

Boundary conditions in supergravity on a manifold with boundary

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ABSTRACT: We explain why it is necessary to use boundary conditions in the proof of supersymmetry of a supergravity action on a manifold with boundary. Working in both boundary (“downstairs”) and orbifold (“upstairs”) pictures, we present a bulk-plus-boundary/brane action for the five-dimensional (on-shell) supergravity which is supersymmetric with the use of *fewer* boundary conditions than were previously employed. The required Gibbons-Hawking-like Y -term and many other aspects of the boundary/orbifold picture correspondence are discussed.

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1. Introduction

In their ground-breaking paper [1], Horava and Witten discussed eleven-dimensional supergravity on a manifold with boundary, that arises as a low energy limit of the strongly coupled heterotic string theory. They explained that there are two possible descriptions of the same theory: the “downstairs” (boundary) picture and the “upstairs” (orbifold) picture. They made a comment, however, that working on the orbifold is technically more convenient, which was, perhaps, the reason why the orbifold picture became the de facto choice for many researchers working in this area.

One of such orbifold constructions is the now famous Randall-Sundrum scenario [2, 3] which is set up in a five-dimensional space-time with a negative cosmological constant. This scenario was supersymmetrized by different groups [4, 5, 6] with somewhat different approaches (see Ref. [7] for the proof of the equivalence of these approaches). The original orbifold construction was used for the supersymmetrization as well.

Over time people came to discover that the boundary (“downstairs” or “interval”) picture is in many respects preferred over the orbifold picture. For example, in Ref. [8] it was demonstrated how the same physical content unambiguously encoded in the boundary picture can be obscured by various “twists” and “jumps” on the orbifold.

In this paper we provide more evidence for simplicity of the boundary picture.

We present a bulk-plus-boundary action with the five-dimensional gauged (on-shell) supergravity in the bulk. It is $N = 2$ (locally) supersymmetric with a boundary condition on the supersymmetry parameter breaking a half of the bulk supersymmetries on the boundary. Supersymmetry of the action requires the use of a *small subset* of all the (natural) boundary conditions encoded in the action itself. Which boundary conditions are necessary is indicated by the supersymmetry algebra.

The boundary action is a sum of two terms. The first one is a Gibbons-Hawking-like term [9] (which we call “Y-term” to acknowledge the work of York [10, 11]). It allows the derivation of Neumann-like boundary conditions (which we call “natural” following Ref. [12]) from the standard variational principle in exactly the same way as equations of motion are derived. (The general variation of the action must vanish for *arbitrary* variations of the fields in the bulk *and* on the boundary.) The second term is (a half of) the brane action which one uses in the orbifold picture.

We present the transition from the boundary to the orbifold picture in detail. We find that the Y-term disappears in the transition. It is represented in the orbifold picture by brane-localized singularities of the bulk Lagrangian.

We explicitly check that the bulk-plus-brane action we obtain for the orbifold picture is supersymmetric upon using the same minimal set of boundary conditions as in the boundary picture. But we find that supersymmetry of the action requires introducing different $\varepsilon(z)$ (sign factor) assignments for odd fields compared to those previously assumed. Instead of the famous $\varepsilon(z)^2\delta(z) = 1/3\delta(z)$ we find that it is necessary to use another equally bizarre relation for a product of distributions: $\varepsilon(z)^{-2}\delta(z) = -\delta(z)$.

The construction of Ref. [7] is obtained from the one presented here by explicitly using the natural boundary conditions (including those outside the minimal set) in the brane action and while performing the supersymmetry variation of the resulting bulk-plus-brane action. We find that in the approach of Ref. [7] the two alternative $\varepsilon(z)$ assignments cannot be distinguished.

We also find that in the orbifold picture *all* local transformations have to be modified by the explicit addition of brane-localized terms such that the resulting transformations are non-singular on the brane. (The hint for this modification appeared already in Refs. [4, 7], where the supersymmetry transformation of ψ_{52} was modified.) Only after this modification the orbifold picture becomes equivalent to the boundary picture.

This paper is a companion to Ref. [13], where a detailed analysis of the Mirabelli and Peskin model [14] in both the boundary and the orbifold pictures is presented.

Our basic conventions are the same as in Ref. [7]. We summarize them in Appendix A. More details are included in Ref. [15].

2. Supersymmetry algebra

In this section we present the (on-shell) supersymmetry algebra of five-dimensional gauged supergravity. The (bulk) supergravity action and supersymmetry transformations are as in Ref. [7]. But in order to show an important feature of the local supersymmetry algebra, we need to include 3-Fermi terms in the supersymmetry transformation of the gravitino.

The complete form of the supersymmetry transformations is ¹

$$\delta_{\mathcal{H}} e_M^A = i \tilde{\mathcal{H}}^i \Gamma^A \Psi_{Mi} \quad (2.1)$$

$$\delta_{\mathcal{H}} B_M = -i \frac{\sqrt{6}}{2} \tilde{\mathcal{H}}^i \Psi_{Mi} \quad (2.2)$$

$$\begin{aligned} \delta_{\mathcal{H}} \Psi_{Mi} = & 2D_M(\hat{\omega})\mathcal{H}_i + \frac{1}{2\sqrt{6}}(\Gamma_M^{NK} + 4\delta_M^N \Gamma^K)\hat{F}_{NK}\mathcal{H}_i \\ & + \lambda Q_i{}^j (\Gamma_M - \sqrt{6}B_M)\mathcal{H}_j, \end{aligned} \quad (2.3)$$

where $Q_i{}^j = i(\vec{q} \cdot \vec{\sigma})_i{}^j$ and

$$\hat{F}_{MN} = F_{MN} + i \frac{\sqrt{6}}{4} \tilde{\Psi}_M^i \Psi_{Ni} \quad (2.4)$$

$$\hat{\omega}_{MAB} = \omega(e)_{MAB} - \frac{i}{4} e_A^N e_B^K \left(\tilde{\Psi}_N^i \Gamma_M \Psi_{Ki} + \tilde{\Psi}_M^i \Gamma_N \Psi_{Ki} - \tilde{\Psi}_M^i \Gamma_K \Psi_{Ni} \right) \quad (2.5)$$

¹The spinors Ψ_{Mi} and \mathcal{H}_i are symplectic Majorana (see Appendix A). The index i can be rotated by $U_i{}^j \in SU(2)$: $\Psi'_i = U_i{}^j \Psi_j$. The (global) $SU(2)$ is the automorphism symmetry group of the algebra when $\lambda \vec{q} = 0$. The real vector $\vec{q} = (q_1, q_2, q_3)$ indicates which $U(1)$ subgroup of the $SU(2)$ has been gauged [16, 7]. One can set it to be a unit vector, $\vec{q}^2 = 1$.

are supercovariant quantities (their supersymmetry variations contain no $\partial_M \mathcal{H}_i$). From these we can derive the (on-shell) supersymmetry algebra,²

$$[\delta_{\text{susy}}(\Xi), \delta_{\text{susy}}(\mathcal{H})] = \delta_{\text{g.c.}}(v^M) + \delta_{\text{loc.L.}}(\omega^{AB}) + \delta_{U(1)}(u) + \delta_{\text{susy}}(\Upsilon_i) . \quad (2.6)$$

The commutator of two supersymmetry transformations with parameters \mathcal{H}_i and Ξ_i gives a general coordinate transformation (instead of a translation in the case of global supersymmetry) as well as other local transformations (in our case these are local Lorentz and $U(1)$ transformations). But what is special to *local* supersymmetry (see, e.g., Ref. [17]), the commutator also gives rise to another supersymmetry transformation! And the 3-Fermi terms in the supersymmetry transformations are essential to identify this feature.

A general coordinate transformation (with parameter v^M) has the same form on e_M^A , B_M and Ψ_{Mi} , since all of them carry the same world index M . Explicitly, on B_M it is given by

$$\delta_v B_M = v^N \partial_N B_M + B_N \partial_M v^N . \quad (2.7)$$

The (local) Lorentz transformation (with parameter $\omega_{AB} = -\omega_{BA}$) is

$$\delta_\omega e_M^A = e_M^B \omega_B^A, \quad \delta_\omega B_M = 0, \quad \delta_\omega \Psi_{Mi} = \frac{1}{4} \omega_{AB} \Gamma^{AB} \Psi_{Mi} . \quad (2.8)$$

And the (local) $U(1)$ transformation (with parameter u) is as follows,

$$\delta_u e_M^A = 0, \quad \delta_u B_M = \partial_M u, \quad \delta_u \Psi_{Mi} = u \frac{\sqrt{6}}{2} \lambda Q_i^j \Psi_{Mj} . \quad (2.9)$$

The parameters v^M , ω_{AB} , u and Υ_i , which appear in the commutator (2.6), are given by

$$\begin{aligned} v^M &= 2i \tilde{\mathcal{H}}^i \Gamma^M \Xi_i \\ \omega^{AB} &= 2i(\tilde{\mathcal{H}}^i \Gamma^K \Xi_i) \hat{\omega}_K^{AB} - 2i \lambda Q_i^j (\tilde{\mathcal{H}}^i \Gamma^{AB} \Xi_j) \\ &\quad - \frac{i}{\sqrt{6}} \tilde{\mathcal{H}}^i \Gamma^{AB}{}_{NK} \Xi_i \hat{F}^{NK} - \frac{4i}{\sqrt{6}} \tilde{\mathcal{H}}^i \Xi_i \hat{F}^{AB} \\ u &= -2i(\tilde{\mathcal{H}}^i \Gamma^K \Xi_i) B_K - i\sqrt{6} \tilde{\mathcal{H}}^i \Xi_i \\ \Upsilon_i &= -i(\tilde{\mathcal{H}}^j \Gamma^K \Xi_j) \Psi_{Ki} . \end{aligned} \quad (2.10)$$

The supersymmetry algebra tells us that in order for a (bulk-plus-boundary) action to be supersymmetric under the indicated supersymmetry transformations, it must also be invariant under the local transformations arising from the commutator (2.6). Namely, the general coordinate transformation, the local Lorentz transformation and the $U(1)$ gauge transformation. This allows one to find the boundary conditions necessary for supersymmetry of the action.

²The algebra closes exactly only on the bosonic fields e_M^A and B_M . For the gravitino, Ψ_{Mi} , additional non-closure terms appear, proportional to its equation of motion. For the off-shell supersymmetry algebra see Ref. [18].

3. Boundary breaks supersymmetry

It is well-known that in the presence of a boundary *half* of the bulk supersymmetries are necessarily broken. The reason for this is the supersymmetry algebra, which generates a general coordinate transformation (a translation in the case of the global supersymmetry) in the direction normal to the boundary.

Indeed, let us consider an action on a manifold \mathcal{M} with boundary $\partial\mathcal{M}$. Its variation under the general coordinate transformation gives rise to a boundary term,

$$\delta_v S_5 = \delta_v \int_{\mathcal{M}} \mathcal{L}_5 = \int_{\mathcal{M}} D_M(v^M \mathcal{L}_5) = \int_{\partial\mathcal{M}} n_M v^M \mathcal{L}_5, \quad (3.1)$$

where n^M is an outward pointing unit vector normal to the boundary. (The measures of integration, $d^5 x e_5$ on \mathcal{M} and $d^4 x e_4^{\text{ind}}$ on $\partial\mathcal{M}$, are implicit.) This is true for *any* action, provided \mathcal{L}_5 is a scalar under the general coordinate transformation. Thus, in the presence of a boundary, the action is *not* invariant under the general coordinate transformation *unless*

$$n_M v^M = 0 \quad \text{on } \partial\mathcal{M}. \quad (3.2)$$

But the supersymmetry algebra shows that a commutator of two supersymmetry transformations generates the general coordinate transformation with

$$v^M = 2i\tilde{\mathcal{H}}^i \Gamma^M \Xi_i. \quad (3.3)$$

And thus a restriction on v^M restricts the allowed supersymmetry transformations.

From now on we assume that the boundary is described by

$$\partial\mathcal{M}: \quad x^5 = \text{const}. \quad (3.4)$$

The allowed general coordinate transformations,

$$x^M \rightarrow x^M - v^M(x), \quad (3.5)$$

with $n_M v^M = 0$ on $\partial\mathcal{M}$, preserve this description, and thus our choice does not limit the general coordinate invariance any further. We also use a (finite) local Lorentz transformation to set $e_m^{\hat{5}} = 0$ on $\partial\mathcal{M}$. In this gauge, which turns out to be very convenient for our discussion,

$$e_m^{\hat{5}} = 0, \quad e_a^{\hat{5}} = 0, \quad e_5^a \neq 0, \quad e_5^m \neq 0 \quad (3.6)$$

and

$$e_m^a e_a^n = \delta_m^n, \quad e_a^m e_m^b = \delta_a^b, \quad e_5^{\hat{5}} e_{\hat{5}}^5 = 1, \quad e_5^m = -e_5^a e_a^m e_{\hat{5}}^5. \quad (3.7)$$

Note also that this is the gauge in which e_m^a is a vierbein of the induced metric on the boundary (which is not true in general) and, therefore,

$$e_4^{\text{ind}} = e_4 = \det e_m^a. \quad (3.8)$$

We also have $e_5 = e_4 e_5^5$ and $n_M = (0, 0, 0, 0, n_5)$ with (see Appendix D)

$$n_5 = -e_5^5. \quad (3.9)$$

After these simplifications and in the two-component spinor notation, Eqs. (3.2) and (3.3) give rise to the following condition,

$$v^5 = 2i\tilde{\mathcal{H}}^i \Gamma^5 \Xi_i = 2e_5^5(\eta_2 \xi_1 - \eta_1 \xi_2) + h.c. = 0 \quad \text{on } \partial\mathcal{M}. \quad (3.10)$$

Here (η_1, η_2) and (ξ_1, ξ_2) are the constituents of \mathcal{H}_i and Ξ_i , respectively. Clearly, its general solution is the following boundary condition on the supersymmetry parameter,

$$\eta_2 = \alpha \eta_1 \quad \text{on } \partial\mathcal{M}, \quad (3.11)$$

where α is (for now) an arbitrary complex function of the boundary coordinates. (We will see, however, that we can find a supersymmetric bulk-plus-boundary action only for $\alpha = \text{const.}$) This is exactly the boundary condition used in Ref. [7]. One linear combination of η_1 and η_2 gets fixed, while the orthogonal combination describes the unbroken supersymmetry transformation. We will first set $\alpha = 0$ to simplify the discussion.³ Our boundary condition is then

$$\eta_2 = 0 \quad \text{on } \partial\mathcal{M}. \quad (3.12)$$

The $N = 2$ supersymmetry gets broken down to $N = 1$ (described by η_1) due to the presence of the boundary.

Note that the breaking is *on the boundary only*. We will find a bulk-plus-boundary action which is invariant under the supersymmetry transformations with arbitrary η_1 and η_2 in the bulk of \mathcal{M} , restricted only by the boundary condition on $\partial\mathcal{M}$. (In contrast, in the case of the *global* supersymmetry, like in the Mirabelli and Peskin model, restricting *constant* η_1 and η_2 on $\partial\mathcal{M}$ is equivalent to restricting them *everywhere*, and thus the $N = 2$ supersymmetry is really broken down to $N = 1$.) However, in the corresponding effective four-dimensional theory one would be able to preserve only $N = 1$ supersymmetry, because the second supersymmetry would be broken by the boundary conditions.

4. Boundary conditions needed for supersymmetry

Preserving the $N = 1$ supersymmetry still requires some effort. In the Mirabelli and Peskin model we found that, in the boundary picture, a particular boundary action is required to preserve the $N = 1$ supersymmetry. Off-shell, no boundary condition was necessary to establish supersymmetry of the bulk-plus-boundary action. On-shell, however, some boundary conditions (which are part of the auxiliary equations of motion) were necessary. In our (on-shell) supergravity case, we can find the boundary conditions that are important for supersymmetry directly from the supersymmetry algebra!

³It is sufficient to consider just the $\alpha = 0$ case. Any other (constant) α is obtainable by a (global) $SU(2)$ rotation Ref. [7]. See Section 7 for details.

4.1 Boundary condition on the gravitino

We found that the commutator of two supersymmetry transformations generates another supersymmetry transformation with parameter

$$\Upsilon_i = -i(\tilde{\mathcal{H}}^j \Gamma^M \Xi_j) \Psi_{Mi} . \quad (4.1)$$

In our gauge ($e_m^{\hat{5}} = e_a^5 = 0$) and with our boundary condition ($\eta_2 = 0$ on $\partial\mathcal{M}$), we have

$$\Upsilon_i = -e_a^m (i\eta_1 \sigma^a \bar{\xi}_1 + h.c.) \Psi_{mi} . \quad (4.2)$$

Writing Υ_i and Ψ_{mi} in terms of their two-component constituents, (ζ_1, ζ_2) and (ψ_{m1}, ψ_{m2}) , respectively, we obtain, in particular,

$$\zeta_2 = -e_a^m (i\eta_1 \sigma^a \bar{\xi}_1 + h.c.) \psi_{m2} . \quad (4.3)$$

Thus, two allowed supersymmetry transformations (with $\eta_2 = 0$ and $\xi_2 = 0$) generate a *forbidden* supersymmetry transformation ($\zeta_2 \neq 0$), *unless* the boundary condition

$$\psi_{m2} = 0 \quad \text{on } \partial\mathcal{M} \quad (4.4)$$

is imposed. (If $\alpha \neq 0$, the boundary condition is $\psi_{m2} = \alpha \psi_{m1}$ on $\partial\mathcal{M}$.)

It is important to note, however, that the supersymmetry transformation in the commutator (2.6) arises *from the 3-Fermi terms* in the supersymmetry variation of the gravitino. Accordingly, we expect that the boundary condition (4.4) is needed to prove supersymmetry of our action to all orders in fermions, *but not* just to quadratic order in fermions. We will show that it is indeed the case (provided an appropriate boundary action is included).

4.2 Boundary condition on the graviphoton

The commutator (2.6) also results in a $U(1)$ gauge transformation with parameter

$$u = -2i(\tilde{\mathcal{H}}^i \Gamma^K \Xi_i) B_K - i\sqrt{6}\tilde{\mathcal{H}}^i \Xi_i . \quad (4.5)$$

In our gauge ($e_m^{\hat{5}} = e_a^5 = 0$) and with our boundary condition ($\eta_2 = 0$ on $\partial\mathcal{M}$), we have

$$u = -2e_a^m (i\eta_1 \sigma^a \bar{\xi}_1 + h.c.) B_m . \quad (4.6)$$

This implies that there are two choices of boundary conditions compatible with supersymmetry of the (bulk-plus-boundary) action:

1. $B_m \neq 0$ on $\partial\mathcal{M}$; the action must be $U(1)$ gauge invariant; the boundary condition (if any) follows from maintaining the $U(1)$ invariance.
2. $B_m = 0$ on $\partial\mathcal{M}$; the $U(1)$ gauge invariance is broken by this boundary condition and thus is not required of the action itself; the gauge invariance must not be broken in the bulk, however, since there the generated u is non-zero.

The first choice appears to be more attractive (gauge invariances lead to more controlled quantum field theories), but the second choice may also be required. In our setup this is the case when $\lambda q_{12} \neq 0$. The reason is that in this case the boundary condition $\eta_2 = 0$ on $\partial\mathcal{M}$ *itself* breaks the $U(1)$ gauge invariance! This happens because the $U(1)$ transformation acts on the supersymmetry parameter \mathcal{H}_i in the same way as it does with Ψ_{Mi} ,

$$\delta_u \mathcal{H}_i = u \frac{\sqrt{6}}{2} \lambda Q_i^j \mathcal{H}_j . \quad (4.7)$$

For the two-component spinors this means

$$\delta_u \eta_1 = i \frac{\sqrt{6}}{2} \lambda u (q_3 \eta_1 - q_{12}^* \eta_2) \quad (4.8)$$

$$\delta_u \eta_2 = i \frac{\sqrt{6}}{2} \lambda u (-q_{12} \eta_1 - q_3 \eta_2) . \quad (4.9)$$

We see that the boundary condition, $\eta_2 = 0$ on $\partial\mathcal{M}$, is not invariant under this transformation, unless $\lambda q_{12} = 0$. (For $\alpha \neq 0$ the condition on \vec{q} is accordingly modified.)

4.3 Are there other boundary conditions?

There are no other boundary conditions which are generated in the way described above. The Lorentz transformation does not generate a boundary condition since any supergravity Lagrangian is Lorentz invariant and, therefore, no boundary term is produced when varying the action.

