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Analytic Solutions for Marginal Deformations in Open String Field Theory

Michael Kiermaier[1], Yuji Okawa[2], Leonardo Rastelli[3], and Barton Zwiebach[1]

[1] *Center for Theoretical Physics*
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
mkiermai@mit.edu, zwiebach@lns.mit.edu

[2] *DESY Theory Group*
Notkestrasse 85
22607 Hamburg, Germany
yuji.okawa@desy.de

[3] *C.N. Yang Institute for Theoretical Physics*
Stony Brook University
Stony Brook, NY 11794, USA
leonardo.rastelli@stonybrook.edu

Abstract

We develop a calculable analytic approach to marginal deformations in open string field theory using wedge states with operator insertions. For marginal operators with regular operator products, we construct analytic solutions to all orders in the deformation parameter. In particular, we construct an exact time-dependent solution that describes D-brane decay and incorporates all α' corrections. For marginal operators with singular operator products, we construct solutions by regularizing the singularity and adding counterterms. We explicitly carry out the procedure to third order in the deformation parameter.

Contents

1 Introduction

Mapping the landscape of vacua is one of the outstanding challenges in string theory. A simpler version of the problem is to characterize the "open string landscape", the set of possible D-brane configurations in a fixed closed string background. In recent years evidence has accumulated that classical open string field theory (OSFT) gives an accurate description of the open string landscape. See [1, 2, 3] for reviews. Much of this evidence is based on numerical work in level truncation, and there remain many interesting questions. Is the correspondence between boundary CFT's and classical OSFT solutions one-to-one? Is the OSFT action of a single D-brane capable of describing configurations of multiple D-branes? Answering these questions is likely to require analytic tools. Important analytic progress was made by Schnabl [4]. He found the exact solution corresponding to the tachyon vacuum by exploiting the simplifications coming from the clever gauge fixing condition

$$B\Psi = 0\,, \tag{1.1}$$

where B is the antighost zero mode in the conformal frame of the sliver. Various aspects of Schnabl's construction have been studied in [5]–[12].

In this paper we describe new analytic solutions of OSFT, corresponding to exactly marginal deformations of the boundary CFT. Previous work on exactly marginal deformations in OSFT [13] was based on solving the level-truncated equations of motion in Siegel gauge. The level-truncated string field was determined as a function of the vacuum expectation value of the exactly marginal mode, fixed to an arbitrary finite value. Level-truncation lifts the flat direction, but it was seen that as the level is increased the flat direction is recovered with better and better accuracy. Instead, our approach will be to expand the solution as $\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n \Psi^{(n)}$, where λ parameterizes the exact flat direction. We solve the equation of motion recursively to find an analytic expression for $\Psi^{(n)}$. Our results are exact in that we are solving the full OSFT equation of motion, but they are perturbative in λ; by contrast, the results of [13] are approximate since the equation of motion has been level-truncated, but they are non-perturbative in the deformation parameter.

The perturbative approach of this paper has certainly been attempted earlier using the Siegel gauge. Analytic work, however, is out of the question because in the Siegel gauge the Riemann surfaces associated with $\Psi^{(n)}$, with $n > 2$, are very complicated. The new insight that makes the problem tractable is to use, as in [4], the remarkable properties of wedge states with insertions [14, 15, 16].

We find qualitatively different results, according to whether the matter vertex operator V that generates the deformation has a regular or singular operator product expansion (OPE) with itself. Sections 2 and 3 of the paper are devoted to the case of regular OPE's and section 4 to the case of singular OPE's. A key technical point is the calculation of the action of B/L, where $L = \{ Q_B, B \}$, on products of string fields.

If V has a regular OPE with itself, the equation of motion can be systematically solved in the Schnabl gauge (1.1). The solution takes a strikingly compact form, given in the CFT language by (3.3) and its geometric picture is presented in Figure 1. The solution $\Psi^{(n)}$ is made of of a wedge state with n insertions of cV on its boundary. The relative separations of the boundary insertions are specified by $n-1$ moduli $\{t_i\}$, with $0 \le t_i \le 1$, which are to be integrated over. Each modulus is accompanied by an antighost line integral $\mathcal{B}$. The explicit evaluation of $\Psi^{(n)}$ in level expansion is straightforward for a specific choice of V.

In §3.2 we apply this general result to the operator $V = e^{\frac{1}{\sqrt{\alpha'}} X^0}$ [17, 18, 19, 20, 21, 22, 23]. This deformation describes a time-dependent tachyon solution that starts at the perturbative vacuum in the infinite past and (if $\lambda < 0$) begins to roll toward the non-perturbative vacuum. The parameter λ can be rescaled by a shift of the origin of time, so the solutions are physically equivalent. The time-dependent tachyon field takes the form

$$T(x^0) = \lambda\, e^{\frac{1}{\sqrt{\alpha'}} x^0} + \sum_{n=2}^{\infty} \lambda^n\, \beta_n\, e^{\frac{1}{\sqrt{\alpha'}} n x^0} \,. \tag{1.2}$$

We derive a closed form integral expression for the coefficients β_n and evaluate them numerically. We find that the coefficients decay so rapidly as n increases that it is plausible that the solution is absolutely convergent for any value of x^0. Our exact result confirms the surprising oscillatory behavior found in the p-adic model [19] and in level-truncation studies of OSFT [19, 23]. The tachyon (1.2) overshoots the non-perturbative vacuum and oscillates with ever-growing amplitude. It has been argued that a field redefinition to the variables of boundary SFT would map this oscillating tachyon to a tachyon field monotonically relaxing to the non-perturbative vacuum [23]. It would be very interesting to calculate the pressure of our exact solution and check whether it tends to zero in the infinite future, as would be expected from Sen's analysis of tachyon matter [24, 1].

In §3.3 we consider the light-cone vertex operator ∂X^+, another example of a marginal vertex operator with a regular OPE. Following [25], we construct the string field solution inspired by the Born-Infeld solution that describes a fundamental string ending on a D-brane [26]. The lightcone direction X^+ is a linear combination of the time direction and a direction normal to the brane, and the vertex operator is dressed by $A(k_i)\, e^{ik_i X^i}$ and integrated over the momenta k_i along the spatial directions on the brane. The solution is not fully self-contained within open string field theory: it requires sources, making the analysis delicate. Sources are also required in the Born Infeld description of the solution.

If the OPE of V with itself is $V(z)V(0) \sim 1/z^2$, the solution presented in Figure 1 is not well defined because divergences arise as the separations t_i of the boundary insertions go to zero. We study the required modifications in section 4. An important example is the Wilson line deformation ∂X. We regularize the divergences by imposing a cut-off in the integration region of the moduli. It turns out that counterterms can be added to achieve a finite $\Psi^{(2)}$ that satisfies the equation of motion. Surprisingly, the result necessarily violates the gauge condition (1.1)! The naive solution $\Psi^{(2)} = -\frac{B}{L}(\Psi^{(1)} * \Psi^{(1)})$ breaks down because the string field $\Psi^{(1)} * \Psi^{(1)}$ contains a component in the kernel of L. This phenomenon is a peculiar quirk of Schnabl gauge that has no counterpart in Siegel gauge. Due to this technical complication, the construction of the higher $\Psi^{(n)}$ becomes quite cumbersome, though still simpler than in Siegel gauge. We argue that for all n, appropriate counterterms can be added to achieve a finite $\Psi^{(n)}$ that solves the equation of motion. We discuss in detail the case of $\Psi^{(3)}$, verifying the non-trivial cancellations that must occur for the construction to succeed. We leave it for future work to achieve simpler closed form expressions for $\Psi^{(n)}$. Such expressions will be needed to investigate the nature of the perturbative series in λ and to make contact with the non-perturbative, but approximate, level-truncation results of [13]. It will also be interesting to understand better the relation between the conditions for exact marginality of boundary CFT [27] and the absence of obstructions in solving the equation of motion of string field theory. The technology developed in this paper will be also useful in open superstring field theory [28].

Independent work by M. Schnabl on the subject of marginal deformations in string field theory appears in [29].

2 The action of B/L

2.1 Solving the equation of motion in the Schnabl gauge

For any matter primary operator V with dimension one, the state $\Psi^{(1)}$ corresponding to the operator $cV(0)$ is BRST closed:

$$Q_B\Psi^{(1)} = 0\,. \tag{2.1}$$

In the context of string field theory, this implies that the linearized equation of motion of string field theory is satisfied. When the marginal deformation associated with V is *exactly* marginal, we expect that a solution of the form

$$\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n\, \Psi^{(n)}\,, \tag{2.2}$$

where λ is a parameter, solves the nonlinear equation of motion

$$Q_B\Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0\,. \tag{2.3}$$

The equation that determines $\Psi^{(n)}$ for $n > 1$ is

$$Q_B\Psi^{(n)} = \Phi^{(n)}\,, \quad \text{with} \quad \Phi^{(n)} = -\sum_{k=1}^{n-1} \Psi^{(n-k)} * \Psi^{(k)}\,. \tag{2.4}$$

For this equation to be consistent, $\Phi^{(n)}$ must be BRST closed. This is easily shown using the equations of motion at lower orders. For example,

$$Q_B\Phi^{(2)} = -Q_B\,(\,\Psi^{(1)} * \Psi^{(1)}\,) = -Q_B\Psi^{(1)} * \Psi^{(1)} + \Psi^{(1)} * Q_B\Psi^{(1)} = 0\,, \tag{2.5}$$

when $Q_B\Psi^{(1)} = 0$. It is crucial that $\Phi^{(n)}$ be BRST *exact* for all $n > 1$, or else we would encounter an obstruction in solving the equations of motion. No such obstruction is expected to arise if the matter operator V is exactly marginal, so we can determine $\Psi^{(n)}$ recursively by solving $Q_B\Psi^{(n)} = \Phi^{(n)}$. This procedure is ambiguous as we can add any BRST closed term to $\Psi^{(n)}$, so we need to choose some prescription. A traditional choice would be to work in Siegel gauge. The solution $\Psi^{(n)}$ is then given by acting with b_0/L_0 on $\Phi^{(n)}$. In practice this is cumbersome since the combination of star products and operators b_0/L_0 in the Schwinger representation generates complicated Riemann surfaces in the CFT formulation.

Inspired by Schnabl's success in finding an analytic solution for tachyon condensation, it is natural to look for a solution Ψ_λ in the Schnabl gauge:

$$B\Psi_\lambda = 0\,. \tag{2.6}$$

Our notation is the same as in [5, 7, 8]. In particular the operator B and L are the zero modes of the antighost and of the stress tensor in the conformal frame of the sliver[1],

$$B \equiv \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} b(\xi) \,, \quad L \equiv \oint \frac{d\xi}{2\pi i} \frac{f(\xi)}{f'(\xi)} T(\xi) \,, \quad f(\xi) \equiv \frac{2}{\pi} \arctan(\xi) \,. \tag{2.7}$$

We define $L^{\pm} \equiv L \pm L^{\star}$, $B^{\pm} \equiv B \pm B^{\star}$, where the superscript $\star$ indicates BPZ conjugation, and we denote with subscripts L and R the left and right parts, respectively, of these operators. Formally, a solution of (2.4) obeying (2.6) can be constructed as follows:

$$\Psi^{(n)} = \frac{B}{L} \Phi^{(n)} \,. \tag{2.8}$$

This can also be written as

$$\Psi^{(n)} = \int_0^\infty dT \, B e^{-TL} \, \Phi^{(n)} \,, \tag{2.9}$$

if the action of e^{-TL} on $\Phi^{(n)}$ vanishes in the limit $T \to \infty$. It turns out that the action of B/L on $\Phi^{(n)}$ is not always well defined. As we discuss in detail in section 4, if the matter primary V has a singular OPE with itself, the formal solution breaks down and the required modification necessarily violates the gauge condition (2.6). On the other hand, if the matter primary has a regular OPE with itself, the formal solution is well defined, as we will confirm later. In the rest of this section, we study the expression (2.9) for $n = 2$ in detail.

2.2 Algebraic preliminaries

We prepare for our work by reviewing and deriving some useful algebraic identities. For further details and conventions the reader can refer to [7, 8].

An important role will be played by the operator $L - L_L^+$, and the antighost analog $B - B_L^+$. These operators are derivations of the star algebra. This is seen by writing the first one, for example, as a sum of two familiar derivations

$$L - L_L^+ = \frac{1}{2} L^- + \frac{1}{2}(L_R^+ + L_L^+) - L_L^+ = \frac{1}{2} L^- + \frac{1}{2}(L_R^+ - L_L^+) = \frac{1}{2}(L^- + K) \,. \tag{2.10}$$

We therefore have

$$(L - L_L^+)\,(\phi_1 * \phi_2) = (L - L_L^+)\,\phi_1 * \phi_2 + \phi_1 * (L - L_L^+)\phi_2 \,. \tag{2.11}$$

Noting that the contributions of L_L^+ on the left-hand side and on the first term of the right-hand side cancel each other, we find

$$L(\phi_1 * \phi_2) \;=\; L\phi_1 * \phi_2 + \; \phi_1 * (L - L_L^+)\,\phi_2 \,, \tag{2.12}$$

$$B(\phi_1 * \phi_2) \;=\; B\phi_1 * \phi_2 + (-1)^{\phi_1} \phi_1 * (B - B_L^+)\phi_2 \,. \tag{2.13}$$

[1] Using reparameterizations, as in [8], it should be straightforward to generalize the discussion to general projectors. In this paper we restrict ourselves to the simplest case of the sliver.

