv -\frac{1}{2}(z_{e_1} + z_{e_2}). \quad (14.2)$$

In the case $c \in \mathcal{C}_\sigma \setminus \mathcal{A}_\sigma$ let us note that there is a unique trinion T_c for which the curve c is the outgoing boundary component. The part of φ_σ contained in T_c is depicted in Figure 22.

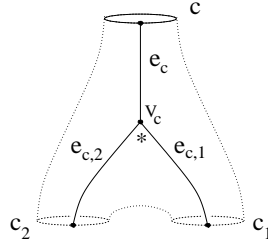


Figure 22: Part of φ_σ contained in T_c

Let $e_{c,2}$ and $e_{c,1}$ be the edges of φ_σ as indicated in Figure 22, and let c_ϵ , $\epsilon = 1, 2$ be the curves which represent the corresponding boundary components of T_p . We shall then define

$$q_c \equiv y_{c_2}, \quad p_c \equiv -y_{c_1}, \quad (14.3)$$

where

$$y_{c_\epsilon} \equiv z_{e_{c,\epsilon}} + \frac{1}{2}f_{c_\epsilon}, \quad \epsilon = 1, 2. \quad (14.4)$$

The same construction yields linear combinations (\hat{p}_c, \hat{q}_c) , $c \in \mathcal{C}_\sigma$ of the Kashaev variables which represent the coordinates (p_c, q_c) within W_{φ_σ} . Note that our construction of the fat graph φ_σ implies that T_c contains a unique vertex $v_c \in \varphi_{\sigma,0}$. It turns out that the variables \hat{q}_c, \hat{p}_c have a simple relation to the Kashaev variables q_{v_c}, p_{v_c} .

Lemma 10.

$$\begin{aligned} \text{(i)} \quad & \hat{q}_c = q_{v_c} + h_{c_2}, \quad \hat{p}_c = p_{v_c} - h_{c_1}. \\ \text{(ii)} \quad & \Omega_\varphi(p_c, q_{c'}) = \delta_{c,c'}, \quad c, c' \in \mathcal{C}_\sigma. \end{aligned}$$

Proof. In order to prove part (i) let us first look at the vertices $v \in \varphi_{\sigma,o}$ which appear on the graph geodesic homotopic to c . The contribution to y_{c_e} of the edges that are incident to v is $\hat{z}_{e'_1} + \hat{z}_{e'_2} + \hat{z}_{e'_3}$ if $e'_i, i = 1, 2, 3$ are the three edges incident to v . It then follows from (7.6) that y_c does not depend on both p_v and q_v . What remains are contributions from the vertex v_c , as well as contributions from the vertices contained in annuli $A_{c'}$ which are determined as follows. Note that the homology class $[c]$ can be decomposed as a linear combination of classes $[c']$ with $c' \in \mathcal{A}_\sigma$. It is straightforward to check that each $c' \in \mathcal{A}_\sigma$ which appears in this decomposition yields a contribution $-h_{c'}$ to \hat{f}_c . These contributions sum up to give $-h_c$. What remains is the contribution from the vertex v_c . Part (i) of the lemma now follows easily by recalling the definition 7.6.

Part (ii) of the lemma is a trivial consequence of part (i). \square

14.3 Quantized Teichmüller spaces for surfaces with holes

The coordinates introduced in the previous subsection make it straightforward to modify the discussion of the quantization of Teichmüller spaces from the case of punctured Riemann surfaces (Section 10) to the case of Riemann surfaces with holes.

Bearing in mind that the Fock variables $z_e, e \in \varphi'_1$ are unconstrained in the present case (see Lemma 5) leads us to identify the algebra of functions on the Teichmüller spaces with the algebras of function of the variables $z_e, e \in \varphi'_1$. A convenient set of coordinates is given by the coordinate functions $(p_c, q_c), c \in \mathcal{C}_\sigma$. Canonical quantization of the Teichmüller spaces is therefore straightforward, and leads to an algebra of operators with generators $(p_c, q_c), c \in \mathcal{C}_\sigma$, which is irreducibly represented (in the sense of Remark 3) on the Hilbert space $\mathcal{H}_z(\sigma) \simeq L^2(\mathbb{R}^{3g-3+2s})$. We will normalize the operators $(p_c, q_c), c \in \mathcal{C}_\sigma$, such that

$$\begin{aligned} \text{(i)} \quad & [p_c, q_{c'}] = (2\pi i)^{-1} \delta_{cc'}, \quad c, c' \in \mathcal{C}_\sigma, \\ \text{(ii)} \quad & [p_c, p_{c'}] = 0 = [q_c, q_{c'}]. \end{aligned} \tag{14.5}$$

Quantization of the Kashaev space W_{φ_σ} produces a *reducible* representation of the algebra (14.5) which is generated by operators (\hat{q}_c, \hat{p}_c) associated to the pairs of variables (\hat{q}_c, \hat{p}_c) . It also yields operators $h_c, c \in H_1(\Sigma, \mathbb{R})$ which represent the quantization of the Poisson vector space H_{φ_σ} with basis $h_c, c \in H_1(\Sigma, \mathbb{R})$. The algebra generated by the $h_c, c \in H_1(\Sigma, \mathbb{R})$ has a center generated by the $h_c, c \in B(\Sigma)$. Following the discussion at the beginning of Subsection 10.3 one constructs a representation of this algebra on the space

$$\mathcal{H}_h(\sigma) \equiv \int_{B'(\Sigma)} d\mathfrak{f} \, \mathcal{H}_{h,\mathfrak{f}}(\sigma),$$

where $\mathcal{H}_{h,\mathfrak{f}}(\sigma) \simeq L^2(\mathbb{R}^g)$ is an irreducible representation of the algebra $i[h_{c_2}, h_{c_1}] = b^2 I(c_2, c_1)$ for $c_2, c_1 \in H_1(\Sigma_{cl}, \mathbb{R})$.

The following Proposition 8 describes how the quantized Teichmüller spaces are related to the quantized Kashaev space.

Proposition 8. *There exists a unitary operator l_σ ,*

$$l_\sigma : \mathcal{K}(\varphi_\sigma) \rightarrow \mathcal{H}_z(\sigma) \otimes \mathcal{H}_h(\sigma)$$

such that

$$\begin{aligned} l_\sigma \cdot \hat{p}_c \cdot l_\sigma^{-1} &= p_c \otimes 1, & \text{and} & & l_\sigma \cdot h_c \cdot l_\sigma^{-1} &= 1 \otimes h'_c, \\ l_\sigma \cdot \hat{q}_c \cdot l_\sigma^{-1} &= q_c \otimes 1, \end{aligned}$$

for any $e \in \sigma_1$ and $c \in H_1(\Sigma, \mathbb{R})$, respectively.

Let us recall that a move $m \in [\tau_m, \sigma_m]$, $m \in \mathcal{M}_1(\Sigma)$ is admissible if both τ_m and σ_m are admissible. Given an element $m \in \mathcal{M}_1^{\text{ad}}(\Sigma)$ it is natural to consider the corresponding fat graphs $\theta_m \equiv \varphi_{\tau_m}$, $\varphi_m \equiv \varphi_{\sigma_m}$ on Σ , to pick a path $\pi_m \in [\theta_m, \varphi_m]$ and consider the operator $u(m) \equiv u(\pi_m)$. The reduction to the quantized Teichmüller spaces proceeds as in Subsection 11.4. By multiplying the operators $l_\sigma \cdot u(m) \cdot l_\sigma^{-1}$ with suitably chosen operators $H(m)$ one gets operators $v(m)$ which factorize as $v(m) = w(m) \otimes 1$. The resulting operators $w(m) : \mathcal{H}_z(\varphi_m) \rightarrow \mathcal{H}_z(\theta_m)$ will then generate a unitary projective representation of the path groupoid of $\mathcal{M}^{\text{ad}}(\Sigma)$.

15. Geodesic length operators

Of fundamental importance for us will be to define and study quantum analogs of the geodesic length functions on the Teichmüller spaces, the geodesic length operators.

15.1 Overview

When trying to define operators which represent the geodesic length functions one has to face the following difficulty: The classical expression for $L_{\varphi,c} \equiv 2 \cosh \frac{1}{2} l_c$ as given by formula 8.6 is a linear combination of monomials in the variables $e^{\pm \frac{z_e}{2}}$ of a very particular form,

$$L_{\varphi,c} = \sum_{\tau \in \mathcal{F}} C_{\varphi,c}(\tau) e^{x(\tau)}, \quad x(\tau) \equiv \sum_{e \in \varphi_1} \tau(e) z_e, \quad (15.1)$$

where the summation is taken over the space \mathcal{F} of all maps $\varphi_1 \ni e \rightarrow \tau(e) \in \frac{1}{2}\mathbb{Z}$. The coefficients $C_{\varphi,c}(\tau)$ are positive integers, and non-vanishing for a finite number of $\tau \in \mathcal{F}$ only.

In the quantum case one is interested in the definition of length operators $L_{\varphi,c}$ which should be representable by expressions similar to (15.1),

$$L_{\varphi,c} = \sum_{\tau \in \mathcal{F}} C_{\varphi,c}^b(\tau) e^{x(\tau)}, \quad x(\tau) \equiv \sum_{e \in \varphi_1} \tau(e) z_e. \quad (15.2)$$

The following properties seem to be indispensable if one wants to interpret an operator of the general form (15.2) as the quantum counterpart of the functions $L_{\varphi,c} = 2 \cosh \frac{1}{2} l_c$:

- (a) **Spectrum:** $L_{\varphi,c}$ is self-adjoint. The spectrum of $L_{\varphi,c}$ is simple and equal to $[2, \infty)$. This is necessary and sufficient for the existence of an operator $l_{\varphi,c}$ - the *geodesic length operator* - such that $L_{\varphi,c} = 2 \cosh \frac{1}{2} l_c$.

(b) **Commutativity:**

$$[\mathbb{L}_{\varphi,c}, \mathbb{L}_{\varphi,c'}] = 0 \quad \text{if } c \cap c' = \emptyset.$$

(c) **Mapping class group invariance:**

$$\mathbf{a}_\mu(\mathbb{L}_{\varphi,c}) = \mathbb{L}_{\mu.\varphi,c}, \quad \mathbf{a}_\mu \equiv \mathbf{a}_{[\mu.\varphi,\varphi]}, \quad \text{for all } \mu \in \text{MC}(\Sigma).$$

(d) **Classical limit:** The coefficients $C_{\varphi,c}^b(\tau)$ which appear in (15.2) should be deformations of the classical coefficients $C_{\varphi,c}(\tau)$ in the sense that

$$\lim_{b \rightarrow 0} C_{\varphi,c}^b(\tau) = C_{\varphi,c}(\tau).$$

Length operators were first defined and studied in the pioneering work [CF2]. It was observed in [CF2] that the necessary deformation of the coefficients $C_{\varphi,c}^b(\tau)$ is indeed nontrivial in general. However, a full proof that the length operators introduced in [CF2] fulfil the requirements (a) and (c) does not seem to be available yet. We will therefore present an alternative approach to this problem, which will allow us to define length operators that satisfy (a)-(d) in full generality.