This means that we should need (at most) two boundary conditions (on the gravitino and the graviphoton) to prove supersymmetry of the total bulk-plus-boundary action. In particular, we should be able to do this *without using any boundary condition for the vielbein!* And we will show explicitly that it is indeed so, provided an appropriate boundary action is found.

We would like to emphasize that despite the limited number of boundary conditions needed to prove supersymmetry of the total action, a dynamical setup should include a full set of boundary conditions to make the boundary value problem well defined. And such a set must itself be supersymmetric (though the use of equations of motion would be required to show the closure of the boundary conditions under supersymmetry if one works with the on-shell formulation of supergravity).

5. Bulk action

In this section we introduce the bulk supergravity action and consider its variation under the $U(1)$ gauge and the supersymmetry transformations.

Our bulk action is the standard gauged supergravity action in five dimensions [19]. We will omit the 4-Fermi terms and will work to quadratic order in fermions. To this order, the

action is

$$\begin{aligned}
S_5 = \int_{\mathcal{M}} d^5x e_5 \Big\{ & -\frac{1}{2}R + \frac{i}{2}\tilde{\Psi}_M^i \Gamma^{M N K} D_N \Psi_{K i} \\
& + 6\lambda^2 \tilde{q}^2 + \frac{i}{2}\tilde{\Psi}_M^i \Gamma^{M N K} \left[\frac{1}{2}\lambda Q_i^j (\Gamma_N - \sqrt{6}B_N) \right] \Psi_{K i} \\
& - i\frac{\sqrt{6}}{16}F_{MN} \left(2\tilde{\Psi}^{M i} \Psi_i^N + \tilde{\Psi}_P^i \Gamma^{M N P Q} \Psi_{Q i} \right) \\
& - \frac{1}{4}F_{MN}F^{MN} - \frac{1}{6\sqrt{6}}\epsilon^{M N P Q K} F_{MN}F_{PQ}B_K \Big\}. \tag{5.1}
\end{aligned}$$

Note that the action with $\lambda \neq 0$ can be obtained from the ungauged action (with $\lambda = 0$) by modifying the covariant derivative on the gravitino,

$$D_M \Psi_{N i} \rightarrow \tilde{D}_M \Psi_{N i} = D_M \Psi_{N i} + \frac{1}{2}\lambda Q_i^j (\Gamma_M - \sqrt{6}B_M) \Psi_{N j}, \tag{5.2}$$

and adding the cosmological constant term $6\lambda^2 \tilde{q}^2$ to the Lagrangian. Similarly, the supersymmetry transformations are obtained by analogous modification of the covariant derivative on \mathcal{H}_i .

The modified derivative is covariant with respect to the $U(1)$ transformation,

$$\delta_u(\tilde{D}_M \Psi_{N i}) = u \frac{\sqrt{6}}{2} \lambda Q_i^j (\tilde{D}_M \Psi_{N j}). \tag{5.3}$$

It is then clear that only the Chern-Simons term in the action is not invariant under the $U(1)$ transformation and, therefore,

$$\begin{aligned}
\delta_u S_5 &= \int_{\mathcal{M}} d^5x e_5 \Big\{ -\frac{1}{6\sqrt{6}} \epsilon^{M N P Q K} F_{MN} F_{PQ} \delta_u B_K \Big\} \\
&= \int_{\mathcal{M}} d^5x e_5 D_K \left[-u \frac{1}{6\sqrt{6}} \epsilon^{M N P Q K} F_{MN} F_{PQ} \right] \\
&= \int_{\partial\mathcal{M}} d^4x e_4 \left[-u \frac{1}{6\sqrt{6}} n_5 \epsilon^{5 M N P Q} F_{MN} F_{PQ} \right]. \tag{5.4}
\end{aligned}$$

In our gauge ($e_m^{\hat{5}} = e_a^5 = 0$, $n_5 = -e_5^{\hat{5}}$) this becomes

$$\delta_u S_5 = \int_{\partial\mathcal{M}} d^4x e_4 \left[u \frac{1}{6\sqrt{6}} \epsilon^{m n p q} F_{mn} F_{pq} \right]. \tag{5.5}$$

Therefore, the bulk action is $U(1)$ gauge invariant ($\delta_u S_5 = 0$), provided the following (gauge invariant) boundary condition is imposed,

$$F_{mn} = 0 \quad \text{on } \partial\mathcal{M}. \tag{5.6}$$

Another way to kill the boundary term is to require $u = 0$ on $\partial\mathcal{M}$. The supersymmetry algebra then leads to a stronger (gauge non-invariant) boundary condition,

$$B_m = 0 \quad \text{on } \partial\mathcal{M}. \tag{5.7}$$

Let us now consider the supersymmetry variation of the bulk action. One can show (see the details in Ref. [15]) that the action varies into a boundary term (the bulk part vanishes, since this action is known to be supersymmetric in the absence of boundary). Explicitly,

$$\delta_{\mathcal{H}} S_5 = \int_{\mathcal{M}} d^5 x e_5 (D_M \tilde{K}^M) = \int_{\partial \mathcal{M}} d^4 x e_4 (n_M \tilde{K}^M) , \quad (5.8)$$

where

$$\begin{aligned} \tilde{K}^M = & -e^{MA} e^{NB} \delta_{\mathcal{H}} \omega_{NAB} - \frac{i}{2} \tilde{\Psi}_N^i \Gamma^{NMK} \delta_{\mathcal{H}} \Psi_{Ki} \\ & - F^{MN} \delta_{\mathcal{H}} B_N - \frac{4}{6\sqrt{6}} \epsilon^{MNPQK} F_{PQ} B_K \delta_{\mathcal{H}} B_N . \end{aligned} \quad (5.9)$$

In our gauge this simplifies to

$$\begin{aligned} \delta_{\mathcal{H}} S_5 = \int_{\partial \mathcal{M}} d^4 x e_4 \Big\{ & -e^{na} \delta_{\mathcal{H}} \omega_{na\hat{5}} + (\psi_{m1} \sigma^{mn} \delta_{\mathcal{H}} \psi_{n2} - \psi_{m2} \sigma^{mn} \delta_{\mathcal{H}} \psi_{n1} + h.c.) \\ & - e_5^{\hat{5}} F^{n5} \delta_{\mathcal{H}} B_n + \frac{4}{6\sqrt{6}} \epsilon^{npqk} F_{pq} B_k \delta_{\mathcal{H}} B_n \Big\} . \end{aligned} \quad (5.10)$$

6. Boundary action for $\eta_2 = 0$

According to our discussion of boundary conditions necessary for supersymmetry of the action, if we

- 1) work to quadratic order in fermions, and
- 2) drop terms with B_M ,

then there should exist a boundary action that makes the total bulk-plus-boundary action supersymmetric *without the use of any boundary conditions* (except the one on the supersymmetry parameter). In this section we present such an action.

6.1 Variational principle

In our analysis [13] of the Mirabelli and Peskin model [14], we found that the boundary action required for supersymmetry is, at the same time, the one which makes the variational principle well defined. Let us now turn this around. We will look for a boundary action which improves the variational principle and then see if it makes the total action supersymmetric.

If we consider the general variation of our action (which one would perform to find the equations of motion), we find

$$\delta S_5 = \int_{\mathcal{M}} d^5 x e_5 \Big\{ D_M K^M + (\text{EOM}) \delta \Phi \Big\} , \quad (6.1)$$

where $\Phi = \{e_M^A, B_M, \Psi_{Mi}\}$ and

$$K^M = -e^{MA}e^{NB}\delta\omega_{NAB} + \frac{i}{2}\tilde{\Psi}_N^i\Gamma^{NMK}\delta\Psi_{Ki} \\ - F^{MN}\delta B_N - \frac{4}{6\sqrt{6}}\epsilon^{MNPQK}F_{PQ}B_K\delta B_N. \quad (6.2)$$

Note that the only difference between K^M and \tilde{K}^M is in the sign in front of the fermionic term. Accordingly, in our gauge ($e_m^{\hat{5}} = e_a^{\hat{5}} = 0$, $n_{\hat{5}} = -e_{\hat{5}}^{\hat{5}}$) we obtain

$$\delta S_5 = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ -e^{na}\delta\omega_{na\hat{5}} - (\psi_{m1}\sigma^{mn}\delta\psi_{n2} - \psi_{m2}\sigma^{mn}\delta\psi_{n1} + h.c.) \right. \\ \left. - e_{\hat{5}}^{\hat{5}}F^{n\hat{5}}\delta B_n + \frac{4}{6\sqrt{6}}\epsilon^{npqk}F_{pq}B_k\delta B_n \right\} + (\text{EOM}). \quad (6.3)$$

Let us now consider the following modified action,

$$S'_5 = S_5 + \int_{\partial\mathcal{M}} d^4x e_4 \left[e^{ma}\omega_{ma\hat{5}} + (\psi_{m1}\sigma^{mn}\psi_{n2} + h.c.) \right]. \quad (6.4)$$

In our gauge $\omega_{ma\hat{5}}$ is simply related to the extrinsic curvature (see Appendix D),

$$K_{ma} = \omega_{ma\hat{5}}, \quad K = e^{ma}\omega_{ma\hat{5}}. \quad (6.5)$$

Therefore, the first term in the boundary action is just the standard Gibbons-Hawking term, which makes the variational problem for the total gravity action

$$-\frac{1}{2}\int_{\mathcal{M}} R + \int_{\partial\mathcal{M}} K \quad (6.6)$$

well defined. In our gauge this is especially easy to see. The general variation of the modified action,

$$\delta S'_5 = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ (K_{ma} - K e_{ma})\delta e^{ma} + (2\psi_{m2}\sigma^{mn}\delta\psi_{n1} + h.c.) \right. \\ \left. - e_{\hat{5}}^{\hat{5}}F^{n\hat{5}}\delta B_n + \frac{4}{6\sqrt{6}}\epsilon^{npqk}F_{pq}B_k\delta B_n \right\} + (\text{EOM}), \quad (6.7)$$

contains only the variation of the induced vierbein e_m^a and not that of its normal derivative, which is exactly the purpose of the Gibbons-Hawking term. In addition, we see that the fermionic boundary term in the modified action (6.4) removes $\delta\psi_{n2}$ from the boundary piece of the general variation. As a result, the boundary condition $\psi_{m2} = 0$ on $\partial\mathcal{M}$ (which is necessary to prove supersymmetry of the action to all orders in fermions) arises from requiring the general variation to vanish under arbitrary variation $\delta\psi_{m1}$ on the boundary.

The expression for the supersymmetry variation of the modified action once again differs from the general variation (6.7) only in the fermionic piece (plus all the bulk terms are absent),

$$\delta_{\mathcal{H}}S'_5 = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ (K_{ma} - K e_{ma})\delta_{\mathcal{H}}e^{ma} + (2\psi_{m1}\sigma^{mn}\delta_{\mathcal{H}}\psi_{n2} + h.c.) \right. \\ \left. - e_{\hat{5}}^{\hat{5}}F^{n\hat{5}}\delta_{\mathcal{H}}B_n + \frac{4}{6\sqrt{6}}\epsilon^{npqk}F_{pq}B_k\delta_{\mathcal{H}}B_n \right\}. \quad (6.8)$$

6.2 Supersymmetry without boundary conditions

Consider the following bulk-plus-boundary action,

$$S = S_5 + \int_{\partial\mathcal{M}} d^4x e_4 \left[K + (\psi_{m1} \sigma^{mn} \psi_{n2} + h.c.) \right] + \int_{\partial\mathcal{M}} d^4x e_4 (-3\lambda_1) , \quad (6.9)$$

where we added an extra tension term to the boundary action. We now omit all B_M terms and verify that the total bulk-plus-boundary action is supersymmetric without the use of any boundary conditions for gravitino or vierbein, but with the restriction on the supersymmetry parameter \mathcal{H}_i ,⁴

$$\eta_2 = 0 \quad \text{on } \partial\mathcal{M} . \quad (6.10)$$

The supersymmetry variation gives

$$\delta_{\mathcal{H}} S = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ - (K^{ma} - K e^{ma} + 3\lambda_1 e^{ma}) \delta_{\mathcal{H}} e_{ma} + (2\psi_{m1} \sigma^{mn} \delta_{\mathcal{H}} \psi_{n2} + h.c.) \right\}, \quad (6.11)$$

where

$$\delta_{\mathcal{H}} e_{ma} = -i\psi_{m1} \sigma^a \bar{\eta}_1 + h.c. \quad (6.12)$$

$$\delta_{\mathcal{H}} \psi_{m2} = -iK_{ma} \sigma^a \bar{\eta}_1 + i\lambda q_3 \sigma_m \bar{\eta}_1 . \quad (6.13)$$

With the help of the following identity,

$$K_{na} \sigma^{mn} \sigma^a = \frac{1}{2} (K_{ma} - K e_{ma}) \sigma^a , \quad (6.14)$$

we obtain

$$\delta_{\mathcal{H}} S = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ 3(\lambda_1 - \lambda q_3) i\psi_{m1} \sigma^m \bar{\eta}_1 + h.c. \right\} . \quad (6.15)$$

Therefore, the bulk-plus-boundary action (6.9) is supersymmetric, provided $\lambda_1 = \lambda q_3$. And no boundary condition for the gravitino or vierbein is needed to prove this.

But the boundary conditions do exist for this bulk-plus-boundary action. If we consider its general variation, we find

$$\begin{aligned} \delta S = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ (K_{ma} - K e_{ma} + 3\lambda_1 e_{ma}) \delta e^{ma} \right. \\ \left. + (2\psi_{m2} \sigma^{mn} \delta \psi_{n1} + h.c.) \right\} + (\text{EOM}) . \end{aligned} \quad (6.16)$$

⁴A similar bulk-plus-boundary action (for a spinning string in the superconformal gauge), supersymmetric with the use of only a boundary condition on the supersymmetry parameter, was obtained in 1982 in Ref. [20]; see their Eq. (5.7). (See also Eq. (5.17) in Ref. [21].) Analogous result for a spinning membrane, presented in a more geometrical setting, appeared in 1989 in Ref. [22]; see their Eq. (5.16).

If we require this to vanish under arbitrary field variations, then in addition to the bulk equations of motion we obtain the following natural boundary conditions,

$$K_{ma} = \lambda_1 e_{ma}, \quad \psi_{m2} = 0. \quad (6.17)$$

If one allows the use of these boundary conditions to prove supersymmetry, then one can also claim that the following bulk-plus-boundary action,

$$S_5 + \int_{\partial\mathcal{M}} d^4x e_4 (+\lambda_1), \quad (6.18)$$

is supersymmetric (for $\lambda_1 = \lambda q_3$). (This action is obtained from (6.9) by implementing the boundary condition in its boundary term.) This is the approach taken in Ref. [8].⁵

Our bulk-plus-boundary action (6.9) has the advantage of giving a *unique* boundary action with which the invariance under supersymmetry is independent of the boundary conditions. At the same time, it generates an acceptable set of natural boundary conditions via the variational principle.

7. Boundary action for $\eta_2 = \alpha\eta_1$

Here we show explicitly that the $\alpha = 0$ case can be rotated into the $\alpha \neq 0$ case and that the rotated bulk-plus-boundary action is once again supersymmetric without the use of any boundary conditions.

7.1 Global $SU(2)$ rotation

We use the fact that the bulk action and the supersymmetry transformations are invariant under the following (global) rotations of the fermions (Ψ_{Mi} and \mathcal{H}_i) and parameters \vec{q} ,

$$\Psi'_{Mi} = U_i^j \Psi_{Mj}, \quad \mathcal{H}'_i = U_i^j \mathcal{H}_j, \quad Q' = U Q U^\dagger, \quad (7.1)$$

where $Q = i(\vec{q} \cdot \vec{\sigma})$ and U is a constant matrix of the $SU(2)$ group. We employ a particular rotation from this class,

$$\eta'_1 = \frac{\eta_1 - \alpha^* \eta_2}{\sqrt{1 + \alpha\alpha^*}}, \quad \eta'_2 = \frac{\alpha\eta_1 + \eta_2}{\sqrt{1 + \alpha\alpha^*}} \quad (7.2)$$

(similarly for (ψ_{m1}, ψ_{m2}) and (ψ_{51}, ψ_{52})), together with

$$q'_{12} = \frac{q_{12} - \alpha^2 q_{12}^* - 2\alpha q_3}{1 + \alpha\alpha^*}, \quad q'_3 = \frac{\alpha q_{12}^* + \alpha^* q_{12} + (1 - \alpha\alpha^*) q_3}{1 + \alpha\alpha^*}. \quad (7.3)$$

The inverse rotation is obtained simply by changing the sign of α . In particular,

$$\eta_2 = \frac{-\alpha\eta'_1 + \eta'_2}{\sqrt{1 + \alpha\alpha^*}}. \quad (7.4)$$

Therefore,

$$\eta_2 = 0 \quad \Rightarrow \quad \eta'_2 = \alpha\eta'_1. \quad (7.5)$$

⁵The statement in Ref. [8] (see the sentence after Eq. (2.12)) that in the downstairs picture no boundary action is required is, in fact, erroneous. We find here that the boundary tension term of Eq. (6.18) is needed. This does not, however, affect the rest of the analysis in Ref. [8].

7.2 Bulk-plus-boundary action

Performing this rotation on the action (6.9) and omitting the primes on the new fields, we obtain

$$S^{(\alpha)} = S_5 + \int_{\partial\mathcal{M}} d^4x e_4 \left[K + (\psi_{m1} \sigma^{mn} \psi_{n2} + h.c.) \right] + \int_{\partial\mathcal{M}} d^4x e_4 \mathcal{L}_B^{(\alpha)}, \quad (7.6)$$

where ⁶

$$\mathcal{L}_B^{(\alpha)} = -3\lambda_1 + (\alpha_{11} \psi_{m1} \sigma^{mn} \psi_{n1} + 2\alpha_{12} \psi_{m1} \sigma^{mn} \psi_{n2} + \alpha_{22} \psi_{m2} \sigma^{mn} \psi_{n2} + h.c.), \quad (7.7)$$

and the parameters are given by

$$\alpha_{11} = \frac{-\alpha}{1 + \alpha\alpha^*}, \quad \alpha_{12} = \frac{-\alpha\alpha^*}{1 + \alpha\alpha^*}, \quad \alpha_{22} = \frac{\alpha^*}{1 + \alpha\alpha^*}. \quad (7.8)$$

We claim that this bulk-plus-boundary action is supersymmetric without the use of any boundary conditions for vierbein or gravitino (omitting the B_M terms and working to second order in fermions), provided only that the supersymmetry parameter \mathcal{H}_i is restricted by the condition

$$\eta_2 = \alpha\eta_1 \quad \text{on } \partial\mathcal{M}. \quad (7.9)$$

7.3 Supersymmetry of the action

The supersymmetry variation gives

$$\begin{aligned} \delta_{\mathcal{H}} S^{(\alpha)} = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & - (K^{ma} - K e^{ma} + 3\lambda_1 e^{ma}) \delta_{\mathcal{H}} e_{ma} \\ & + 2 \left[(\alpha_{11} \psi_{m1} + \alpha_{12} \psi_{m2}) \sigma^{mn} \delta_{\mathcal{H}} \psi_{n1} \right. \\ & \left. + (\alpha_{22} \psi_{m2} + (\alpha_{12} + 1) \psi_{m1}) \sigma^{mn} \delta_{\mathcal{H}} \psi_{n2} + h.c. \right] \Big\}, \quad (7.10) \end{aligned}$$

where

$$\delta e_m^a = -i(\psi_{m1} \sigma^a \bar{\eta}_1 + \psi_{m2} \sigma^a \bar{\eta}_2) + h.c. \quad (7.11)$$

$$\delta \psi_{m1} = 2\hat{D}_m \eta_1 + iK_{ma} \sigma^a \bar{\eta}_2 + i\lambda \sigma_m (q_3 \bar{\eta}_2 + q_{12}^* \bar{\eta}_1) \quad (7.12)$$

$$\delta \psi_{m2} = 2\hat{D}_m \eta_2 - iK_{ma} \sigma^a \bar{\eta}_1 + i\lambda \sigma_m (q_3 \bar{\eta}_1 - q_{12} \bar{\eta}_2). \quad (7.13)$$

Using $\eta_2 = \alpha\eta_1$ (and assuming $\alpha = \text{const}$),⁷ we can bring the variation to the following form,

$$\begin{aligned} \delta_{\mathcal{H}} S^{(\alpha)} = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & 4C_1 \psi_{m1} \sigma^{mn} \hat{D}_n \eta_1 + 4C_2 \psi_{m2} \sigma^{mn} \hat{D}_n \eta_1 \\ & + C_3 (K_a^m - K e_a^m) i \psi_{m1} \sigma^a \eta_1 + C_4 (K_a^m - K e_a^m) i \psi_{m2} \sigma^a \eta_1 \\ & - 3C_5 i \psi_{m1} \sigma^m \bar{\eta}_1 - 3C_6 i \psi_{m2} \sigma^m \bar{\eta}_1 + h.c. \Big\}, \quad (7.14) \end{aligned}$$

⁶For a discussion of a boundary Lagrangian with general fermionic mass terms of the form $\alpha_{ij} \psi_i \psi_j$ see also Ref. [23].