Here and in what follows, a string field in the exponent of -1 denotes its Grassmann property: it is 0 mod 2 for a Grassmann-even string field and 1 mod 2 for a Grassmann-odd string field. ¿From this we immediately deduce the generalization of (2.12) and (2.13) for products of multiple string fields. For B, for example, we have

$$B(\phi_1 * \phi_2 * \dots \phi_n) = (B\phi_1) * \dots * \phi_n + \sum_{m=2}^{n} (-1)^{\sum_{k=1}^{m-1} \phi_k} \, \phi_1 * \dots * (B - B_L^+)\phi_m * \dots * \phi_n \,. \tag{2.14}$$

Exponentiation of (2.12) gives

$$e^{-TL}(\phi_1 * \phi_2) = e^{-TL}\phi_1 * e^{-T(L-L_L^+)}\phi_2 \,. \tag{2.15}$$

¿From the familiar commutators

$$[L, L^+] = L^+ \,, \quad [B, L^+] = B^+ \,, \tag{2.16}$$

we deduce

$$[L, L_L^+] = L_L^+ \,, \quad [B, L_L^+] = B_L^+ \,. \tag{2.17}$$

See section 2 of [7] for a careful analysis of this type of manipulations. We will need to reorder exponentials of the derivation $L - L_L^+$. We claim that

$$e^{-T(L-L_L^+)} = e^{(1-e^{-T})L_L^+} \, e^{-TL} \,. \tag{2.18}$$

The above is a particular case of the Baker-Campbell-Hausdorff formula for a two-dimensional Lie algebra with generators x and y and commutation relation $[\, x, y \,] = y$. In the adjoint representation we can write

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} , \quad y = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} . \tag{2.19}$$

It follows that as two-by-two matrices, $x^2 = x$, $xy = y$, $yx = 0$, and $y^2 = 0$. One then verifies that

$$e^{\alpha x + \beta y} = e^{\frac{\beta}{\alpha}(e^{\alpha}-1)y} \, e^{\alpha x} \,, \quad \text{when} \quad [\, x, y \,] = y \,. \tag{2.20}$$

With $\alpha = -\beta = -T$, $x = L$, and $y = L_L^+$, (2.20) reproduces (2.18).

2.3 The action of B/L and its geometric interpretation

We are now ready to solve the equation for $\Psi^{(2)}$. The state $\Psi^{(1)}$ satisfies

$$Q_B \Psi^{(1)} = 0 \,, \qquad B\Psi^{(1)} = 0 \,, \qquad L\Psi^{(1)} = 0 \,. \tag{2.21}$$

We will use correlators in the sliver frame to represent states made of wedge states and operator insertions. The state $\Psi^{(1)}$ can be described as follows:

$$\langle\, \phi, \Psi^{(1)} \,\rangle = \langle\, f \circ \phi(0) \, cV(1) \,\rangle_{\mathcal{W}_1} \,. \tag{2.22}$$

Note that cV is primary with dimension zero so that there is no associated conformal factor. Here and in what follows we use ϕ to denote a generic state in the Fock space and $\phi(0)$ to denote its corresponding operator. The surface $\mathcal{W}_\alpha$ is the one associated with the wedge state W_α in the sliver conformal frame. We use the doubling trick in calculating correlators. We define the oriented straight lines $V_\alpha^\pm$ by

$$\begin{aligned} &V_\alpha^\pm = \Big\{ z \,\Big|\, \text{Re}(z) = \pm \frac{1}{2}(1+\alpha) \Big\}, \\ &\text{orientation} \,:\, \pm \frac{1}{2}(1+\alpha) - i\,\infty \to \pm \frac{1}{2}(1+\alpha) + i\,\infty\,. \end{aligned} \tag{2.23}$$

The surface $\mathcal{W}_\alpha$ can be represented as the region between V_0^- and $V_{2\alpha}^+$, where V_0^- and $V_{2\alpha}^+$ are identified by translation.

A formal solution to the equation $Q_B \Psi^{(2)} = -\Psi^{(1)} * \Psi^{(1)}$ is

$$\Psi^{(2)} = -\int_0^\infty dT\, B e^{-TL} \big[\, \Psi^{(1)} * \Psi^{(1)} \big]\,. \tag{2.24}$$

By construction, $B\Psi^{(2)} = 0$. Using the identities (2.15) and (2.13), we have

$$\Psi^{(2)} = -\int_0^\infty dT\, \Big[\, B\, e^{-TL}\, \Psi^{(1)} * e^{-T(L-L_L^+)}\, \Psi^{(1)} - e^{-TL}\, \Psi^{(1)} * (B - B_L^+)\, e^{-T(L-L_L^+)}\, \Psi^{(1)} \Big]\,. \tag{2.25}$$

Because of the properties of $\Psi^{(1)}$ in (2.21), the first term vanishes and the second reduces to

$$\Psi^{(2)} = \int_0^\infty dT\, \Psi^{(1)} * (B - B_L^+)\, e^{-T(L-L_L^+)}\, \Psi^{(1)}\,. \tag{2.26}$$

We further use the identity (2.18) together with $L\Psi^{(1)} = 0$ to find

$$\Psi^{(2)} = \int_0^\infty dT\, \Psi^{(1)} * (B - B_L^+)\, e^{(1-e^{-T})L_L^+}\, \Psi^{(1)}\,. \tag{2.27}$$

It follows from $[\,B, L_L^+\,] = B_L^+$ that $[\,B,\, g(L_L^+)\,] = B_L^+\, g'(L_L^+)$ for any analytic function g. Using this formula with $B\Psi^{(1)} = 0$, we find

$$\Psi^{(2)} = -\int_0^\infty dT\, e^{-T}\, \Psi^{(1)} * e^{(1-e^{-T})L_L^+}\, B_L^+ \Psi^{(1)}\,. \tag{2.28}$$

Using the change of variables $t = e^{-T}$, we obtain the following final expression of $\Psi^{(2)}$:

$$\Psi^{(2)} = \int_0^1 dt\, \Psi^{(1)} * e^{-(t-1)L_L^+}\, (-B_L^+)\, \Psi^{(1)}\,. \tag{2.29}$$

There is a simple geometric picture for $\Psi^{(2)}$. Let us represent $\langle\, \phi, \Psi^{(2)}\,\rangle$ in the CFT formulation. The exponential action of L_L^+ on a generic string field A can be written as

$$e^{-\alpha L_L^+} A = e^{-\alpha L_L^+} (\mathcal{I} * A) = e^{-\alpha L_L^+} \mathcal{I} * A = W_\alpha * A\,. \tag{2.30}$$

Here we have recalled the familiar expression of the wedge state $W_\alpha = e^{-\frac{\alpha}{2}L^+}\mathcal{I} = e^{-\alpha L_L^+}\mathcal{I}$ [4], where $\mathcal{I}$ is the identity string field. We thus learn that $e^{-\alpha L_L^+}$ with $\alpha > 0$ creates a semi-infinite strip with a width of α in the sliver frame, while $e^{-\alpha L_L^+}$ with $\alpha < 0$ deletes a semi-infinite strip with a width of $|\alpha|$. The inner product $\langle\, \phi, \Psi^{(2)} \,\rangle$ is thus represented by a correlator on $\mathcal{W}_{2-|t-1|} = \mathcal{W}_{1+t}$. In other words, the integrand in (2.29) is made of the wedge state W_{1+t} with operator insertions. The state ϕ is represented by the region between V_0^- and V_0^+ with the operator insertion $f \circ \phi(0)$ at the origin. The left factor of $\Psi^{(1)}$ in (2.29) can be represented by the region between V_0^+ and V_2^+ with an insertion of cV at $z = 1$. For $t = 1$ the right factor of $\Psi^{(1)}$ can be represented by the region between V_2^+ and V_4^+ with an insertion of cV at $z = 2$. For $0 < t < 1$, the region is shifted to the one between $V^+_{2-2|t-1|} = V^+_{2t}$ and $V^+_{4-2|t-1|} = V^+_{2+2t}$, and the insertion of cV is at $z = 2 - |t-1| = 1+t$. Finally, the operator $(-B_L^+)$ is represented by an insertion of $\mathcal{B}$ [8] defined by

$$\mathcal{B} = \int \frac{dz}{2\pi i}\, b(z)\,, \tag{2.31}$$

where the contour of the integral can be taken to be $-V_\alpha^+$ with $1 < \alpha < 1+2t$. We thus have

$$\langle\, \phi, \Psi^{(2)} \,\rangle = \int_0^1 dt\, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t)\, \rangle_{\mathcal{W}_{1+t}}\,. \tag{2.32}$$

As $t \to 0$ the pair of cV's collide, and at $t = 1$ they attain the maximum separation.

The state $\Psi^{(2)}$ should formally solve the equation of motion by construction. Let us examine the BRST transformation of $\Psi^{(2)}$ more carefully based on the expression (2.32). The BRST operator in $\langle\, \phi, Q_B \Psi \,\rangle$ can be represented as an integral of the BRST current on $V^+_{2(1+t)} - V_0^+$:[2]

$$\langle\, \phi, Q_B \Psi^{(2)} \,\rangle = \int_0^1 dt \Big\langle\, f \circ \phi(0) \int_{-V_0^+ + V^+_{2(1+t)}} \frac{dz}{2\pi i}\, j_B(z)\; cV(1)\, \mathcal{B}\, cV(1+t) \,\Big\rangle_{\mathcal{W}_{1+t}}\,, \tag{2.33}$$

where j_B is the BRST current. Since cV is BRST closed, the only nontrivial action of the BRST operator is to change the insertion of the b ghost to that of the energy-momentum tensor:

$$\langle\, \phi, Q_B \Psi^{(2)} \,\rangle = -\int_0^1 dt\, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{L}\, cV(1+t)\, \rangle_{\mathcal{W}_{1+t}}\,, \tag{2.34}$$

where

$$\mathcal{L} = \int \frac{dz}{2\pi i}\, T(z)\,, \tag{2.35}$$

and the contour of the integral can be taken to be $-V_\alpha^+$ with $1 < \alpha < 1+2t$. The minus sign on the right-hand side of (2.34) is from anticommuting the BRST current with the left cV. Since

[2] To derive this we first use the relation $\langle\, \phi, Q_B \Psi^{(2)} \,\rangle = -(-1)^\phi \,\langle\, Q_B \phi, \Psi^{(2)} \,\rangle$, where Q_B on the right-hand side is an integral of the BRST current j_B over a contour that encircles the origin counterclockwise, with the operator j_B placed to the left of $f \circ \phi(0)$ in the correlator. Using the identification of the surface $\mathcal{W}_{1+t}$, the contour can be deformed to $-V^+_{2(1+t)} + V_0^+$. In the correlator, we move the BRST current from the left of $f \circ \phi(0)$ to the right of it. This cancels $(-1)^\phi$, and the additional minus sign is canceled by reversing the orientation of the contour.

$\partial_t e^{-tL_L^+} = -L_L^+ e^{-tL_L^+}$ and $-L_L^+$ corresponds to $\mathcal{L}$ in the correlation function, an insertion of $\mathcal{L}$ is equivalent to taking a derivative with respect to t [5]. We thus find

$$\langle \phi, Q_B \Psi^{(2)} \rangle = -\int_0^1 dt \, \frac{\partial}{\partial t} \langle f \circ \phi(0) \, cV(1) \, cV(1+t) \rangle_{\mathcal{W}_{1+t}} \, . \tag{2.36}$$

The surface term from $t = 1$ gives $-\Psi^{(1)} * \Psi^{(1)}$. The equation of motion is therefore satisfied if the surface term from $t = 0$ vanishes. The surface term from $t = 0$ vanishes if

$$\lim_{t \to 0} cV(0) \, cV(t) = 0 \, . \tag{2.37}$$

Therefore, $\Psi^{(2)}$ defined by (2.32) does solve the equation $Q_B \Psi^{(2)} + \Psi^{(1)} * \Psi^{(1)} = 0$ when V satisfies (2.37). Since $\Psi^{(1)} * \Psi^{(1)}$ is a finite state, the equation guarantees that $Q_B \Psi^{(2)}$ is also finite. However, it is still possible that $\Psi^{(2)}$ has a divergent term which is BRST closed. The ghost part of $\Psi^{(2)}$ is finite since it is given by an integral of ψ_t over t from $t = 0$ to $t = 1$, where ψ_n is the key ingredient in the tachyon vacuum solution [4]:

$$\langle \phi, \psi_n \rangle = \langle f \circ \phi(0) \, c(1) \, \mathcal{B} \, c(1+n) \rangle_{\mathcal{W}_{1+n}} \, , \tag{2.38}$$

and the contour of the integral for $\mathcal{B}$ can be taken to be $-V_\alpha$ with $1 < \alpha < 2n + 1$. When the operator product of V with itself is regular, the condition (2.37) is satisfied and $\Psi^{(2)}$ itself is finite. Note that $V(0) \, V(t)$ in the limit $t \to 0$ can be finite or can be vanishing. We construct $\Psi^{(n)}$ for marginal operators with regular operator products in the next section. When the operator product of V with itself is singular, the formal solution $\Psi^{(2)}$ is not well defined. We discuss this case in section 4.

3 Solutions for marginal operators with regular operator products

In the previous section we constructed a well-defined solution to the equation $Q_B \Psi^{(2)} + \Psi^{(1)} * \Psi^{(1)} = 0$ when V has a regular operator product. In this section we generalize it to $\Psi^{(n)}$ for any n.

We then present the solution that corresponds to the decay of an unstable D-brane in §3.2. In §3.3 we study marginal deformations in the light-cone direction and discuss the application to the solution that represents a string ending on a D-brane.

3.1 Solution

Once we understand how $\Psi^{(2)}$ in the form of (2.32) satisfies the equation of motion, it is easy to construct $\Psi^{(n)}$ satisfying $Q_B \Psi^{(n)} = \Phi^{(n)}$. It is given by

$$\begin{aligned} \langle \phi, \Psi^{(n)} \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \, \langle f \circ \phi(0) \, cV(1) \, \mathcal{B} \, cV(1+t_1) \, \mathcal{B} \, cV(1+t_1+t_2) \, \ldots \\ \times \mathcal{B} \, cV(1+t_1+t_2+\ldots+t_{n-1}) \rangle_{\mathcal{W}_{1+t_1+t_2+\ldots+t_{n-1}}} \, . \end{aligned} \tag{3.1}$$

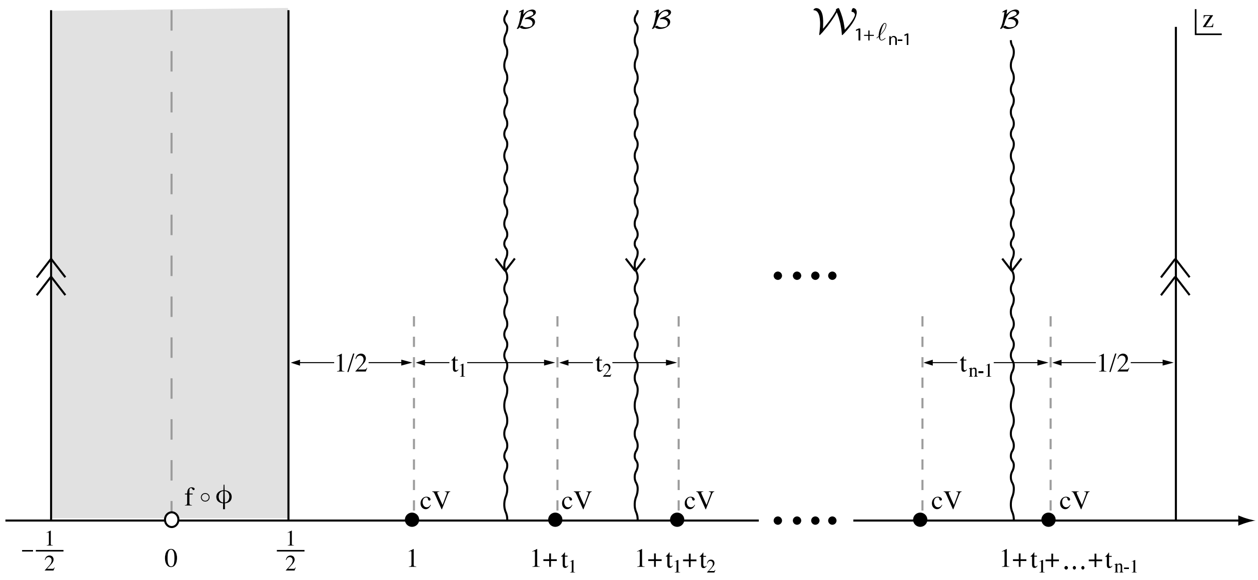

Figure 1: The surface $\mathcal{W}_{1+\ell_{n-1}}$ with the operator insertions used to construct the solution $\Psi^{(n)}$ given in (3.3). The parameters $t_1, t_2, \ldots t_{n-1}$ must all be integrated from zero to one. The leftmost and rightmost vertical lines with double arrows are identified.