15.2 Construction of the length operators

Our construction of the length operators will proceed in two steps. First, we will construct length operators $\mathbb{L}_{\sigma,c}$ in the case that the fat graph φ under consideration equals φ_σ . This will facilitate the verification of the properties (a)-(d) formulated above. In order to define the length operators $\mathbb{L}_{\varphi,c}$ in the general case we shall then simply pick any marking σ such that the given curve c is contained in the cut system \mathcal{C}_σ , and define

$$\mathbb{L}_{\varphi,c} \equiv \mathbf{a}_{[\varphi,\varphi_\sigma]}(\mathbb{L}_{\sigma,c}). \quad (15.3)$$

Independence of this construction from the choice of σ will follow from Theorem 6 below. Definition (15.3) reduces the proof of properties (a), (b) to the proof of the corresponding statements for the length operators $\mathbb{L}_{\sigma,c}$ which will be given below. Property (c) follows from $\mathbf{a}_{[\mu.\varphi,\varphi]} \circ \mathbf{a}_{[\varphi,\varphi_\sigma]} = \mathbf{a}_{[\mu.\varphi,\varphi_\sigma]}$.

In order to prepare for our construction of length operators it is useful us recall the construction of the fat graph φ_σ in section 14.

Definition 6.

Case $c \in \mathcal{A}_\sigma$: Let A be an annular neighborhood of c such that the part of φ_σ which is contained in A is isotopic to the model depicted in Figure 21. We will then define

$$\mathbb{L}_{\varphi,c} \equiv e^{-2\pi b q_c} + 2 \cosh 2\pi b p_c. \quad (15.4)$$

Case $c \notin \mathcal{A}_\sigma$: If a curve c is not contained in \mathcal{A}_σ , it is necessarily the outgoing boundary component of a trinion T_p (cf. Figure 18). Let $c_\epsilon, \epsilon = 1, 2$ be the curves which represent the incoming boundary components of T_p as indicated in Figure 22. Given that $\mathbb{L}_{\sigma,c_i}, i = 1, 2$ are already defined we will define $\mathbb{L}_{\sigma,c}$ by

$$\mathbb{L}_{\sigma,c} = 2 \cosh(y_{c_2} + y_{c_1}) + e^{-y_{c_2}} \mathbb{L}_{\sigma,c_1} + e^{y_{c_1}} \mathbb{L}_{\sigma,c_2} + e^{y_{c_1} - y_{c_2}}, \quad (15.5)$$

where $y_{c_\epsilon}, \epsilon = 1, 2$ are defined as $y_{c_2} = 2\pi b q_c, y_{c_1} = -2\pi b p_c$.

It is easy to see that this recursively defines length operators for all remaining $c \in \mathcal{C}_\sigma$.

Remark 9. Let us note that (14.5) implies that $[y_{c_\epsilon}, L_{\sigma, c_{\epsilon'}}] = 0$ for $\epsilon, \epsilon' \in \{1, 2\}$. We therefore do not have an issue of operator ordering in (15.5).

Proposition 9. *The length operators $L_{\sigma, c}, L_{\sigma, c'}$ associated to different curves $c, c' \in \mathcal{C}_\sigma$ commute with each other.*

Proof. Let us recall that cutting the surface Σ along all of the curves $c \in \mathcal{A}_\sigma$, yields a set of connected components which all have genus zero. For a given curve $c \in \mathcal{C}_\sigma \setminus \mathcal{A}_\sigma$ let Σ_c be the connected component which contains c . Cutting Σ_c along c produces two connected components. The component which has c as its outgoing boundary component will be denoted Σ'_c . It follows from Definition 6 that $L_{\sigma, c}$ is an operator function of the operators p_d and q_d , where $d \in \mathcal{C}_\sigma$ is contained in Σ'_c . The claim therefore follows immediately from (14.5) if Σ'_c and $\Sigma'_{c'}$ are disjoint.

Otherwise we have the situation that one of $\Sigma'_c, \Sigma'_{c'}$, say $\Sigma'_{c'}$ is a subsurface of the other. The crucial point to observe is that the resulting expression for $L_{\sigma, c}$ depends on the variables p_d and q_d associated to the subsurface $\Sigma'_{c'}$ exclusively via $L_{\sigma, c'}$. The claim therefore again follows from (14.5). \square

The following theorem expresses the consistency of our definition with the automorphisms induced by a change of the marking σ .

Theorem 6. *For a given curve c let $\sigma_i, i = 1, 2$ be markings such that c is contained in both cut systems \mathcal{C}_{σ_1} and \mathcal{C}_{σ_2} . We then have*

$$a_{[\varphi_{\sigma_2}, \varphi_{\sigma_1}]}(L_{\sigma_1, c}) = L_{\sigma_2, c}. \quad (15.6)$$

On the proof of Theorem 6: The description of the modular groupoid in terms of generators and relations given in section 13 reduces the proof of Theorem 6 to the case that σ_2 and σ_1 are connected by one of the elementary moves m defined in section 13.2. In order to reduce the proof of Theorem 6 to a finite number of verifications one would need to have simple standard choices for the paths $\pi_m \in [\varphi_{\sigma_2}, \varphi_{\sigma_1}]$ for all elementary moves m . Existence of such standard paths π_m turns out to be nontrivial, though. The task to find suitable paths π_m is particularly simple for a subclass of moves m which is defined as follows.

Definition 7.

- (i) *Let $\pi \in [\varphi', \varphi]$ be a path in the complex $\mathcal{Pt}(\Sigma)$ which is described by a sequence $S_\pi \equiv (\varphi' \equiv \varphi_n, \dots, \varphi_1 \equiv \varphi)$ of fat graphs such that consecutive elements of S_π are connected by edges in $\mathcal{Pt}_1(\Sigma)$. We will say that $\pi \in [\varphi', \varphi]$ is realized locally in a subsurface $\Sigma' \hookrightarrow \Sigma$ if the restrictions of φ_i to $\Sigma \setminus \Sigma'$ coincide for all $i = 1, \dots, n$.*
- (ii) *We will say that the move $m = [\tau_m, \sigma_m] \in \mathcal{M}'_1(\Sigma)$ can be realized locally if there exists a path $\pi \in [\theta_m, \varphi_m]$ in the complex $\mathcal{Pt}(\Sigma)$ that is realized locally in Σ_m in the sense of (i).*

For moves $m = [\tau_m, \sigma_m] \in \mathcal{M}_1(\Sigma)'$ that can be realized locally we may choose essentially the same path $\pi_m \in [\theta_m, \varphi_m]$ for all surfaces Σ into which Σ_m can be embedded. It is then crucial to observe the following fact

Proposition 10. *If $m \in \mathcal{M}'_1(\Sigma)$ is admissible, but can not be realized locally, there always exists a path ϖ_m which is (i) homotopic to m within $\mathcal{M}^{\text{ad}}(\Sigma)$, and (ii) takes the form*

$$\varpi_m = Y_m \circ m \circ Y_m^{-1}, \quad (15.7)$$

where Y_m is a chain composed out of Z_p -moves and F_{pq} -moves which can all be realized locally.

The proof of Proposition 10 is given in Appendix D. It therefore suffices to prove Theorem 6 in the case that σ_2 and σ_1 are connected by any elementary move m that can be realized locally. This amounts to a finite number of verifications which can be carried out by straightforward, but tedious calculations. Some details are given in the Appendix E. \square

15.3 Spectrum

Theorem 7. *The spectrum of $L_{\sigma,c}$ is simple and equal to $[2, \infty)$.*

Proof. To begin with, let us consider the following simple model for the length operators:

$$L \equiv 2 \cosh 2\pi b p + e^{-2\pi b q}, \quad (15.8)$$

where p, q are operators on $L^2(\mathbb{R})$ which satisfy the commutation relations $[p, q] = (2\pi i)^{-1}$.

A basic fact is that L is self-adjoint. Indeed, being a sum of two *positive* self-adjoint operators L is self-adjoint on the intersection of the domains of the summands. The main spectral properties of this operator are summarized in the following proposition.

Proposition 11. — [Ka4] —

(i) *We have $\text{Spec}(L) = (2, \infty)$.*

(ii) *The spectrum of L in $L^2(\mathbb{R})$ is simple.*

Validity of Theorem 7 in the case $c \in \mathcal{A}_\sigma$ is a direct consequence of Proposition 11. It remains to treat the case $c \notin \mathcal{A}_\sigma$. We will keep the notations introduced in Definition 6. The main ingredient will be an operator $C_{\sigma,c} : \mathcal{H}(\sigma) \rightarrow \mathcal{H}(\sigma)$ which maps all length operators $L_{\sigma,c}$ to the simple standard form $L_{\sigma,c}^{\text{st}}$,

$$L_{\sigma,c}^{\text{st}} = 2 \cosh 2\pi b p_c + e^{-2\pi b q_c} \quad (15.9)$$

in the sense that the following commutation relations are satisfied:

$$C_{\sigma,c} \cdot L_{\sigma,c} = L_{\sigma,c}^{\text{st}} \cdot C_{\sigma,c}. \quad (15.10)$$

Definition 8. *Let the unitary operator $C_{\sigma,c}$ be defined by*

$$C_{\sigma,c}^{-1} = e_b(q_c - s_2) \frac{s_b(s_1 - p_c)}{s_b(s_1 + p_c)} e^{2\pi i s_2 q_c}, \quad (15.11)$$

where

$$s_\epsilon = (2\pi b)^{-1} \text{arccosh} \frac{1}{2} L_{\sigma,c_\epsilon} \quad \epsilon = 1, 2,$$

and the special function $s_b(x)$ is a close relative of $e_b(x)$ defined in the Appendix A.

Lemma 11. *The unitary operator $C_{\sigma,p}$ satisfies (15.10).*

Proof. Proposition 11 is proven by means of a direct calculation using the explicit form of $L_{\sigma,p}$ given in (15.5) and the functional equations (A.3) and (A.9). \square

The proof of Theorem 7 is thereby reduced to Proposition 11. \square

15.4 Relation with the Dehn twist generator

To round off the picture we shall now discuss, following [Ka3, Ka4], the relation between the length operators $L_{\varphi,c}$ and the operator $D_{\varphi,c}$ which represents the Dehn twist D_c along c .

A closed curve c will be called a curve of simple type if the connected components of $\Sigma \setminus c$ all have more than one boundary component. It is not hard to see (using the construction in Section 14, for example) that for curves of simple type there always exists a fat graph φ and an annular neighborhood A_c of c in which φ takes the form depicted in Figure 23.

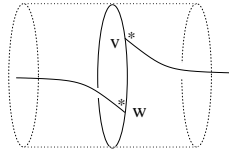


Figure 23: Annulus A_c and fat graph φ on A .