⁷We were unable to find a bulk-plus-boundary action which would be supersymmetric for $\eta_2 = \alpha\eta_1$ with α being a *function* of boundary coordinates. We can allow only $\alpha = \text{const}$, despite the fact that the supersymmetry algebra does not explain this limitation.

where the coefficients are

$$\begin{aligned}
C_1 &= \alpha_{11} + \alpha(\alpha_{12} + 1) \\
C_2 &= \alpha_{12} + \alpha\alpha_{22} \\
C_3 &= \alpha_{11}\alpha^* - \alpha_{12} \\
C_4 &= \alpha^*(\alpha_{12} + 1) - \alpha_{22} \\
C_5 &= -\lambda_1 + \lambda A_1 \\
C_6 &= -\lambda_1\alpha^* + \lambda A_2
\end{aligned} \tag{7.15}$$

with

$$A_1 = \alpha_{11}(q_3\alpha^* + q_{12}^*) + (\alpha_{12} + 1)(q_3 - q_{12}\alpha^*) \tag{7.16}$$

$$A_2 = \alpha_{12}(q_3\alpha^* + q_{12}^*) + \alpha_{22}(q_3 - q_{12}\alpha^*) . \tag{7.17}$$

Our bulk-plus-boundary action is supersymmetric without the use of boundary conditions for the fields if all the coefficients C_i vanish. And indeed, using our expressions (7.8) for the parameters α_{ij} , we find

$$C_1 = C_2 = C_3 = C_4 = 0 . \tag{7.18}$$

We also find $A_2 = \alpha^* A_1$ as well as

$$A_1 = -\frac{\alpha q_{12}^* + \alpha^* q_{12} + (\alpha\alpha^* - 1)q_3}{1 + \alpha\alpha^*} . \tag{7.19}$$

Therefore, the remaining conditions, $C_5 = 0$ and $C_6 = 0$, are satisfied provided we choose λ_1 in the boundary Lagrangian (7.7) to be

$$\lambda_1 = -\frac{\alpha q_{12}^* + \alpha^* q_{12} + (\alpha\alpha^* - 1)q_3}{1 + \alpha\alpha^*} \lambda . \tag{7.20}$$

This is exactly the relation found in Ref. [7]. (Note also that it is just the rotated version of $\lambda_1 = \lambda q_3$ for $\alpha = 0$.) But, unlike Ref. [7] or [8], we did not have to use boundary conditions to prove supersymmetry of the total action.

7.4 Boundary conditions

The general variation of the bulk-plus-boundary action (7.6) gives

$$\begin{aligned}
\delta S = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & (K_{ma} - K e_{ma} + 3\lambda_1 e_{ma}) \delta e^{ma} \\
& + 2 \Big[(\alpha_{11}\psi_{m1} + (\alpha_{12} + 1)\psi_{m2}) \sigma^{mn} \delta\psi_{n1} \\
& + (\alpha_{12}\psi_{m1} + \alpha_{22}\psi_{m2}) \sigma^{mn} \delta\psi_{n2} + h.c. \Big] \Big\} + (\text{EOM}) . \tag{7.21}
\end{aligned}$$

One could worry that having both $\delta\psi_{m1}$ and $\delta\psi_{m2}$ in the boundary piece of the general variation could interfere with the application of the variational principle. And indeed, if we require the variation to vanish for arbitrary δe^{ma} , $\delta\psi_{m1}$ and $\delta\psi_{m2}$ on the boundary, we would obtain *two* fermionic boundary conditions,

$$\psi_{m2} = -\frac{\alpha_{11}}{\alpha_{12} + 1}\psi_{m1}, \quad \psi_{m2} = -\frac{\alpha_{12}}{\alpha_{22}}\psi_{m1} . \quad (7.22)$$

For general α_{ij} this would overdetermine the boundary value problem. However, for our special choice (7.8), the two boundary conditions reduce to one! This saves the variational principle.

Our bulk-plus-boundary action leads to the following natural boundary conditions,

$$K_{ma} = \lambda_1 e_{ma}, \quad \psi_{m2} = \alpha\psi_{m1} . \quad (7.23)$$

These, once again, coincide with the boundary conditions used in Refs. [7] and [8]. And if we plug these boundary conditions in our boundary Lagrangian (7.7), we find

$$\mathcal{L}_B^{(\alpha)} = -3\lambda_1 - (\alpha\psi_{m1}\sigma^{mn}\psi_{n1} + h.c.) , \quad (7.24)$$

which is (a half of) the brane Lagrangian in Ref. [7].

7.5 $U(1)$ gauge invariance

We now investigate when the boundary action and/or boundary conditions are gauge invariant. The $U(1)$ gauge transformation on the fermions is

$$\begin{aligned} \delta_u \eta_1 &= iw(q_3 \eta_1 - q_{12}^* \eta_2) \\ \delta_u \eta_2 &= iw(-q_3 \eta_2 - q_{12} \eta_1) , \end{aligned} \quad (7.25)$$

where $w \equiv i\frac{\sqrt{6}}{2}\lambda u \in \mathbb{R}$; the transformation on the $\psi_{m1,2}$ is similar.

The boundary condition $\eta_2 = \alpha\eta_1$ is gauge invariant if

$$\delta_u(\eta_2 - \alpha\eta_1) = -iw[(q_{12} + \alpha q_3) - \alpha(\alpha q_{12}^* - q_3)] = 0 . \quad (7.26)$$

This leads to the following quadratic equation for α ,

$$q_{12} - \alpha^2 q_{12}^* + 2\alpha q_3 = 0 . \quad (7.27)$$

It is equivalent to two linear equations,

$$\alpha q_{12}^* - q_3 = \pm 1, \quad q_{12} + \alpha q_3 = \pm \alpha , \quad (7.28)$$

where the signs \pm correspond to the two solutions of the quadratic equation [7]. (We assumed here the normalization condition $\bar{q}^2 = 1$ of Ref. [7].)

The fermionic part of the boundary Lagrangian (*including* the $\psi_{m1}\sigma^{mn}\psi_{n2}$ term) is

$$\mathcal{L}_{BF} = \frac{1}{1 + \alpha\alpha^*} \left[-\alpha\psi_1\psi_1 + (1 - \alpha\alpha^*)\psi_1\psi_2 + \alpha^*\psi_2\psi_2 \right], \quad (7.29)$$

where we used a shorthand notation $\psi_i\psi_j \equiv \psi_{mi}\sigma^{mn}\psi_{nj}$ and dropped the “+h.c.”. (Note that \mathcal{L}_{BF} *vanishes* when the boundary condition $\psi_{m2} = \alpha\psi_{m1}$ is used!) Its variation under the gauge transformation gives

$$\delta_u \mathcal{L}_{BF} = \frac{iw}{1 + \alpha\alpha^*} \left[W\psi_1\psi_1 + 2(\alpha q_{12}^* - \alpha^* q_{12})\psi_1\psi_2 + W^*\psi_2\psi_2 \right], \quad (7.30)$$

where

$$W = \alpha(\alpha^* q_{12} - q_3) - (\alpha q_3 + q_{12}). \quad (7.31)$$

We see that the variation vanishes when α and \vec{q} are related as in Eq. (7.28). We conclude, therefore, that the (full) boundary Lagrangian and the boundary condition $\eta_2 = \alpha\eta_1$ are *both* gauge invariant when the equation (7.28) is satisfied!

For future reference, we note that if we do not include the $\psi_{m1}\sigma^{mn}\psi_{n2}$ term, that is if we consider the variation of

$$\mathcal{L}_{BF}^{(\alpha)} = \frac{1}{1 + \alpha\alpha^*} \left[-\alpha\psi_1\psi_1 - 2\alpha\alpha^*\psi_1\psi_2 + \alpha^*\psi_2\psi_2 \right] \quad (7.32)$$

under the gauge transformation, we find

$$\delta_u \mathcal{L}_{BF}^{(\alpha)} = \frac{iw}{1 + \alpha\alpha^*} \left[\widetilde{W}\psi_1\psi_1 + 2(\alpha q_{12}^* - \alpha^* q_{12})\psi_1\psi_2 + \widetilde{W}^*\psi_2\psi_2 \right], \quad (7.33)$$

where

$$\widetilde{W} = 2\alpha(\alpha^* q_{12} - q_3). \quad (7.34)$$

Therefore, when the boundary condition $\eta_2 = \alpha\eta_1$ is gauge invariant, i.e. the equation (7.28) is satisfied, the α -dependent part of the boundary Lagrangian, $\mathcal{L}_B^{(\alpha)}$, is *not* gauge invariant! (Unless $\alpha = 0$, in which case $\mathcal{L}_{BF}^{(\alpha)} = 0$; or $\lambda = 0$, in which case the $U(1)$ does not act on the fermions.)

8. Fate of B_M terms

In this section we will keep all B_M and F_{MN} terms (as well as e_5^a and e_5^m), and show that, with an appropriate addition to the boundary action, the bulk-plus-boundary action is supersymmetric provided we use the (gauge non-invariant) boundary condition $B_m = 0$ on $\partial\mathcal{M}$.

8.1 Old action

Consider the modified bulk action (6.4),

$$S'_5 = S_5 + \int_{\partial\mathcal{M}} d^4x e_4 \left[K + (\psi_{m1} \sigma^{mn} \psi_{n2} + h.c.) \right]. \quad (8.1)$$

Its general variation is (6.7),

$$\begin{aligned} \delta S'_5 = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & - (K_a^m - K e_a^m) \delta e_m^a + (2\psi_{m2} \sigma^{mn} \delta \psi_{n1} + h.c.) \\ & - e_5^{\hat{5}} F^{n5} \delta B_n + \frac{4}{6\sqrt{6}} \epsilon^{npqk} F_{pq} B_k \delta B_n \Big\} + (\text{EOM}) , \end{aligned} \quad (8.2)$$

and its supersymmetry variation is given by (6.8),

$$\begin{aligned} \delta_{\mathcal{H}} S'_5 = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & - (K_a^m - K e_a^m) \delta_{\mathcal{H}} e_m^a + (2\psi_{m1} \sigma^{mn} \delta_{\mathcal{H}} \psi_{n2} + h.c.) \\ & - e_5^{\hat{5}} F^{n5} \delta_{\mathcal{H}} B_n + \frac{4}{6\sqrt{6}} \epsilon^{npqk} F_{pq} B_k \delta_{\mathcal{H}} B_n \Big\} . \end{aligned} \quad (8.3)$$

This is true for any \mathcal{H}_i .

8.2 Supersymmetry with B_M terms

Let us first discuss the case $\alpha = 0$, so that the boundary condition on the supersymmetry parameter \mathcal{H}_i is

$$\eta_2 = 0 \quad \text{on } \partial\mathcal{M} . \quad (8.4)$$

The supersymmetry transformations then are

$$\delta_{\mathcal{H}} e_m^a = -i\psi_{m1} \sigma^a \bar{\eta}_1 + h.c. \quad (8.5)$$

$$\delta_{\mathcal{H}} B_m = i \frac{\sqrt{6}}{2} \psi_{m2} \eta_1 + h.c. \quad (8.6)$$

$$\begin{aligned} \delta_{\mathcal{H}} \psi_{m2} = & -iK_{ma} \bar{\eta}_1 + i\lambda q_3 \sigma_m \bar{\eta}_1 \\ & + i\sqrt{6} \lambda q_{12} \eta_1 B_m - \frac{1}{2\sqrt{6}} (i\epsilon_m^{nkl} \sigma_l + 4\delta_m^n \sigma^k) \bar{\eta}_1 F_{nk} . \end{aligned} \quad (8.7)$$

Therefore,

$$\begin{aligned} \delta_{\mathcal{H}} S'_5 = \int_{\partial\mathcal{M}} d^4x e_4 \Big\{ & - 3i\lambda q_3 (\psi_{m1} \sigma^m \bar{\eta}_1) \\ & + 2i\sqrt{6} \lambda q_{12} (\psi_{m1} \sigma^{mn} \eta_1) B_n - \frac{3i}{2\sqrt{6}} (\psi_{m1} \sigma_n \bar{\eta}_1) \epsilon^{mpqn} F_{pq} \\ & - i \frac{\sqrt{6}}{2} (\psi_{n2} \eta_1) e_5^{\hat{5}} F^{n5} + \frac{i}{3} (\psi_{n2} \eta_1) \epsilon^{npqk} F_{pq} B_k + h.c. \Big\} . \end{aligned} \quad (8.8)$$

The term with λq_3 can be compensated by a boundary tension term (see Eq. (6.9)). Most of the other terms can be killed by application of the boundary condition for B_m ,

$$F_{mn} = 0 \quad \text{on } \partial\mathcal{M} \quad (8.9)$$

if $\lambda q_{12} = 0$ and the gauge invariance is preserved on the boundary; or

$$B_m = 0 \quad \text{on } \partial\mathcal{M} \quad (8.10)$$

if $\lambda q_{12} \neq 0$ and the gauge invariance on the boundary is broken by the boundary condition on the supersymmetry parameter. However, we are still left with

$$\delta_{\mathcal{H}} S'_5 = \int_{\partial\mathcal{M}} d^4 x e_4 \left\{ -i \frac{\sqrt{6}}{2} (\psi_{m2} \eta_1) e_5^{\hat{5}} F^{n5} \right\}. \quad (8.11)$$

This remaining term can, in principle, be canceled by the boundary condition on the gravitino,

$$\psi_{m2} = 0 \quad \text{on } \partial\mathcal{M}. \quad (8.12)$$

However, our analysis of the supersymmetry algebra indicates that we *should not need* this boundary condition for supersymmetry to *quadratic order* in fermions. Therefore, there should exist a boundary action which lets us avoid using this boundary condition.

Another reason in favor of modifying our action is that, at the moment, the B_m boundary condition we need ($B_m = 0$ or $F_{mn} = 0$ on $\partial\mathcal{M}$) is not the same as the natural boundary condition arising from the bulk-plus-boundary action,

$$e_5^{\hat{5}} F^{n5} - \frac{4}{6\sqrt{6}} \epsilon^{npqk} F_{pq} B_k = 0 \quad \text{on } \partial\mathcal{M}. \quad (8.13)$$

8.3 New action

In order to

- 1) avoid using the $\psi_{m2} = 0$ boundary condition in proving supersymmetry (to quadratic order in fermions), and
- 2) have the boundary condition for B_m ($B_m = 0$ or $F_{mn} = 0$ on $\partial\mathcal{M}$) appear as a natural boundary condition,

we need to add an appropriate boundary action. Such a boundary action exists, but it turns out that it itself breaks gauge invariance! That is, for agreement with the supersymmetry algebra we have to break the gauge invariance (only on the boundary) by hand. However, we will find that the same boundary action is also needed for the correct transition to the orbifold picture.

The required boundary action is easy to find. Indeed, consider the following bulk-plus-boundary action,

$$S_5'' = S_5' + \int_{\partial\mathcal{M}} d^4x e_4 \left\{ e_5^{\hat{5}} F^{n5} B_n \right\}. \quad (8.14)$$

Its general variation is,

$$\begin{aligned} \delta S_5'' = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ -[K_a^m - K e_a^m - (B_n C^n) e_a^m] \delta e_m^a + (2\psi_{m2} \sigma^{mn} \delta\psi_{n1} + h.c.) \right. \\ \left. + B_m \delta C^m + \frac{4}{6\sqrt{6}} \epsilon^{npqk} F_{pq} B_k \delta B_n \right\} + (\text{EOM}), \end{aligned} \quad (8.15)$$

where we defined

$$\begin{aligned} C^m \equiv e_5^{\hat{5}} F^{m5} &= e_5^{\hat{5}} [g^{mn} g^{k5} F_{nk} + (g^{mn} g^{55} - g^{m5} g^{n5}) F_{n5}] \\ &= \gamma^{mn} F_{n\hat{5}} \equiv e_a^m e^{na} (e_5^{\hat{5}} F_{n5} + e_5^{\hat{k}} F_{nk}). \end{aligned} \quad (8.16)$$

The natural boundary conditions corresponding to *independent* variations of B_m and C^m (they are independent since C^m involves $\partial_5 B_n$ whose value on $\partial\mathcal{M}$ is independent of the value of B_n) coincide on the following boundary condition,

$$B_m = 0 \quad \text{on } \partial\mathcal{M}. \quad (8.17)$$

This is exactly the (gauge non-invariant) boundary condition dictated by the supersymmetry algebra. (Note that the B_m and C^m fields are analogous to, respectively, the Φ and $D = X_3 - \partial_5 \Phi$ fields of the Mirabelli and Peskin model [14]. There, in order to derive the boundary condition for Φ , we need a boundary term ΦD [13].)

The result of the supersymmetry variation, compared to Eq. (8.11), is now

$$\delta_{\mathcal{H}} S_5'' = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ B_n \delta_{\mathcal{H}} C^n + (B_n C^n) e_a^m \delta_{\mathcal{H}} e_m^a \right\}. \quad (8.18)$$

And, therefore, the bulk-plus-boundary action (8.14) is supersymmetric upon using only the $B_m = 0$ boundary condition, but not the $\psi_{m2} = 0$ one.

8.4 Extension to the $\alpha \neq 0$ case

The generalization to the $\alpha \neq 0$ case is straightforward. Since B_M is not rotated under the $SU(2)$, we do not get new boundary terms for the $\alpha \neq 0$ case. The boundary condition remains $B_m = 0$ on $\partial\mathcal{M}$. The only terms we should consider, therefore, are those containing F_{m5} .

The variation (7.10) can also be written as follows,

$$\begin{aligned} \delta_{\mathcal{H}} S^{(\alpha)} = \int_{\partial\mathcal{M}} d^4x e_4 \left\{ -(K^{ma} - K e^{ma} + 3\lambda_1 e^{ma}) \delta_{\mathcal{H}} e_{ma} \right. \\ \left. + 2 \left[\frac{\psi_{m1} + \alpha^* \psi_{m2}}{1 + \alpha \alpha^*} \sigma^{mn} (\delta_{\mathcal{H}} \psi_{n2} - \alpha \delta_{\mathcal{H}} \psi_{n1}) + h.c. \right] \right\}. \end{aligned} \quad (8.19)$$

The terms with F_{m5} appear only in the variation of the gravitino,

$$\delta_{\mathcal{H}}\psi_{m1} = \frac{-4i}{2\sqrt{6}}(\sigma_m{}^n + \delta_m^n)\eta_1(e_5^5 F_{n5}) + \dots \quad (8.20)$$

$$\delta_{\mathcal{H}}\psi_{m2} = \frac{-4i}{2\sqrt{6}}(\sigma_m{}^n + \delta_m^n)\eta_2(e_5^5 F_{n5}) + \dots, \quad (8.21)$$

where dots represent the terms already considered. But since $\eta_2 = \alpha\eta_1$ on $\partial\mathcal{M}$, the F_{m5} terms cancel in the combination $\delta_{\mathcal{H}}\psi_{n2} - \alpha\delta_{\mathcal{H}}\psi_{n1}$! (Note that in the orbifold picture the cancellation of the F_{m5} terms is less straightforward, as we will see.) This completes our explicit check of the fact that the rotated action,

$$S^{(\alpha)} + \int_{\partial\mathcal{M}} d^4x e_4 \left\{ e_5^5 F^{n5} B_n \right\}, \quad (8.22)$$

is supersymmetric (using only the $\eta_2 = \alpha\eta_1$ and $B_m = 0$ boundary conditions).