Introducing the length parameters

$$\ell_i \equiv \sum_{k=1}^{i} t_k \,, \tag{3.2}$$

the solution can be written more compactly as

$$\boxed{\langle\, \phi, \Psi^{(n)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big] \,\Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} .} \tag{3.3}$$

See Figure 1. The solution obeys the Schnabl gauge condition. It is remarkably simple contrasted with the expression one would obtain in Siegel gauge.

Let us now prove that the equation of motion is satisfied for (3.3). It is straightforward to generalize the calculation of $\langle\, \phi, Q_B \Psi^{(2)} \,\rangle$ in the previous section to that of $\langle\, \phi, Q_B \Psi^{(n)} \,\rangle$. The BRST operator in $\langle\, \phi, Q_B \Psi^{(n)} \,\rangle$ can be represented as an integral of the BRST current on $V^+_{2(1+\ell_{n-1})} - V^+_0$. Since cV is BRST closed, the BRST operator acts only on the insertions of $\mathcal{B}$'s:

$$\begin{aligned} \langle\, \phi, Q_B \Psi^{(n)} \,\rangle = \, - \sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big] \\ \times\, \mathcal{L}\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \,\Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} . \end{aligned} \tag{3.4}$$

An insertion of $\mathcal{L}$ between $cV(1+\ell_{j-1})$ and $cV(1+\ell_j)$ corresponds to taking a derivative with respect to t_j. We thus have

$$
\begin{aligned}
\langle\, \phi, Q_B \Psi^{(n)} \,\rangle = & - \sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1}\, \partial_{t_j} \Big\langle\, f \circ \phi(0)\, cV(1) \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big] \\
& \qquad \times cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}}} \\
= & - \sum_{j=1}^{n-1} \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{j-1} \int_0^1 dt_{j+1} \ldots \int_0^1 dt_{n-1} \Big\langle\, f \circ \phi(0)\, cV(1) \\
& \qquad \times \prod_{i=1}^{j-1} \Big[\, \mathcal{B}\, cV(1+\ell_i) \Big]\, cV(1+\ell_j) \prod_{k=j+1}^{n-1} \Big[\, \mathcal{B}\, cV(1+\ell_k) \Big] \Big\rangle_{\mathcal{W}_{2+\ell_{n-1}}} \Big|_{t_j=1} \\
= & - \sum_{j=1}^{n-1} \langle\, \phi, \Psi^{(j)} * \Psi^{(n-j)} \,\rangle \,.
\end{aligned}
\tag{3.5}
$$

The equation of motion is thus satisfied. Note that the surface term from $t_j = 0$ vanishes because the condition (2.37) is satisfied.

We can also derive this expression of $\Psi^{(n)}$ by acting with B/L on $\Phi^{(n)}$. It is in fact interesting to see how the region of the integrals over $t_1, t_2, \ldots, t_{n-1}$ is reproduced. Let us demonstrate it taking the case of $\Psi^{(3)}$ as an example. Using the Schwinger representation of B/L, the expression (2.26) for $\Psi^{(2)}$, and the identities (2.15) and (2.13), we have

$$
\begin{aligned}
\Psi^{(3)} & = - \int_0^\infty dT_2\, B\, e^{-T_2 L}\, [\, \Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)} \,] \\
& = - \int_0^\infty dT_2 \int_0^\infty dT_1\, B\, e^{-T_2 L}\, [\, \Psi^{(1)} * \Psi^{(1)} * (B - B_L^+)\, e^{-T_1 (L - L_L^+)}\, \Psi^{(1)} \\
& \qquad + \Psi^{(1)} * (B - B_L^+)\, e^{-T_1 (L - L_L^+)}\, \Psi^{(1)} * \Psi^{(1)} \,] \\
& = \int_0^\infty dT_1 \int_0^\infty dT_2\, [\, \Psi^{(1)} * (B - B_L^+)\, e^{-T_2 (L - L_L^+)}\, \Psi^{(1)} * (B - B_L^+)\, e^{-(T_1+T_2)(L - L_L^+)}\, \Psi^{(1)} \\
& \qquad + \Psi^{(1)} * (B - B_L^+)\, e^{-(T_1+T_2)(L - L_L^+)}\, \Psi^{(1)} * (B - B_L^+)\, e^{-T_2 (L - L_L^+)}\, \Psi^{(1)} \,]
\end{aligned}
\tag{3.6}
$$

By changing variables as $\tau_1 = T_2$ and $\tau_2 = T_1 + T_2$ for the first term and as $\tau_2 = T_2$ and $\tau_1 = T_1 + T_2$ for the second term, the two terms combine into

$$
\Psi^{(3)} = \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, \Psi^{(1)} * (B - B_L^+)\, e^{-\tau_1 (L - L_L^+)}\, \Psi^{(1)} * (B - B_L^+)\, e^{-\tau_2 (L - L_L^+)}\, \Psi^{(1)} \,. \tag{3.7}
$$

The same manipulations we performed with $\Psi^{(2)}$ give

$$
\Psi^{(3)} = \int_0^1 dt_1 \int_0^1 dt_2\, \Psi^{(1)} * e^{-(t_1 - 1) L_L^+} (-B_L^+)\, \Psi^{(1)} * e^{-(t_2 - 1) L_L^+} (-B_L^+)\, \Psi^{(1)} \,, \tag{3.8}
$$

and the following expression in the CFT formulation:

$$
\langle\, \phi, \Psi^{(3)} \,\rangle = \int_0^1 dt_1 \int_0^1 dt_2\, \langle\, f \circ \phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2) \,\rangle_{\mathcal{W}_{1+t_1+t_2}} \,, \tag{3.9}
$$

in agreement with (3.3). It is not difficult to use induction to prove that for all n (3.3) follows from the action of B/L on $\Phi^{(n)}$.

We conclude the subsection by writing other forms of the solution that are suitable for explicit calculations. We represent the surface $\mathcal{W}_\alpha$ between the region V_2^- and $V_{2(\alpha-1)}^+$. The operator $cV(1+\ell_{n-1})$ in (3.3) is then mapped to $cV(-1)$. We further transform $\langle\, \phi, \Psi^{(n+1)}\, \rangle$ in the following way:

$$
\begin{aligned}
\langle\, \phi, \Psi^{(n+1)}\, \rangle &= \int_0^1 dt_1 \ldots \int_0^1 dt_n \Big\langle cV(-1)\, f\circ\phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[\mathcal{B}\, cV(1+\ell_i)\Big]\mathcal{B}\Big\rangle_{\mathcal{W}_{1+\ell_n}} \\
&= \int_0^1 dt_1 \ldots \int_0^1 dt_n \Big\langle cV(-1)\, f\circ\phi(0)\, cV(1) \prod_{i=1}^{n-1} \Big[V(1+\ell_i)\Big]\mathcal{B}\Big\rangle_{\mathcal{W}_{1+\ell_n}} \\
&= -\int_0^1 dt_1 \ldots \int_0^1 dt_n \, \frac{1}{2+\ell_n} \\
&\quad \times \Big\langle \int_{V_{2\ell_n}^+ - V_2^-} \frac{dz}{2\pi i}\, z\, b(z) \Big[\, cV(-1)\, f\circ\phi(0)\, cV(1)\Big] \prod_{i=1}^{n-1} \Big[\, V(1+\ell_i)\Big] \Big\rangle_{\mathcal{W}_{1+\ell_n}} .
\end{aligned}
\tag{3.10}
$$

First we recursively used the relation $\mathcal{B}\, c(z)\, \mathcal{B} = \mathcal{B}$, which follows from $\{\, \mathcal{B},\, c(z)\,\} = 1$ and $\mathcal{B}^2 = 0$. In the last step, we used the identity:

$$
\int_{V_{2(\alpha-1)}^+ - V_2^-} \frac{dz}{2\pi i}\, z\, b(z) = (\alpha+1) \int_{V_{2(\alpha-1)}^+} \frac{dz}{2\pi i}\, b(z) \quad \text{on} \quad \mathcal{W}_\alpha . \tag{3.11}
$$

This follows from

$$
\int_{V_2^-} \frac{dz_-}{2\pi i}\, z_-\, b(z_-) = \int_{V_{2(\alpha-1)}^+} \frac{dz_+}{2\pi i} \Big\{\, z_+ - (\alpha+1)\,\Big\}\, b(z_+) \quad \text{on} \quad \mathcal{W}_\alpha , \tag{3.12}
$$

where the coordinate z_- for V_2^- and the coordinate z_+ for $V_{2(\alpha-1)}^+$ are identified by $z_+ = z_- + \alpha + 1$. The contour $V_{2\ell_n}^+ - V_2^-$ can be deformed to encircle $cV(-1)$, $f\circ\phi(0)$, and $cV(1)$, and we obtain

$$
\begin{aligned}
\langle\, \phi, \Psi^{(n+1)}\, \rangle = \int_0^1 dt_1 \ldots \int_0^1 dt_n \, \frac{1}{2+\ell_n} \Big\langle\, \Big\{ & V(-1)\, f\circ\phi(0)\, cV(1) + cV(-1)\, f\circ\phi(0)\, V(1) \\
& + cV(-1) \Big[\oint \frac{dz}{2\pi i}\, z\, b(z)\, f\circ\phi(0)\Big]\, cV(1) \Big\} \prod_{i=1}^{n-1} V(1+\ell_i) \Big\rangle_{\mathcal{W}_{1+\ell_n}} ,
\end{aligned}
\tag{3.13}
$$

where the contour in the last line encircles the origin counterclockwise.

When $\phi(0)$ factorizes into a matter part $\phi_m(0)$ and a ghost part $\phi_g(0)$ we can use the matter-ghost factorization of the correlator to give an alternative form of (3.3):

$$
\begin{aligned}
\langle\, \phi, \Psi^{(n)}\, \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} & \Big\langle\, f\circ\phi_m(0) \prod_{i=0}^{n-1} V(1+\ell_i) \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}},\, m} \\
& \times \Big\langle\, f\circ\phi_g(0)\, c(1)\, \mathcal{B}\, c(1+\ell_{n-1}) \Big\rangle_{\mathcal{W}_{1+\ell_{n-1}},\, g} ,
\end{aligned}
\tag{3.14}
$$

where $\ell_0 \equiv 0$ and we denoted matter and ghost correlators by subscripts m and g, respectively. The ghost correlator in the above expression is $\langle\, \phi_g, \psi_{\ell_{n-1}}\,\rangle$ in (2.38). The algorithm for its calculation has been developed in [4, 8].

3.2 Rolling tachyon marginal deformation to all orders

We can now apply the general solution (3.13) to the special case of a marginal deformation corresponding to a rolling tachyon. For this purpose we pick the operator

$$V(z,\bar{z}) = e^{\frac{1}{\sqrt{\alpha'}} X^0(z,\bar{z})}\,, \tag{3.15}$$

restricted to the boundary $z = \bar{z} = y$ of the upper half plane $\mathbb{H}$, where we write it as[3]

$$V(y) = e^{\frac{1}{\sqrt{\alpha'}} X^0(y)}\,, \qquad X^0(y) \equiv X^0(y,y)\,. \tag{3.16}$$

The operator $e^{ik\cdot X(y)}$ has dimension $\alpha' k^2$ and we can write

$$V(y) = e^{ik\cdot X(y)}\,, \quad \text{with} \quad k^\mu = \frac{i}{\sqrt{\alpha'}}\big(1,\vec{0}\,\big) \;\to\; \alpha' k^2 = 1\,, \tag{3.17}$$

showing that V is a dimension one matter primary. We also have

$$V(y)V(0) \sim\, |y|^2 V(0)^2\,, \tag{3.18}$$

so the OPE of the matter operators satisfies the requisite regularity condition. We will also use exponential operators of X^0 with different exponents. We thus record the following transformation law and ordering results:

$$f \circ e^{\frac{1}{\sqrt{\alpha'}}\, n X^0(y)} = \Big|\frac{df}{dy}\Big|^{n^2} e^{\frac{1}{\sqrt{\alpha'}}\, n X^0(f(y))} \tag{3.19}$$

$$e^{\frac{1}{\sqrt{\alpha'}}\, m X^0(y)} e^{\frac{1}{\sqrt{\alpha'}}\, n X^0(y')} = |y-y'|^{2mn} : e^{\frac{1}{\sqrt{\alpha'}}\, m X^0(y)} e^{\frac{1}{\sqrt{\alpha'}}\, n X^0(y')} : \tag{3.20}$$

Physically, deformation by cV represents a rolling tachyon solution in which the state of the system at time $x^0 = -\infty$ is the perturbative vacuum. We have set the marginal operator to be

$$\Psi^{(1)} = e^{\frac{1}{\sqrt{\alpha'}} X^0(0)}\, c_1|0\rangle\,, \tag{3.21}$$

and we are to calculate the higher $\Psi^{(n)}$ which, by momentum conservation, must take the form

$$\Psi^{(n)} = e^{\frac{1}{\sqrt{\alpha'}}\, n X^0(0)} \Big[\beta_n\, c_1|0\rangle + \ldots\Big]\,, \quad n \geq 2\,. \tag{3.22}$$

[3]We use the metric $(-,++\ldots+)$. For a point $z = \bar{z} = y$ on the boundary of $\mathbb{H}$ we write $X^\mu(y) \equiv X^\mu(y,y)$. The two-point function is $\langle X^\mu(y) X^\nu(y')\rangle = -2\alpha'\eta^{\mu\nu} \ln|y-y'|$ and the mode expansion for a Neumann coordinate reads $i\partial_y X^\mu(y) = \sqrt{2\alpha'}\sum_m \frac{\alpha_m^\mu}{y^{m+1}}$. The basic correlator is $\langle e^{ik\cdot X(y)} e^{ik'\cdot X(y')}\rangle = (2\pi)^D \delta^D(k+k')|y-y'|^{2\alpha' k\cdot k'}$. The operator $e^{ik\cdot X(y)}$ has dimension $\alpha' k^2$ and transforms as $f\circ e^{ik\cdot X(y)} = |\frac{df}{dy}|^{\alpha' k^2} e^{ik\cdot X(f(y))}$. We do not use the doubling trick for the matter sector in §3.2 and §3.3. In these subsections, $\partial X^\mu \equiv \partial_z X^\mu + \partial_{\bar{z}} X^\mu$ when μ is a direction along the D-brane and $\partial X^\mu \equiv \partial_z X^\mu - \partial_{\bar{z}} X^\mu$ when μ is a direction transverse to the D-brane.