Lemma 12. *Let Σ be a surface with genus g and s holes. The pure mapping class group $MC(\Sigma)$ is generated by the Dehn twists along geodesics of simple type.*

Proof. For $g \geq 2$ it is known that the Dehn twists along non-separating closed curves suffice to generate the mapping class group $MC(\Sigma)$ [Ge]. Closed curves which are non-separating are always of simple type.

For $g = 0$ any closed curve is of simple type. In the remaining case $g = 1$ one may note that the only closed curve c which is not of simple type is the one which separates a one-holed torus from the rest of Σ . It is then well-known that the Dehn twists along a - and b -cycles of the one-holed torus generate the Dehn-twist along c . If we supplement these generators by the Dehn twists along the remaining closed curves (which are all of simple type), we get a complete system of generators for $MC(\Sigma)$ [Ge]. \square

We may therefore use a fat graph which in an annular neighborhood of c takes the form depicted in Figure 23. It is easy to see that the action of the Dehn twist D_c on the fat graph φ can be undone by a single flip ω_{vw} . As the representative $F_{\varphi,c}$ for the Dehn twist D_c on $\mathcal{K}(\varphi)$ we may therefore choose $F_{\varphi,c} = T_{vw}$.

The operator $h_{\varphi,c}$ associated to the homology cycle c is $h_{\varphi,c} \equiv \frac{1}{2}(p_v + q_w)$. It is not very difficult to verify that the operator

$$D_{\varphi,c} = \zeta^{-6} \exp(2\pi i h_{\varphi,c}^2) F_{\varphi,c}. \quad (15.12)$$

commutes with all h_c , $c \in H_1(\Sigma, \mathbb{R})$. The prefactor ζ^{-6} , $\zeta = e^{\pi i b^2/3}$ was inserted to define a convenient normalization. We then have the following result.

Proposition 12. [Ka4] — D_c coincides with the following function of the length operator l_c :

$$D_c = \zeta^{-6} \exp\left(i \frac{l_c^2}{8\pi b^2}\right).$$

This should be compared with the classical result that the geodesic length functions are the Hamiltonian generators of the Fenchel-Nielsen twist flow, which reproduces the Dehn twist for a twist angle of 2π .

16. Passage to the length representation

16.1 The length representation

Our aim is to define a representation for $\mathcal{H}(\Sigma)$ which is such that the length operators associated to a cut system are all realized as multiplication operators. For a surface Σ and a marking σ on Σ let

$$\mathcal{H}_L(\sigma) \equiv \mathcal{K}_{\text{sp}}^{\otimes \sigma_1}, \quad (16.1)$$

where $\mathcal{K}_{\text{sp}} \simeq L^2(\mathbb{R}_+, \eta)$, and η is the spectral measure of the operator $l_A = 2\text{arcosh}\frac{1}{2}L_A$. The numbering of the edges associated to σ defines canonical isomorphisms

$$\mathcal{H}_L(\sigma) \simeq \mathcal{H}_L(\Sigma) \equiv L^2(\mathfrak{L}, d\eta_{\mathfrak{L}}), \quad (16.2)$$

where $\mathfrak{L} \simeq \mathbb{R}_+^{3g-3+2s}$ and $d\eta_{\mathfrak{L}} = d\eta(l_1) \wedge \cdots \wedge d\eta(l_{3g-3+2s})$ is the corresponding product measure. For $e \in \sigma_1$ and $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ let us define the multiplication operator $m_e[f]$ by

$$(m_e[f]\Psi)(\lambda_\sigma) \equiv f(l_e)\Psi(\lambda_\sigma). \quad (16.3)$$

The operator $m_e[f]$ will represent a bounded operator $m_e[f] : \mathcal{H}_L(\sigma) \rightarrow \mathcal{H}_L(\sigma)$ iff $f \in L^\infty(\mathbb{R}^+, d\eta)$.

For the rest of this section let us consider an admissible marking σ and the associated fat graph φ_σ . The Definition 6 yields a family of mutually commuting length operators $L_{\sigma,c}$ associated to the curves c which constitute the cut system \mathcal{C}_σ . It follows from the spectral theorem⁹ for the family $\{L_e; e \in \sigma_1\}$ of self-adjoint operators that there exists a unitary operator

$$D(\sigma) : \mathcal{H}_z(\sigma) \longrightarrow \mathcal{H}_L(\sigma)$$

that diagonalizes the set of length operators $L_{\sigma,e}$, $e \in \sigma_1$ in the sense that

$$D(\sigma) \cdot L_{\sigma,e} = m_e[2 \cosh \frac{1}{2}l] \cdot D(\sigma). \quad (16.4)$$

Our aim for the rest of this section will be to give a recursive construction for such an operator $D(\sigma)$ in terms of operators C_p which are associated to the vertices $p \in \sigma_\circ$ of σ .

⁹See Appendix B for the precise statement.

16.2 Construction of the operator $D(\sigma)$

The main ingredient will be an operator $C_\sigma : \mathcal{H}(\sigma) \rightarrow \mathcal{H}(\sigma)$ which maps all length operators $L_{\sigma,c}$ to the simple standard form L_c^{st} ,

$$L_{\sigma,c}^{\text{st}} = e^{-2\pi b q_c} + 2 \cosh 2\pi b p_c. \quad (16.5)$$

The operators $C_{\sigma,c}$ that were defined in Definition 8 solve this task locally for every curve $c \in \mathcal{C}_\sigma \setminus \mathcal{A}_\sigma$. If the trinion T_p has c_p as the outgoing boundary component we will define

$$C_{\sigma,p} \equiv C_{\sigma,c_p}. \quad (16.6)$$

We are now ready to define

$$C_\sigma \equiv \prod_{p \in \sigma_0} C_{\sigma,p}. \quad (16.7)$$

Let us note that we do not have to specify the order in which the operators $C_{\sigma,p}$ appear thanks to the following lemma.

Lemma 13. *We have $C_{\sigma,p} C_{\sigma,q} = C_{\sigma,q} C_{\sigma,p}$.*

Proof. This follows from Proposition 9 and Definition 6, keeping in mind (14.5). \square

In order to construct the sought-for operator $D(\sigma)$ it now remains to map the length operators $L_{\sigma,e}^{\text{st}}$ to multiplication operators. Proposition 11 ensures existence of an operator

$$d_{\sigma,e} : L^2(\mathbb{R}) \rightarrow \mathcal{K}_{\text{sp}} \quad \text{such that} \quad d_{\sigma,e} \cdot L_e^{\text{st}} = m_e [2 \cosh \tfrac{1}{2} l] \cdot d_{\sigma,e}.$$

Out of E_σ we may finally define the operator $D(\sigma)$ as

$$D(\sigma) \equiv d_\sigma \cdot C_\sigma, \quad d_\sigma \equiv \bigotimes_{e \in \sigma_1} d_{\sigma,e}, \quad (16.8)$$

where l_σ is the operator introduced in Proposition 4. It is straightforward to verify that the operator $D(\sigma)$ indeed satisfies the desired property (16.4).

17. Realization of $M(\Sigma)$

17.1 Two constructions for the generators

Our aim is to define operators $U(m)$ associated to the edges $m \in \mathcal{M}_1(\Sigma)$. We will give two constructions for these operators, each of which makes certain properties manifest. The proof of the equivalence of these two constructions will be the main difficulty that we will have to deal with.

The first construction. Let us recall that a move $m \in [\tau_m, \sigma_m]$, $m \in \mathcal{M}_1(\Sigma)$ is admissible if both τ_m and σ_m are admissible. Given an element $m \in \mathcal{M}_1^{\text{ad}}(\Sigma)$ it is natural to consider the corresponding fat graphs $\theta_m \equiv \varphi_{\tau_m}$, $\varphi_m \equiv \varphi_{\sigma_m}$ on Σ , to pick a path $\pi_m \in [\theta_m, \varphi_m]$ and define operators $\tilde{U}(m) : \mathcal{H}_L(\sigma_m) \rightarrow \mathcal{H}_L(\tau_m)$ as

$$\tilde{U}(m) \equiv D(\tau_m) \cdot u(\pi_m) \cdot D(\sigma_m)^\dagger. \quad (17.1)$$

The second construction. We note that for all $m \in \mathcal{M}_1(\Sigma)$ the markings τ_m and σ_m will coincide outside of a subsurface $\Sigma_m \hookrightarrow \Sigma$. Let us therefore consider the restrictions τ'_m and σ'_m of τ_m and σ_m to Σ_m respectively. Admissibility of τ'_m and σ'_m is obvious for $m \in \mathcal{M}_1(\Sigma)$, allowing us to use the first construction in order to define an operator

$$U'(m) : \mathcal{H}_L(\sigma'_m) \rightarrow \mathcal{H}_L(\tau'_m).$$

Out of $U'(m)$ we may then construct the sought-for operator $U(m)$ by acting with $U'(m)$ non-trivially only on those tensor factors of $\mathcal{H}_L(\sigma) = \mathcal{K}_{\text{sp}}^{\otimes \sigma_1}$ which correspond to the subsurface $\Sigma_m \hookrightarrow \Sigma$. More precisely, let $\mathbb{E}_m \subset \sigma_1$ be the set of edges in σ_1 which have nontrivial intersection with Σ_m . Out of $U'(m)$ let us then define the operator $U(m) : \mathcal{K}(\sigma_m) \rightarrow \mathcal{K}(\tau_m)$ by applying definition (B.5) to the case $\mathcal{O} \equiv U'(m)$ and $\mathbb{J} \equiv \mathbb{E}_m$.

Comparison. The crucial difference between $\tilde{U}(m)$ and $U(m)$ is that the latter is manifestly acting *locally* in $\mathcal{H}_L(\sigma)$, in the sense that it acts only on the tensor factor of $\mathcal{H}_L(\sigma)$ which corresponds to the subsurface Σ_m . This is not obvious in the case of $\tilde{U}(m)$.

The length operators $L_{\sigma,c}$ associated to the boundary components $c \in A(\Sigma)$ form a commutative family of operators. The joint spectral decomposition for this family of operators leads us to represent $\mathcal{H}_L(\Sigma)$ as

$$\mathcal{H}_L(\Sigma) \simeq \int_{\mathfrak{L}} d\eta_{\Sigma}(\mathfrak{c}) \mathcal{H}_L(\Sigma, \mathfrak{c}), \quad (17.2)$$

where the integration is extended over the set $\mathfrak{L} \simeq \mathbb{R}_+^s$ of all colorings \mathfrak{c} of the boundary by elements of \mathbb{R}_+ . It follows from Theorem 6 that the operators $U(m)$, $\tilde{U}(m)$ commute (up to permutations of the boundary components) with the length operators $L_{\sigma,c}$, $c \in A(\Sigma)$. Within the representation (17.2) one may therefore¹⁰ represent $U(m)$ and $\tilde{U}(m)$ by families of operators $(U(m, \mathfrak{c}))_{\mathfrak{c} \in \mathfrak{L}}$ and $(\tilde{U}(m, \mathfrak{c}))_{\mathfrak{c} \in \mathfrak{L}}$, respectively.

17.2 The main result

The following theorem is the main result of this paper.