9. From boundary to orbifold picture

In this section we will show that that the generalized Gibbons-Hawking term [9] (which we call “Y-term” to honor the work of York [10, 11]) matches onto the brane-localized singularities of the bulk Lagrangian in the orbifold picture. This explains why the Y-term appears only in the bulk-plus-boundary action (in the boundary picture), but not in the bulk-plus-brane action (in the orbifold picture).

9.1 Summary of the boundary picture discussion

Our total bulk-plus-boundary action is

$$S = \int_{\mathcal{M}} d^5x e_5 \mathcal{L}_5 + \int_{\partial\mathcal{M}} d^4x e_4 Y + \int_{\partial\mathcal{M}} d^4x e_4 \mathcal{L}_B^{(\alpha)}, \quad (9.1)$$

where the α -independent boundary term Y is

$$Y = K + (\psi_{m1}\sigma^{mn}\psi_{n2} + h.c.) + e_5^5 F^{n5} B_n \quad (9.2)$$

and $\mathcal{L}_B^{(\alpha)}$ is given by Eq. (7.7) together with Eqs. (7.8) and (7.20). We showed that the action is supersymmetric for $\eta_2 = \alpha\eta_1$ on $\partial\mathcal{M}$ provided we use just one more boundary condition: $B_m = 0$ on $\partial\mathcal{M}$.

The Y-term is a generalization of the Gibbons-Hawking boundary term for our bulk action. It allows us to derive (natural) boundary conditions by requiring that the general variation of the action vanish for arbitrary field variations in the bulk and on the boundary. The boundary conditions thus obtained are

$$B_m = 0, \quad K_{ma} = \lambda_1 e_{ma}, \quad \psi_{m2} = \alpha\psi_{m1} \quad \text{on } \partial\mathcal{M}. \quad (9.3)$$

They are consistent with supersymmetry, as was shown in Ref. [7]. Their supersymmetry variations also produce other (secondary) boundary conditions. In on-shell formulation, one can at most expect that the full system of boundary conditions closes under supersymmetry only *up to equations of motion* [13]. (See also Ref. [24].)

9.2 Lifting to the orbifold

Let us now lift our results to the \mathbb{R}/\mathbb{Z}_2 orbifold. The orbifold can be constructed from two copies of our space-time manifold with boundary,

$$\mathcal{M}_- = \mathbb{R}^{1,3} \times (-\infty, 0] \quad \text{and} \quad \mathcal{M}_+ = \mathbb{R}^{1,3} \times [0, +\infty) . \quad (9.4)$$

(We use the notation valid for the flat case [13], but keep in mind that we are actually on a general curved manifold.) Since the boundaries of \mathcal{M}_+ and \mathcal{M}_- coincide, we denote $\Sigma = \partial\mathcal{M}_+ = \partial\mathcal{M}_-$ and call this hypersurface a “brane”.

We use a \mathbb{Z}_2 symmetry of the bulk action to impose the following parity assignments [7] (“even/odd” means $\phi(-z) = \pm\phi(+z)$),

$$\begin{aligned} \text{even} : & e_m^a e_5^{\hat{5}} B_5 \psi_{m1} \psi_{52} \eta_1 q_{12} \lambda \\ \text{odd} : & e_m^{\hat{5}} e_5^a B_m \psi_{m2} \psi_{51} \eta_2 q_3 . \end{aligned} \quad (9.5)$$

It follows that $K = e^{ma}\omega_{mg\hat{5}}$ is odd and F^{m5} is even. Therefore, the Y -term is odd.

Since we used $n_5 = -e_5^{\hat{5}}$, our bulk-plus-boundary action is appropriate for \mathcal{M}_+ . For \mathcal{M}_- , we should use $n_5 = +e_5^{\hat{5}}$ (see Appendix C) and accordingly change the sign of the Y -term. The boundary conditions, which our actions for \mathcal{M}_+ and \mathcal{M}_- should reproduce, are

$$B_m^{(+)} = 0, \quad K_{ma}^{(+)} = +\lambda_1^{(+)} e_{ma}, \quad \psi_{m2}^{(+)} = +\alpha^{(+)} \psi_{m1} \quad (9.6)$$

$$B_m^{(-)} = 0, \quad K_{ma}^{(-)} = -\lambda_1^{(+)} e_{ma}, \quad \psi_{m2}^{(-)} = -\alpha^{(+)} \psi_{m1} . \quad (9.7)$$

(The superscripts (\pm) mean “evaluated on the \mathcal{M}_\pm side of the brane Σ ”.) This is equivalent to using Eq. (9.3) on both sides of Σ if we set $\alpha^{(-)} = -\alpha^{(+)}$ and $\lambda_1^{(-)} = -\lambda_1^{(+)}$. From the expression for λ_1 , Eq. (7.20), we see that

$$\alpha^{(-)} = -\alpha^{(+)}, \quad q_3^{(-)} = -q_3^{(+)} \quad \Rightarrow \quad \lambda_1^{(-)} = -\lambda_1^{(+)} . \quad (9.8)$$

We, therefore, find that our boundary Lagrangian $\mathcal{L}_B^{(\alpha)}$ is *odd*. (In the Mirabelli and Peskin model, the boundary Lagrangian \mathcal{L}_4 includes only even bulk fields and is itself even. Our $\mathcal{L}_B^{(\alpha)}$, however, includes also an odd bulk field, ψ_{m2} , and odd parameters, α and q_3 .)

The correct actions for the both sides then are

$$S_\pm = \int_{\mathcal{M}_\pm} d^5 x e_5 \mathcal{L}_5 \pm \int_{\partial\mathcal{M}_\pm} d^4 x e_4 Y^{(\pm)} \pm \int_{\partial\mathcal{M}_\pm} d^4 x e_4 \mathcal{L}_B^{(\alpha)(\pm)} . \quad (9.9)$$

The orbifold action is the sum of these two,

$$S = \int_{\mathcal{M}_+ \cup \mathcal{M}_-} d^5 x e_5 \mathcal{L}_5 + \int_\Sigma d^4 x e_4 [2Y^{(+)}] + \int_\Sigma d^4 x e_4 [2\mathcal{L}_B^{(\alpha)(+)}] . \quad (9.10)$$

By analogy with our analysis of the Mirabelli and Peskin model [13], we expect that the Y -term matches onto the brane-localized terms produced by the bulk Lagrangian \mathcal{L}_5 . We will show now that the match is (almost) perfect.

9.3 Boundary Y -term vs. Orbifold singular terms

For each odd field, we can write

$$\Phi(x, z) = \varepsilon(z)\Phi^{(+)}(x, |z|), \quad \partial_5\Phi = (\partial_5\Phi)^{(+)} + 2\Phi^{(+)}\delta(z), \quad (9.11)$$

where $\varepsilon(z) = \pm 1$ on \mathcal{M}_\pm . In particular,

$$F_{m5} = F_{m5}^{(+)} - 2B_m^{(+)}\delta(z). \quad (9.12)$$

This allows us to separate the Σ -localized terms in \mathcal{L}_5 explicitly.

The relevant part of the bulk Lagrangian is

$$\begin{aligned} \mathcal{L}_5 = & -\frac{1}{2}R + \frac{i}{2}\tilde{\Psi}_M^i\Gamma^{M5K}\partial_5\Psi_{Ki} \\ & -\frac{1}{4}F_{MN}F^{MN} - \frac{1}{6\sqrt{6}}\epsilon^{MNPQK}F_{MN}F_{PQ}B_K + \dots \end{aligned} \quad (9.13)$$

The analysis simplifies a lot in our gauge ($e_m^5 = e_a^5 = 0$).⁸ In particular, a HUGE advantage of this gauge is that

there are no $\delta(z)$ -terms in ω_{MAB} !

Therefore,

$$\begin{aligned} R = & -2e^{5\hat{5}}e^{ma}\partial_5\omega_{ma\hat{5}} + \dots \\ = & -4e^{5\hat{5}}e^{ma}\omega_{ma\hat{5}}^{(+)}\delta(z) + \dots \\ = & -4e_5^5K^{(+)}\delta(z) + \dots, \end{aligned} \quad (9.14)$$

where the dots denote non-singular terms. Next,

$$\begin{aligned} \frac{i}{2}\tilde{\Psi}_M^i\Gamma^{M5K}\partial_5\Psi_{Ki} = & e_5^5[\psi_{m1}\sigma^{mn}\partial_5\psi_{n2} + h.c.] + \dots \\ = & e_5^5[2\psi_{m1}\sigma^{mn}\psi_{n2}^{(+)}\delta(z) + h.c.] + \dots \end{aligned} \quad (9.15)$$

The Chern-Simons term is straightforward to consider,

$$\begin{aligned} \epsilon^{MNPQK}F_{MN}F_{PQ}B_K = & -4e_5^5\epsilon^{mpqk}F_{m5}F_{pq}B_k + \dots \\ = & 8e_5^5\epsilon^{mpqk}F_{pq}B_kB_m^{(+)}\delta(z) + \dots \\ = & 8e_5^5\epsilon^{mpqk}F_{pq}^{(+)}B_k^{(+)}B_m^{(+)}\varepsilon(z)^2\delta(z) + \dots \\ = & 0 + \dots, \end{aligned} \quad (9.16)$$

⁸Note that we keep $e_5^a \neq 0$, $e_5^m \neq 0$. See Appendix D.

whereas the B_M kinetic term is the trickiest to analyze. We have

$$-\frac{1}{4}F_{MN}F^{MN} = -\frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}F_{m5}F^{m5} . \quad (9.17)$$

Both terms contain $\delta(z)$ -terms. Indeed, let us introduce

$$\alpha^{mnk} = g^{mn}g^{k5}, \quad \beta^{mn} = g^{mn}g^{55} - g^{m5}g^{n5} . \quad (9.18)$$

Then

$$F^{m5} = \alpha^{mnk}F_{nk} + \beta^{mn}F_{n5}, \quad F^{mn} = g^{mk}g^{nl}F_{kl} + (\alpha^{mkn} - \alpha^{nkm})F_{k5} , \quad (9.19)$$

and we find

$$-\frac{1}{4}F_{mn}F^{mn} = (\alpha^{mkn}F_{mn}B_k)^{(+)}\varepsilon(z)^2\delta(z) + \dots \quad (9.20)$$

$$\begin{aligned} -\frac{1}{2}F_{m5}F^{m5} &= 2\beta^{mn}F_{m5}^{(+)}B_n^{(+)}\delta(z) + (\alpha^{mkn}F_{mn}B_k)^{(+)}\varepsilon(z)^2\delta(z) \\ &\quad - 2\beta^{mn}B_m^{(+)}B_n^{(+)}\delta(z)^2 + \dots \end{aligned} \quad (9.21)$$

Finally, combining the pieces, we obtain the following expression for the singular part of the bulk Lagrangian (the $e_5^{\hat{5}}$ is taken from $e_5 = e_4 e_5^{\hat{5}}$),

$$\begin{aligned} e_5^{\hat{5}}\mathcal{L}_5 &= 2K^{(+)}\delta(z) + [2\psi_{m1}\sigma^{mn}\psi_{n2}^{(+)} + h.c.]\delta(z) \\ &\quad + e_5^{\hat{5}}\left\{ 2\beta^{mn}F_{m5}^{(+)}B_n^{(+)}\delta(z) + 2(\alpha^{mkn}F_{mn}B_k)^{(+)}\varepsilon(z)^2\delta(z) \right. \\ &\quad \left. - 2\beta^{mn}B_m^{(+)}B_n^{(+)}\delta(z)^2 \right\} + \dots \end{aligned} \quad (9.22)$$

This is to be compared with

$$\begin{aligned} 2Y^{(+)}\delta(z) &= 2K^{(+)}\delta(z) + 2[\psi_{m1}\sigma^{mn}\psi_{n2} + h.c.]^{(+)}\delta(z) \\ &\quad + 2e_5^{\hat{5}}\left[\beta^{mn}F_{m5}B_n + \alpha^{mkn}F_{mn}B_k\right]^{(+)}\delta(z) . \end{aligned} \quad (9.23)$$

9.4 Auxiliary boundary condition

We see that we definitely do not match the $\delta(z)^2$ terms. To do so, one would have to put $\delta(0)$ terms on the boundary which we consider unnatural. Instead, we refer to the discussion of the Mirabelli and Peskin model [13], where it was found that the $\delta(z)^2$ terms are taken care of by the auxiliary fields upon going on-shell.

How could this help if there are no auxiliary fields, but $\delta(z)^2$ terms are present? The point is that “going on-shell” in the boundary picture means not only eliminating the auxiliary fields, but also using some boundary conditions which are a part of the auxiliary equations of motion [13].

We conjecture that $B_m = 0$ on $\partial\mathcal{M}$ is exactly such an “auxiliary boundary condition”. (This is so if, in the Y -term for the *off-shell* supergravity action, the B_m appears multiplied by an auxiliary field.) Using this boundary condition takes care of the discrepancy in the $\delta(z)^2$ terms.

9.5 Different $\varepsilon(z)$ for different fields

The other mismatch is in the term with $\varepsilon(z)^2$,

$$2(\alpha^{mkn}F_{mn}B_k)^{(+)}\varepsilon(z)^2\delta(z) \in \mathcal{L}_5. \quad (9.24)$$

Setting $\varepsilon(z)^2 = 1$ would eliminate the discrepancy, but we are not allowed to do so, since ⁹

$$\varepsilon(z)^2\delta(z) = \frac{1}{3}\delta(z). \quad (9.25)$$

One could find various “excuses” for neglecting this term. One could use the $B_m = 0$ boundary condition to kill it. But this also kills the $\beta^{mn}F_{m5}B_n$ term which matches perfectly. Or one could argue that it is “of higher order in the brane coupling”. Indeed, this term is special in the sense that it is a product of three odd fields (g^{m5} , F_{mn} and B_m) evaluated on $\partial\mathcal{M}_+$. If the brane action is such that these fields acquire non-zero boundary conditions, this term becomes proportional to g^3 , with g being a coupling constant in front of the boundary action. (This is exactly the type of expansion used by Horava and Witten in Ref. [1]. The role of g is played there by $\kappa^{2/3}$.)

But there is, actually, another way to eliminate the mismatch. And it can be motivated as follows. Note that the orbifold construction may correspond to a discontinuous limit of some smooth supergravity realization (when the brane sources are smoothed out into the bulk). The $\varepsilon(z)$ would then correspond to a smooth warp-factor. But then, why would all the odd fields have the same warp-factor?

Let us, therefore, introduce different $\varepsilon(z)$ for different odd fields! We numerate them as follows,

$$\eta_2 = \varepsilon_1\eta_2^{(+)}, \quad \psi_{m2} = \varepsilon_2\psi_{m2}^{(+)}, \quad K_{ma} = \varepsilon_3K_{ma}^{(+)}, \quad q_3 = \varepsilon_4q_3^{(+)} \quad (9.26)$$

$$B_m = \varepsilon_5B_m^{(+)}, \quad e_{5a} = \varepsilon_6e_{5a}^{(+)} \quad (9.27)$$

They have to satisfy $\varepsilon_i(z) = \pm 1$ on \mathcal{M}_\pm , but if we have one such $\varepsilon(z)$, then we can write many functions of it¹⁰ still satisfying this property, ¹¹

$$\varepsilon(z), \quad \frac{1}{\varepsilon(z)}, \quad \frac{2\varepsilon(z)}{1 + \varepsilon(z)^2}, \quad \text{etc.} \quad (9.28)$$

⁹This relation was first noticed by Conrad in Ref. [25]. See also Ref. [26]. The key to its understanding is the fact that the “sign function” $\varepsilon(z)$ must be treated as a *distribution*, just like the delta function $\delta(z)$. One way to define it is via a limit of a sequence of regular (smooth) functions: $\varepsilon(z) = \lim \varepsilon_n(z)$. Accordingly, $\delta(z) = \lim \delta_n(z)$. The product of distributions is ill-defined *unless* we relate the two sequences. We require $\varepsilon'_n(z) = 2\delta_n(z)$. Then $\lim \int dz \varepsilon_n^2(z) \delta_n(z) f(z) = (1/3) \lim \int dz \delta_n(z) f(z)$ for any (smooth) test function $f(z)$. This gives precise meaning to the distributional equality $\varepsilon(z)^2\delta(z) = (1/3)\delta(z)$.

¹⁰We can define a function $w(\varepsilon)$ of the distribution $\varepsilon(z) = \lim \varepsilon_n(z)$ by $w(\varepsilon(z)) = \lim w(\varepsilon_n(z))$.

¹¹The possible appearance of the sign factors of this type in the orbifold constructions was mentioned before; see, e.g., Refs. [27, 28].

The $\varepsilon(z)^2$ in Eq. (9.24) now changes as

$$\varepsilon(z)^2 \longrightarrow \varepsilon_5(z)\varepsilon_6(z) . \quad (9.29)$$

Therefore, by choosing $\varepsilon_5(z) = \varepsilon(z)$ and $\varepsilon_6(z) = 1/\varepsilon(z)$, or vice versa, we eliminate the mismatch!

One should be careful, however, because such a modification can change some of the relations used before. Namely, we used $\varepsilon'(z) = 2\delta(z)$. It turns out, however, that we are safe since¹²

$$\left(\frac{1}{\varepsilon(z)}\right)' = -\frac{1}{\varepsilon(z)^2}\varepsilon'(z) = -2\frac{1}{\varepsilon(z)^2}\delta(z) = 2\delta(z) . \quad (9.30)$$

We used here the following relation,

$$\varepsilon(z)^{-2}\delta(z) = -\delta(z) , \quad (9.31)$$

which can be proven in the same way as Eq. (9.25). Namely,

$$\int_{-a}^{+a} \varepsilon(z)^{-2}\delta(z)dz = \frac{1}{2} \int_{-a}^{+a} \varepsilon^{-2}d\varepsilon = -\frac{1}{2}\varepsilon(z)^{-1}\Big|_{-a}^{+a} = -\frac{1}{2}(1 - (-1)) = -1 . \quad (9.32)$$

We will find a more convincing proof of the necessity to introduce different $\varepsilon_i(z)$ for different fields when checking supersymmetry of our action in the orbifold picture.

9.6 Another addition to the Y -term

We see now that the presence of the $e_3^5 F^{n5} B_n$ term in the boundary action follows most easily from the requirement that the Y -term match singularities of the bulk Lagrangian. We will now use this approach to find another term which should be included in the Y -term.

In our expressions for the boundary terms of the general and supersymmetry variations of the bulk action, we ignored a contribution from the following term,

$$-i\frac{\sqrt{6}}{16}F_{MN}\left(2\tilde{\Psi}^{Mi}\Psi_i^N + \tilde{\Psi}_P^i\Gamma^{MNPQ}\Psi_{Qi}\right) \in \mathcal{L}_5 . \quad (9.33)$$

The reason was that its contribution to the boundary term of the supersymmetry variation (if it is at all non-zero) comes from $\delta_{\mathcal{H}}B_M$ and thus is *quartic* in fermions, which is of higher order than we consider. But its contribution to the boundary term of the general variation is of quadratic order in fermions and thus should be included. (Note that the variation $\delta_{\mathcal{H}}\Psi_{Mi}$ in Eq. (9.33) does not contribute to the boundary term of the supersymmetry variation as the explicit calculation of \tilde{K}_M in Eq. (5.8) shows [15].)

¹²Note that although $\varepsilon'(z) = 2\delta(z)$ and $(1/\varepsilon(z))' = 2\delta(z)$ are both true in the distributional sense, the *functional* relation $\varepsilon'_n(z) = 2\delta_n(z)$ does not hold between $1/\varepsilon_n(z)$ and $\delta_n(z)$.

This part of \mathcal{L}_5 produces brane-localized terms because $F_{m5} \ni -2B_m^{(+)}\delta(z)$. Since the singular part of $e_5^{\hat{5}}\mathcal{L}_5$ should match onto $2Y^{(+)}\delta(z)$, we find the following contribution to the Y -term,

$$Y^{(+)} \ni \frac{\sqrt{6}}{4}i\gamma^{mk}B_k^{(+)}(\psi_{m2}\psi_{\hat{5}1} - \psi_{m1}\psi_{\hat{5}2}) - \frac{\sqrt{6}}{8}i\varepsilon^{mnpq}B_m^{(+)}(\psi_{p2}\sigma_n\bar{\psi}_{q2} + \psi_{p1}\sigma_n\bar{\psi}_{q1}) + h.c. , \quad (9.34)$$

where $\psi_{\hat{5}1,2} = e_5^{\hat{5}}\psi_{51,2} + e_5^m\psi_{m1,2}$. This contribution, however, does not change our previous analysis.