In the above we have separated out the tachyon component – higher level fields are indicated by dots. The tachyon field T profile is determined by the coefficients β_n that we aim to calculate:

$$T(x^0) = \lambda\, e^{\frac{1}{\sqrt{\alpha'}}x^0} + \sum_{n=2}^{\infty} \beta_n\, \lambda^n e^{\frac{1}{\sqrt{\alpha'}}\, n x^0}\,. \tag{3.23}$$

Since the solution (for every component field) depends only on the combination $\lambda e^{\frac{1}{\sqrt{\alpha'}}x^0}$ a scaling of λ can be absorbed by a shift of x^0. We can therefore focus on the case $\lambda = \mp 1$. As we will see, all coefficients β_n are positive, so the sign of λ makes a physical difference. In our conventions the tachyon vacuum lies at some $T < 0$, so $\lambda = -1$ corresponds to a tachyon that begins to roll in the direction of the tachyon vacuum – the solution we are mostly interested in. For $\lambda = +1$ the tachyon begins to roll towards the unbounded region of the potential. All in all, we write

$$T(x^0) = \mp\, e^{\frac{1}{\sqrt{\alpha'}}x^0} + \sum_{n=2}^{\infty} (\mp 1)^n \beta_n\, e^{\frac{1}{\sqrt{\alpha'}}\, n x^0}\,. \tag{3.24}$$

In order to extract the coefficients β_n from the solution we introduce test states ϕ_n and their BPZ duals:

$$|\phi_n\rangle = e^{-\frac{1}{\sqrt{\alpha'}}\, nX^0(0)}\, c_0 c_1 |0\rangle\,, \qquad \langle\phi_n| = \lim_{y\to\infty} \langle 0| c_{-1} c_0 e^{-\frac{1}{\sqrt{\alpha'}}\, nX^0(y)}\, \frac{1}{|y|^{2n^2}}\,. \tag{3.25}$$

The state ϕ_n has dimension $n^2 - 1$. Using (3.22) we find

$$\langle \phi_n, \Psi^{(n)}\rangle = \langle\phi_n|\Psi^{(n)}\rangle = \beta_n \cdot (\text{vol})\,, \qquad \text{vol} = (2\pi)^D \delta^D(0)\,. \tag{3.26}$$

The spacetime volume (vol) factors always cancel out, so we will simply use vol$=1$ in the following. We now use (3.13) to write $\beta_{n+1} = \langle\phi_{n+1}, \Psi^{(n+1)}\rangle$ as

$$\begin{aligned}\beta_{n+1} = \int_0^1 dt_1 \ldots \int_0^1 dt_n\, \frac{1}{2+\ell_n} \Big\langle\, \Big\{ & e^{\frac{1}{\sqrt{\alpha'}}X^0(-1)}\, f\circ (\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}\,(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(1) \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-1)\, f\circ (\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}\,(n+1)X^0}(0)\, e^{\frac{1}{\sqrt{\alpha'}}X^0(1)} \\ & + c e^{\frac{1}{\sqrt{\alpha'}}X^0}(-1)\, f\circ c e^{-\frac{1}{\sqrt{\alpha'}}\,(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}}X^0}(1) \Big\} \prod_{i=1}^{n-1} e^{\frac{1}{\sqrt{\alpha'}}X^0(1+\ell_i)} \Big\rangle_{\mathcal{W}_{1+\ell_n}}\,.\end{aligned} \tag{3.27}$$

In the last term, due to the simple structure of ϕ_{n+1}, the antighost line integral acts as b_0 and simply removes the c_0 part of the state. We must now evaluate the correlator in the above right hand side.

This computation requires the map from the wedge $\mathcal{W}_{1+\ell_n}$ to the upper half plane. We recall that a wedge of unit width can be mapped to the upper half plane by the function

$$g(z) = \frac{1}{2}\tan(\pi z)\,. \tag{3.28}$$

Due to the periodicity $g(z+1) = g(z)$, this map works independent of the position of the wedge along the real axis. Consequently, we merely need to rescale our wedge region $\mathcal{W}_{1+\ell_n}$ to a unit wedge by

$z \to \frac{z}{2+\ell_n}$ and then map to the upper half plane by acting with g. The overall conformal transformation on the test states is therefore the map h:

$$h(\xi) \equiv g\Big(\frac{1}{2+\ell_n}\, f(\xi)\Big). \tag{3.29}$$

which satisfies $h'(0) = \frac{1}{2+\ell_n}$. All other vertex operators are mapped with $g\big(\frac{1}{2+\ell_n}\, z\big)$. It is therefore natural to define

$$g_i \equiv g\Big(\frac{1+\ell_i}{2+\ell_n}\Big)\,, \qquad g_i' \equiv g'\Big(\frac{1+\ell_i}{2+\ell_n}\Big) \quad i = 0, 1, \ldots, n\,, \quad \ell_0 \equiv 0\,. \tag{3.30}$$

With these abbreviations, the correlator on the upper half plane reads

$$\begin{aligned}\beta_{n+1} = \int d^n t\, \frac{h'(0)^{(n+1)^2-1}}{2+\ell_n} \Big\langle \Big\{ \frac{g_0'}{2+\ell_n} \Big(e^{\frac{1}{\sqrt{\alpha'}} X^0(-g_0)}\, (\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}} X^0}(g_0) \\ + c e^{\frac{1}{\sqrt{\alpha'}} X^0}(-g_0)\, (\partial c) c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, e^{\frac{1}{\sqrt{\alpha'}} X^0(g_0)} \Big) \\ + c e^{\frac{1}{\sqrt{\alpha'}} X^0}(-g_0)\, c e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0}(0)\, c e^{\frac{1}{\sqrt{\alpha'}} X^0}(g_0) \Big\} \prod_{i=1}^{n-1} \frac{g_i'}{2+\ell_n} e^{\frac{1}{\sqrt{\alpha'}} X^0(g_i)} \Big\rangle_{\mathbb{H}}\,,\end{aligned} \tag{3.31}$$

where we have defined $\int d^n t \equiv \int_0^1 dt_1 \ldots \int_0^1 dt_n$. We can now factor this into a matter and a ghost correlator

$$\begin{aligned}\beta_{n+1} = \int d^n t\, (2+\ell_n)^{-(n+1)^2} \Big\langle e^{\frac{1}{\sqrt{\alpha'}} X^0(-g_0)}\, e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)}\, e^{\frac{1}{\sqrt{\alpha'}} X^0(g_0)} \prod_{i=1}^{n-1} \frac{g_i'}{2+\ell_n} e^{\frac{1}{\sqrt{\alpha'}} X^0(g_i)} \Big\rangle_m \\ \Big\langle \frac{g_0'}{2+\ell_n} \Big((\partial c) c(0)\, c(g_0) + c(-g_0)\, (\partial c) c(0) \Big) + c(-g_0)\, c(0)\, c(g_0) \Big\rangle_g\,.\end{aligned} \tag{3.32}$$

The ghost correlator can be evaluated using $\big\langle c(-z) c(0) c(z) \big\rangle_g = -2z^3$ and $\big\langle \partial c\, c(0)\, c(z) \big\rangle_g = z^2$. Using also $-g_0 = g_n$ and $g_0' = g_n'$, we find

$$\beta_{n+1} = 2 \int d^n t\, (2+\ell_n)^{-n(n+3)} \Big(\frac{g_0'}{2+\ell_n} - g_0 \Big) g_0^2 \prod_{i=1}^{n-1} \Big[g_i' \Big] \Big\langle e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)} \prod_{i=0}^{n} e^{\frac{1}{\sqrt{\alpha'}} X(g_i)} \Big\rangle_m\,. \tag{3.33}$$

Computing the matter correlator we obtain our final result for the coefficients of the rolling tachyon solution

$$\boxed{\beta_{n+1} = 2 \int d^n t\, (2+\ell_n)^{-n(n+3)} \Big(\frac{g_0'}{2+\ell_n} - g_0 \Big) \frac{g_0^2}{g_0'^2} \Big[\prod_{i=0}^{n} \frac{g_i'}{g_i^{2(n+1)}} \Big] \prod_{0 \le i < j \le n} (g_i - g_j)^2.} \tag{3.34}$$

Another way to derive (3.34) is to use (3.14). The ghost correlator, which gives the tachyon coefficient of ψ_{ℓ_n}, has been calculated in [4, 8]:

$$\begin{aligned}\langle\, f \circ (\partial c) c(0)\, c(1)\, \mathcal{B}\, c(1+\ell_n)\, \rangle_{\mathcal{W}_{1+\ell_n}, g} &= 2\,(2+\ell_n)\, \frac{g_0^2}{g_0'} \left(1 - \frac{(2+\ell_n)\, g_0}{g_0'} \right) \\ &= \frac{2+\ell_n}{\pi} \left[1 - \frac{2+\ell_n}{2\pi} \sin \frac{2\pi}{2+\ell_n} \right] \sin^2 \frac{\pi}{2+\ell_n}\,.\end{aligned} \tag{3.35}$$

The calculation of the matter correlator is straightforward:

$$
\begin{aligned}
&\Big\langle f \circ e^{-\frac{1}{\sqrt{\alpha'}}(n+1)X^0(0)} \prod_{i=0}^{n} e^{\frac{1}{\sqrt{\alpha'}}X^0(1+\ell_i)} \Big\rangle_{\mathcal{W}_{1+\ell_n},\,m} \\
&= \Big(\frac{2}{\pi}\Big)^{(n+1)^2} \Bigg[\prod_{i=0}^{n} \frac{(2+\ell_n)^{-2(n+1)}}{\pi^{-2(n+1)}} \sin^{-2(n+1)} \frac{\pi(1+\ell_i)}{2+\ell_n}\Bigg] \prod_{0\le i<j\le n} \frac{(2+\ell_n)^2}{\pi^2} \sin^2 \frac{\pi(\ell_i-\ell_j)}{2+\ell_n} \\
&= (2+\ell_n)^{-(n+1)(n+2)} \Bigg[\prod_{i=0}^{n} \frac{g_i'}{g_i^{2(n+1)}}\Bigg] \prod_{0\le i<j\le n} (g_i-g_j)^2\,.
\end{aligned}
\tag{3.36}
$$

It is easy to see that (3.34) is reproduced.

The integrand in (3.34) is manifestly positive since $g'(z)>0$ and $\frac{g_0'}{2+\ell_n}-g_0>0$. It follows that all β_n coefficients are positive. For $n=1$ we find

$$
\beta_2 = 8\int_0^1 dt\, \frac{\frac{g_0'}{2+t}-g_0}{(2+t)^4 g_0^4} = 8\int_0^1 dt \left(\frac{2\cot\big(\frac{\pi}{2+t}\big)}{2+t}\right)^4 \Big(\frac{\pi}{2(2+t)\cos^2\big(\frac{\pi}{2+t}\big)} - \frac{1}{2}\tan\Big(\frac{\pi}{2+t}\Big)\Big)\,. \tag{3.37}
$$

Surprisingly, analytic evaluation of the integral is possible using *Mathematica*:

$$
\beta_2 = \frac{64}{243\sqrt{3}}\,. \tag{3.38}
$$

For $n=2$ the final integral must be done numerically:

$$
\beta_3 = 8\int_0^1 dt_1 \int_0^1 dt_2\, \frac{\Big(\frac{g_0'}{2+t_1+t_2}-g_0\Big) g_1' \big(g_0^2-g_1^2\big)^2}{(2+t_1+t_2)^{10} g_0^8 g_1^6} \simeq 2.14766\cdot 10^{-3} \tag{3.39}
$$

The results for the first few β_n are summarized in table 1. The resulting tachyon profile (3.24) takes the form

$$
\begin{aligned}
T(x^0) = \mp\, e^{\frac{1}{\sqrt{\alpha'}}x^0} +\ 0.15206\ e^{\frac{1}{\sqrt{\alpha'}}2x^0} &\mp 2.148\cdot 10^{-3}\, e^{\frac{1}{\sqrt{\alpha'}}3x^0} \\
+\,2.619\cdot 10^{-6}\, e^{\frac{1}{\sqrt{\alpha'}}4x^0} &\mp 2.791\cdot 10^{-10}\, e^{\frac{1}{\sqrt{\alpha'}}5x^0} \\
+\,2.801\cdot 10^{-15}\, e^{\frac{1}{\sqrt{\alpha'}}6x^0} &\mp 2.729\cdot 10^{-21}\, e^{\frac{1}{\sqrt{\alpha'}}7x^0} + \ldots
\end{aligned}
\tag{3.40}
$$

The top sign gives us the physical solution: the tachyon rolls towards the tachyon vacuum, it overshoots it, and then begins to perform larger and larger oscillations. The coefficients in the solution decrease so rapidly that it suggests that the series is actually absolutely convergent, for any value of $\frac{x^0}{\sqrt{\alpha'}}$. Indeed, the n-th term T_n in the above series appears to take the form

$$
|T_n| \sim 2.7\cdot 10^{-\frac{1}{2}n(n-1)}\, e^{\frac{1}{\sqrt{\alpha'}}nx^0}\,. \tag{3.41}
$$

One then finds that the ratio of consecutive coefficients is

$$
\Big|\frac{T_{n+1}}{T_n}\Big| \sim 10^{-n} e^{\frac{1}{\sqrt{\alpha'}}x^0} \simeq e^{-2.303\,n} e^{\frac{1}{\sqrt{\alpha'}}x^0}\,. \tag{3.42}
$$

n	β_n
2	$\frac{64}{243\sqrt{3}} \approx 0.152059$
3	$2.14766 \cdot 10^{-3}$
4	$2.61925 \cdot 10^{-6}$
5	$2.79123 \cdot 10^{-10}$
6	$2.80109 \cdot 10^{-15}$
7	$2.72865 \cdot 10^{-21}$

Table 1: Numerical values of the rolling tachyon profile coefficients

For any value of $\frac{x^0}{\sqrt{\alpha'}}$ the ratio becomes smaller than one for sufficiently large n, ensuring absolute convergence. It would be useful to do analytic estimates of β_n using (3.34) to confirm the above speculation.