Theorem 8. *The operators $U(m, \mathfrak{c})$, $\mathfrak{c} \in \mathfrak{L}$, generate a tower of projective unitary representations of the modular groupoids $M(\Sigma)$.*

The proof of Theorem 8 will take up the rest of this subsection.

To begin with, let us note that the necessary structure (12.2) of the Hilbert spaces follows trivially from our definition of the length representation in Section 16, where in the present case we simply have $\mathcal{H}(S_3, \mathfrak{c}_3) \simeq \mathbb{C}$. It is furthermore clear that the operators $\tilde{U}(m, \mathfrak{c})$, $\mathfrak{c} \in \mathfrak{L}$ generate a unitary projective representation of the modular groupoid $M(\Sigma)$ for each surface Σ within the considered class. Let us finally note that the naturality properties formulated in Subsection 12.3 clearly hold for the operators $U(m, \mathfrak{c})$, $\mathfrak{c} \in \mathfrak{L}$. Our main task is therefore to show that $\tilde{U}(m, \mathfrak{c}) = U(m, \mathfrak{c})$, as will be established in Proposition 13 below.

¹⁰According to Proposition 14 in Appendix B

Proposition 13. *For all $m \in \mathcal{M}_1^{\text{ad}}(\Sigma)$ there exists a path $\pi_m \in [\theta_m, \varphi_m]$ such that we have*

$$\tilde{U}(m, \mathfrak{c}) = U(m, \mathfrak{c}).$$

Proof. When we compare the respective definitions of $U(m)$ and $\tilde{U}(m)$, we observe that there are two main discrepancies that we need to deal with. First, it is not always true that the path π_m can be realized locally in the sense of Definition 7. It may therefore not be clear a priori why there should exist a simple relation between $U(m)$ and $\tilde{U}(m)$.

Second, we may observe that the definition of the operators $U(m)$ and $\tilde{U}(m)$ involves products of operators $C_{\sigma,p}$, where the set of vertices p that the product is extended over is generically much smaller in the case of $U(m)$. This means that most of the factors $C_{\sigma,p}$ must ultimately cancel each other in the expression for $\tilde{U}(m)$. The first step will be to prove Proposition 13 in the case that m can be realized locally.

Lemma 14. *We have*

$$\tilde{U}(m) = U(m) \tag{17.3}$$

whenever $m = [\tau_m, \sigma_m] \in \mathcal{M}'_1(\Sigma)$ can be realized locally.

Proof. It follows from (15.11) and (14.5) that $C_{\sigma,p}$ can be represented as a function of the following operators

$$C_{\sigma,p} = C_{\sigma,p}(\mathfrak{p}_{v_p}, \mathfrak{q}_{v_p}; L_{c_2}, L_{c_1}),$$

where v_p is the vertex of φ_σ contained in the trinion T_p and $c_\epsilon \equiv c_\epsilon(p)$, $\epsilon = 1, 2$ are the curves which represent the outgoing and incoming boundary components of T_p respectively. Let us recall that the length operators L_{τ_m, c_ϵ} and L_{σ_m, c_ϵ} $\epsilon = 1, 2$ satisfy (15.6). These observations imply that

$$u(\pi_m) \cdot C_{\sigma_m,p} = C_{\tau_m,p} \cdot u(\pi_m), \tag{17.4}$$

whenever the operators $\mathfrak{p}_{\sigma,p}$ and $\mathfrak{q}_{\sigma,p}$ commute with $u(\pi_m)$. Our task is therefore to determine the set of all $p \in \sigma_\circ$ for which this is the case. The condition that m can be realized locally implies that (17.4) will hold unless p is located within Σ_m . The claim now follows straightforwardly from these observations. \square

In order to treat the general case let us recall that Proposition 10 implies that π_m may be chosen as $\pi_m = y_m \circ \hat{\pi}_m \circ y_m^{-1}$, where y_m is uniquely defined by the factorization of Y_m into elementary moves that can be realized locally, and $\hat{\pi}_m$ is the (fixed) path which was chosen to represent m in the case that m can be realized locally. This leads to the following representation for $\tilde{U}(m)$:

$$\begin{aligned} \tilde{U}(m) &= D(\tau_m) \cdot u(\pi_m) \cdot D(\sigma_m)^\dagger \\ &= D(\tau_m) \cdot u(y_m) \cdot u(\hat{\pi}_m) \cdot u(y_m^{-1}) \cdot D(\sigma_m)^\dagger. \end{aligned} \tag{17.5}$$

By using Lemma 14 one may deduce from (17.5) that the following holds:

$$\begin{aligned} \tilde{U}(m) &= \tilde{U}(Y_m) \cdot U(m) \cdot \tilde{U}(Y_m^{-1}) \\ &= U(Y_m) \cdot U(m) \cdot U(Y_m)^\dagger. \end{aligned} \tag{17.6}$$

It remains to observe that

Lemma 15. *We have $U(m_1) \cdot U(m_2) \cdot U(m_1)^\dagger = U(m_2)$ whenever $\text{supp}(m_1) \cap \text{supp}(m_2) = \emptyset$.*

Recall that we had defined the notation $\text{supp}(m)$ by $\text{supp}(m) = \{p\}$ if $m = Z_p, B_p, S_p$ and $\text{supp}(m) = \{p, q\}$ if $m = (pq), F_{pq}$.

Proof. Let us factorize $\mathcal{H}_L(\Sigma)$ as $\mathcal{H}_L(\Sigma) = \mathcal{K}_{\text{sp}}^{\otimes \beta_1} \otimes \mathcal{K}_{\text{sp}}^{\otimes \sigma_1 \setminus \beta_1}$, where $\beta_1 \subset \sigma_1$ is the set of all edges which end in boundary components of Σ . The representation 17.2 may be rewritten as

$$\mathcal{H}_L(\Sigma) \simeq \int_{\mathbb{R}_+^s}^\oplus d\eta_s(\mathbf{c}) \mathcal{H}_L(\Sigma, \mathbf{c}), \quad (17.7)$$

where $\mathcal{H}_L(\Sigma, \mathbf{c}) \simeq \mathcal{K}_{\text{sp}}^{\otimes \sigma_1 \setminus \beta_1}$ for all $\mathbf{c} \in \mathbb{R}_+^s$. Let $U(m, \mathbf{c})$, $\mathbf{c} \in \mathbb{R}_+^{s(m)}$ be the unitary operators on $\mathcal{H}_L(\Sigma_m, \mathbf{c})$ which represent the operators $U(m)$ in the representation (17.7). Within this representation it becomes almost trivial to complete the proof of Proposition 13. Let $\beta_{21} = \beta_2 \cup \beta_1$, where $\beta_j \subset \sigma_1$, $j = 1, 2$ are the sets of edges which correspond to boundary components of Σ_{m_j} . Let $\delta_j \subset \sigma_1$, $j = 1, 2$ be the sets of edges that are fully contained in the interior of Σ_{m_j} respectively. Let finally $\sigma'_1 = \sigma_1 \setminus (\beta_{21} \cup \delta_2 \cup \delta_1)$. We may then factorize $\mathcal{H}_L(\Sigma)$ in the following way:

$$\mathcal{H}_L(\Sigma) = \mathcal{K}_{\text{sp}}^{\otimes \beta_{21}} \otimes \mathcal{K}_{\text{sp}}^{\otimes \delta_2} \otimes \mathcal{K}_{\text{sp}}^{\otimes \delta_1} \otimes \mathcal{K}_{\text{sp}}^{\otimes \sigma'_1}, \quad (17.8)$$

where we define $\mathcal{K}_{\text{sp}}^{\otimes \delta} = \mathbb{C}$ if $\delta = \emptyset$. This may be rewritten as

$$\mathcal{H}_L(\Sigma) \simeq \int_{\mathbb{R}_+^{s_{21}}} d\mu(\mathbf{c}_{21}) \mathcal{H}_L(\Sigma, \mathbf{c}_{21}), \quad (17.9)$$

where $\mathcal{H}_L(\Sigma, \mathbf{c}_{21}) \simeq \mathcal{K}_{\text{sp}}^{\otimes \delta_2} \otimes \mathcal{K}_{\text{sp}}^{\otimes \delta_1} \otimes \mathcal{K}_{\text{sp}}^{\otimes \sigma'_1}$ for all $\mathbf{c}_{21} \in \mathbb{R}_+^{s_{21}}$, $s_{21} = \text{card}(\beta_{21})$. In the representation (17.9) we may represent $U(m_j)$, $j = 1, 2$ by families of operators $U(m_j, \mathbf{c}_{21})$ which take the form $U(m_2, \mathbf{c}_{21}) \simeq U_2(m_2, \mathbf{c}_{21}) \otimes \text{id} \otimes \text{id}$ and $U(m_1, \mathbf{c}_{21}) \simeq \text{id} \otimes U_1(m_1, \mathbf{c}_{21}) \otimes \text{id}$ respectively. The Lemma follows easily from these observations. \square

It follows from the Lemma that $U(Y_m) \cdot U(m) \cdot U(Y_m)^\dagger = U(m)$, which completes the proof of Proposition 13. \square

Taken together our results show that the quantization of Teichmüller spaces yields a tower of unitary projective representations of the modular groupoid in the sense of §12.3.

17.3 Concluding remarks

Let us recall that the mapping class group is a group of symmetries for both complexes $\mathcal{Pt}(\Sigma)$ and $\mathcal{M}(\Sigma)$. Having projective unitary representations of the associated groupoids $\mathcal{Pt}(\Sigma)$ and $\mathcal{MC}(\Sigma)$ will therefore induce corresponding representations of the mapping class group by means of the construction in §11.2. It is not hard to see It follows quite easily from Proposition 7 that these two representations are equivalent.

Finally it is clearly of interest to calculate the phases in the relations $u(\pi_\varpi) = \zeta^{\nu(\varpi)}$.

Conjecture 1. *There exists a definition for the operators $U(m)$, $m \in \mathcal{M}_1(\Sigma)$ such that the phases $\zeta^{\nu(\varpi)}$ which appear in the relations $u(\pi_\varpi) = \zeta^{\nu(\varpi)}$ are trivial for all but one $\varpi \in$*

$\mathcal{M}_2(\Sigma)$, which can be chosen as the relation (13.9),a). The phase which appears in the relation (13.9),a) is given as

$$\zeta^{\nu(\varpi)} = e^{\frac{\pi i}{2} c_L}, \quad c_L \equiv 1 + 6(b + b^{-1})^2.$$

We now believe to have a proof of this conjecture. Details will appear elsewhere. Our conjecture is also strongly supported by the calculation in [Ka2] which establishes a similar result for the realization of the *pure* mapping class group on the quantized Teichmüller spaces.