Indeed, because of the $e_5^{\hat{5}}F^{n5}B_n$ term, the gauge invariance of the bulk-plus-boundary action is broken (on the boundary) and we have to use the $B_m^{(+)} = 0$ boundary condition. The terms in Eq. (9.34) are, therefore, harmless for the supersymmetry variation unless we vary B_m itself, but this is of higher order in fermions. They do modify the natural boundary conditions, making $B_m \sim \mathcal{O}(\psi^2)$, but this is again of higher order in our approximation.

Also, by construction, these terms match singularities of the bulk Lagrangian and thus do not appear in the bulk-plus-brane action of the orbifold picture.

9.7 Result: the orbifold action

We found that the Y -term of the boundary picture matches (with some subtleties) onto the brane-localized terms arising from the singularities of the bulk Lagrangian \mathcal{L}_5 . As a result, the total action (9.10) reduces to

$$S = \int_{\mathcal{M}_5} d^5x e_5 \mathcal{L}_5 + \int_{\Sigma} d^4x e_4 \mathcal{L}_4 , \quad (9.35)$$

where $\mathcal{M}_5 = \mathbb{R}^{1,4}$ is the (curved) space-time *without* boundary, with $z \in (-\infty, +\infty)$, and Σ denotes the brane at $z = 0$. The brane Lagrangian \mathcal{L}_4 is *twice* the boundary Lagrangian (not including the Y -term) evaluated on the \mathcal{M}_+ side of Σ ,

$$\mathcal{L}_4 = 2\mathcal{L}_B^{(\alpha)(+)} . \quad (9.36)$$

10. Supersymmetry in the orbifold picture

Here we check explicitly whether the bulk-plus-brane action, constructed starting from the boundary picture, is in fact supersymmetric in the orbifold picture. In the process, we find that using different $\varepsilon_i(z)$ for different odd fields is essential and that checking supersymmetry *without* the use of the boundary conditions fixes the $\varepsilon_i(z)$ *uniquely*.

10.1 Bulk-plus-brane action

Our bulk-plus-brane action is

$$S = S_5 + S_4 = \int d^5x e_5 \mathcal{L}_5 + \int d^5x e_4 \delta(z) \mathcal{L}_4 , \quad (10.1)$$

where S_5 is the bulk supergravity action (5.1), and \mathcal{L}_4 is the following brane Lagrangian,

$$\mathcal{L}_4 = -6\lambda_1 + 2\left[\alpha_{11}\psi_{m1}\sigma^{mn}\psi_{n1} + 2\alpha_{12}\psi_{m1}\sigma^{mn}\psi_{n2}^{(+)} + \alpha_{22}\psi_{m2}^{(+)}\sigma^{mn}\psi_{n2}^{(+)} + h.c.\right]. \quad (10.2)$$

The parameters are fixed in terms of λ , \vec{q} and $\alpha \equiv \alpha^{(+)}$,

$$\alpha_{11} = \frac{-\alpha}{1 + \alpha\alpha^*}, \quad \alpha_{12} = \frac{-\alpha\alpha^*}{1 + \alpha\alpha^*}, \quad \alpha_{22} = \frac{\alpha^*}{1 + \alpha\alpha^*} \quad (10.3)$$

$$\lambda_1 = -\frac{\alpha q_{12}^* + \alpha^* q_{12} + (\alpha\alpha^* - 1)q_3^{(+)}}{1 + \alpha\alpha^*}\lambda. \quad (10.4)$$

This bulk-plus-brane action was derived starting from the boundary picture. (Note that setting $\alpha = 0$ kills all the fermionic terms in \mathcal{L}_4 . Therefore, in the orbifold picture, the transition between the $\alpha = 0$ and $\alpha \neq 0$ cases is not as straightforward as in the boundary picture.)

We would now like to check explicitly that the action is supersymmetric in the orbifold picture under the local $N = 2$ supersymmetry restricted on the brane by the boundary condition

$$\eta_2^{(+)} = \alpha\eta_1 \quad \text{on } \Sigma. \quad (10.5)$$

As we found in the previous section, it may be necessary to use the freedom of defining different $\varepsilon(z)$ for different fields. We therefore set

$$\eta_2 = \varepsilon_1\eta_2^{(+)}, \quad \psi_{m2} = \varepsilon_2\psi_{m2}^{(+)}, \quad K_{ma} = \varepsilon_3 K_{ma}^{(+)}, \quad q_3 = \varepsilon_4 q_3^{(+)}, \quad B_m = \varepsilon_5 B_m^{(+)}. \quad (10.6)$$

We will see that ε_i are either $\varepsilon(z)$ or $1/\varepsilon(z)$, so that we can freely use $\varepsilon'_i(z) = 2\delta(z)$.

10.2 Supersymmetry variation of the bulk action

The supersymmetry variation of the bulk Lagrangian produces a total derivative term, Eq. (5.8), which was important in the boundary picture but integrates to zero on the orbifold. But on the orbifold we get additional brane-localized contributions from the bulk action due to the discontinuities in the fields and parameters.

First, we promoted the parameter q_3 to a function,

$$q_3(z) = \varepsilon_4(z)q_3^{(+)}, \quad (10.7)$$

where $q_3^{(+)}$ is a constant. Performing the supersymmetry variation of S_5 without assuming the parameters to be constant, we find [7] (dropping the total derivative term),

$$\delta_{\mathcal{H}}^{(1)} S_5 = \int d^5 x e_5 \left\{ -3i\tilde{\Psi}_M^i \Gamma^{MN} \mathcal{H}_j \partial_N (\lambda Q_i^j) - i\sqrt{6}\tilde{\Psi}_M^i \Gamma^{MNK} \mathcal{H}_j B_K \partial_N (\lambda Q_i^j) \right\}. \quad (10.8)$$

In our case only $\partial_5 q_3 \neq 0$. Going into our gauge ($e_m^5 = e_a^5 = 0$), we obtain

$$\begin{aligned} \delta_{\mathcal{H}}^{(1)} S_5 = \int d^5 x e_5 e_5^5 \left[2\lambda q_3^{(+)} \delta(z) \right] & \left\{ 3i(-\psi_{m1} \sigma^m \bar{\eta}_1 + \psi_{m2} \sigma^m \bar{\eta}_2) \right. \\ & \left. + 2\sqrt{6}i(\psi_{m2} \sigma^{mn} \eta_1 + \psi_{m1} \sigma^{mn} \eta_2) B_n + h.c. \right\}, \quad (10.9) \end{aligned}$$

where we used $q_3'(z) = 2q_3^{(+)} \delta(z)$.

Second, because we have ∂_5 hitting odd fields in our supersymmetry transformations, the transformations are *singular* and thus *not well-defined on the brane*. In our analysis of the Mirabelli and Peskin model [13], we showed that in order for the (on-shell) supersymmetry algebra to close onto the (singular) orbifold equations of motion we need to modify the supersymmetry transformations by adding appropriate $\delta(z)$ -terms. The modifications should be such that when the (natural) boundary conditions are taken into account the supersymmetry transformations *become non-singular* on the brane. This approach was already used in Ref. [7].

By inspection of our supersymmetry transformations, we see that ∂_5 hits η_2 in $\delta\psi_{52}$. Therefore, we modify the supersymmetry transformations by *adding* to $\delta\psi_{52}$ a new piece,

$$\delta_{\mathcal{H}}^{(2)} \psi_{52} = -4\eta_2^{(+)} \delta(z) = -4\alpha\eta_1 \delta(z), \quad (10.10)$$

which subtracts the singular piece in $\delta\psi_{52}$. This modification produces an additional brane-localized contribution to the supersymmetry variation of the bulk action,

$$\begin{aligned} \delta_{\mathcal{H}}^{(2)} S_5 = \int d^5 x e_5 e_5^5 [-4\delta(z)] & \left\{ -2\psi_{m1} \sigma^{mn} \widehat{D}_n \eta_2^{(+)} + iK_{na} \psi_{m2} \sigma^{mn} \sigma^a \bar{\eta}_2^{(+)} \right. \\ & \left. - \frac{3}{2} \lambda i (q_3 \psi_{m2} + q_{12} \psi_{m1}) \sigma^m \bar{\eta}_2^{(+)} + \frac{\sqrt{6}}{4} i \gamma^{mk} F_{k5} \psi_{m1} \eta_2^{(+)} + h.c. \right\}, \quad (10.11) \end{aligned}$$

where $\gamma^{mn} = e^{ma} e_a^n$ and $F_{m5} = e_5^5 F_{m5} + e_5^n F_{mn}$. (The F_{mn} terms not appearing in F_{m5} , as well as the B_m terms, are not shown here.)

All other ∂_5 in the supersymmetry transformations appear only via F_{m5} . However, as we will explain in detail later, when the natural boundary condition on B_m is $B_m^{(+)} = 0$ on Σ , no further modifications to the supersymmetry transformations are necessary.

We will return to the discussion of the B_m and F_{mn} terms in the next section. For now we simply set $B_m = 0$ and $F_{mn} = 0$. (But we will keep F_{m5} .)

10.3 Supersymmetry variation of the brane action

The supersymmetry variation of the brane action gives

$$\begin{aligned} \delta_{\mathcal{H}} S_4 = \int d^5 x e_4 \delta(z) & \left\{ -6\lambda_1 e_a^m \delta_{\mathcal{H}} e_m^a + 4 \left[(\alpha_{11} \psi_{m1} + \alpha_{12} \psi_{m2}^{(+)}) \sigma^{mn} \delta_{\mathcal{H}} \psi_{n1} \right. \right. \\ & \left. \left. + (\alpha_{22} \psi_{m2}^{(+)} + \alpha_{12} \psi_{m1}) \sigma^{mn} \delta_{\mathcal{H}} \psi_{n2}^{(+)} + h.c. \right] \right\}. \quad (10.12) \end{aligned}$$

The (induced on the brane) supersymmetry transformations are

$$\begin{aligned}
\delta_{\mathcal{H}} e_m^a &= -i(\psi_{m1} \sigma^a \bar{\eta}_1 + \varepsilon_1 \varepsilon_2 \psi_{m2}^{(+)} \sigma^a \bar{\eta}_2^{(+)}) + h.c. \\
\delta_{\mathcal{H}} \psi_{m1} &= 2\hat{D}_m \eta_1 + i\varepsilon_1 \varepsilon_3 K_{ma}^{(+)} \bar{\eta}_2^{(+)} + i\lambda \sigma_m (q_{12}^* \bar{\eta}_1 + \varepsilon_1 \varepsilon_4 q_3^{(+)} \bar{\eta}_2^{(+)}) \\
&\quad - \frac{4i}{2\sqrt{6}} (\sigma_m{}^n + \delta_m^n) \eta_1 F_{n\hat{5}} \\
\delta_{\mathcal{H}} \psi_{m2}^{(+)} &= 2\hat{D}_m \eta_2^{(+)} - iK_{ma}^{(+)} \bar{\eta}_1 + i\lambda \sigma_m (q_3^{(+)} \bar{\eta}_1 - q_{12} \bar{\eta}_2^{(+)}) \\
&\quad - \frac{4i}{2\sqrt{6}} (\sigma_m{}^n + \delta_m^n) \eta_2^{(+)} F_{n\hat{5}} , \quad (10.13)
\end{aligned}$$

where the $\varepsilon_i(z)$ factors have been explicitly shown. (Note that the variations of the even fields contain products of ε_i , whereas the variation of $\psi_{m2}^{(+)}$ contains no ε_i and simply corresponds to evaluating the bulk transformation of the odd field on the positive side of the brane.)

10.4 Supersymmetry variation of the total action

Writing all the $\varepsilon_i(z)$ factors explicitly in the expressions for the supersymmetry variation of the bulk action, we find

$$\delta_{\mathcal{H}}^{(1)} S_5 = \int d^5 x e_4 \delta(z) \left\{ 6i\lambda q_3^{(+)} \left(-\psi_{m1} \sigma^m \bar{\eta}_1 + \varepsilon_1 \varepsilon_2 \psi_{m2}^{(+)} \sigma^m \bar{\eta}_2^{(+)} \right) + h.c. \right\} \quad (10.14)$$

and

$$\begin{aligned}
\delta_{\mathcal{H}}^{(2)} S_5 &= \int d^5 x e_4 \delta(z) \left\{ 8\psi_{m1} \sigma^{mn} \hat{D}_n \eta_2^{(+)} - 4i\varepsilon_2 \varepsilon_3 K_{na}^{(+)} \psi_{m2}^{(+)} \sigma^{mn} \sigma^a \bar{\eta}_2^{(+)} \right. \\
&\quad + 6\lambda i \left(\varepsilon_2 \varepsilon_4 q_3^{(+)} \psi_{m2}^{(+)} + q_{12} \psi_{m1} \right) \sigma^m \bar{\eta}_2^{(+)} \\
&\quad \left. - \sqrt{6} i \gamma^{mk} F_{k\hat{5}} \psi_{m1} \eta_2^{(+)} + h.c. \right\} . \quad (10.15)
\end{aligned}$$

Adding all three contributions and setting $\eta_2^{(+)} = \alpha \eta_1$, we obtain the following expression for the supersymmetry variation of our bulk-plus-brane action,

$$\begin{aligned}
\delta_{\mathcal{H}} S &= \delta_{\mathcal{H}}^{(1)} S_5 + \delta_{\mathcal{H}}^{(2)} S_5 + \delta_{\mathcal{H}} S_4 = \int d^5 x e_4 \delta(z) \left\{ 8 \left[\tilde{C}_1 \psi_{m1} + \tilde{C}_2 \psi_{m2}^{(+)} \right] \sigma^{mn} \hat{D}_n \eta_1 \right. \\
&\quad + 4i K_{na}^{(+)} \left[\tilde{C}_3 \psi_{m1} + \tilde{C}_4 \psi_{m2}^{(+)} \right] \sigma^{mn} \sigma^a \bar{\eta}_1 - 6i \left[\tilde{C}_5 \psi_{m1} + \tilde{C}_6 \psi_{m2}^{(+)} \right] \sigma^m \bar{\eta}_1 \\
&\quad \left. - \sqrt{6} i \gamma^{mk} F_{k\hat{5}} \left[\tilde{C}_1 \psi_{m1} + \tilde{C}_2 \psi_{m2}^{(+)} \right] \eta_1 + h.c. \right\} , \quad (10.16)
\end{aligned}$$

where the coefficients are

$$\begin{aligned}
\tilde{C}_1 &= \alpha_{11} + \alpha(\alpha_{12} + 1) \\
\tilde{C}_2 &= \alpha_{12} + \alpha\alpha_{22} \\
\tilde{C}_3 &= \varepsilon_1 \varepsilon_3 \alpha_{11} \alpha^* - \alpha_{12} \\
\tilde{C}_4 &= \alpha^* (\varepsilon_1 \varepsilon_3 \alpha_{12} - \varepsilon_2 \varepsilon_3) - \alpha_{22} \\
\tilde{C}_5 &= -\lambda_1 + \lambda \tilde{A}_1 \\
\tilde{C}_6 &= -\varepsilon_1 \varepsilon_2 \lambda_1 \alpha^* + \lambda \tilde{A}_2 \quad (10.17)
\end{aligned}$$

with

$$\begin{aligned}\tilde{A}_1 &= \alpha_{11}(\varepsilon_1 \varepsilon_4 q_3^{(+)} \alpha^* + q_{12}^*) + (\alpha_{12} + 1)(q_3^{(+)} - q_{12} \alpha^*) \\ \tilde{A}_2 &= \alpha_{12}(\varepsilon_1 \varepsilon_4 q_3^{(+)} \alpha^* + q_{12}^*) + \alpha_{22}(q_3^{(+)} - q_{12} \alpha^*) - (\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_4) q_3^{(+)} \alpha^* .\end{aligned}\quad (10.18)$$

The total action is supersymmetric (subject only to the $\eta_2 = \alpha \eta_1$ and $B_m = 0$ boundary conditions) if all \tilde{C}_i vanish. Comparing \tilde{C}_i with C_i in Eq. (7.15), we see that this happens if and only if

$$\varepsilon_1 \varepsilon_3 = 1, \quad \varepsilon_2 \varepsilon_3 = -1, \quad \varepsilon_1 \varepsilon_2 = 1, \quad \varepsilon_1 \varepsilon_4 = 1, \quad \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_4 = 0 . \quad (10.19)$$

But this must be true *when multiplied by $\delta(z)$* ! Since we know that

$$\varepsilon^2 \delta(z) = \frac{1}{3} \delta(z), \quad \varepsilon^{-2} \delta(z) = -\delta(z) , \quad (10.20)$$

we see that our bulk-plus-brane action is supersymmetric provided we choose

$$\varepsilon_1 = \varepsilon(z), \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1}{\varepsilon(z)} . \quad (10.21)$$

10.5 Connection with earlier work

Since this assignment differs from

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon(z) , \quad (10.22)$$

assumed in Ref. [7], let us reproduce that calculation in which the boundary conditions

$$K_{ma}^{(+)} = \lambda e_{ma}, \quad \psi_{m2}^{(+)} = \alpha \psi_{m1} \quad \text{on } \Sigma \quad (10.23)$$

were used in checking supersymmetry of the bulk-plus-brane action.

If we use these boundary conditions in the supersymmetry variation of the action, we find

$$\begin{aligned}\delta_{\mathcal{H}} S &= \int d^5 x e_4 \delta(z) \left\{ (\tilde{C}_1 + \alpha \tilde{C}_2) \left[8 \psi_{m1} \sigma^{mn} \hat{D}_n \eta_1 - \sqrt{6} i \gamma^{mk} F_{k\hat{5}} \psi_{m1} \eta_1 \right] \right. \\ &\quad \left. + 6 i \tilde{M} \psi_{m1} \sigma^m \bar{\eta}_1 + h.c. \right\} ,\end{aligned}\quad (10.24)$$

where

$$\tilde{M} = -\lambda_1 (\tilde{C}_3 + \alpha \tilde{C}_4) - (\tilde{C}_5 + \lambda \tilde{C}_6) . \quad (10.25)$$

We already saw in Eq. (7.24) that using the ψ_{m2} boundary condition in the brane action reduces the coefficients α_{ij} as follows,

$$(\alpha_{11}, \alpha_{12}, \alpha_{22}) \longrightarrow (-\alpha, 0, 0) . \quad (10.26)$$

This makes

$$\begin{aligned}
\tilde{C}_1 &= \tilde{C}_2 = 0, & \tilde{C}_3 &= -\varepsilon_1 \varepsilon_3 \alpha \alpha^*, & \tilde{C}_4 &= -\varepsilon_2 \varepsilon_3 \alpha^* \\
\tilde{C}_5 &= -\lambda_1 - \lambda \alpha (\varepsilon_1 \varepsilon_4 q_3 \alpha^* + q_{12}^*) + \lambda (q_3 - q_{12} \alpha^*) \\
\tilde{C}_6 &= -\varepsilon_1 \varepsilon_2 \lambda_1 \alpha^* - \lambda (\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_4) q_3 \alpha^*
\end{aligned} \tag{10.27}$$

and, therefore,

$$\begin{aligned}
\widetilde{M} &= \lambda_1 \left[1 + (\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3) \alpha \alpha^* \right] \\
&\quad + \lambda \left\{ q_{12} \alpha^* + q_{12}^* \alpha + q_3 [(\varepsilon_1 \varepsilon_4 + \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_4) \alpha \alpha^* - 1] \right\}.
\end{aligned} \tag{10.28}$$

The action is supersymmetric provided $\widetilde{M} \delta(z) = 0$, which is equivalent to

$$(\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3) \delta(z) = \delta(z) \tag{10.29}$$

$$(\varepsilon_1 \varepsilon_4 + \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_4) \delta(z) = \delta(z), \tag{10.30}$$

when Eq. (10.4) is taken into account. We see that *both* choices of $\varepsilon_i(z)$, Eqs. (10.21) and (10.22), satisfy these conditions!