It is interesting to compare with the results of the p-adic model [19]. The relevant solution is discussed in §4.2.2 of that paper and has the same qualitative behavior as the solution presented here: the tachyon rolls towards the minimum, overshoots it, and then performs ever growing oscillations. The solution is of the form

$$\phi(t) = 1 - \sum_{n=1}^{\infty} a_n \, e^{\sqrt{2}nt}\,, \quad a_1 = 1\,. \tag{3.43}$$

The coefficients a_n can be calculated exactly with a simple recursion and fall off very rapidly, but an analytic expression for their large n behavior is not known. A fit of the values of a_n for $n = 3, \ldots, 13$ gives: $\ln a_n \simeq -0.1625 + 1.506\, n - 1.389\, n^2$ (a fit with an n^3 term returns a very small coefficient for this term). The fit implies that the ratio of two consecutive terms in the solution is

$$\Big|\frac{a_{n+1}}{a_n}\Big| e^{\sqrt{2}\,t} \sim e^{-2.778\,n + 0.117} e^{\sqrt{2}\,t} \simeq 1.125 \cdot 16^{-n} e^{\sqrt{2}\,t}\,. \tag{3.44}$$

This result suggests that the p-adic rolling solution is also absolutely convergent.

A low level solution of the string theory rolling tachyon in Siegel gauge was also obtained in [19], where significant similarities with the p-adic solution were noted. The higher-level Siegel gauge analysis of the rolling tachyon in [23] confirmed the earlier analysis and added much confidence to the validity of the oscillatory solution. The exact analytic solution presented here settles the issue convincingly.

3.3 Lightcone-like deformations

Another simple example of a marginal operator with regular OPE is provided by the lightcone-like operator

$$V(z) = \frac{i}{\sqrt{2\alpha'}}\, \partial X^+\,, \tag{3.45}$$

as usual, inserted at $z = \bar{z} = y$. Here $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)$, is a lightcone coordinate (we could have also chosen $X^- = \frac{1}{\sqrt{2}}(X^0 - X^1)$). For this operator one has the regularity $\lim_{z\to 0} V(z)V(0) = V(0)^2$. The operator is dimension one and cV is BRST closed. We can therefore construct a solution using the above $V(z)$ and our general result (3.13). If we consider some Dp-brane, we could choose x^1 to be a direction normal to the brane and the above matter deformation would correspond to giving constant expectation values to the time component of the gauge field on the brane and to the scalar field on the brane that represents displacements of the brane.

To make the analysis a bit more nontrivial we consider the discussion of Michishita [25] who discussed the Callan-Maldacena solution [26] for a string ending on a brane in the framework of OSFT. We therefore choose

$$V(y) = \int dk_i\, A(k_i)\, \frac{i}{\sqrt{2\alpha'}}\, \partial X^+ e^{ik_i X^i}(y)\,, \tag{3.46}$$

where X^i's are the spatial directions on the brane. This operator has a non-singular OPE with itself: the exponentials $e^{ik_i X^i}(y)$ giving positive powers of distances given that k_i is spacelike. It follows that our prescription result (3.13) would be well-defined. The operator $cV(y)$, however, has dimension $\alpha' k^2$, so unless $k_i = 0$ it is not BRST closed and does not provide a solution. But it is not too far from a solution: if one chooses $A(k) \sim 1/k^2$, the action of Q_B on cV represents a delta function in position space.

We thus take $\Psi_A^{(1)} = V(0)c_1|0\rangle$ and, following [25], take its failure to be annihilated by Q_B to define the source term $J^{(1)}$ that hopefully would arise independently in a complete theory: $Q_B\Psi_A^{(1)} = J^{(1)}$. We can then calculate a $\Psi_A^{(2)}$ which will satisfy $Q_B\Psi_A^{(2)} + \Psi_A^{(1)} * \Psi_A^{(1)} = J^{(2)}$. While $BJ^{(1)} \neq 0$, we demand $BJ^{(n)} = 0$ for $n \geq 2$, in analogy to the Siegel gauge solution [25]. Acting with B on the above equation for $\Psi_A^{(2)}$ we find

$$L\Psi_A^{(2)} + B(\Psi_A^{(1)} * \Psi_A^{(1)}) = 0 \quad \rightarrow \quad \Psi_A^{(2)} = -\frac{B}{L}(\Psi_A^{(1)} * \Psi_A^{(1)})\,. \tag{3.47}$$

Acting with Q_B on the solution, one confirms that

$$Q_B\Psi_A^{(2)} = -\Psi_A^{(1)} * \Psi_A^{(1)} + \frac{B}{L}\Big(J^{(1)} * \Psi_A^{(1)} - \Psi_A^{(1)} * J^{(1)}\Big)\,, \tag{3.48}$$

so that the source term $J^{(2)}$ is indeed annihilated by B.

Given (3.47) and the fact that $L\Psi_A^{(1)} \neq 0$, we need to slightly generalize our result (2.29) to find the action of B/L on a string field product $\chi * \chi'$ where χ and χ' are not annihilated by L but instead satisfy

$$B\chi = B\chi' = 0\,, \qquad L\chi = l_\chi\, \chi\,, \qquad L\chi' = l_{\chi'}\, \chi'\,. \tag{3.49}$$

The steps leading to (2.29) can be carried out analogously for this case and we pick up extra l_χ and $l_{\chi'}$ dependent factors:

$$\frac{B}{L}(\chi * \chi') = \int_0^1 dt\, t^{(l_\chi + l_{\chi'})}\, \chi * e^{-(t-1)L_L^+}\,(-B_L^+)\,\chi'\,. \tag{3.50}$$

To construct $\Psi_A^{(2)}$, we need to express states of the type $\frac{B}{L}(\chi * \chi')$ as CFT correlators. As χ and χ' are conformal primaries of non-vanishing dimension, we also pick up extra conformal factors in the sliver frame expression for these states. Defining a shift function $s_l(z) = z + l$, we can express the generalization of (2.32) that accounts for these extra factors as

$$\begin{aligned} \langle\, \phi, \frac{B}{L}(\chi * \chi')\,\rangle &= -\int_0^1 dt\, t^{(l_\chi + l_{\chi'})}\, \langle\, f\circ\phi(0)\;\; s_1\circ f\circ\chi(0)\;\mathcal{B}\;\; s_{1+t}\circ f\circ\chi'(0)\,\rangle_{\mathcal{W}_{1+t}} \\ &= -\int_0^1 dt\, \big(tf'(0)\big)^{(l_\chi + l_{\chi'})}\, \langle\, f\circ\phi(0)\;\chi(1)\;\mathcal{B}\;\chi'(1+t)\,\rangle_{\mathcal{W}_{1+t}}\,. \end{aligned} \tag{3.51}$$

Here we have explicitly carried out the conformal maps of χ and χ' to the sliver frame and used $s_l'(z) = 1$. It is now straight-forward to carry out the construction of $\Psi_A^{(2)}$ in generalization of (3.13). This yields

$$\begin{aligned} \langle\phi, \Psi_A^{(2)}\rangle = \int dk_i dk_i' A(k_i) A(k_i') \int_0^1 dt\, \frac{-\big(tf'(0)\big)^{\alpha'(k^2+k'^2)}}{(2+t)2\alpha'} \Big\langle\, \Big\{ &\partial X^+ e^{ik_iX^i}(-1)\, f\circ\phi(0)\, c\partial X^+ e^{ik_i'X^i}(1) \\ &+ c\partial X^+ e^{ik_iX^i}(-1) \Big[\oint \frac{dz}{2\pi i} z b(z) f\circ\phi(0)\Big]\, c\partial X^+ e^{ik_i'X^i}(1) \\ &+ c\partial X^+ e^{ik_i'X^i}(-1)\, f\circ\phi(0)\, \partial X^+ e^{ik_iX^i}(1)\Big\}\,\Big\rangle_{\mathcal{W}_{1+t}}. \end{aligned} \tag{3.52}$$

To obtain a Fock space expression of $\Psi_A^{(2)}$, we follow the same steps leading to (5.50) of [8]. The map we need to perform on the correlator is $I\circ g$, so the total map on the test state ϕ is $I\circ h$. Here we have used g and h as defined above in (3.28) and (3.29) and we have defined $I(z) = -\frac{1}{z}$. Let us further define

$$\hat{B} = \oint \frac{dz}{2\pi i}\, \frac{g^{-1}(z)}{(g^{-1})'(z)} b(z). \tag{3.53}$$

Then we can start by mapping the correlator to the upper half plane through g. Again, we will suppress all arguments of g and abbreviate

$$g \equiv g\Big(\frac{1}{2+t}\Big) = -g\Big(-\frac{1}{2+t}\Big) \qquad\qquad g' \equiv g'\Big(\frac{1}{2+t}\Big) = g'\Big(-\frac{1}{2+t}\Big). \tag{3.54}$$

With this we find

$$\begin{aligned} \langle\phi, \Psi_A^{(2)}\rangle = \int dk_i dk_i' A(k_i) A(k_i') \int_0^1 dt\, \frac{-1}{(2+t)2\alpha'} \Big(\frac{tf'(0)g'}{2+t}\Big)^{\alpha'(k^2+k'^2)} \\ \Big\langle \frac{g'}{2+t}\Big\{ \partial X^+ e^{ik_iX^i}(-g)\, h\circ\phi(0)\, c\partial X^+ e^{ik_i'X^i}(g) + c\partial X^+ e^{ik_i'X^i}(-g)\, h\circ\phi(0)\, \partial X^+ e^{ik_iX^i}(g)\Big\} \\ + c\partial X^+ e^{ik_iX^i}(-g)\, \Big[\hat{B}\, h\circ\phi(0)\Big]\, c\partial X^+ e^{ik_i'X^i}(g)\,\Big\rangle_{\mathbb{H}}. \end{aligned} \tag{3.55}$$

Here we used the fact that the operator $c\partial X^+ e^{ik_iX^i}$ has conformal dimension $\alpha' k^2$. We notice that the two terms in paranthesis can be tranformed into each other through the map $g \to -g$. Therefore, we can drop one of them and simply take the g-even part of the other. We can now perform the remaining transformation with I to obtain an operator expression for $\Psi_A^{(2)}$:

$$\begin{aligned}\Psi_A^{(2)} =& \int dk_i dk'_i\, A(k_i)A(k'_i) \int_0^1 dt\, \frac{-1}{(2+t)2\alpha'}\Big(\frac{tf'(0)g'}{(2+t)g^2}\Big)^{\alpha'(k^2+k'^2)} \\ & U_h^\star\Big[\Big\{\frac{2g'}{(2+t)g^2}\,\partial X^+ e^{ik_iX^i}\big(-1/g\big)\, c\partial X^+ e^{ik'_iX^i}\big(1/g\big)\Big\}_{\text{g-even}} \\ & \qquad + \hat{B}^\star\, c\partial X^+ e^{ik_iX^i}\big(-1/g\big)\, c\partial X^+ e^{ik'_iX^i}\big(1/g\big)\Big]|0\rangle \\ \equiv& \int dk_i dk'_i\, A(k_i)A(k'_i)\, \Psi_{k,k'}^{(2)}\,.\end{aligned} \tag{3.56}$$

We would now like to determine the level expansion of $\Psi_A^{(2)}$, or equivalently of its momentum decomposition $\Psi_{k,k'}^{(2)}$. We can either attempt a direct level expansion of the operator result (3.56), or we can use the test state formalism that we carried in §3.2. It is straightforward to carry out the first method for the case of vanishing momentum $k = k' = 0$, so we will start with this approach. We will then use the test state method to find the level expansion with full momentum dependence.

Let us start by level expanding $\Psi_{k,k'}^{(2)}$ from (3.56). Again we use the results of [8], this time of §6.1, to obtain the following useful expansions:

$$\hat{B}^\star = b_0 + \frac{8}{3}b_{-2} + \dots \qquad\qquad U_h^\star = (2+t)^{-L_0} + \dots \tag{3.57}$$

Here the dots denote higher level corrections. We notice that the self-contractions of the ∂X^+vanish as $\eta^{++} = 0$. We end up with the following mode expansions for the matter and ghost fields:

$$\begin{aligned}\frac{-1}{2\alpha'}\,\partial X^+(-1/g)\partial X^+(1/g)|0\rangle &= \sum_{i<0,j<0} (-)^{i+1}(\alpha_i^+\alpha_j^+)g^{i+j+2}|0\rangle\,, \\ c(\pm 1/g) = \sum_m c_m\big(\pm g\big)^{m-1}\,,\quad \partial c(\pm 1/g) &= -\sum_m (m-1)c_m\big(\pm g\big)^m\,.\end{aligned} \tag{3.58}$$

Let us now level expand $\Psi_{(k=0,k'=0)}^{(2)}$ of (3.56) for vanishing momentum to leading order:

$$\begin{aligned}\Psi_{(k=0,k'=0)}^{(2)} =& \int_0^1 dt(2+t)^{-L_0-1}\Big[\frac{2g'}{(2+t)g^2}(\alpha_{-1}^+)^2c_1 - b_0(\alpha_{-1}^+)^2\frac{2}{g}c_0c_1\Big]|0\rangle \\ =& 2\int_0^1 dt\frac{\frac{g'}{2+t}-g}{(2+t)^2g^2}(\alpha_{-1}^+)^2c_1|0\rangle = \frac{4}{3\sqrt{3}}(\alpha_{-1}^+)^2c_1|0\rangle\,.\end{aligned} \tag{3.59}$$

The above component of the string field solution is exact to all orders in λ, as it cannot receive contributions from any higher $\Psi^{(n)}$. The coefficient was determined analytically with *Mathematica*.