A. The special functions $e_b(x)$ and $s_b(x)$

The function $s_b(x)$ may be defined with the help of the following integral representation.

$$\log s_b(x) = \frac{1}{i} \int_0^\infty \frac{dt}{t} \left(\frac{\sin 2xt}{2 \sinh bt \sinh b^{-1}t} - \frac{x}{t} \right). \quad (\text{A.1})$$

This function, or close relatives of it like

$$e_b(x) = e^{\frac{\pi i}{2}x^2} e^{-\frac{\pi i}{24}(2-Q^2)} s_b(x), \quad (\text{A.2})$$

have appeared in the literature under various names like “Quantum Dilogarithm” [FK], “Hyperbolic G-function” [Ru], “Quantum Exponential Function” [Wor] and “Double Sine Function”, we refer to the appendix of [KLS] for a useful collection of properties of $s_b(x)$ and further references. The most important properties for our purposes are

$$(i) \text{ Functional equation: } s_b(x - i\frac{b}{2}) = 2 \cosh \pi b x s_b(x + i\frac{b}{2}). \quad (\text{A.3})$$

$$(ii) \text{ Analyticity: } s_b(x) \text{ is meromorphic,}$$

$$\text{poles: } x = c_b + i(nb - mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}. \quad (\text{A.4})$$

$$\text{zeros: } x = -c_b - i(nb - mb^{-1}), n, m \in \mathbb{Z}^{\geq 0}.$$

$$(iii) \text{ Self-duality: } s_b(x) = s_{1/b}(x). \quad (\text{A.5})$$

$$(iv) \text{ Inversion relation: } s_b(x) s_b(-x) = 1. \quad (\text{A.6})$$

$$(v) \text{ Unitarity: } \overline{s_b(x)} = 1/s_b(\bar{x}). \quad (\text{A.7})$$

$$(vi) \text{ Residue: } \text{res}_{x=c_b} s_b(x) = e^{-\frac{\pi i}{12}(1-4c_b^2)} (2\pi i)^{-1}. \quad (\text{A.8})$$

The function $e_b(x)$ clearly has very similar properties as $s_b(x)$. We list the properties that get modified compared to $s_b(x)$ below.

$$(i)' \text{ Functional equation: } e_b(x - i\frac{b}{2}) = (1 + e^{2\pi b x}) e_b(x + i\frac{b}{2}). \quad (\text{A.9})$$

$$(iv)' \text{ Inversion relation: } e_b(x) e_b(-x) = e^{\pi i x^2} e^{-\frac{\pi i}{6}(1+2c_b^2)}. \quad (\text{A.10})$$

$$(vi)' \text{ Residue: } \text{res}_{x=c_b} s_b(x) = (2\pi i)^{-1}. \quad (\text{A.11})$$

Among the most remarkable properties satisfied by the function $e_b(x)$ is the so-called pentagonal relation (A.12) which underlies the validity of the pentagonal relation (11.34),

$$e_b(p) \cdot e_b(q) = e_b(q) \cdot e_b(p+q) \cdot e_b(p), \quad (\text{A.12})$$

Relation (A.12) is valid if q and p represent $[p, q] = (2\pi i)^{-1}$ on $L^2(\mathbb{R})$. Equation (A.12) in turn is equivalent to the following property of the function $E_b(x) \equiv e_b(-\frac{x}{2\pi b})$:

$$E_b(U) \cdot E_b(V) = E_b(U+V), \quad (\text{A.13})$$

where $U = e^{2\pi b q}$, $V = e^{2\pi b p}$. Proofs of (A.12) and (A.13) can be found in [Wor, FKV, BT, Vo].

B. Operator-theoretical background

Let \mathcal{H} be a separable Hilbert space. The algebra of bounded operators in \mathcal{H} will be denoted by $\mathbf{B}(\mathcal{H})$. We will only need some of the most basic notions and results from functional analysis and the theory of operator algebras as summarized e.g. in [Wa, Chapter 14]. For the reader's convenience and to fix some notations we shall formulate the results that we need below.

Theorem 9. — [Wa, Theorem 14.8.14] —

Let \mathbf{C} be a commutative von Neumann subalgebra of $\mathbf{B}(\mathcal{H})$, where \mathcal{H} is a separable Hilbert space. Then there exists a compact, separable Hausdorff space X , a Radon measure μ on X , a measurable family of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$, and a unitary bijection

$$\mathbf{U} : \mathcal{H} \rightarrow \mathcal{H}_{\mathbf{C}} \equiv \int_X d\mu(s) \mathcal{H}_s \quad (\text{B.1})$$

such that

$$\mathbf{U} \cdot \mathbf{C} \cdot \mathbf{U}^\dagger = \{ \mathbf{m}_f ; f \in L^\infty(x, d\mu) \}, \quad (\text{B.2})$$

where \mathbf{m}_f is the multiplication operator defined by

$$\mathbf{m}_f : \mathcal{H}_{\mathbf{C}} \ni \{ \Psi_x \}_{x \in X} \rightarrow \{ f(x) \Psi_x \}_{x \in X} \in \mathcal{H}_{\mathbf{C}}. \quad (\text{B.3})$$

If \mathbf{C} is a commutative von Neumann subalgebra of $\mathbf{B}(\mathcal{H})$ we will call an operator \mathbf{O} on \mathcal{H} \mathbf{C} -decomposable if there exists a family of bounded operators \mathbf{O}_x , defined on \mathcal{H}_x for μ -almost every $x \in X$, such that

$$\mathbf{U} \cdot \mathbf{O} \cdot \mathbf{U}^\dagger = \int_X d\mu(s) \mathbf{O}_s. \quad (\text{B.4})$$

Let $\mathbf{D}_{\mathbf{C}}$ be the algebra of all \mathbf{C} -decomposable operators.

Proposition 14. — [Wa, Proposition 14.8.8] — $\mathbf{D}_{\mathbf{C}}$ is the commutant of \mathbf{C} within $\mathbf{B}(\mathcal{H})$.

The unbounded operators that we will have to deal with will all be self-adjoint. We will freely use standard functional calculus for self-adjoint operators. When we say that two unbounded self-adjoint operators commute,

$$[A, B] = 0,$$

we will always mean commutativity of the spectral projections. Let $\mathcal{F} = \{A_i ; i \in \mathcal{I}\}$ be a family of commuting self-adjoint operators defined on dense domains in a separable Hilbert space \mathcal{H} . Standard functional calculus for commuting self-adjoint operators associates to \mathcal{F} a commutative von Neumann subalgebra $\mathbf{C}_{\mathcal{F}}$ of $\mathbf{B}(\mathcal{H})$. Theorem 9 applied to $\mathbf{C}_{\mathcal{F}}$ yields the existence of a common spectral decomposition for the family \mathcal{F} , where X represents the one-point compactification of the spectrum of \mathcal{F} .

Tensor product notation

For a given finite set I we will often use the notation $\mathcal{H}^{\otimes \mathbb{I}}$ instead of $\mathcal{H}^{\otimes \text{card}(\mathbb{I})}$, where $\text{card}(\mathbb{I})$ is the number of elements in \mathbb{I} . In order to avoid fixing a numbering of the elements of \mathbb{I} we will

find it useful to employ the following “leg-numbering” notation. To a given subset $\mathbb{J} \subset \mathbb{I}$ we may associate the canonical permutation operators

$$P_{\mathbb{J}} : \mathcal{H}^{\otimes \mathbb{I}} \rightarrow \mathcal{H}^{\otimes \mathbb{J}} \otimes \mathcal{H}^{\otimes \mathbb{I} \setminus \mathbb{J}}.$$

To an operator $O \in \mathbf{B}(\mathcal{H}^{\otimes \mathbb{I}})$ we may then associate an operator $O_{\mathbb{J}} \in \mathbf{B}(\mathcal{H}^{\otimes \mathbb{I}})$ via

$$O_{\mathbb{J}} \equiv P_{\mathbb{J}}^{-1} \cdot (O \otimes \text{id}) \cdot P_{\mathbb{J}}. \quad (\text{B.5})$$

If $\mathbb{J} = \{i_1, i_2, \dots\}$ we will sometimes write $O_{i_1 i_2 \dots}$ instead of $O_{\mathbb{J}}$. We will also abbreviate $O_{\mathbb{J}_1 \cup \mathbb{J}_2 \cup \dots}$ to $O_{\mathbb{J}_1 \mathbb{J}_2 \dots}$.

C. On the proof of Theorem 2

A similar statement is known [CF1] for the closely related groupoid $\widetilde{\text{Pt}}(\Sigma)$ whose elements are the moves $[\tilde{\varphi}_2, \tilde{\varphi}_1]$ between fat graphs $\tilde{\varphi}_2, \tilde{\varphi}_1$ which do not have the decoration introduced in Subsection 5.1, but which have a numbering of the edges instead.

Theorem 2’. *The groupoid $\widetilde{\text{Pt}}(\Sigma)$ is the path groupoid of the complex $\widetilde{\text{Pt}}(\Sigma)$ which has vertices $\tilde{\varphi}$ and the following generators and relations:*

Generators:

- (i) *Flips F_e along the edges $e \in \tilde{\varphi}_1$ (see Figure (7)).*
- (ii) *The exchanges (ef) of the numbers assigned to edges e and f .*

Relations:

- (a) *There is no vertex that both edges e and f are incident to: $F_f \circ F_e = F_e \circ F_f$.*
- (b) *The edges e and f are incident to the same vertex: $F_e \circ F_f \circ F_e \circ F_f \circ F_e = (ef)$.*
- (c) *$(ef) \circ F_e = F_f \circ (ef)$*
- (d) *$F_e \circ F_e = \text{id}$.*

It was observed in [P3, CF1, CP] that Theorem 2’ follows quite easily from the fact [P1, Ko] that the fat graphs $\tilde{\varphi}$ can be used to label cells in certain cell decompositions [Ha, P1] of $\mathcal{T}(\Sigma) \times \mathbb{R}_+^s$. This allows one to associate a path $\hat{\pi}$ in $\mathcal{T}(\Sigma) \times \mathbb{R}_+^s$ to each path $\tilde{\pi} \in [\tilde{\varphi}_2, \tilde{\varphi}_1]$ in $\widetilde{\text{Pt}}(\Sigma)$. Each closed path $\hat{\pi}$ in $\mathcal{T}(\Sigma) \times \mathbb{R}_+^s$ can be deformed into small circles around the codimension two faces of the cell decompositions from [Ha, P1]. The latter are easily identified with the relations listed in Theorem 2’. This implies that no relations other than those listed in Theorem 2’ are needed to contract a closed path $\tilde{\pi} \in [\tilde{\varphi}, \tilde{\varphi}]$ to the identity.