We, therefore, conclude that checking supersymmetry with the boundary conditions taken into account is *insufficient* to distinguish between the different $\varepsilon_i(z)$ assignments. Without the use of the boundary conditions, supersymmetry of the bulk-plus-brane action provides more consistency checks and requires the assignment in Eq. (10.21). Namely,

$$\eta_2 = \varepsilon(z) \eta_2^{(+)}, \quad \psi_{m2} = \frac{1}{\varepsilon(z)} \psi_{m2}^{(+)}, \quad K_{ma} = \frac{1}{\varepsilon(z)} K_{ma}^{(+)}, \quad q_3 = \frac{1}{\varepsilon(z)} q_3^{(+)}. \tag{10.31}$$

11. Fate of B_m terms on the orbifold

In this section we will write down the B_m and F_{mn} terms appearing in the supersymmetry variation of the bulk-plus-brane action explicitly. We will find that they generally do not cancel so that the use of the $B_m^{(+)} = 0$ boundary condition appears to be necessary for supersymmetry.

11.1 F_{mn} terms in the supersymmetry variation

We consider first the F_{mn} terms. Those that appear in the supersymmetry transformations and in the bulk action in the combination $F_{m\hat{5}} = e_5^{\hat{5}} F_{m5} + e_5^n F_{mn}$, go through the supersymmetry variation in this combination and cancel just as the F_{m5} terms we considered in the previous section. We will, therefore, omit them here. Among the remaining F_{mn} terms, we need only the following terms in the bulk Lagrangian,

$$\mathcal{L}_5 = i \frac{\sqrt{6}}{8} \varepsilon^{mnpq} F_{pq} (\psi_{m2} \sigma_n \bar{\psi}_{52}) e_5^{\hat{5}} + h.c., \tag{11.1}$$

and only the following terms in the supersymmetry transformations,

$$\begin{aligned}\delta_{\mathcal{H}}\psi_{m1} &= +\frac{1}{2\sqrt{6}}(i\epsilon_m{}^{nkl}\sigma_l + 4\delta_m^n\sigma^k)\bar{\eta}_2 F_{nk} \\ \delta_{\mathcal{H}}\psi_{m2} &= -\frac{1}{2\sqrt{6}}(i\epsilon_m{}^{nkl}\sigma_l + 4\delta_m^n\sigma^k)\bar{\eta}_1 F_{nk} .\end{aligned}\quad (11.2)$$

There are no “ $q_3 \cdot F_{MN}$ ” terms, thus $\delta_{\mathcal{H}}^{(1)} S_5 = 0$. The modification $\delta_{\mathcal{H}}^{(2)} \psi_{52} = -4\eta_2^{(+)} \delta(z)$ gives

$$\delta_{\mathcal{H}}^{(2)} S_5 = \int d^5 x e_4 \delta(z) \left[-i \frac{\sqrt{6}}{2} \varepsilon^{mnpq} F_{pq} (\psi_{m2} \sigma_n \bar{\eta}_2^{(+)} + h.c.) \right]. \quad (11.3)$$

Using a σ -matrix identity ($\gamma^{mn} \equiv e_a^m e^{na}$),

$$\sigma^{mp}(i\epsilon_p{}^{nkl}\sigma_l + 4\delta_p^n\sigma^k) = \frac{3}{2}i\epsilon^{mnkl}\sigma_l + (\gamma^{km}\sigma^n - \gamma^{kn}\sigma^m), \quad (11.4)$$

we find the following expression for the supersymmetry variation of the brane action,

$$\begin{aligned}\delta_{\mathcal{H}} S_4 &= \int d^5 x e_4 \delta(z) \frac{4}{2\sqrt{6}} \left\{ (\alpha_{11}\psi_{m1} + \alpha_{12}\psi_{m2}^{(+)}) \left(\frac{3}{2}i\epsilon^{mnkl}\sigma_l + \gamma^{km}\sigma^n \right) \bar{\eta}_2 F_{nk} \right. \\ &\quad \left. - (\alpha_{12}\psi_{m1} + \alpha_{22}\psi_{m2}^{(+)}) \left(\frac{3}{2}i\epsilon^{mnkl}\sigma_l + \gamma^{km}\sigma^n \right) \bar{\eta}_1 F_{nk}^{(+)} \right\} + h.c. \end{aligned} \quad (11.5)$$

Employing the $\varepsilon_i(z)$ assignments and using the $\eta_2^{(+)} = \alpha\eta_1$ boundary condition, we obtain the following expression for the supersymmetry variation of our bulk-plus-brane action (showing only the F_{mn} terms),

$$\begin{aligned}\delta_{\mathcal{H}} S &= \int d^5 x e_4 \delta(z) \frac{4}{2\sqrt{6}} F_{nk}^{(+)} \left\{ \frac{3}{2}i\epsilon^{mnkl} [Z_1\psi_{m1} + Z_2\psi_{m2}^{(+)}] \sigma_l \bar{\eta}_1 \right. \\ &\quad \left. + \gamma^{km} [Z_3\psi_{m1} + Z_4\psi_{m2}^{(+)}] \sigma^n \bar{\eta}_1 \right\} + h.c. , \end{aligned} \quad (11.6)$$

where

$$\begin{aligned}Z_1 &= \varepsilon_1 \varepsilon_5 \alpha_{11} \alpha^* - \alpha_{12}, & Z_2 &= \varepsilon_1 \varepsilon_5 \alpha_{12} \alpha^* - \alpha_{22} - \varepsilon_2 \varepsilon_5 \alpha^* \\ Z_3 &= \varepsilon_1 \varepsilon_5 \alpha_{11} \alpha^* - \alpha_{12}, & Z_4 &= \varepsilon_1 \varepsilon_5 \alpha_{12} \alpha^* - \alpha_{22} .\end{aligned} \quad (11.7)$$

We find that we can make $Z_1 = Z_2 = Z_3 = 0$ by choosing

$$\varepsilon_5 = \frac{1}{\varepsilon(z)} \quad \Longleftrightarrow \quad B_m = \frac{1}{\varepsilon(z)} B_m^{(+)} . \quad (11.8)$$

But we are still left with $Z_4 = -\alpha^*$. In order to cancel the remaining piece in the supersymmetry variation,

$$\delta_{\mathcal{H}} S = \int d^5 x e_4 \delta(z) \left\{ -\frac{4}{2\sqrt{6}} \alpha^* \gamma^{km} F_{nk}^{(+)} (\psi_{m2}^{(+)} \sigma^n \bar{\eta}_1) \right\} + h.c. , \quad (11.9)$$

we have to use the boundary condition $B_m^{(+)} = 0$ on Σ . We note, however, that we need it here only in its (seemingly) gauge invariant form: $F_{nk}^{(+)} = 0$ on Σ .

11.2 B_m terms in the supersymmetry variation

Repeating the above steps, we find that the B_m terms give the following contributions,

$$\delta_{\mathcal{H}}^{(1)} S_5 = \int d^5 x e_4 \delta(z) \left\{ 4\sqrt{6}i\lambda q_3^{(+)} (\psi_{m2} \sigma^{mn} \eta_1 + \psi_{m1} \sigma^{mn} \eta_2) B_n + h.c. \right\} \quad (11.10)$$

$$\delta_{\mathcal{H}}^{(2)} S_5 = \int d^5 x e_4 \delta(z) \left\{ 4\sqrt{6}i\lambda B_n [q_3 \psi_{m1} - q_{12}^*] \psi_{m1} \sigma^{mn} \eta_2^{(+)} + h.c. \right\} \quad (11.11)$$

$$\begin{aligned} \delta_{\mathcal{H}} S_4 = \int d^5 x e_4 \delta(z) 4\sqrt{6}i\lambda \left\{ (\alpha_{11} \psi_{m1} + \alpha_{12} \psi_{m2}^{(+)}) \sigma^{mn} (q_{12}^* \eta_2 - q_3 \eta_1) B_n \right. \\ \left. - (\alpha_{12} \psi_{m1} + \alpha_{22} \psi_{m2}^{(+)}) \sigma^{mn} (q_3^{(+)} \eta_2^{(+)} + q_{12} \eta_1) B_n^{(+)} \right\} + h.c. \end{aligned} \quad (11.12)$$

The total contribution to the supersymmetry variation of our bulk-plus-brane action is the sum of these three,

$$\delta_{\mathcal{H}} S = \int d^5 x e_4 \delta(z) 4\sqrt{6}i\lambda B_n^{(+)} \left\{ W_1 \psi_{m1} \sigma^{mn} \eta_1 + W_2 \psi_{m2}^{(+)} \sigma^{mn} \eta_1 + h.c. \right\}, \quad (11.13)$$

where

$$\begin{aligned} W_1 &= \alpha \alpha_{11} q_{12}^* + \alpha_{12} q_{12} + q_3^{(+)} [\varepsilon_1 \varepsilon_5 \alpha + \varepsilon_4 \varepsilon_5 \alpha - \varepsilon_4 \varepsilon_5 \alpha_{11} + \alpha \alpha_{12}] \\ W_2 &= \alpha \alpha_{12} q_{12}^* + \alpha_{22} q_{12} + q_3^{(+)} [\varepsilon_2 \varepsilon_5 - \varepsilon_4 \varepsilon_5 \alpha_{12} + \alpha \alpha_{22}]. \end{aligned} \quad (11.14)$$

With our $\varepsilon_i(z)$ assignments,

$$\varepsilon_1 = \varepsilon(z), \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \frac{1}{\varepsilon(z)}, \quad (11.15)$$

the coefficients simplify to

$$\begin{aligned} W_1 &= \alpha \alpha_{11} q_{12}^* + \alpha_{12} q_{12} - \alpha q_3^{(+)} \\ W_2 &= \alpha \alpha_{12} q_{12}^* + \alpha_{22} q_{12} - q_3^{(+)} \end{aligned} \quad (11.16)$$

and can also be rewritten as

$$\begin{aligned} W_1 &= \frac{-\alpha}{1 + \alpha \alpha^*} \left[(\alpha q_{12}^* + q_3) + \alpha^* (q_{12} + \alpha q_3) \right] \\ W_2 &= \frac{1}{1 + \alpha \alpha^*} \left[(\alpha^* q_{12} - q_3) - \alpha \alpha^* (\alpha q_{12}^* + q_3) \right]. \end{aligned} \quad (11.17)$$

These coefficients do not vanish unless $\alpha = 0$ and $q_3^{(+)} = 0$ (or $\lambda = 0$ for any α). In a general case, we need to use the $B_m^{(+)} = 0$ boundary condition to cancel this part of the supersymmetry variation.

This completes our check of supersymmetry of the bulk-plus-brane action, Eq. (10.1). (We remind that we work only to quadratic order in fermions.) We found that besides the boundary condition on the supersymmetry parameter, $\eta_2^{(+)} = \alpha \eta_1$, we need to use only one other boundary condition: $B_m^{(+)} = 0$.

In order to understand if the use of this boundary condition is forced on us by the supersymmetry algebra, we have to understand when the $U(1)$ gauge invariance is broken. We discuss this in the next section.

12. Gauge transformations on the orbifold

In this section we discuss the breaking of the general coordinate and the $U(1)$ gauge invariances on the orbifold. We come to the conclusion that in order to reproduce the results of the boundary picture discussion, we need to modify all the gauge transformations on the orbifold by making them *non-singular*! We show how the supersymmetry transformations are modified for a general B_m boundary condition, and find that the algebra of the modified transformations closes as in the boundary picture.

12.1 Breaking of the general coordinate invariance

The variation of the bulk Lagrangian under the general coordinate transformation is a total derivative,

$$\delta_v \mathcal{L}_5 = D_M(v^M \mathcal{L}_5) . \quad (12.1)$$

In the boundary picture, this produces a boundary term, Eq. (3.1), the vanishing of which requires $v^5 = 0$ on the boundary, and in turn leads to the restriction $\eta_2 = \alpha\eta_1$ on the supersymmetry parameters. But on the orbifold the total derivative integrates to zero! And it appears that the restriction $(v^5)^{(+)} = 0$ on the brane does not arise.

If we take the point of view that the orbifold picture should reproduce all the major results of the boundary picture (such as the breaking of a gauge invariance), we are forced to make some modifications.

The necessary modification comes naturally from the requirement that the gauge transformations be the same in the both pictures both in the bulk *and* on the brane/boundary. But if we take the transformations of the boundary picture and assume them to be *literally* the same on the orbifold, we find that they are in general *singular* (and thus not well-defined) on the brane! Indeed,

$$\begin{aligned} \delta_v B_m &= v^n \partial_n B_m + v^5 \partial_5 B_m + B_n \partial_m v^n + B_5 \partial_m v^5 = 2v^5 B_m^{(+)} \delta(z) + \dots \\ \delta_v B_5 &= v^n \partial_n B_5 + v^5 \partial_5 B_5 + B_n \partial_5 v^n + B_5 \partial_5 v^5 = 2(v^5)^{(+)} B_5 \delta(z) + \dots , \end{aligned} \quad (12.2)$$

where the dots represent non-singular terms. (We used here the fact that on the orbifold v^5 is odd, whereas v^m is even.) In order for the transformations in the both pictures to agree on the boundary, we have to *modify* the transformations for the orbifold picture by subtracting the singular pieces! For the B_M field, the modified general coordinate transformations are as follows,

$$\begin{aligned} \delta'_v B_m &= \delta_v B_m - 2v^5 B_m^{(+)} \delta(z) \\ \delta'_v B_5 &= \delta_v B_5 - 2(v^5)^{(+)} B_5 \delta(z) . \end{aligned} \quad (12.3)$$

The transformations for other fields are appropriately modified. It is clear, that the variation of the bulk Lagrangian under the modified general coordinate transformation produces now additional brane-localized terms which vanish only when $(v^5)^{(+)} = 0$ on the brane. Therefore, all the related conclusions of the boundary picture are now reproduced.

12.2 Breaking of the $U(1)$ gauge invariance

In the boundary picture, the variation of the bulk Lagrangian under the $U(1)$ gauge transformation is a total derivative (arising from the Chern-Simons term),

$$\delta_u \mathcal{L}_5 = D_K \left[-u \frac{1}{6\sqrt{6}} \epsilon^{MNPQK} F_{MN} F_{PQ} \right]. \quad (12.4)$$

It produces a boundary term, Eq. (5.5), which tells us that the bulk action is gauge invariant only if some boundary condition is imposed. Namely, $u = 0$ or $F_{mn} = 0$ on the boundary.

On the orbifold, the total derivative is also generated, but it integrates to zero. It turns out, however, that under the original $U(1)$ gauge transformation, Eq. (2.9),

$$\delta_u B_M = \partial_M u, \quad \delta_u \Psi_{Mi} = u \frac{\sqrt{6}}{2} \lambda Q_i^j \Psi_{Mj}, \quad (12.5)$$

we now do get brane-localized terms in the variation of the bulk Lagrangian.

The variation receives a new contribution on the orbifold because the modified covariant derivative in Eq. (5.2) is *not* covariant under the $U(1)$ gauge transformation when the parameters are not constant,

$$\delta_u (\tilde{D}_M \Psi_{Ni}) = u \frac{\sqrt{6}}{2} \lambda Q_i^j (\tilde{D}_M \Psi_{Nj}) + u \frac{\sqrt{6}}{2} \partial_M (\lambda Q_i^j) \Psi_{Nj}. \quad (12.6)$$

The variation of the Lagrangian relevant in the orbifold picture is, therefore,

$$\delta_u \mathcal{L}_5 = \frac{i}{2} \tilde{\Psi}_M^i \Gamma^{MNK} \left[u \frac{\sqrt{6}}{2} \partial_N (\lambda Q_i^j) \Psi_{Kj} \right]. \quad (12.7)$$

Since only q_3 is not a constant, we obtain

$$\delta_u S_5 = \int d^5 x e_4 \delta(z) \frac{\sqrt{6}}{2} \lambda u \left[-4i q_3^{(+)} (\psi_{m1} \sigma^{mn} \psi_{n2}) + h.c. \right]. \quad (12.8)$$

Therefore, the bulk action is not gauge invariant if $\lambda q_3^{(+)} \neq 0$.

Our brane Lagrangian is $\mathcal{L}_4 = 2\mathcal{L}_B^{(\alpha)(+)}$, where $\mathcal{L}_B^{(\alpha)}$ is the boundary Lagrangian of the boundary picture *without* the Y -term. Its variation under the $U(1)$ gauge transformation is given in Eq. (7.33). Therefore, the brane Lagrangian is by itself gauge invariant only when $\alpha = 0$ (so that the Lagrangian vanishes) or when $\lambda = 0$ (so that the gauge transformation does not act on the fermions).

It is easy to check that the bulk-plus-brane action is gauge invariant only when the bulk and the brane actions are *separately* gauge invariant. (That is the sum of the two contributions still vanishes only when $\alpha = 0$ and $q_3^{(+)} = 0$. Or when $\lambda = 0$.)

On the other hand, the boundary condition $\eta_2^{(+)} = \alpha \eta_1$ is gauge invariant when

$$\lambda \left[(q_{12} + \alpha q_3^{(+)}) - \alpha (\alpha q_{12}^* - q_3^{(+)}) \right] = 0. \quad (12.9)$$

It follows that the $U(1)$ gauge invariance in the orbifold picture is broken by either the bulk-plus-brane action or the fermionic boundary condition unless $\lambda = 0$!

Since this is drastically different from the way the $U(1)$ gauge invariance is broken in the boundary picture, we have to make some modifications if we would like the two pictures to describe the same physics.

12.3 Modified $U(1)$ gauge transformation

We found that taking the $U(1)$ gauge transformation in the orbifold picture to be *literally* the same as in the boundary picture leads to very different conclusions about the breaking of the gauge invariance in the both pictures. Therefore, as in the case of the general coordinate invariance, we are led to modify the $U(1)$ gauge transformation in the orbifold picture. The modification affects only B_5 ,

$$\delta'_u B_5 = \partial_5 u - 2u^{(+)} \delta(z) . \quad (12.10)$$

(The parameter u is odd.) The modified transformation is *non-singular* on the brane and coincides with the $U(1)$ transformation induced on the boundary in the boundary picture,

$$\delta_u B_5 = \partial_5 u \quad \text{on } \partial\mathcal{M} \quad \Longleftrightarrow \quad \delta'_u B_5 = \partial_5 u^{(+)} \quad \text{on } \Sigma . \quad (12.11)$$

The variation of the bulk action under the modified $U(1)$ gauge transformation produces the following brane-localized term (from the variation of the Chern-Simons term in the bulk Lagrangian),

$$\delta'_u S_5 \ni \int d^5 x e_4 \delta(z) \left\{ u^{(+)} \frac{2}{6\sqrt{6}} \epsilon^{mnpq} F_{mn} F_{pq} \right\} , \quad (12.12)$$

which is the orbifold version of Eq. (5.5).