Let us now use the test state approach to determine this coefficient for general momentum dependence $A(k)$. In other words, we are trying to determine $\beta_{k,k'}$ in

$$\Psi_{k,k'}^{(2)} = \beta_{k,k'}e^{i(k_i+k'_i)X^i(0)}(\alpha_{-1}^+)^2c_1|0\rangle + \dots \tag{3.60}$$

As always, the dots denote higher level contributions. The appropriate test state $\phi_{k,k'}$, such that $\langle\phi_{k,k'},\Psi^{(2)}_{k,k'}\rangle = \beta_{k,k'}\cdot(\mathrm{vol})$ and its BPZ-conjugate are given by

$$\begin{aligned}|\phi_{k,k'}\rangle &= \frac{1}{2}e^{-i(k_i+k'_i)X^i(0)}\,(\alpha^-_{-1})^2c_0c_1|0\rangle = \frac{1}{2}\Big(\frac{-1}{2\alpha'}\Big)\,\partial X^-\partial X^- e^{-i(k_i+k'_i)X^i}(0)\,|0\rangle\\ \langle\phi_{k,k'}| &= \frac{1}{2}\lim_{y\to\infty}\langle 0|(\alpha^-_1)^2c_{-1}c_0e^{-i(k_i+k'_i)X^i(y)}\,\frac{1}{|y|^{2\alpha'(k+k')^2}}\,.\end{aligned} \tag{3.61}$$

The state $\phi_{k,k'}$ has weight $\alpha'(k+k')^2+1$. We can now evaluate the correlator of $\Psi^{(2)}_{k,k'}$ with this test state in the spirit of §3.2:

$$\begin{aligned}\beta_{k,k'} = &\int_0^1 dt\,\frac{-1}{(2+t)2\alpha'}\Big(\frac{tf'(0)g'}{2+t}\Big)^{\alpha'(k^2+k'^2)}\Big\langle\frac{g'}{2+t}\Big\{\partial X^+e^{ik_iX^i}(-g)\,h\circ\phi_{k,k'}(0)\,c\partial X^+e^{ik'_iX^i}(g)\\ &\quad + c\partial X^+e^{ik_iX^i}(-g)\,h\circ\phi_{k,k'}(0)\,\partial X^+e^{ik'_iX^i}(g)\Big\}\\ &\quad + c\partial X^+e^{ik_iX^i}(-g)\,\Big[\hat{B}\,h\circ\phi_{k,k'}(0)\Big]\,c\partial X^+e^{ik'_iX^i}(g)\,\Big\rangle_{\mathbb{H}}\\ =&\frac{1}{2}\Big(\frac{1}{2\alpha'}\Big)^2\int_0^1 dt\,\frac{h'(0)^{\alpha'(k+k')^2+1}}{(2+t)}\Big(\frac{tf'(0)g'}{2+t}\Big)^{\alpha'(k^2+k'^2)}\\ &\quad\cdot\Big\langle\partial X^+e^{ik_iX^i}(-g)\;\partial X^-\partial X^-e^{-i(k+k')_iX^i}(0)\;\partial X^+e^{ik'_iX^i}(g)\Big\rangle_m\\ &\quad\cdot\Big\langle\frac{g'_0}{2+\ell_n}\,\Big((\partial c)c(0)\,c(g_0)+c(-g_0)\,(\partial c)c(0)\Big)+c(-g_0)\,c(0)\,c(g_0)\Big\rangle_g\end{aligned} \tag{3.62}$$

where we have again factored the correlator into a ghost and a matter contribution. The matter contribution vanishes unless the ∂X^+ and ∂X^- contract pairwise and the ghost correlator is the same as in §3.2. Therefore

$$\begin{aligned}\beta_{k,k'} =&\Big(\frac{1}{2\alpha'}\Big)^2\int_0^1 dt\,\frac{h'(0)^{\alpha'(k+k')^2+1}}{(2+t)}\Big(\frac{tf'(0)g'}{2+t}\Big)^{\alpha'(k^2+k'^2)}2\Big(\frac{g'}{2+t}-g\Big)g^2\\ &\cdot\Big\langle e^{ik_iX^i}(-g)\;e^{-i(k+k')_iX^i}(0)\;e^{ik'_iX^i}(g)\Big\rangle_m\cdot\Big(\frac{(2\alpha')\eta^{+-}}{g^2}\Big)^2.\end{aligned} \tag{3.63}$$

We can now plug in $h'(0)$ determined above in §3.2, use $f'(0)=\frac{2}{\pi}$ and evaluate the remaining familiar matter correlator:

$$\beta_{k,k'} =2\int_0^1 dt\,(2+t)^{-\alpha'(k+k')^2-2}\Big(\frac{2tg'}{\pi(2+t)}\Big)^{\alpha'(k^2+k'^2)}\Big(\frac{g'}{2+t}-g\Big)\frac{(2g)^{2\alpha'k\cdot k'}}{g^{2+2\alpha'(k+k')^2}}. \tag{3.64}$$

For general momenta the integral is very complicated, but for vanishing momentum, we recover the result from the operator expansion: $\beta_{k=0,k'=0}=\frac{4}{3\sqrt{3}}$. We can summarize our solution as

$$\begin{aligned}\Psi &= \lambda\int dk_i\,A(k_i)e^{ik_iX^i(0)}\alpha^+_{-1}c_1|0\rangle\\ &\quad+\lambda^2\Big(\int dk_idk'_i\,A(k_i)A(k'_i)\beta_{k,k'}e^{i(k_i+k'_i)X^i(0)}(\alpha^+_{-1})^2c_1|0\rangle+\ldots\Big)+\mathcal{O}(\lambda^3)\end{aligned} \tag{3.65}$$

with $\beta_{k,k'}$ given in (3.64).

4 Solutions for marginal operators with singular operator products

In the previous section, we constructed analytic solutions for marginal deformations when the operator V has a regular operator product with itself. In this section we generalize the construction to the case where V has a singular OPE with itself that takes the form

$$V(z)\,V(w) \sim \frac{1}{(z-w)^2} + \text{regular}. \tag{4.1}$$

4.1 Construction of $\Psi^{(2)}$

The string field $\Psi^{(2)}$ in (2.32) is not well defined when V has a singular OPE with itself. Let us define a regularized string field $\Psi_0^{(2)}$ as follows:

$$\langle\, \phi, \Psi_0^{(2)} \,\rangle = \int_{2\epsilon}^{1} dt\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t)\,\rangle_{\mathcal{W}_{1+t}}\,. \tag{4.2}$$

The equation of motion is no longer satisfied by $\Psi_0^{(2)}$ because the surface term at $t = 2\epsilon$ in (2.36) is nonvanishing. The BRST transformation of $\Psi_0^{(2)}$ is given by

$$\langle\, \phi, Q_B\Psi_0^{(2)} \,\rangle = -\,\langle\, \phi, \Psi^{(1)} * \Psi^{(1)} \,\rangle + \langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}}\,, \tag{4.3}$$

and we see that the second term on the right-hand side violates the equation of motion. Using the OPE

$$cV(-\epsilon)\, cV(\epsilon) = \frac{1}{2\epsilon}\, c\partial c(0) + O(\epsilon)\,, \tag{4.4}$$

the term violating the equation of motion can be written as

$$\langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} = \frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c\partial c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} + O(\epsilon)\,. \tag{4.5}$$

Since the operator $c\partial c$ is the BRST transformation of c, we recognize that up to terms vanishing as $\epsilon \to 0$, the offending term is BRST exact. This crucial property makes it possible to satisfy the equation of motion by adding a counterterm to the regularized string field $\Psi_0^{(2)}$. We define the counterterm $\Psi_1^{(2)}$ by

$$\langle\, \phi, \Psi_1^{(2)} \,\rangle = -\,\frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}}\,. \tag{4.6}$$

The sum of $\Psi_0^{(2)}$ and $\Psi_1^{(2)}$ then solves the equation of motion in the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\,\langle\, \phi, Q_B\,(\Psi_0^{(2)} + \Psi_1^{(2)}) + \Psi^{(1)} * \Psi^{(1)} \,\rangle = 0\,. \tag{4.7}$$

This is not yet the end of the story, as we must also require that the solution be finite as $\epsilon \to 0$. Since $\Psi^{(1)} * \Psi^{(1)}$ is a finite state, $Q_B\,(\Psi_0^{(2)} + \Psi_1^{(2)})$ is also finite in the limit $\epsilon \to 0$. This implies that while the state $\Psi_0^{(2)} + \Psi_1^{(2)}$ can be divergent, the divergent terms must be BRST closed. It follows that we can finally achieve a finite solution by simply subtracting the divergent terms from $\Psi_0^{(2)} + \Psi_1^{(2)}$. Let us isolate the divergent terms in $\Psi_0^{(2)}$. Using the anticommutation relation $\{\,\mathcal{B}, c(z)\,\} = 1$, the operator insertions in $\Psi_0^{(2)}$ can be written as:

$$\begin{aligned} cV(1)\,\mathcal{B}\,cV(1+t) &= cV(1)\,V(1+t) - cV(1)\,cV(1+t)\,\mathcal{B} \\ &= \frac{1}{t^2}\,c(1) - \frac{1}{t}\,c\partial c(1)\,\mathcal{B} + O(t^0)\,. \end{aligned} \tag{4.8}$$

Using the formula

$$\begin{aligned} \langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\rangle_{\mathcal{W}_{\alpha+\delta\alpha}} &= \langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\rangle_{\mathcal{W}_\alpha} \\ &\quad + \delta\alpha\,\langle\, \mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2)\,\ldots\,\mathcal{O}_n(z_n)\,\mathcal{L}\,\rangle_{\mathcal{W}_\alpha} + O(\delta\alpha^2)\,, \end{aligned} \tag{4.9}$$

valid for any set of operators $\mathcal{O}_i$, we find

$$\begin{aligned} \langle f\circ\phi(0)\,cV(1)\,\mathcal{B}\,cV(1+t)\rangle_{\mathcal{W}_{1+t}} &= \frac{1}{t^2}\,\langle f\circ\phi(0)\,c(1)\rangle_{\mathcal{W}_1} + \frac{1}{t}\,\langle f\circ\phi(0)\big[c(1)\mathcal{L} - c\partial c(1)\mathcal{B}\big]\rangle_{\mathcal{W}_1} + O(t^0) \\ &= \frac{1}{t^2}\,\langle\, f\circ\phi(0)\,c(1)\,\rangle_{\mathcal{W}_1} + \frac{1}{t}\,\langle\,\phi,\,\psi_0'\,\rangle + O(t^0)\,, \end{aligned} \tag{4.10}$$

where in the last equality we have recalled the expression for ψ_0' [5, 8]. The first term on the right-hand side is not BRST closed. After integration over t, it gives a divergent $O(1/\epsilon)$ term that is precisely canceled by the divergent term from $\Psi_1^{(2)}$, as expected. The integral over t of the second term gives a divergent $O(\ln\epsilon)$ term which is not canceled but, as expected, is BRST closed. (It is in fact BRST exact.) If we define the counterterm $\Psi_2^{(2)}$ by

$$\Psi_2^{(2)} = \ln(2\epsilon)\,\psi_0'\,, \tag{4.11}$$

we finally assemble a string field $\Psi^{(2)}$ that is finite and satisfies the equation of motion:

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\,\Big[\,\Psi_0^{(2)} + \Psi_1^{(2)} + \Psi_2^{(2)}\,\Big]\,. \tag{4.12}$$

We can also write the solution as

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\left[\,\Psi_0^{(2)} - \frac{1}{\pi\epsilon}\,c_1|0\rangle + \ln(2\epsilon)\,\psi_0' + \frac{1}{\pi}\,L^+\,c_1|0\rangle\,\right], \tag{4.13}$$

using the following operator expression for $\Psi_1^{(2)}$:

$$\Psi_1^{(2)} = -\frac{1}{\pi\epsilon}\,e^{-\epsilon L^+}c_1|0\rangle = -\frac{1}{\pi\epsilon}\,c_1|0\rangle + \frac{1}{\pi}\,L^+\,c_1|0\rangle + O(\epsilon)\,. \tag{4.14}$$

Our construction of $\Psi^{(2)}$ did not rely on any property of V other than the OPE (4.1). The OPE (4.1) is more restrictive than the generic OPE of a dimension one primary. For example, we may have

$$V(z)\,V(w) \sim \frac{1}{(z-w)^2} + \frac{1}{z-w}\,U(w)\,, \tag{4.15}$$

where $U(w)$ is some matter primary of dimension one. In this case, V would not be exactly marginal. Indeed, a dimension one primary $\bar{U}$ must exist such that $\langle \bar{U}(z)U(0)\rangle = 1/z^2$. The OPE (4.15) then implies that the three-point function $\langle V\,V\,\bar{U}\rangle$ is non-vanishing, while a necessary condition for the exact marginality of V is the vanishing of $\langle VVW\rangle$ for all dimension one primaries W (see *e.g.* [27]). Thus we expect that our construction of $\Psi^{(2)}$ should not go through if the OPE takes the form (4.15). Let us see this explicitly. In this case (4.5) is replaced by

$$\begin{aligned}\langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} = \frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c\partial c(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} \\ +\langle\, f\circ\phi(0)\, c\partial c U(1+\epsilon)\,\rangle_{\mathcal{W}_{1+2\epsilon}} + O(\epsilon)\,.\end{aligned} \tag{4.16}$$

The second term on the right-hand side is finite in the limit $\epsilon\to 0$. The operator $c\partial cU$ is BRST closed, but it is *not* BRST exact. Therefore the equation of motion cannot be satisfied by adding a counterterm.