It remains to show that Theorem 2 follows from Theorem 2’. To this aim it is important to observe that $\widetilde{\text{Pt}}(\Sigma)$ imbeds into $\text{Pt}(\Sigma)$ by means of the following construction: Given the numbered fat graph $\tilde{\varphi}$ let us define a decorated fat graph φ according to the following rule: Pick any numbering of the vertices of $\tilde{\varphi}$. For each vertex v let us distinguish among the edges which emanate from v the one with the smallest number assigned to it. For each move F_e let us choose a lift ϕ_e of the form $\phi_e = \delta'_e \circ \omega_{u_e v_e} \circ \delta_e$, where the vertices u_e and v_e represent are the ends of e and δ'_e, δ_e change the decoration only. It is then crucial to check that the relations of $\text{Pt}(\Sigma)$ ensure that the image of $\widetilde{\text{Pt}}(\Sigma)$ within $\text{Pt}(\Sigma)$ is simply connected. This follows if the images of the relations listed in Theorem 2’ are contractible within $\text{Pt}(\Sigma)$. We have drawn a

particular example for such an image in Figure 24 below. It is easy to verify that the image of the corresponding path in $\text{Pt}(\Sigma)$ is closed. More generally one needs to consider all relations obtained by changes of the numbering of the edges. These relations are obtained by inserting $\eta^{-1} \circ \eta$ between the moves which occur in the image of the relation under consideration, where η represents the change of decoration induced by a change of numbering. In a similar way one may convince oneself that all the relations listed in Theorem 2' are mapped to closed paths in $\text{Pt}(\Sigma)$, which completes the proof that $\widetilde{\text{Pt}}(\Sigma)$ embeds into $\text{Pt}(\Sigma)$.

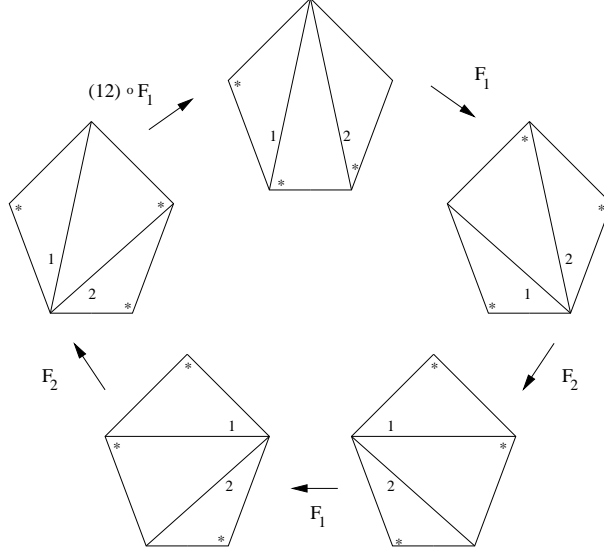


Figure 24: Image of the pentagon in $\text{Pt}(\Sigma)$.

It remains to show that each closed path in $\text{Pt}(\Sigma)$ is homotopic to a closed path in the image of $\widetilde{\text{Pt}}(\Sigma)$. Let us consider a closed path $\pi \in [\varphi, \varphi] \in \text{Pt}(\Sigma)$. Write $\pi = \omega_n \circ \dots \circ \omega_1$, where each ω_k is of the form $\omega_k = \delta'_k \circ \omega_{u_k v_k} \circ \delta_k$, with δ_k, δ'_k being composed out of moves ρ_w and permutations only. Choose any numbering for the edges of φ and denote the resulting numbered fat graph by $\tilde{\varphi}$. The path π then defines a path $\tilde{\pi}$ in $\widetilde{\text{Pt}}(\Sigma)$ by means of the following construction: Substitute the moves ω_r for $k = 1, \dots, n$ by the corresponding flips along the edges which connect u_k and v_k , and then multiply the result by the necessary permutations of the numbers assigned to the edges of $\tilde{\varphi}$. $\tilde{\pi}$ has the form $\tilde{\pi} = \Phi_n \circ \dots \circ \Phi_1$, where $\Phi_k \in [\tilde{\varphi}_{k+1}, \tilde{\varphi}_k]$ factorize into a single flip F_k times a change of numbering P_k , $\Phi_k = P_k \circ F_k$.

The path $\tilde{\pi}$ is mapped to a path π' in $\text{Pt}(\Sigma)$ by means of the construction above. π' has the form $\pi' = \phi'_n \circ \dots \circ \phi'_1$ with $\phi'_k \in [\varphi_{k+1}, \varphi_k]$. On the other hand let us note that the path $\pi'' = \phi''_n \circ \dots \circ \phi''_1$ with $\phi''_k \equiv \eta_{k+1}^{-1} \circ \omega_k \circ \eta_k$ is clearly homotopic to π for all changes of decorations η_k which satisfy $\eta_{n+1} = \eta_1$. For suitable choice of the η_k one gets $\phi''_k \in [\varphi_{k+1}, \varphi_k]$. It is then easy to see that ϕ'_k and ϕ''_k are homotopic. \square

D. Proof of Proposition 10

One may easily convince oneself that the moves Z_p and B_p can generically *not* be realized locally. There are simple sufficient criteria for a move m to be realizable locally.

Lemma 16.

- (i) A move $m = [\tau_m, \sigma_m] \in \mathcal{M}'_1(\Sigma)$ can always be realized locally if all but one of the boundary components of Σ_m are contained in $\{c_e; e \in \mathcal{A}_\sigma\}$.
- (ii) The moves B_p, B'_p, T_p can be realized locally if the curve $c_{p,1}$ which represents the boundary component of \mathfrak{t}_p assigned number 1 in Figure 12 is contained in \mathcal{A}_σ .
- (iii) The moves $F_{pq}, A_{pq}, S_p, S_{pq}$ and B_{pq} can always be realized locally.

Proof. Straightforward verifications. □

Proposition 15. (\equiv **Proposition 10**) If $m \in \mathcal{M}'_1(\Sigma)$ is admissible, but can not be realized locally, there always exists a path ϖ_m which is (i) homotopic to m within $\mathcal{M}'_1(\Sigma)$, and (ii) takes the form

$$\varpi_m = Y_m \circ m \circ Y_m^{-1}, \quad (\text{D.6})$$

where Y_m is a chain composed out of Z_p -moves and F_{pq} -moves which can all be realized locally.

Proof. The claim will follow easily from another auxiliary result that we will formulate as a separate lemma. As a preparation, let us recall that cutting a surface Σ along the curves $c(\sigma, e)$, $e \in \mathcal{A}_\sigma$ produces a surface Σ^\dagger , the connected components of which all have genus zero. Let us call a marking σ *irreducible* if the set of connected components of Σ^\dagger has only one element. It is easy to see that a marking σ is irreducible iff it has precisely one outgoing external edge.

Lemma 17. Let σ be a marking on a surface Σ that is irreducible. For any chosen boundary component b of Σ there exists a chain of Z_p -moves and F_{pq} -moves that (i) preserves admissibility in each step, (ii) consists only of moves that can be realized locally, and (iii) transforms σ to a irreducible marking σ' whose outgoing external edge ends in the chosen component b .

Sketch of proof. In order to check the following arguments it may be useful to think of an irreducible marking as being obtained from the corresponding marking σ^\dagger on the surface Σ^\dagger of genus zero by identifying the appropriate boundary circles. To each incoming boundary component of Σ^\dagger there corresponds a unique element of \mathcal{A}_σ .

The following claim is easy to verify. By means of F_{pq} -moves one may transform σ to a marking $\tilde{\sigma}$ which has the following two properties. First, for each vertex $p \in \tilde{\sigma}_\circ$ at least one of the edges that are incident at p is contained in \mathcal{A}_σ . Second, there is no edge $e \in \sigma_1$ which connects a vertex p to itself. These two properties insure that for each $p \in \tilde{\sigma}_\circ$ either Z_p or Z_p^{-1} preserves the admissibility of the marking. It is then not very hard to construct a sequence of Z_p -moves that (i) can all be realized locally and (ii) which transform $\tilde{\sigma}$ to a marking $\tilde{\sigma}'$ whose outgoing external edge ends in the chosen boundary component b .

By means of a chain of F_{pq} -moves one may finally transform $\tilde{\sigma}'$ back to a marking σ' that differs from the original marking only by the desired change of decoration. □

End of proof of Lemma 15. Lemma 16 allows us to restrict attention to the cases $m = Z_p, B_p, B'_p, T_p$. We need to transform the original marking to another one which has the property that both incoming boundary components of the trinion \mathfrak{t}_p are contained in \mathcal{A}_σ . Given a curve $c \in \mathcal{C}_\sigma \setminus \mathcal{A}_\sigma$ there is a unique subsurface $\Sigma_c \hookrightarrow \Sigma$ with marking σ_c such that (i) c is the unique

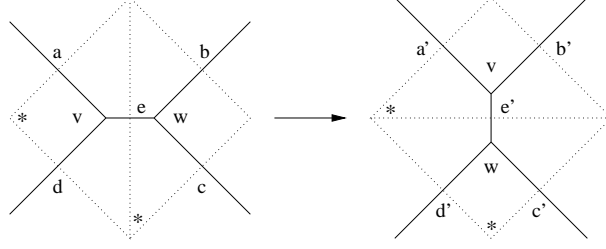


Figure 25: Labelling of the edges which are relevant for the description of a flip.

outgoing curve in the boundary of Σ_c , and (ii) the incoming boundary curves of Σ_c are contained in \mathcal{A}_σ . In order to infer the existence of the chain Y_p it clearly suffices to apply Lemma 17 to the subsurfaces Σ_{c_1} and Σ_{c_2} respectively, where c_1 and c_2 are the incoming boundary components of the trinion with label p . \square

Lemma 15 implies that π_m may be chosen as $\pi_m = y_m \circ \hat{\pi}_m \circ y_m^{-1}$, where y_m is uniquely defined by the factorization of Y_m into elementary moves that can be realized locally, and $\hat{\pi}_m$ is the (fixed) path which was chosen to represent m in the case that m can be realized locally.

E. On the proof of Theorem 6

E.1 A technical preliminary

To begin with, let us present a technical result that facilitates the explicit computations. The operator functions we will be interested in are of the form

$$\mathcal{O} \equiv \mathcal{O}(\{z_i; i \in \varphi_1\}) = \sum_{\tau \in \mathcal{F}} C_\tau e^{x(\tau)}, \quad x(\tau) \equiv \sum_{e \in \varphi_1} \tau(e) z_e, \quad (\text{E.7})$$

where the summation is taken over the space \mathcal{F} of all maps $\varphi_1 \ni e \rightarrow \tau(e) \in \frac{1}{2}\mathbb{Z}$, and the coefficients C_τ are assumed to be non-vanishing for a finite number of $\tau \in \mathcal{F}$ only, in which case we assume $C_\tau \in \mathbb{R}_+$. These operators are densely defined and positive self-adjoint due to the self-adjointness of $x(\tau)$. The cone generated by operator functions of the form (E.7) will be denoted $\mathbf{C}_+(\varphi)$.