For the covariant derivative defined in Eq. (5.2) we now obtain, instead of Eq. (12.6),

$$\begin{aligned} \delta_u (\tilde{D}_5 \Psi_{Ni}) &= u \frac{\sqrt{6}}{2} \lambda Q_i^j (\tilde{D}_5 \Psi_{Nj}) \\ &+ u \frac{\sqrt{6}}{2} \partial_5 (\lambda Q_i^j) \Psi_{Nj} + \sqrt{6} u^{(+)} \delta(z) (\lambda Q_i^j) \Psi_{Nj} . \end{aligned} \quad (12.13)$$

The variation of the the fermionic part of the bulk action now gets two contributions. (Let us use the shorthand notation $\psi_i \psi_j \equiv \psi_{mi} \sigma^{mn} \psi_{nj}$ for the following.) The first contribution, arising from the jumping parameter $q_3 = \varepsilon_4 q_3^{(+)}$, is

$$\delta_u^{(1)} S_5 = \int d^5 x e_4 \delta(z) \frac{\sqrt{6}}{2} \lambda u \left[-4i q_3^{(+)} \psi_1 \psi_2 + h.c. \right] , \quad (12.14)$$

and the second one, arising from the extra piece $\delta'_u B_5 = -2u^{(+)} \delta(z)$ in the modified gauge transformation, is

$$\delta_u^{(2)} S_5 = \int d^5 x e_4 \delta(z) \frac{\sqrt{6}}{2} \lambda u^{(+)} \left[-4i q_3 \psi_1 \psi_2 - 2i q_{12} (\psi_1 \psi_1 + \bar{\psi}_2 \bar{\psi}_2) + h.c. \right] . \quad (12.15)$$

The variation of the brane action is *twice* that in Eq. (7.33), with ψ_{m2} and q_3 evaluated on the \mathcal{M}_+ side of Σ . Namely,

$$\delta'_u S_4 = \int d^5 x e_4 \delta(z) \frac{\sqrt{6} i \lambda u^{(+)}}{1 + \alpha \alpha^*} \left[X_1 \psi_1 \psi_1 + X_2 \psi_1 \psi_2^{(+)} + X_3 \psi_2^{(+)} \psi_2^{(+)} \right] + h.c. , \quad (12.16)$$

where

$$X_1 = X_3^* = 2\alpha(\alpha^* q_{12} - q_3^{(+)}) , \quad X_2 = 2(\alpha q_{12}^* - \alpha^* q_{12}) . \quad (12.17)$$

The total variation of the fermionic part of the bulk-plus-brane action under the (modified) $U(1)$ gauge transformation is the sum of the three contributions,

$$\delta'_u S \ni \int d^5 x e_4 \delta(z) \frac{\sqrt{6} i \lambda u^{(+)}}{1 + \alpha \alpha^*} \left[\tilde{X}_1 \psi_1 \psi_1 + \tilde{X}_2 \psi_1 \psi_2^{(+)} + \tilde{X}_3 \psi_2^{(+)} \psi_2^{(+)} \right] + h.c. , \quad (12.18)$$

where

$$\begin{aligned} \tilde{X}_1 &= X_1 - q_{12}(1 + \alpha \alpha^*) \\ \tilde{X}_3 &= X_3 + q_{12}^*(1 + \alpha \alpha^*) \varepsilon_2 \varepsilon_2 \\ \tilde{X}_2 &= X_2 - 2q_3^{(+)}(1 + \alpha \alpha^*) [\varepsilon_u \varepsilon_2 + \varepsilon_2 \varepsilon_4] \end{aligned} \quad (12.19)$$

after all the $\varepsilon_i(z)$ factors are separated. The ε_u is such a factor for the odd parameter u ,

$$u(x, z) = \varepsilon_u(z) u^{(+)}(x, |z|) . \quad (12.20)$$

With our $\varepsilon_i(z)$ assignments, Eq. (10.21), we have $\varepsilon_2 \varepsilon_2 \delta(z) = -\delta(z)$. Therefore,

$$\tilde{X}_1 = \tilde{X}_3^* = \alpha(\alpha^* - q_3^{(+)}) - (\alpha q_3^{(+)} + q_{12}) , \quad (12.21)$$

which is exactly the coefficient W in Eq. (7.30)! More than that, if we choose

$$\varepsilon_u = \varepsilon(z) , \quad (12.22)$$

we get $[\varepsilon_u \varepsilon_2 + \varepsilon_2 \varepsilon_4] \delta(z) = 0$, so that the equation (7.30) is reproduced completely! This means that the fermionic part of our bulk-plus-brane action is now gauge invariant with the same restriction on the parameters \vec{q} and α , Eq. (7.28), as is necessary for the gauge invariance of the boundary condition $\eta_2^{(+)} = \alpha \eta_1$!

After this choice of \vec{q} and α is made, the variation of the bulk-plus-brane action under the $U(1)$ gauge transformation has only one uncanceled piece, Eq. (12.12),

$$\delta'_u S = \int d^5 x e_4 \delta(z) \left\{ u^{(+)} \frac{2}{6\sqrt{6}} \epsilon^{mnpq} F_{mn} F_{pq} \right\} , \quad (12.23)$$

which is the orbifold picture analog of Eq. (5.5). Therefore, our modification of the gauge transformations in the orbifold picture, Eq. (12.10), leads to agreement with conclusions of the boundary picture.

12.4 Modified supersymmetry transformations

The modification of the supersymmetry transformation for ψ_{52} , Eq. (10.10), now becomes just a part of the general modification of *all* the gauge transformations in the orbifold picture. Indeed,

$$\delta'_{\mathcal{H}}\psi_{52} = \delta_{\mathcal{H}}\psi_{52} - 4\eta_2^{(+)}\delta(z) \quad (12.24)$$

is an analog of Eq. (12.10) which makes the supersymmetry transformations *non-singular* on the brane, so that the induced on the brane transformations are exactly the same as those in the boundary picture.

The supersymmetry transformations should also be modified when the boundary condition on B_m is no longer $B_m^{(+)} = 0$, but instead $B_m^{(+)} = J_m$. This happens, for example, when one couples the B_m field to some brane-localized matter. The J_m is then a composite of the brane matter fields. (The coupling of brane-localized matter to the bulk supergravity in five dimensions is discussed in Refs. [29, 30, 31, 32, 33, 34].)

The necessary modifications in the supersymmetry transformations can be obtained simply by the following substitution,

$$F_{m5} \longrightarrow F_{m5} + 2J_m\delta(z) , \quad (12.25)$$

which makes the (modified) transformations non-singular *when the boundary condition $B_m^{(+)} = J_m$ is taken into account*. (From our analysis [13] of the Mirabelli and Peskin model, we know that we need this substitution, and *not* $F_{m5} \rightarrow F_{m5} + 2B_m^{(+)}\delta(z)$. The reason is that the supersymmetry variations of $B_m^{(+)}$ and J_m are *different*, which plays a role when the supersymmetry algebra is calculated.)

Explicitly, the modified supersymmetry transformations are

$$\begin{aligned} \delta'_{\mathcal{H}}\psi_{m1} &= \delta_{\mathcal{H}}\psi_{m1} - \frac{8i}{2\sqrt{6}}(\sigma_m^n + \delta_m^n)\eta_1 J_n e_5^5 \delta(z) \\ \delta'_{\mathcal{H}}\psi_{m2} &= \delta_{\mathcal{H}}\psi_{m2} - \frac{8i}{2\sqrt{6}}(\sigma_m^n + \delta_m^n)\eta_2 J_n e_5^5 \delta(z) \\ \delta'_{\mathcal{H}}\psi_{51} &= \delta_{\mathcal{H}}\psi_{51} - \frac{8}{2\sqrt{6}}\sigma^n \bar{\eta}_2 J_n \delta(z) + \frac{8i}{2\sqrt{6}}e_5^m (\sigma_m^n + \delta_m^n)\eta_1 J_n \delta(z) \\ \delta'_{\mathcal{H}}\psi_{52} &= \delta_{\mathcal{H}}\psi_{52} + \frac{8}{2\sqrt{6}}\sigma^n \bar{\eta}_1 J_n \delta(z) + \frac{8i}{2\sqrt{6}}e_5^m (\sigma_m^n + \delta_m^n)\eta_2 J_n \delta(z) - 4\eta_2^{(+)}\delta(z) . \end{aligned} \quad (12.26)$$

It remains to see whether these are the correct modifications for a particular model. What we can check at the moment is the closure of the supersymmetry algebra on the bosonic fields.

Let us consider the commutator of the two (modified) supersymmetry transformations on B_5 . We find,

$$\begin{aligned}
[\delta'_\Xi, \delta'_\mathcal{H}]B_5 &= i\frac{\sqrt{6}}{2}(\eta_1\delta'_\Xi\psi_{52} - \eta_2\delta'_\Xi\psi_{51}) \\
&= [\delta_\Xi, \delta_\mathcal{H}]B_5 + \left\{ 2i\sqrt{6}(\eta_2^{(+)}\xi_1 - \eta_1\xi_2^{(+)})\delta(z) \right. \\
&\quad \left. + 4i(\eta_1\sigma^n\bar{\xi}_1 + \eta_2\sigma^n\bar{\xi}_2)J_n\delta(z) + 4(\eta_2\xi_1 - \eta_1\xi_2)e_5^n J_n\delta(z) + h.c. \right\} .
\end{aligned} \tag{12.27}$$

From the (original) supersymmetry algebra, Eq. (2.6), we know that

$$\begin{aligned}
[\delta_\Xi, \delta_\mathcal{H}]B_5 &= \delta_v B_5 + \delta_u B_5 \\
&= v^n \partial_n B_5 + v^5 \partial_5 B_5 + B_n \partial_5 v^n + B_5 \partial_5 v^5 + \partial_5 u ,
\end{aligned} \tag{12.28}$$

where (see Eq. (2.10))

$$\begin{aligned}
u &= -v^m B_m - v^5 B_5 + u_0 \\
v^m &= 2i(\eta_1\sigma^m\bar{\xi}_1 + \eta_2\sigma^m\bar{\xi}_2) + 2e_5^m(\eta_2\xi_1 - \eta_1\xi_2) + h.c. \\
u_0 &= -i\sqrt{6}(\eta_2\xi_1 - \eta_1\xi_2) + h.c.
\end{aligned} \tag{12.29}$$

We can, therefore, write

$$[\delta_\Xi, \delta_\mathcal{H}]B_5 = v^n F_{n5} + \partial_5 u_0 , \tag{12.30}$$

whereas for the commutator of the modified supersymmetry transformations we find

$$[\delta'_\Xi, \delta'_\mathcal{H}]B_5 = v^n [F_{n5} + 2J_n\delta(z)] + [\partial_5 u_0 - 2u_0^{(+)}\delta(z)] . \tag{12.31}$$

This is one explicit check of the fact that the commutator of the modified supersymmetry transformations closes onto the modified gauge transformations. (It is, actually, obvious. The commutator of two *non-singular* transformations must be *non-singular*!)

But in order that the algebra of the modified gauge transformations close *without* the use of the boundary conditions, we should correct our modified general coordinate transformations, Eq. (12.3), by replacing there $B_m^{(+)}$ with J_m (and similarly for other odd fields). Then,

$$\delta'_v B_m = \delta_v B_m - 2v^5 J_m \delta(z) . \tag{12.32}$$

The modified transformation is, therefore, non-singular *only when the boundary condition $B_m^{(+)} = J_m$ is taken into account*. If the natural boundary condition for B_m is $B_m^{(+)} = 0$, then no modification is necessary.

13. Brane-localized matter

In this section we argue that the preservation of the bulk $U(1)$ gauge invariance is *necessary* if the brane-localized matter is to provide a non-zero boundary condition for the bulk B_m field. We show that a seemingly gauge non-invariant boundary condition $B_m^{(+)} = J_m$ can in fact be gauge invariant, if the brane fields transform appropriately under the bulk $U(1)$ transformation. We also find that a similar boundary condition exists in the Horava and Witten model.

13.1 Preserving the bulk $U(1)$ gauge invariance

Let us discuss the addition of the brane-localized matter a little bit further. As in Section 12.4, all we will use is that the boundary condition for B_m is modified to $B_m^{(+)} = J_m$, where J_m is a composite of the brane-localized fields.

We found that we need to make a modification of our gauge transformations by making them non-singular. The modified transformations coincide with those of the boundary picture *exactly*, both in the bulk and on the brane/boundary. Therefore, all the conclusions of the boundary picture discussion hold in the orbifold picture as well. In particular, if the $U(1)$ gauge invariance is broken, then the closure of the supersymmetry algebra requires (because of Eq. 4.6) the $B_m^{(+)} = 0$ boundary condition. But now this boundary condition is *inconsistent* with the natural boundary condition $B_m^{(+)} = J_m$. The only way out is to *preserve* the $U(1)$ gauge invariance.

We can try the following approach. In parallel with the modification of the supersymmetry transformations, we make the same substitution (12.25),

$$F_{m5} \longrightarrow F'_{m5} = F_{m5} + 2J_m \delta(z) , \quad (13.1)$$

in the bulk action. The modified action then has $B_m^{(+)} = J_m$ as its natural boundary condition. Indeed, the B_M equation of motion is now

$$D_M F^{MK} - \frac{3}{6\sqrt{6}} \epsilon^{MNPQK} F_{MN} F_{PQ} = 0 \quad (13.2)$$

(omitting the 2-Fermi terms), where F_{m5} should be replaced by F'_{m5} . The cancellation of the singular terms requires F'_{m5} to be non-singular, which determines the jump of B_m across the brane. The parity assignment (B_m is odd) then implies the boundary condition $B_m^{(+)} = J_m$.

Under the modified $U(1)$ gauge transformation, Eq. (12.10),

$$\delta'_u B_5 = \partial_5 u - 2u^{(+)} \delta(z) , \quad (13.3)$$

the original F_{m5} is not invariant, whereas the modified F'_{m5} can be made invariant, if we choose a special transformation for J_m ,

$$\delta'_u J_m = \partial_m u^{(+)} \quad \Rightarrow \quad \delta'_u F'_{m5} = 0 . \quad (13.4)$$

Since the gauge transformation for B_m is unmodified, $\delta'_u B_m = \partial_m u$, its restriction on the brane is

$$\delta'_u B_m^{(+)} = \partial_m u^{(+)} \quad \text{on } \Sigma , \quad (13.5)$$

which means that the boundary condition $B_m^{(+)} = J_m$ is now *gauge invariant*!

But the modified bulk-plus-brane action is *not* yet gauge invariant. Its variation under the modified $U(1)$ gauge transformation is still given by Eq. (12.23). It appears that there is no way to cancel it, at least *without* the use of the boundary condition for B_m . (Note that adding any term with B_m in the brane action would now break the construction, giving a different boundary condition for B_m .) But if we use the $B_m^{(+)} = J_m$ boundary condition, we can write Eq. (12.23) as follows,

$$\delta'_u S = \int d^5 x e_4 \delta(z) \left\{ u^{(+)} \frac{2}{6\sqrt{6}} \epsilon^{mnpq} J_{mn} J_{pq} \right\} , \quad (13.6)$$

where $J_{mn} = \partial_m J_n - \partial_n J_m$. This, in principle, can be canceled by the variation of the brane action (excluding terms which combine into $F'_{m5} = F_{m5} + 2J_m \delta(z)$).

For example, suppose that the matter fields on the brane include a scalar ϕ and two vectors, A_m and C_m . Take their transformations under the bulk $U(1)$ to be as follows,

$$\delta'_u \phi = u^{(+)} , \quad \delta'_u A_m = \partial_m u^{(+)} , \quad \delta'_u C_m = 0 . \quad (13.7)$$

Let the brane action (before coupling) contain ϕ and A_m only via $A_{mn} \equiv \partial_m A_n - \partial_n A_m$ and $\mathcal{D}_m \phi = \partial_m \phi - A_m$, so that it is gauge invariant. Now couple the brane fields to the bulk supergravity in the way described above, taking

$$J_m = \partial_m \phi + C_m , \quad (13.8)$$

which has the correct gauge transformation, $\delta'_u J_m = \partial_m u^{(+)}$. The resulting action is not yet gauge invariant, because Eq. (13.6) gives

$$\delta'_u S = \int d^5 x e_4 \delta(z) \left\{ u^{(+)} \frac{2}{6\sqrt{6}} \epsilon^{mnpq} C_{mn} C_{pq} \right\} , \quad (13.9)$$

where $C_{mn} \equiv \partial_m C_n - \partial_n C_m$. But it is now easy to make it gauge invariant by adding a term of the form $\phi \epsilon^{mnpq} C_{mn} C_{pq}$ to the brane action.

We conclude, therefore, that adding brane-localized matter can help restore the invariance of the bulk-plus-brane action (and of the associated with it natural boundary conditions) under the bulk $U(1)$ gauge transformation. The preservation of this invariance is *necessary* for supersymmetry as we argued based on the closure of the supersymmetry algebra.

We also would like to emphasize that it appears to be impossible to maintain the gauge invariance *without* the use of the B_m boundary condition.

13.2 Modified Bianchi identity

The modified field strength, F'_{MN} , which we introduced satisfies the following modified Bianchi identity,

$$(dF')_{5nk} = \frac{1}{3}(\partial_5 F'_{nk} + \partial_n F'_{k5} - \partial_k F'_{n5}) = \frac{2}{3}J_{nk}\delta(z), \quad (13.10)$$

where $(dF)_{MNK} \equiv \partial_{[M}F_{NK]}$ and $J_{mn} \equiv \partial_m J_n - \partial_n J_m$. This is not a surprise, since our construction in the previous subsection is analogous to the construction used by Horava and Witten [1]. But while they came to the modification of the bulk field strength in order to preserve the *brane-localized gauge invariance*, we need it to preserve the *bulk gauge invariance*. And our modification is *forced* on us by the supersymmetry algebra!

Note that Horava and Witten write their boundary condition, Eq. (2.20) in Ref. [1], in a form which appears to be manifestly gauge invariant (under both the brane *and* the bulk gauge invariances). The form is analogous to our $F_{mn}^{(+)} = J_{mn}$. However, this relation is *derivative* from the basic boundary condition $B_m^{(+)} = J_m$. There is a similar not manifestly gauge invariant boundary condition in the Horava and Witten model.

The boundary condition $B_m^{(+)} = J_m$ follows from our (modified) action as a natural boundary condition, since now F'_{m5} must be non-singular to avoid uncanceled singularities in the B_M equation of motion. It is exactly what Horava and Witten say after Eq. (2.18) in Ref. [1]. Their Eq. (2.13) then implies the boundary condition $C_{ABC}^{(+)} \sim \omega_{ABC}$, from which their Eq. (2.20) follows. This boundary condition can be made invariant under the brane gauge invariance, provided that C_{ABC} transforms under the brane gauge transformation like ω_{ABC} in their Eq. (2.14). However, it is unclear how to achieve the *bulk gauge invariance* ($\delta C_{IJK} = \partial_{[I}\Lambda_{JK]}$ with *arbitrary* Λ_{IJ}) of this boundary condition.

The boundary condition $C_{ABC}^{(+)} \sim \omega_{ABC}$ was also found in Refs. [35, 36] (although there it was not derived as a natural boundary condition following from the action, but simply imposed for consistency). The transformation of C_{ABC} necessary to preserve the brane gauge invariance is provided there, but the preservation of the bulk gauge invariance is not discussed.

14. Summary and Conclusions

In this paper we constructed a bulk-plus-boundary action with the five-dimensional gauged (on-shell) supergravity in the bulk which is supersymmetric upon the use of the minimum set of boundary conditions dictated by the supersymmetry algebra. In a general case when the supersymmetry parameter \mathcal{H}_i is restricted by $\eta_2 = \alpha\eta_1$ on the boundary, we found that only the boundary condition $B_m = 0$ have to be used to prove supersymmetry to second order in fermions. Other boundary conditions following from the action, $K_{ma} = \lambda_1 e_{ma}$ and $\psi_{m2} = \alpha\psi_{m1}$ on $\partial\mathcal{M}$, are not needed in the proof of supersymmetry of the action.

The necessary ingredient of our boundary action is the Gibbons-Hawking-like Y -term presented in Eq. (9.2),

$$Y = K + (\psi_{m1}\sigma^{mn}\psi_{n2} + h.c.) + \gamma^{mn}F_{m\hat{5}}B_n \quad (14.1)$$

(where we used $e_5^{\hat{5}}F^{m5} = \gamma^{mn}F_{n\hat{5}}$ as follows from Eq. (D.10)). It includes

- 1) the standard Gibbons-Hawking term (the trace of the extrinsic curvature) which allows the derivation of the boundary condition $K_{ma} = \lambda_1 e_{ma}$ as a natural boundary condition corresponding to the variation δe_{ma} [12];
- 2) a fermionic term, which leads to the derivation of $\psi_{m2} = \alpha\psi_{m1}$ as a natural boundary condition for $\delta\psi_{m1}$;
- 3) another bosonic term, which lets us derive the $B_m = 0$ boundary condition as a natural boundary condition.

We argued that the Y -term can be derived most easily from the fact that it must match onto the brane-localized singularities of the bulk Lagrangian. (This can be used to derive appropriate additions to the Y -term when higher order Fermi terms are considered.)

In the transition to the orbifold picture, the Y -term disappears. The rest of the boundary action becomes (after the multiplication by 2) the brane action with which supersymmetry of the bulk-plus-brane action in the orbifold picture can once again be proven using only the minimum set of the boundary conditions. We found, however, that one also has to choose unconventional $\varepsilon(z)$ assignments for the odd fields and parameters, Eqs. (10.21),

$$\eta_2 = \varepsilon(z)\eta_2^{(+)}, \quad \psi_{m2} = \frac{1}{\varepsilon(z)}\psi_{m2}^{(+)}, \quad K_{ma} = \frac{1}{\varepsilon(z)}K_{ma}^{(+)}, \quad q_3 = \frac{1}{\varepsilon(z)}q_3^{(+)}. \quad (14.2)$$

and use the property $\varepsilon(z)^{-2}\delta(z) = -\delta(z)$.