4.2 Gauge condition, L eigenstates, and divergence structure

All the terms of $\Psi^{(2)}$ in (4.13) are annihilated by B, except $L^+c_1|0\rangle$:

$$BL^+c_1|0\rangle = [B,L^+]c_1|0\rangle = B^+c_1|0\rangle \neq 0\,. \tag{4.17}$$

Thus, rather curiously, $\Psi^{(2)}$ violates the Schnabl gauge condition. It appears that this violation is intrinsic. While we can add an arbitrary BRST closed state Z to $\Psi^{(2)}$, we believe that no choice of Z can restore the Schnabl gauge condition. Indeed, assume that such a Z exists:

$$B(L^+c_1|0\rangle + Z) = 0\,,\quad Q_BZ = 0\,. \tag{4.18}$$

Acting with Q_B on this equation, we find that Z must satisfy

$$L\,Z = -Q_BBL^+c_1|0\rangle\,. \tag{4.19}$$

Note that while the left hand side is in the image of L, the right-hand side is in the kernel of L because $[L,Q_B]=[L,B]=0$ and $LL^+c_1|0\rangle = 0$. We believe that (4.19) has no solution for Z, though we do not have a proof.[4]

This obstruction in preserving the Schnabl gauge condition when V has singular OPE with itself is rather unexpected. To gain some insight, let us reconsider the situation in Siegel gauge. In Siegel gauge the equations of motion (2.4) are solved by setting

$$\Psi^{(n)} = \frac{b_0}{L_0}\Phi^{(n)}\,. \tag{4.20}$$

[4]If an operator is diagonalizable, its kernel and its image have no non-trivial overlap. Since L is non-hermitian, it is not a priori clear that it can be diagonalized. In principle a Z solving (4.19) may exist if L has a suitable Jordan structure, but we find this unlikely.

It turns out that the right-hand side is well defined and thus manifestly obeys the gauge condition because $\Phi^{(n)}$ has no overlap with states in the kernel of L_0. When the equations of motion have a solution, $\Phi^{(n)}$ is a BRST exact state of ghost number two. The only BRST exact state of ghost number two in the kernel of L_0 is $Q_B c_0|0\rangle = 2c_1c_{-1}|0\rangle$. We are claiming that $\Phi^{(n)}$ has no overlap with $c_1c_{-1}|0\rangle$. This is shown using twist symmetry in the ghost sector. For a generic state in the Fock space

$$|\phi\rangle = \{\text{matter oscillators}\}\, b_{-m_j}\cdots b_{-m_1}c_{-n_k}\cdots c_{-n_1}|0\rangle\,, \qquad m_i \geq 2\,,\ n_i \geq -1\,, \tag{4.21}$$

the ghost twist eigenvalue is defined as

$$1 + \sum_{i=1}^{j} m_i + \sum_{i=1}^{k} n_i \pmod 2\,. \tag{4.22}$$

The linearized solution $\Psi^{(1)}$ is even under ghost twist, which implies that $\Phi^{(2)} = -\Psi^{(1)} * \Psi^{(1)}$ is also even. On the other hand, the problematic state $c_1c_{-1}|0\rangle$ is odd. This shows that $\Phi^{(2)}$ has no overlap with it. A little inductive argument can be used to extend this result to the higher $\Phi^{(n)}$. Assuming that all the lower $\Psi^{(k)}$, $k < n$, are even, we see that $\Phi^{(n)}$, which consists of symmetrized star products of the lower $\Psi^{(k)}$'s, is also even. Hence there is no obstruction in finding $\Psi^{(n)} = \frac{b_0}{L_0}\Phi^{(n)}$. The operator b_0/L_0 preserves twist, so $\Psi^{(n)}$ is even, and the induction can proceed to the next step.

We now perform a similar analysis for the case of Schnabl gauge. The formal solution

$$\Psi^{(n)} = \frac{B}{L}\Phi^{(n)} \tag{4.23}$$

is well defined if and only if $\Phi^{(n)}$ has no overlap with states in the kernel of L. While we do not have a complete understanding of the spectrum of L, we will find a consistent picture by assuming that $\Phi^{(n)}$ can be expanded in a sum of L eigenstates of integer eigenvalues ≥ -1. We can systematically enumerate the L eigenstates that have ghost number two and are BRST exact within a subspace of states which can appear in the expansion of $\Phi^{(n)}$. It will be sufficient to focus on states with $L \leq 0$. We believe that the only such states are:

- $L = -1$: the state $c_1c_0|0\rangle = Q_B c_1|0\rangle$.
- $L = 0$: the state $c_1c_{-1}|0\rangle = \frac{1}{2}Q_B c_0|0\rangle$.
- $L = 0$: the state $L^+c_1c_0|0\rangle = Q_B L^+ c_1|0\rangle$.

Contrasting the kernel of L with the kernel of L_0, we see the surprising appearance of the extra state $L^+c_1c_0|0\rangle$. Since this state is *even* under ghost twist, it can a priori appear in $\Phi^{(n)}$. The first state with $L = 0$ cannot appear, by the same arguments we made for the Siegel gauge. We can write the following ansatz for a finite $\Phi^{(n)}$:

$$\Phi^{(n)} = \alpha^{(n)}c_1c_0|0\rangle + \beta^{(n)}L^+c_1c_0|0\rangle + \Phi^{(n)}_{>}\,, \tag{4.24}$$

where $\Phi^{(n)}_{>}$ contains only $L > 0$ eigenstates. The most general $\Psi^{(n)}$ that satisfies the equation $Q_B\Psi^{(n)} = \Phi^{(n)}$ is the manifestly finite string field

$$\Psi^{(n)} = \alpha^{(n)} c_1|0\rangle + \beta^{(n)} L^+ c_1|0\rangle + \frac{B}{L}\Phi^{(n)}_{>} + (Q_B \text{ closed})\,. \tag{4.25}$$

If $\beta^{(n)} \neq 0$, the term $L^+c_1|0\rangle$ violates the gauge condition. In the following we will not write the Q_B closed term that plays no role.

We are now going to establish a precise relationship between the violation of the gauge condition and the divergences that arise in the Schwinger representation of the action of B/L whenever the matter operator has singular OPE with itself. When acting on $\Phi^{(n)}_{>}$, we can use for B/L the Schwinger representation

$$\frac{B}{L} = \lim_{\Lambda\to\infty}\int_0^\Lambda dt\, Be^{-tL} = \frac{B}{L} - \lim_{\Lambda\to\infty} e^{-\Lambda L}\frac{B}{L}\,, \tag{4.26}$$

since the boundary term vanishes in the limit. Thus we rewrite (4.25) as

$$\begin{aligned}\Psi^{(n)} &= \alpha^{(n)} c_1|0\rangle + \beta^{(n)} L^+ c_1|0\rangle + \lim_{\Lambda\to\infty}\int_0^\Lambda dt\, Be^{-tL}(\Phi^{(n)} - \alpha^{(n)} c_1c_0|0\rangle - \beta^{(n)} L^+c_1c_0|0\rangle)\\ &= \lim_{\Lambda\to\infty}\left[\left(\int_0^\Lambda dt\, Be^{-tL}\Phi^{(n)}\right) + e^\Lambda\alpha^{(n)}c_1|0\rangle - \Lambda\beta^{(n)}BL^+c_1c_0|0\rangle\right] + \beta^{(n)}L^+c_1|0\rangle\,. \end{aligned}\tag{4.27}$$

Note that we have

$$BL^+c_1c_0|0\rangle = \pi\psi_0'\,. \tag{4.28}$$

Since the string field $\Psi^{(n)}$ is finite, we see that

$$\int_0^\Lambda dt\, Be^{-tL}\Phi^{(n)} = -e^\Lambda\alpha^{(n)}c_1|0\rangle + \Lambda\,\pi\beta^{(n)}\,\psi_0' + \text{finite}\,. \tag{4.29}$$

We have thus learned that the divergences of the integral on the left-hand side, which performs the naive inversion of Q_B on $\Phi^{(n)}$, are directly related to the $L = -1$ and $L = 0$ eigenstates in the decomposition of $\Phi^{(n)}$. Moreover, the coefficient of the $O(\Lambda)$ divergence is correlated with the coefficient of the Schnabl-gauge violating term $L^+c_1|0\rangle$.

The divergences in (4.29) can only arise from the collision of the cV insertions on the boundary of the worldsheet. If V has a non-singular OPE with itself, all integrals are manifestly finite, $\alpha^{(n)} = \beta^{(n)} = 0$ for any n, $\Psi^{(n)}$ satisfies the Schnabl gauge condition, and the naive prescription $Q_B^{-1} = B/L$ is adequate to handle this case, as discussed in section 3. On the other hand, if V has a singular OPE with itself, (4.27) severely constrains the structure of the result. Let us look at the case of $\Psi^{(2)}$. To begin, note that the integral

$$\int_0^\Lambda dt\, Be^{-tL}\Phi^{(2)} \tag{4.30}$$

is in fact the regularized $\Psi^{(2)}_0$, with the identification $\Lambda = -\log(2\epsilon)$. Substituting this in (4.27), our general analysis predicts

$$\Psi^{(2)} = \lim_{\epsilon\to 0}\left[\Psi^{(2)}_0 + \frac{\alpha^{(2)}}{2\epsilon}\,c_1|0\rangle + \ln(2\epsilon)\pi\,\beta^{(2)}\,\psi_0' + \beta^{(2)}\,L^+\,c_1|0\rangle\right]\,, \tag{4.31}$$

in complete agreement with the explicit result (4.13), with $\alpha^{(2)} = -2/\pi$, $\beta^{(2)} = 1/\pi$.

The analysis can be extended for higher n. An interesting simplification occurs if $V = i\sqrt{\frac{2}{\alpha'}}\partial X$. Since the number of ∂X is conserved mod 2 under Wick contractions, the coefficients $\alpha^{(n)}$ and $\beta^{(n)}$ are zero for n odd. It follows that for odd n the integral (4.29) is finite. In particular, we expect that for $V = i\sqrt{\frac{2}{\alpha'}}\partial X$the most general $\Psi^{(3)}$ is given by

$$\Psi^{(3)} = -\lim_{\Lambda\to\infty}\int_0^\Lambda dt\, Be^{-tL}\left(\Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)}\right) + (Q_B \text{ closed}), \tag{4.32}$$

where the $\Lambda \to \infty$ limit is guaranteed to be finite.

While $\Psi^{(3)}$ may be obtained this way (setting to zero the arbitrary BRST closed terms and performing the integral by brute force), in the following subsection we will follow a route analogous to the one in §4.1. We will start with a regularized $\Psi_0^{(3)}$ and systematically look for counterterms such that the final state $\Psi^{(3)}$ satisfies the equation of motion and is finite.

The arguments in this section suggest strongly that a finite string field $\Psi^{(n)}$ satisfying the equation of motion exists for all n and it can be written as a regularized string field plus counterterms.

4.3 Construction of $\Psi^{(3)}$

In this subsection we perform an explicit construction of $\Psi^{(3)}$ for V with the OPE (4.1). The first step is to regularize (3.3) and define $\Psi_0^{(3)}$ by

$$\langle \phi, \Psi_0^{(3)}\rangle = \int_{2\epsilon}^1 dt_1 \int_{2\epsilon}^1 dt_2\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2)\,\rangle_{\mathcal{W}_{1+t_1+t_2}}\,. \tag{4.33}$$

The BRST transformation of $\Psi_0^{(3)}$ is given by

$$\langle \phi, Q_B\Psi_0^{(3)}\rangle = -\langle\, \phi, \Psi^{(1)} * \Psi_0^{(2)} + \Psi_0^{(2)} * \Psi^{(1)}\,\rangle + R_1 + R_2\,, \tag{4.34}$$

where

$$\begin{aligned} R_1 &= \int_{2\epsilon}^1 dt_2\, \langle\, f\circ\phi(0)\, cV(1)\, cV(1+2\epsilon)\, \mathcal{B}\, cV(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\,, \\ R_2 &= \int_{2\epsilon}^1 dt_1\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, cV(1+t_1+2\epsilon)\,\rangle_{\mathcal{W}_{1+t_1+2\epsilon}}\,. \end{aligned} \tag{4.35}$$

As in the case of $Q_B\Psi_0^{(2)}$, the contributions R_1 and R_2 from the surface terms at $t_1 = 2\epsilon$ and at $t_2 = 2\epsilon$, respectively, are nonvanishing. We also need to reproduce $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$ and $-\Psi^{(1)} * \Psi_2^{(2)} - \Psi_2^{(2)} * \Psi^{(1)}$ to satisfy the equation of motion. It is not difficult to realize that the BRST transformation of $\Psi_1^{(3)}$ defined by

$$\Psi_1^{(3)} = -\int_{2\epsilon}^1 dt_1\, \Psi^{(1)} * B_L^+\, e^{(1-t_1)L_L^+}\, \Psi_1^{(2)} - \int_{2\epsilon}^1 dt_2\, \Psi_1^{(2)} * B_L^+\, e^{(1-t_2)L_L^+}\, \Psi^{(2)} \tag{4.36}$$

cancels the divergent terms from the OPE's of $cV(1)\,cV(1+2\epsilon)$ in R_1 and of $cV(1+t_1)\,cV(1+t_1+2\epsilon)$ in R_2 and reproduces $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$. We also introduce $\Psi_2^{(3)}$ defined by

$$\Psi_2^{(3)} = -\int_{2\epsilon}^{1} dt_1\, \Psi^{(1)} * B_L^+\, e^{(1-t_1)L_L^+}\, \Psi_2^{(2)} - \int_{2\epsilon}^{1} dt_2\, \Psi_2^{(2)} * B_L^+\, e^{(1-t_2)L_L^+}\, \Psi^{(2)} \tag{4.37}$$

so that its BRST transformation reproduces $-\Psi^{(1)} * \Psi_2^{(2)} - \Psi_2^{(2)} * \Psi^{(1)}$.

However, this is not the whole story. First, when t_2 in R_1 is of $O(\epsilon)$, three V's are simultaneously close so that we cannot simply replace two of them by the most singular term of the OPE. The same remark applies to R_2 when t_1 is of $O(\epsilon)$. Secondly, while the contributions from the surface terms at $t_1 = 2\epsilon$ or at $t_2 = 2\epsilon$ in the calculation of $Q_B\Psi_2^{(3)}$ turn out to vanish in the limit $\epsilon \to 0$, the corresponding contributions in the calculation of $Q_B\Psi_1^{(3)}$ turn out to be *finite* and *not* BRST exact. These contributions have to be canceled in order for the equation of motion to be satisfied.