Proposition 16. *Let φ, φ' be two fat graphs that are related by $\rho \equiv \omega_{vw} \in \mathcal{Pt}(\Sigma)$, and let us adopt the labelling of the relevant edges given by Figure 25. For each $\mathcal{O} \in \mathbf{C}_+(\varphi)$ one has*

$$a_\rho(\mathcal{O}) \equiv u(\pi) \cdot \mathcal{O} \cdot u(\pi)^{-1} = E_b(z_{e'}) \cdot \mathcal{O}' \cdot (E_b(z_{e'}))^{-1}, \quad (\text{E.8})$$

where $E_b(z) \equiv e_b(-\frac{z}{2\pi b})$, and \mathcal{O}' is related to \mathcal{O} via

$$\begin{aligned} \mathcal{O}' &\equiv \mathcal{O}(z_{a'}, z_{b'} + z_{e'}, z_{c'}, z_{d'} + z_{e'}, -z_{e'}, \{z_i; i \in \varphi_1 \setminus \{a', b', c', d', e'\}\}) \text{ if} \\ \mathcal{O} &= \mathcal{O}(z_a, z_b, z_c, z_d, z_e, \{z_i; i \in \varphi_1 \setminus \{a, b, c, d, e\}\}). \end{aligned} \quad (\text{E.9})$$

Proof. Given the explicit expression for T_{vw} we only have to verify that

$$e^{-2\pi i p_v q_w} \cdot \mathcal{O} \cdot e^{-2\pi i p_v q_w} = \mathcal{O}',$$

with O and O' being related by (E.9). Keeping in mind (7.6) we notice that the operators z_f may be represented in the form $z_f = z_{f,v} + z_{f,w}$, where $v, w \in \varphi_o$ are the vertices connected by the edge $f \in \varphi_o$. Let us label the vertices in φ_o and φ'_o such that the edges a and d are incident to the vertices v_a and v_d besides to the vertex v respectively, and similarly for the vertices v_b, v_c and w . This leads to the following representations for the relevant Fock variables:

$$\begin{aligned} z_e &= 2\pi b(q_v - p_v + p_w), & z_{e'} &= 2\pi b(q'_w - q'_v - p'_w), \\ z_a &= 2\pi b p_v + z_{a,v_a}, & z_{a'} &= 2\pi b p'_v + z_{a',v_a}, \\ z_b &= 2\pi b(q_w - p_w) + z_{b,v_b}, & z_{b'} &= 2\pi b(q'_v - p'_v) + z_{b',v_b}, \\ z_c &= 2\pi b(-q_w) + z_{c,v_c}, & z_{c'} &= 2\pi b(-q'_w) + z_{c',v_c}, \\ z_d &= 2\pi b(-q_v) + z_{d,v_d}, & z_{d'} &= 2\pi b p'_w + z_{d',v_d}, \end{aligned} \quad (E.10)$$

To complete the proof of Proposition 16 is now the matter of a straightforward calculation. \square

Remark 10. This yields in particular the formulae

$$\begin{aligned} e^{-z_{a'}} &= e^{-\frac{1}{2}z_a}(1 + e^{-z_e})e^{-\frac{1}{2}z_a}, & e^{-z_{b'}} &= e^{+\frac{1}{2}z_b}(1 + e^{+z_e})e^{+\frac{1}{2}z_b}, \\ e^{+z_{d'}} &= e^{+\frac{1}{2}z_d}(1 + e^{+z_e})e^{+\frac{1}{2}z_d}, & e^{+z_{c'}} &= e^{-\frac{1}{2}z_c}(1 + e^{-z_e})e^{-\frac{1}{2}z_c}. \end{aligned} \quad z_{e'} = -z_e,$$

It is quite obvious that these transformations reduce to their classical counterparts (6.3) in the limit $b \rightarrow 0$.

As an example let us consider the monomials $M_{n_a n_c}^{n_b n_d} \equiv e^{\frac{1}{2}(n_a z_a + n_b z_b + n_c z_c + n_d z_d)}$, where $n_b, b \in \{a, b, c, d\}$ are restricted by the requirement that $2N \equiv n_a + n_c - n_b - n_d$ must be even. One then finds certain simplifications on the right hand side of (E.8):

$$\begin{aligned} E_b(z'_e) \cdot M_{n_a n_c}^{n_b n_d} \cdot (E_b(z'_e))^{-1} &= \sqrt{M_{n_a n_c}^{n_b n_d}} \frac{E_b(z'_e + \pi i b^2 N)}{E_b(z'_e - \pi i b^2 N)} \sqrt{M_{n_a n_c}^{n_b n_d}} \\ &= \sqrt{M_{n_a n_c}^{n_b n_d}} \left[\prod_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} (1 + q^{2m} e^{-z'_e}) \right] \sqrt{M_{n_a n_c}^{n_b n_d}}. \end{aligned} \quad (E.11)$$

E.2 Invariance of length operators

In the main text we have explained how to reduce the proof of Theorem 6 to the following Lemma:

Lemma 18. *Assume that $m \in \mathcal{M}'_1(\Sigma)$ can be realized locally. We then have the relation*

$$u(\pi_m) \cdot L_{\sigma_m, c} = L_{\tau_m, c} \cdot u(\pi_m) \quad (E.12)$$

for all curves c such that $c \in \mathcal{C}_{\sigma_m}$ and $c \in \mathcal{C}_{\tau_m}$.

On the proof of Lemma 18. It is easy to see that our recursive Definition 6 of the length operators $L_{\sigma, c}$ reduces the task of proving Lemma 18 to the curves that represent a boundary component of Σ_m . Let us write $L_{\sigma, e} \equiv L_{\sigma, c(\sigma, e)}$.

In the cases $m = F_{pq}, S_p, T_p, Z_p$ we will need to verify Lemma 18 by direct calculations. Let us begin with the case $m = F_{pq}$. The claim is again trivial for the length operators associated to the incoming boundary components of Σ_m . Let us therefore focus on the length operators $L_{\sigma_m, f}, L_{\tau_m, f}$ assigned to the outgoing boundary component of Σ_m . The Definition 6 yields the following expressions for the length operators $L_{\sigma_m, f}$ and $L_{\tau_m, f}$ respectively.

$$\begin{aligned}
L_{\sigma_m, f} &= e^{z_{21}+y_3+y_2+y_1} + e^{z_{21}-y_3+y_2+y_1} + e^{-z_{21}-y_3-y_2-y_1} \\
&\quad + e^{y_1-y_2-y_3} + e^{y_1+y_2-y_3} + e^{-y_1-y_2-y_3} \\
&\quad + e^{z_{21}+y_2+y_1} L_3 + e^{y_1-y_3} L_2 + e^{-y_2-y_3} L_1, \\
L_{\tau_m, f} &= e^{z'_{32}+y'_3+y'_2+y'_1} + e^{-z'_{32}-y'_3-y'_2-y'_1} + e^{-z'_{32}-y'_3-y'_2+y'_1} \\
&\quad + e^{y'_1+y'_2+y'_3} + e^{y'_1-y'_2-y'_3} + e^{y'_1+y'_2-y'_3} \\
&\quad + e^{y'_2+y'_1} L'_3 + e^{y'_1-y'_3} L'_2 + e^{-z'_{32}-y'_2-y'_3} L'_1.
\end{aligned} \tag{E.13}$$

We have used the abbreviations $y_\iota \equiv y_{\varphi_\sigma, e_\iota}$, $y'_\iota \equiv y_{\varphi_\tau, e_\iota}$, $L_\iota \equiv L_{\sigma, e_\iota}$ and $L'_\iota \equiv L_{\tau, e_\iota}$ for $\iota \in \{1, 2, 3\}$, as well as $z_\iota \equiv z_{\varphi_\sigma, e_\iota}$, $z'_\iota \equiv z_{\varphi_\tau, e_\iota}$ for $\iota \in \{32, 21\}$. The labelling of the edges is the one introduced in Figure 26. According to Proposition 16 we need to calculate

$$\begin{aligned}
E_b(z'_{32}) \cdot &\left(e^{y'_3+y'_2+y'_1+z_{32}} + e^{y'_1-y'_2-y'_3} + \right. \\
&+ e^{y'_2+y'_1} L'_3 + e^{y'_1-y'_3} L'_2 + e^{-z'_{32}-y'_2-y'_3} L'_3 + \\
&+ e^{\frac{1}{2}(y'_1+y'_2-y'_3)} (1 + e^{-z'_{32}}) e^{\frac{1}{2}(y'_1+y'_2-y'_3)} + \\
&\left. + e^{-\frac{1}{2}(y'_1+y'_2+y'_3+z'_{32})} (1 + e^{-z'_{32}}) e^{-\frac{1}{2}(y'_1+y'_2+y'_3+z'_{32})} \right) \cdot E_b(z'_{32})^\dagger.
\end{aligned} \tag{E.14}$$

We finally need to apply equation (E.11). The terms in the first line of (E.14) have $N = 1$, those in the second line $N = 0$, and all other terms have $N = -1$. Straightforward application of equation (E.11) shows that the expression given in (E.14) equals $L_{\tau_m, f}$, as claimed.

The next case we will consider is $m = S_p$. It clearly suffices to restrict attention to the case that $\Sigma_m = \Sigma_{1,1}$, the one-holed torus. On Σ_m let us consider the fat graphs φ_i , $i = 1, 2, 3$ depicted in Figure 27.

We have $\varphi_1 = \tilde{\varphi}'_m$ and $\varphi_3 = \theta'_m$ respectively. The sequence of fat graph $(\varphi_3, \varphi_2, \varphi_1)$ defines the path $\pi_m \in [\theta'_m, \tilde{\varphi}'_m]$ which we will use. The only relevant curve is the curve β which represents the boundary of Σ_m . Definition 6 yields the following expressions for the length

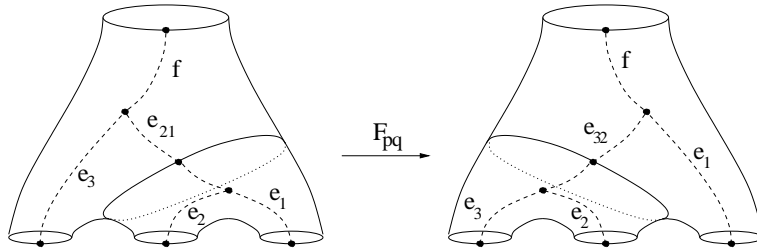
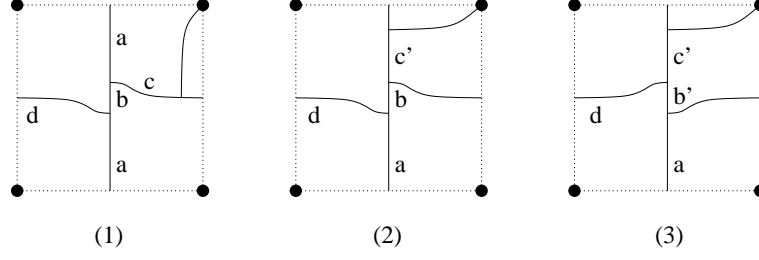


Figure 26: Labelling of the edges of σ_m for $m = F_{pq}$.

Figure 27: Fat graphs φ_i on the one-holed torus.

operators $L_{\varphi_1, \beta}$ and $L_{\varphi_3, \beta}$ respectively:

$$\begin{aligned}
 L_{\varphi_1, \beta} &= e^{z_d - z_c} + 2 \cosh(z_c + z_d + z_a + z_b) \\
 &\quad + e^{z_d} (1 + e^{z_b} + e^{z_b + z_a}) + e^{-z_c} (1 + e^{-z_a} + e^{-z_a - z_b}), \\
 L_{\varphi_3, \beta} &= e^{z_{c'} - z_a} + 2 \cosh(z_a + z_d + z_{c'} + z_{b'}) \\
 &\quad + e^{z_{c'}} (1 + e^{z_{b'}} + e^{z_{b'} + z_d}) + e^{-z_a} (1 + e^{-z_d} + e^{-z_d - z_{b'}}).
 \end{aligned} \tag{E.15}$$

The labelling of the relevant edges is the one introduced in Figure 27. With the help of Proposition 16 one may calculate

$$\begin{aligned}
 L_{\varphi_2, \beta} \equiv a_{[\varphi_2, \varphi_1]}(L_{\varphi_1, \beta}) &= e^{z_{c'} - z_a} + 2 \cosh(z_a + z_d + z_{c'} + z_b) \\
 &\quad + e^{z_{c'}} (1 + e^{z_d} + e^{z_d + z_b}) + e^{-z_a} (1 + e^{-z_b} + e^{-z_b - z_b}).
 \end{aligned} \tag{E.16}$$

One may then apply Proposition 16 once more to calculate $a_{[\varphi_3, \varphi_2]}(L_{\varphi_2, \beta})$. The result is $L_{\varphi_3, \beta}$ as claimed.

We have furthermore verified by direct calculations that Lemma 18 holds in the cases $m = T_p, Z_p, B_p, B'_p$ respectively. These calculations proceed along very similar lines as in the previous two cases, which motivates us to omit the details.

In the remaining cases $m = A_{pq}$, $m = B_{pq}$ and $m = S_{pq}$ one may use the factorization of the chains π_m which follows from the definitions (13.5), (13.7) and (13.8) respectively in order to reduce the proof to the cases where the result is already established. This completes our discussion of the proof of Lemma 18. \square

References

- [ADPW] S. Axelrod, S. Della Pietra, E. Witten, *Geometric quantization of Chern-Simons gauge theory*. J. Differential Geom. **33** (1991) 787–902.
- [AGS] A.Yu. Alekseev; H. Grosse, Harald; V. Schomerus, *Combinatorial quantization of the Hamiltonian Chern-Simons theory*. I. Comm. Math. Phys. **172** (1995) 317–358, II. Comm. Math. Phys. **174** (1996) 561–604,
A.Yu. Alekseev, V. Schomerus, *Representation theory of Chern-Simons observables*. Duke Math. J. **85** (1996) 447–510
- [AMR] J.E. Andersen; J. Mattes; N. Reshetikhin, *Quantization of the algebra of chord diagrams*. Math. Proc. Cambridge Philos. Soc. **124** (1998) 451–467

- [Ba] E.W. Barnes: *Theory of the double gamma function*, Phil. Trans. Roy. Soc. **A196** (1901) 265–388
- [BK1] B. Bakalov, A. Kirillov jun., *On the Lego-Teichmüller Game*, Transform. Groups **5** (2000) 207–244 [arXiv: math.GT/9809057]
- [BK2] B. Bakalov, A. Kirillov jun., *Lectures on tensor categories and modular functors*. University Lecture Series, 21. American Mathematical Society, Providence, RI, 2001
- [BR] E. Buffenoir, Ph. Roche, *Link invariants and combinatorial quantization of Hamiltonian Chern-Simons theory*. Comm. Math. Phys. **181** (1996) 331–365.
- [BT] A. Bytsko, J. Teschner, R-operator, co-product and Haar-measure for the Modular Double of $U_q(\mathfrak{sl}(2, \mathbb{R}))$, Comm. Math. Phys. **240** (2003) 171–196, [arXiv: math.QA/0208191]
- [CF1] L.O. Checkov, V. Fock: *A quantum Teichmüller space*, Theor. Math. Phys. **120** (1999) 1245–1259
- [CF2] L.O. Checkov, V. Fock: *Quantum modular transformations, the pentagon relation, and geodesics*, Proc. Steklov Inst. Math. **226** (1999) 149–163
- [CP] L.O. Checkov, R. Penner: *On Quantizing Teichmüller and Thurston theories*, Preprint [arXiv:math.AG/0403247]
- [D] M. Dehn: *Die Gruppe der Abbildungsklassen*, Acta Math. **69** (1938) 135–206
- [Fa] L. Faddeev: *Modular double of a quantum group*. Math. Phys. Stud. **21** (2000) 149–156 [arXiv:math.QA/9912078]
- [FG] V.V. Fock, A.B. Goncharov: *Cluster ensembles, quantization and the dilogarithm*, Preprint [arXiv:math.AG/0311245]
- [FuG] L. Funar, R. Gelca: *On the groupoid of transformations of rigid structures on surfaces*, J.Math.Sci.Univ.Tokyo **6** (1999) 599–646.
- [FK] L.D. Faddeev and R.M. Kashaev: *Quantum dilogarithm*, Mod. Phys. Lett. **A9** (1994) 427–434 [arXiv: hep-th/9310070]
- [FKV] L. Faddeev, R. Kashaev, A. Volkov: *Strongly coupled discrete Liouville theory: Algebraic approach and duality*, Comm. Math. Phys. **219** (2001) 199–219 [arXiv:hep-th/0006156]
- [Fo] V.V. Fock: *Dual Teichmüller spaces*, [arXiv: hep-th/9702018]
- [FR] V.V. Fock, A.A. Rosly, *Poisson structure on moduli of flat connections on Riemann surfaces and the r -matrix*, in: Morozov, A. Yu. (ed.) et al., Moscow Seminar in mathematical physics. Providence, RI: American Mathematical Society. Transl., Ser. 2, Am. Math. Soc. 191 (43), 67–86

- [Ge] S. Gervais: *Presentation and central extension of mapping class groups*, Trans. AMS **348** (1996) 3097-3132
- [Go1] W.M. Goldman, *The symplectic nature of fundamental groups of surfaces*. Adv. in Math. **54** (1984) 200–225.
- [Go2] W.M. Goldman, *Topological components of spaces of representations*. Invent. Math. **93** (1988), no. 3, 557–607
- [GS] V. Guillemin, S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1990
- [Ha] J. Harer: *The virtual cohomological dimension of the mapping class group of an orientable surface*, Inv. Math. **84** (1986) 157-176
- [Hi1] N. J. Hitchin, *The self-duality equations on a Riemann surface*. Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126
- [Hi2] N. J. Hitchin, *Flat connections and geometric quantization*. Comm. Math. Phys. **131** (1990) 347–380
- [IT] Y. Iwayoshi, M. Taniguchi: *An introduction to Teichmüller spaces*, Springer 1992
- [Ka1] R.M. Kashaev: *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. **43** (1998) 105-115, q-alg/9705021,
- [Ka2] R.M. Kashaev: *Liouville central charge in quantum Teichmüller theory*, Proc. of the Steklov Inst. of Math. Vol. **226** (1999) 63-71, hep-th/9811203
- [Ka3] R.M. Kashaev: *On the spectrum of Dehn twists in quantum Teichmüller theory*, Physics and combinatorics, 2000 (Nagoya), 63–81, World Sci. Publishing, River Edge, NJ, 2001.
- [Ka4] R.M. Kashaev, *The quantum dilogarithm and Dehn twists in quantum Teichmüller theory*. Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 211–221, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001.
- [KaV] R.M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*. Lett. Math. Phys. **39** (1997) 269–275
- [KLS] S. Kharchev, D. Lebedev and M. Semenov-Tian-Shansky: *Unitary representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$, the modular double, and the multiparticle q -deformed Toda chains*, Comm. Math. Phys. **225** (2002) 573–609 [arXiv: hep-th/0102180]
- [Ko] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992) 1-23
- [La] Y. Laszlo, *Hitchin's and WZW connections are the same*. J. Differential Geom. **49** (1998) 547–576

- [Lu] F. Luo: *A Presentation of the mapping class groups*, Math. Res. Lett. **4** (1997) 735-739
- [MS] G. Moore, N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989) 177-254
- [P1] R.C. Penner: *The Decorated Teichmüller Space of Punctured Surfaces*, Comm. Math. Phys. **113** (1987) 299-339
- [P2] R.C. Penner: *Weil-Petersson Volumes* J. Differential Geometry **35** (1992) 559-608
- [P3] R.C. Penner: *Universal Constructions in Teichmüller Theory*, Adv. Math. **98** (1993) 143-215
- [PT] B.Ponsot and J.Teschner. *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U_q(\mathfrak{sl}(2, \mathbb{R}))$* , Comm. Math. Phys. **224** (2001) 613-655
- [RT] N. Reshetikhin, V.G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*. Invent. Math. **103** (1991) 547–597
- [Ru] S.N.M. Ruijsenaars: *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997) 1069–1146.
- [Se] I.E. Segal: *Transforms for operators and symplectic automorphisms over a locally compact abelian group*, Math. Scand. **13** (1963) 31-43.
- [Sh] T. Shintani: *On a Kronecker limit formula for real quadratic fields*, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. **24** (1977) 167–199
- [So] C. Sorger, *La formule de Verlinde*. Séminaire Bourbaki, Vol. 1994/95. Astérisque No. **237** (1996), Exp. No. 794, 3, 87–114
- [TT] L.A. Takhtajan, I.-P. Teo: *Quantum Liouville theory in the background field formalism I. Compact Riemann surfaces* Commun.Math.Phys. 268 (2006) 135-197
- [TL1] J. Teschner, *Liouville theory revisited*. Classical Quantum Gravity **18** (2001), no. 23, R153–R222
- [TL2] J. Teschner, *A lecture on the Liouville vertex operators.*, Int. J. Mod. Phys. A, Supplement, Vol. **19** (2004) 436-458
- [TT2] J. Teschner, *An analog of a modular functor from quantized Teichmüller theory, II*, in preparation
- [Tu] V. Turaev, *Quantum Invariants of Knots and 3-manifolds*, W. de Gruyter, Berlin 1994.
- [V] H. Verlinde, *Conformal field theory, two-dimensional quantum gravity and quantization of Teichmüller space*. Nuclear Phys. B **337** (1990) 652–680.
- [Vo] A.Yu. Volkov: *Noncommutative hypergeometry*, Comm. Math. Phys. **258** (2005) 257–273

- [Wa] N. Wallach, *Real Reductive Groups II*, Academic Press, Inc., Boston, MA, 1992.
- [Wi] E. Witten, *Quantum field theory and the Jones polynomial*. Comm. Math. Phys. **121** (1989) 351–399
- [Wo1] S.A. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface* Ann. of Math. **117** (1983) 207–234
- [Wo2] S.A. Wolpert, *On the Weil-Petersson geometry of the moduli space of curves* Amer. J. Math **107** (1985) 969–997
- [Wo3] S.A. Wolpert, *Geodesic length functions and the Nielsen problem* J. Differential Geom. **25** (1987) 275–296
- [Wor] S.L. Woronowicz: *Quantum exponential function*, Rev. Math. Phys. **12** (2000) 873–920