We thus need to calculate R_1, R_2, $Q_B\Psi_1^{(3)}$, and $Q_B\Psi_2^{(3)}$. The calculations of $Q_B\Psi_1^{(3)}$ and $Q_B\Psi_2^{(3)}$ are universal for any V which has the OPE (4.1), while those of R_1 and R_2 are not. Let us begin with $Q_B\Psi_1^{(3)}$. It is convenient to use the CFT description of $\Psi_1^{(3)}$ given by

$$\begin{aligned} \langle \phi, \Psi_1^{(3)} \rangle = &-\frac{1}{2\epsilon}\int_{2\epsilon}^{1} dt_1\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, c(1+t_1+\epsilon)\,\rangle_{\mathcal{W}_{1+t_1+2\epsilon}} \\ &-\frac{1}{2\epsilon}\int_{2\epsilon}^{1} dt_2\, \langle\, f\circ\phi(0)\, c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\,. \end{aligned} \tag{4.38}$$

The BRST transformation of $\Psi_1^{(3)}$ is

$$\langle \phi, Q_B\Psi_1^{(3)} \rangle = -\langle\, \phi, \Psi^{(1)} * \Psi_1^{(2)} + \Psi_1^{(2)} * \Psi^{(1)}\,\rangle + \widetilde{R}_1 + \widetilde{R}_2 + \widetilde{R}_3\,, \tag{4.39}$$

where

$$\begin{aligned} \widetilde{R}_1 &= -\frac{1}{2\epsilon}\int_{2\epsilon}^{1} dt_2\, \langle\, f\circ\phi(0)\, c\partial c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\,, \\ \widetilde{R}_2 &= -\frac{1}{2\epsilon}\int_{2\epsilon}^{1} dt_1\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, c\partial c(1+t_1+\epsilon)\,\rangle_{\mathcal{W}_{1+t_1+2\epsilon}}\,, \\ \widetilde{R}_3 &= -\frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, cV(1)\, c(1+3\epsilon)\,\rangle_{\mathcal{W}_{1+4\epsilon}} - \frac{1}{2\epsilon}\,\langle\, f\circ\phi(0)\, c(1+\epsilon)\, cV(1+4\epsilon)\,\rangle_{\mathcal{W}_{1+4\epsilon}}\,. \end{aligned} \tag{4.40}$$

As we mentioned earlier, the BRST transformation of $\Psi_1^{(3)}$ reproduces $-\Psi^{(1)} * \Psi_1^{(2)} - \Psi_1^{(2)} * \Psi^{(1)}$, and $\widetilde{R}_1$ and $\widetilde{R}_2$ cancel part of R_1 and R_2, respectively. The last term $\widetilde{R}_3$ is finite in the limit $\epsilon \to 0$ and not BRST exact:

$$\widetilde{R}_3 = -3\,\langle\, f\circ\phi(0)\, c\partial cV(1)\,\rangle_{\mathcal{W}_1} + O(\epsilon)\,. \tag{4.41}$$

Let us next calculate $\Psi_2^{(3)}$. It is again convenient to use the CFT description of $\Psi_2^{(3)}$:

$$\begin{aligned} \langle\, \phi, \Psi_2^{(3)}\,\rangle = &\ \ln(2\epsilon)\int_{2\epsilon}^{1} dt_1\, \langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\ Q_B\cdot[\,\mathcal{B}\, c(1+t_1)\,]\,\rangle_{\mathcal{W}_{1+t_1}} \\ &+ \ln(2\epsilon)\int_{2\epsilon}^{1} dt_2\, \langle\, f\circ\phi(0)\ Q_B\cdot[\,\mathcal{B}\, c(1)\,]\ \mathcal{B}\, cV(1+t_2)\,\rangle_{\mathcal{W}_{1+t_2}}\,. \end{aligned} \tag{4.42}$$

The BRST transformation of $\Psi_2^{(3)}$ is given by

$$\begin{aligned}\langle\,\phi, Q_B\Psi_2^{(3)}\,\rangle = -\ \langle\,\phi,\ \Psi^{(1)} * \Psi_2^{(2)} + \Psi_2^{(2)} * \Psi^{(1)}\,\rangle \\ - \ln(2\epsilon)\,\langle\, f\circ\phi(0)\ Q_B\cdot[\,cV(1)\,\mathcal{B}\,c(1+2\epsilon)\,]\,\rangle_{\mathcal{W}_{1+2\epsilon}} \\ + \ln(2\epsilon)\,\langle\, f\circ\phi(0)\ Q_B\cdot[\,\mathcal{B}\,c(1)\,cV(1+2\epsilon)\,]\,\rangle_{\mathcal{W}_{1+2\epsilon}}\,.\end{aligned} \tag{4.43}$$

Since the BRST transformations of $cV(1)\,\mathcal{B}\,c(1+2\epsilon)$ and $\mathcal{B}\,c(1)\,cV(1+2\epsilon)$ are both of $O(\epsilon)$, the last two terms vanish in the limit $\epsilon \to 0$. We have thus shown that

$$\lim_{\epsilon\to 0}\,\langle\,\phi, Q_B\Psi_2^{(3)} + \Psi^{(1)} * \Psi_2^{(2)} + \Psi_2^{(2)} * \Psi^{(1)}\,\rangle = 0\,. \tag{4.44}$$

To summarize, we have seen that the BRST transformation of $\Psi_0^{(3)} + \Psi_1^{(3)} + \Psi_2^{(3)}$ reproduces $-\Psi^{(1)} * \Psi^{(2)} - \Psi^{(2)} * \Psi^{(1)}$ with $\Psi^{(2)} = \Psi_0^{(2)} + \Psi_1^{(2)} + \Psi_2^{(2)}$, and there are remaining terms R_1, R_2, $\widetilde{R}_1$, $\widetilde{R}_2$, and $\widetilde{R}_3$. We now calculate R_1 and R_2. These terms involve a triple operator product of V's and the results depend on V. We choose

$$V(z) = i\sqrt{\frac{2}{\alpha'}}\,\partial X(z)\,, \tag{4.45}$$

which is *exactly* marginal. With this choice of V, the triple operator product of V's on $\mathcal{W}_{n-1}$ is

$$\begin{aligned}V(z_1)V(z_2)V(z_3) = G_{n-1}(z_1-z_2)\,V(z_3) + G_{n-1}(z_1-z_3)\,V(z_2) + G_{n-1}(z_2-z_3)\,V(z_1) \\ + :V(z_1)V(z_2)V(z_3):\,,\end{aligned} \tag{4.46}$$

where G_{n-1} is the propagator on $\mathcal{W}_{n-1}$:

$$G_{n-1}(z) = \frac{\pi^2}{n^2}\left[\sin\frac{\pi z}{n}\right]^{-2} = \frac{1}{z^2} + O(z^0)\,. \tag{4.47}$$

The normal-ordered term in (4.46) does not contribute in the calculations of R_1 and R_2 in the limit $\epsilon \to 0$. The term with $V(1)$ and $V(1+2\epsilon)$ contracted in R_1 cancels $\widetilde{R}_1$:

$$\lim_{\epsilon\to 0}\left[\int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(2\epsilon)\,\langle\, f\circ\phi(0)\,c(1)\,c(1+2\epsilon)\,\mathcal{B}\,cV(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}} + \widetilde{R}_1\right] = 0\,. \tag{4.48}$$

The remaining two terms are finite in the limit $\epsilon \to 0$:

$$\begin{aligned}\lim_{\epsilon\to 0}\Bigg[\int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(t_2)\,\langle\, f\circ\phi(0)\,cV(1)\,c(1+2\epsilon)\,\mathcal{B}\,c(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \\ + \int_{2\epsilon}^{1} dt_2\, G_{1+t_2+2\epsilon}(t_2+2\epsilon)\,\langle\, f\circ\phi(0)\,c(1)\,cV(1+2\epsilon)\,\mathcal{B}\,c(1+2\epsilon+t_2)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}}\Bigg] \\ = \frac{3}{2}\,\langle\, f\circ\phi(0)\,c\partial cV(1)\,\rangle_{\mathcal{W}_1}.\end{aligned} \tag{4.49}$$

We therefore have

$$\lim_{\epsilon\to 0}\left[R_1 + \widetilde{R}_1\right] = \frac{3}{2}\,\langle\, f\circ\phi(0)\,c\partial cV(1)\,\rangle_{\mathcal{W}_1}. \tag{4.50}$$

The calculation of R_2 is parallel, and we obtain

$$\lim_{\epsilon\to 0}\Big[\,R_2+\widetilde{R}_2\,\Big]=\frac{3}{2}\,\langle\, f\circ\phi(0)\, c\partial c V(1)\,\rangle_{\mathcal{W}_1}. \tag{4.51}$$

The sum of the five remaining terms vanishes in the limit $\epsilon\to 0$:

$$\lim_{\epsilon\to 0}\Big[\,R_1+R_2+\widetilde{R}_1+\widetilde{R}_2+\widetilde{R}_3\,\Big]=0\,. \tag{4.52}$$

We have thus shown

$$\lim_{\epsilon\to 0}\langle\,\phi, Q_B\,[\,\Psi_0^{(3)}+\Psi_1^{(3)}\,]+\Psi^{(1)}*[\,\Psi_0^{(2)}+\Psi_1^{(2)}\,]+[\,\Psi_0^{(2)}+\Psi_1^{(2)}\,]*\Psi^{(1)}\,\rangle=0 \tag{4.53}$$

and

$$\lim_{\epsilon\to 0}\langle\,\phi, Q_B\,[\,\Psi_0^{(3)}+\Psi_1^{(3)}+\Psi_2^{(3)}\,]+\Psi^{(1)}*\Psi^{(2)}+\Psi^{(2)}*\Psi^{(1)}\,\rangle=0\,. \tag{4.54}$$

The sum of the five terms did not have to vanish in the limit $\epsilon\to 0$, but it had to be BRST exact to satisfy the equation of motion by adding a counterterm. In particular, the coefficient of $\langle\, f\circ\phi(0)\, c\partial cV(1)\,\rangle_{\mathcal{W}_1}$ had to vanish. We found that $\widetilde{R}_3$ from $\Psi_1^{(3)}$ is nontrivially canceled by contributions from $\Psi_0^{(3)}$.

Let us next study the divergent terms of $\Psi_0^{(3)}$. The triple operator product of V's in (4.33) can be written as follows:

$$\begin{aligned}
&V(1)\,V(1+t_1)\,V(1+t_1+t_2)\\
&=G_{1+t_1+t_2}(t_2)\,V(1)+G_{1+t_1+t_2}(t_1)\,V(1+t_1+t_2)\\
&\quad+G_{1+t_1+t_2}(t_1+t_2)\,V(1+t_1)+\,:V(1)\,V(1+t_1)\,V(1+t_1+t_2):\,.
\end{aligned} \tag{4.55}$$

Note that no further divergence appears when remaining operators collide. The contribution from the normal-ordered product in the last line is obviously finite. The divergent terms from the first two terms on the right-hand side are canceled by the divergent terms from $\Psi_1^{(3)}$ and $\Psi_2^{(3)}$. The contribution from the third term on the right-hand side is

$$\begin{aligned}
&\int_{2\epsilon}^{1}dt_1\int_{2\epsilon}^{1}dt_2\,\left(\frac{\pi}{t_1+t_2+2}\right)^2\left[\sin\frac{\pi\,(t_1+t_2)}{t_1+t_2+2}\right]^{-2}\\
&\qquad\qquad\times\langle\, f\circ\phi(0)\,c(1)\,\mathcal{B}\,cV(1+t_1)\,\mathcal{B}\,c(1+t_1+t_2)\,\rangle_{\mathcal{W}_{1+t_1+t_2}}.
\end{aligned} \tag{4.56}$$

This contains a divergent term $-\ln(4\epsilon)\,\langle\, f\circ\phi(0)\,cV(1)\,\rangle_{\mathcal{W}_1}$, which comes from the most singular term $1/(t_1+t_2)^2$ in the region where t_1 and t_2 are simultaneously of $O(\epsilon)$. Note that the divergent term is proportional to $\Psi^{(1)}$ and thus BRST closed, as expected. Therefore, if we define

$$\Psi^{(3)}=\lim_{\epsilon\to 0}\,[\,\Psi_0^{(3)}+\Psi_1^{(3)}+\Psi_2^{(3)}+\Psi_3^{(3)}\,]\,, \tag{4.57}$$

where

$$\Psi_3^{(3)}=\ln(4\epsilon)\,\Psi^{(1)}\,, \tag{4.58}$$

$\Psi^{(3)}$ is finite and satisfies the equation of motion:

$$\langle\, \phi,\, Q_B \Psi^{(3)} + \Psi^{(1)} * \Psi^{(2)} + \Psi^{(2)} * \Psi^{(1)} \,\rangle = 0\,. \tag{4.59}$$

An explicit form of $\Psi^{(3)}$ is given by

$$\begin{aligned}
\langle\, \phi, \Psi^{(3)} \,\rangle = \lim_{\epsilon\to 0} \Bigg[& \int_{2\epsilon}^{1} dt_1 \int_{2\epsilon}^{1} dt_2 \,\langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, cV(1+t_1)\, \mathcal{B}\, cV(1+t_1+t_2)\,\rangle_{\mathcal{W}_{1+t_1+t_2}} \\
& - \frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_1 \,\langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\, c(1+t_1+\epsilon)\,\rangle_{\mathcal{W}_{1+t_1+2\epsilon}} \\
& - \frac{1}{2\epsilon} \int_{2\epsilon}^{1} dt_2 \,\langle\, f\circ\phi(0)\, c(1+\epsilon)\, \mathcal{B}\, cV(1+t_2+2\epsilon)\,\rangle_{\mathcal{W}_{1+t_2+2\epsilon}} \\
& + \ln(2\epsilon) \int_{2\epsilon}^{1} dt_1 \,\langle\, f\circ\phi(0)\, cV(1)\, \mathcal{B}\; Q_B \cdot [\, \mathcal{B}\, c(1+t_1)\,]\,\rangle_{\mathcal{W}_{1+t_1}} \\
& + \ln(2\epsilon) \int_{2\epsilon}^{1} dt_2 \,\langle\, f\circ\phi(0)\; Q_B \cdot [\, \mathcal{B}\, c(1)\,]\; \mathcal{B}\, cV(1+t_2)\,\rangle_{\mathcal{W}_{1+t_2}} \\
& + \ln(4\epsilon)\, \langle\, f\circ\phi(0)\, cV(1)\,\rangle_{\mathcal{W}_1} \Bigg]\,.
\end{aligned} \tag{4.60}$$

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