The reason for such $\varepsilon(z)$ assignments is unclear. (Perhaps, the explanation can come from a smooth realization of the supersymmetric Randall-Sundrum scenario.) We can only observe that together with the Eq. (12.22), $u = \varepsilon(z)u^{(+)}$ (where u is the odd parameter of the $U(1)$ gauge transformation), there is an indication that odd parameters of local transformations come with $\varepsilon(z)$, whereas other fields and parameters come with $1/\varepsilon(z)$. (There is a slight

problem with such a conclusion, because the equation (11.8) and the discussion in Section 9.5 seem to indicate that

$$B_m = \frac{1}{\varepsilon(z)} B_m^{(+)}, \quad e_{5a} = \varepsilon(z) e_{5a}^{(+)}. \quad (14.3)$$

But the evidence provided for these assignments is not on a very firm footing.)

Another important conclusion of this work is that in the orbifold picture *all* local transformations have to be modified by the addition of brane-localized terms. The modifications must be such that the modified transformations become *non-singular* on the brane when the natural boundary conditions (encoded in the action) are used. This is the reason both for the modification of $\delta_{\mathcal{H}}\psi_{52}$ in Refs. [4, 7] and for the “modification of the Bianchi identity” in the Horava-Witten model [1].

We also note that our results (concerning the boundary picture) are in agreement with the recent work of Moss [35, 36], who did a similar analysis for the eleven-dimensional supergravity. The use of the fermionic boundary condition there is necessary, according to our discussion, precisely because higher order fermionic terms are considered. Our approach to the Y -term can be employed there to derive the boundary condition for C_{ABC} from the bulk-plus-boundary action (instead of just postulating it for consistency).

Finally, it would be interesting to see how the analysis presented here for the case of on-shell supergravity can be done for *off-shell* supergravity of Zucker [18]. One question this could answer is whether the boundary condition $B_m = 0$ is, actually, an “auxiliary boundary condition” (see Section 9.4) similar to $\Phi = 0$ in the Mirabelli and Peskin model (in the absence of brane-localized matter) [13].

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A. Conventions

We follow conventions of Ref. [7]. Our indices are

$$\begin{array}{llll} M, N, P, Q, K & \text{curved space} & M = \{m, 5\} & m = \{0, 1, 2, 3\} \\ A, B, C, D, E & \text{tangent space} & A = \{a, \hat{5}\} & a = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}\} \\ i, j & SU(2) & i = \{1, 2\}. \end{array} \quad (\text{A.1})$$

We denote the determinant of an n -bein by e_n : $e_5 = \det e_M^A$, $e_4 = \det e_m^a$. We use the fünfbein e_M^A to relate the two types of indices, e.g.

$$g_{MN} = e_M^A e_N^B \eta_{AB}, \quad \epsilon^{MNPQK} = e_A^M e_B^N e_C^P e_D^Q e_E^K \epsilon^{ABCDE}. \quad (\text{A.2})$$

It also defines the torsion-free connection,

$$\omega(e)_{MAB} = \frac{1}{2} e_A^N e_B^K (C_{MNK} + C_{NMK} - C_{KMN}), \quad (\text{A.3})$$

where $C_{MNK} = e_{MC}(\partial_N e_K^C - \partial_K e_N^C)$. The covariant derivative is defined to act as follows,

$$D(\omega)_M \Psi_N^A = \partial_M \Psi_N^A + \omega_{MC}{}^A \Psi_N^C - \Gamma(\omega)_{MN}^K \Psi_K^A + \frac{1}{4} \omega_{MBC} \Gamma^{BC} \Psi_N^A, \quad (\text{A.4})$$

where the spinor indices on Ψ and Γ^{BC} are implicit. The Christoffel connection is made dependent by imposing $D_M e_N^A = 0$, which implies

$$\Gamma_{MN}^K = \omega_{MN}{}^K + e_A^K \partial_M e_N^A. \quad (\text{A.5})$$

The curvature tensor is defined by

$$R_{MNAB} = \partial_M \omega_{NAB} - \partial_N \omega_{MAB} + \omega_{NA}{}^C \omega_{MCB} - \omega_{MA}{}^C \omega_{NCB}, \quad (\text{A.6})$$

and the scalar curvature is $R = e^{MA} R_{MA} = e^{MA} e^{NB} R_{MNAB}$.

The gamma matrices obey the following relations,

$$\begin{aligned} \{\Gamma^A, \Gamma^B\} &= -2\eta^{AB}, & \Gamma^{ABCDE} &= -\epsilon^{ABCDE} \\ \Gamma^{ABCD} &= \epsilon^{ABCDE} \Gamma_E, & \Gamma^{ABC} &= \frac{1}{2} \epsilon^{ABCDE} \Gamma_{DE}, \end{aligned} \quad (\text{A.7})$$

where $\Gamma^{A_1 \dots A_n}$ are antisymmetrized with “strength one”, e.g. $\Gamma^{AB} = \frac{1}{2}(\Gamma^A \Gamma^B - \Gamma^B \Gamma^A)$. The metric and the Levi-Civita tensor are determined by

$$\eta_{AB} = \text{diag}(-+++), \quad \epsilon^{\hat{0}\hat{1}\hat{2}\hat{3}\hat{5}} = +1, \quad \epsilon^{abcd\hat{5}} = \epsilon^{abcd}. \quad (\text{A.8})$$

In reduction to the two-component notation [37] we use the following representation of the gamma matrices,

$$\Gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}, \quad \Gamma^{ab} = 2 \begin{pmatrix} \sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}, \quad \Gamma^{abc} = i\epsilon^{abcd} \begin{pmatrix} 0 & \sigma_d \\ -\bar{\sigma}_d & 0 \end{pmatrix} \quad (\text{A.9})$$

$$\Gamma^{\hat{5}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \Gamma^{a\hat{5}} = i \begin{pmatrix} 0 & \sigma^a \\ -\bar{\sigma}^a & 0 \end{pmatrix}, \quad \Gamma^{ab\hat{5}} = 2i \begin{pmatrix} -\sigma^{ab} & 0 \\ 0 & \bar{\sigma}^{ab} \end{pmatrix}. \quad (\text{A.10})$$

A four-component Dirac spinor Ψ , its Dirac conjugate $\bar{\Psi}$ and its Majorana conjugate $\tilde{\Psi}$ are written in terms of two-component spinors ψ_1 and ψ_2 as follows,

$$\Psi = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}, \quad \bar{\Psi} = (\psi_2, \bar{\psi}_1), \quad \tilde{\Psi} = (-\psi_1, \bar{\psi}_2). \quad (\text{A.11})$$

A symplectic Majorana spinor Ψ_i satisfies $\tilde{\Psi}^i = \bar{\Psi}_i$, where index i can be raised and lowered with an antisymmetric tensor ε_{ij} . We use the following representation,

$$\Psi_1 = -\Psi^2 = \begin{pmatrix} \psi_1 \\ \bar{\psi}_2 \end{pmatrix}, \quad \Psi_2 = \Psi^1 = \begin{pmatrix} -\psi_2 \\ \bar{\psi}_1 \end{pmatrix}. \quad (\text{A.12})$$

The following identities are satisfied,

$$\tilde{\Psi} \vec{\Gamma} \mathcal{H} = \tilde{\mathcal{H}} \overleftarrow{\Gamma} \Psi, \quad \tilde{\Psi}^i \vec{\Gamma} \mathcal{H}_i = -\tilde{\mathcal{H}}^i \overleftarrow{\Gamma} \Psi_i, \quad Q_i^j \tilde{\Psi}^i \vec{\Gamma} \mathcal{H}_j = Q_i^j \tilde{\mathcal{H}}^i \overleftarrow{\Gamma} \Psi_j, \quad (\text{A.13})$$

where $\vec{\Gamma} = \Gamma^{A_1} \Gamma^{A_2} \dots \Gamma^{A_n}$, $\overleftarrow{\Gamma} = \Gamma^{A_n} \dots \Gamma^{A_2} \Gamma^{A_1}$ and

$$Q_i^j = i\vec{q} \cdot \vec{\sigma} = i \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix}. \quad (\text{A.14})$$

Also, for arbitrary symplectic Majorana spinors we have

$$\begin{aligned} i\tilde{\Psi}^i \vec{\Gamma} \mathcal{H}_i &= i\bar{\Psi}_1 \vec{\Gamma} \mathcal{H}_1 + h.c. \\ i\tilde{\Psi}^i \vec{\Gamma} Q_i^j \mathcal{H}_j &= -q_3 \bar{\Psi}_1 \vec{\Gamma} \mathcal{H}_1 - q_{12} \bar{\Psi}_2 \vec{\Gamma} \mathcal{H}_1 + h.c., \end{aligned} \quad (\text{A.15})$$

where $q_{12} = q_1 + iq_2$. These identities allow a straightforward reduction of the action and supersymmetry transformations to the two-component expressions. The following set of expressions is especially helpful,

$$\begin{aligned} \bar{\Psi} \mathcal{H} &= \psi_2 \eta_1 + \bar{\psi}_2 \bar{\eta}_2, & \bar{\Psi} \Gamma^{\hat{5}} \mathcal{H} &= -i(\psi_2 \eta_1 - \bar{\psi}_1 \bar{\eta}_2) \\ \bar{\Psi} \Gamma^a \mathcal{H} &= \psi_2 \sigma^a \bar{\eta}_2 + \bar{\psi}_1 \bar{\sigma}^a \eta_1, & \bar{\Psi} \Gamma^{a\hat{5}} \mathcal{H} &= i(\psi_2 \sigma^a \bar{\eta}_2 - \bar{\psi}_1 \bar{\sigma}^a \eta_1) \\ \bar{\Psi} \Gamma^{ab} \mathcal{H} &= 2(\psi_2 \sigma^{ab} \eta_1 + \bar{\psi}_1 \bar{\sigma}^{ab} \bar{\eta}_2), & \bar{\Psi} \Gamma^{ab\hat{5}} \mathcal{H} &= -2i(\psi_2 \sigma^{ab} \eta_1 - \bar{\psi}_1 \bar{\sigma}^{ab} \bar{\eta}_2). \end{aligned} \quad (\text{A.16})$$

Finally,

$$\bar{\Psi} \Gamma^{abcd} \mathcal{H} = \epsilon^{abcd} \bar{\Psi} \Gamma^{\hat{5}} \mathcal{H}, \quad \bar{\Psi} \Gamma^{abc\hat{5}} \mathcal{H} = -\epsilon^{abcd} \bar{\Psi} \Gamma_d \mathcal{H}. \quad (\text{A.17})$$

B. Gibbons-Hawking boundary term

We define the extrinsic curvature as ¹³

$$K_{MN} = P_M^K P_N^L D_K n_L , \quad (\text{B.1})$$

where n^M is the (outward pointing) unit vector normal to the boundary $\partial\mathcal{M}$, and

$$P_M^N = \delta_M^N - n_M n^N \quad (\text{B.2})$$

is a projector onto the boundary, $P_M^N n_N = 0$. The trace of the extrinsic curvature is

$$K = g^{MN} K_{MN} . \quad (\text{B.3})$$

One can show [15] that its general variation gives

$$\delta K = K_{MN} e_A^M \delta e^{NA} + n_M (e^{MA} e^{NB} \delta \omega_{NAB}) + P_M^K D_K (e_A^L P_L^M n_N \delta e^{NA}) . \quad (\text{B.4})$$

The last term is a total tangential derivative which vanishes upon integrating over $\partial\mathcal{M}$. Since

$$\delta \int_{\mathcal{M}} \left(-\frac{1}{2} R \right) = \int_{\mathcal{M}} \left(R_A^M - \frac{1}{2} R e_A^M \right) \delta e_M^A + \int_{\partial\mathcal{M}} (-n_M e^{MA} e^{NB} \delta \omega_{NAB}) , \quad (\text{B.5})$$

we see that the Einstein-Hilbert action with the Gibbons-Hawking boundary term,

$$S_{EH+GH} = -\frac{1}{2} \int_{\mathcal{M}} R + \int_{\partial\mathcal{M}} K , \quad (\text{B.6})$$

under the general variation gives

$$\delta S_{EH+GH} = \int_{\mathcal{M}} \left(R_A^M - \frac{1}{2} R e_A^M \right) \delta e_M^A + \int_{\partial\mathcal{M}} (K_{MN} - K P_{MN}) e_A^N \delta e^{MA} . \quad (\text{B.7})$$

The Gibbons-Hawking term makes only the variation of the metric (vielbein e^{MA}) appear in the boundary term of the general variation of the total action. This improves the variational principle, allowing both the use of the Dirichlet boundary conditions for the metric,

$$\delta e_M^A = 0 \quad \text{on } \partial\mathcal{M} , \quad (\text{B.8})$$

and the derivation of the “natural” (generalized Neumann) boundary conditions,

$$K_{MN} - K P_{MN} = S_{MN} \quad \text{on } \partial\mathcal{M} , \quad (\text{B.9})$$

where S_{MN} represents a contribution from a boundary action.

¹³For a detailed discussion of the extrinsic curvature see Refs. [38, 39, 40].

C. Why we choose $n_5 = -e_5^{\hat{5}}$

The Stokes's theorem states (see, e.g., Ref. [40])

$$\int_{\mathcal{M}} d^5x e_5 (D_M K^M) = \int_{\mathcal{M}} d^5x \partial_M (e_5 K^M) = \int_{\partial\mathcal{M}} d^4x e_4^{\text{ind}} (n_M K^M) , \quad (\text{C.1})$$

where (denoting by g_{mn}^{ind} the induced four-dimensional metric on $\partial\mathcal{M}$)

$$e_5 = \det e_{MA} = \sqrt{|\det g_{MN}|} \quad (\text{C.2})$$

$$e_4^{\text{ind}} = \det e_{ma}^{\text{ind}} = \sqrt{|\det g_{mn}^{\text{ind}}|} , \quad (\text{C.3})$$

and n_M 1) is orthogonal to $\partial\mathcal{M}$; 2) has the unit norm, $g^{MN} n_M n_N = 1$; 3) is outward pointing. With our description of $\partial\mathcal{M}$ as a hypersurface $x^5 = \text{const}$, the first condition implies that only $n_5 \neq 0$, the second says

$$n_5 = \pm \frac{1}{\sqrt{g^{55}}} , \quad (\text{C.4})$$

and the third has to do with choosing one of the two signs.

In our gauge ($e_m^{\hat{5}} = 0$), we have $g^{55} = e_5^{\hat{5}} e_5^{\hat{5}}$ and $e_5^{\hat{5}} e_5^{\hat{5}} = 1$, thus

$$n_5 = \pm e_5^{\hat{5}} . \quad (\text{C.5})$$

In this gauge we also have $g_{mn}^{\text{ind}} = e_m^a e_{na}$ and, therefore, we can choose the induced vierbein as $e_{ma}^{\text{ind}} = e_{ma}$ and obtain $e_5 = e_4 e_5^{\hat{5}}$.

Let us assume that our \mathcal{M} is a strip $x^5 \equiv z \in [z_1, z_2]$ and Σ denotes an $x^5 = \text{const}$ hypersurface. The Stokes's theorem can then be written as

$$\int_{\Sigma} d^4x \int_{z_1}^{z_2} dz \partial_5 (e_4 e_5^{\hat{5}} K^5) = \int_{\partial\mathcal{M}} d^4x e_4 (n_5 K^5) , \quad (\text{C.6})$$

and, therefore,

$$(e_5^{\hat{5}} K^5)|_{z_2} - (e_5^{\hat{5}} K^5)|_{z_1} = (n_5 K^5)|_{z_2} + (n_5 K^5)|_{z_1} . \quad (\text{C.7})$$

This means that for the Stokes's theorem to hold (that is for n_M to be “outward pointing”), we should choose

$$n_5 = -e_5^{\hat{5}} \text{ at } z_1 \quad \text{and} \quad n_5 = +e_5^{\hat{5}} \text{ at } z_2 . \quad (\text{C.8})$$

This choice coincides with the intuitive one when $e_5^{\hat{5}} > 0$.

When $\mathcal{M} = \mathcal{M}_+ = \mathbb{R}^{1,3} \times [0, +\infty)$, the outward pointing n_M at $z = 0$ has $n_5 = -e_5^{\hat{5}}$. With this choice, the boundary conditions we obtain on $\partial\mathcal{M}$ coincide with the boundary conditions “on the positive side of the brane” (that is “at $z = +0$ ”) and thus directly correspond to the boundary conditions in Refs. [7] and [8].

D. Our gauge

Our gauge choice is $e_m^{\hat{5}} = 0$. Thus,

$$e_m^{\hat{5}} = 0, \quad e_a^{\hat{5}} = 0, \quad e_5^a \neq 0, \quad e_5^m \neq 0. \quad (\text{D.1})$$

Since e_A^M is the inverse to e_M^A ,

$$e_M^A e_A^N = \delta_M^N, \quad e_A^M e_M^B = \delta_A^B, \quad (\text{D.2})$$

in this gauge we have

$$e_m^a e_a^n = \delta_m^n, \quad e_a^m e_m^b = \delta_a^b, \quad e_5^{\hat{5}} e_5^{\hat{5}} = 1, \quad e_5^m = -e_5^a e_a^m e_5^{\hat{5}}. \quad (\text{D.3})$$

D.1 Metric tensor

For the metric tensor g_{MN} we obtain

$$g_{mn} = \gamma_{mn}, \quad g_{m5} = g_{5m} = N_m, \quad g_{55} = \gamma^{mn} N_m N_n + N^2, \quad (\text{D.4})$$

where we defined

$$\gamma_{mn} \equiv e_m^a e_{na}, \quad N_m \equiv e_5^a e_{ma}, \quad N = e_5^{\hat{5}}, \quad (\text{D.5})$$

and γ^{mn} is the inverse to γ_{mn} ,

$$\gamma^{mn} \equiv e^{ma} e_a^n, \quad \gamma_{mk} \gamma^{kn} = \delta_m^n. \quad (\text{D.6})$$

Defining

$$N^m \equiv \gamma^{mn} N_n = e_5^a e_a^m = -N e_5^m, \quad (\text{D.7})$$

the inverse five dimensional metric tensor g^{MN} can be written as

$$g^{mn} = \gamma^{mn} + N^{-2} N^m N^n, \quad g^{m5} = g^{5m} = -N^{-2} N^m, \quad g^{55} = N^{-2}. \quad (\text{D.8})$$

D.2 Field strength

Components of $F^{MN} = g^{MK} g^{NL} F_{KL}$ for the field strength $F_{MN} = \partial_M B_N - \partial_N B_M$ are

$$\begin{aligned} F^{mn} &= g^{mk} g^{nl} F_{kl} + (g^{mk} g^{n5} - g^{nk} g^{m5}) F_{m5} \\ F^{m5} &= g^{mn} g^{k5} F_{nk} + (g^{mn} g^{55} - g^{m5} g^{n5}) F_{n5}. \end{aligned} \quad (\text{D.9})$$

We can write this in a more convenient form,

$$F^{mn} = \gamma^{mk} \gamma^{nl} F_{kl} + \gamma^{mk} e_5^n F_{k\hat{5}} - \gamma^{nk} e_5^m F_{k\hat{5}}, \quad F^{m5} = \gamma^{mn} e_5^{\hat{5}} F_{n\hat{5}}, \quad (\text{D.10})$$

where we defined

$$F_{m\hat{5}} \equiv e_5^{\hat{5}} F_{m5} + e_5^n F_{mn}. \quad (\text{D.11})$$

D.3 Spin connection

We use the following spin connection,

$$\omega_{MAB} = e_A^N e_B^K \omega_{MNK}, \quad \omega_{MNK} = \frac{1}{2}(C_{MNK} + C_{NMK} - C_{KMN}), \quad (\text{D.12})$$

where

$$C_{MNK} = e_{MC}(\partial_N e_K^C - \partial_K e_N^C). \quad (\text{D.13})$$

We find that in our gauge

$$\begin{aligned} C_{mnk} &= \hat{C}_{mnk}, & C_{mn5} &= u_{mn} \\ C_{5mn} &= N^k \hat{C}_{kmn}, & C_{5n5} &= N^k u_{kn} + N \partial_n N, \end{aligned} \quad (\text{D.14})$$

where

$$\hat{C}_{mnk} \equiv e_{mc}(\partial_n e_k^c - \partial_k e_n^c), \quad u_{mn} \equiv e_{mc}(\partial_n e_5^c - \partial_5 e_n^c). \quad (\text{D.15})$$

The spin connection coefficients are given by

$$\omega_{mab} = \frac{1}{2} e_a^n e_b^k (\hat{C}_{mnk} + \hat{C}_{nmk} - \hat{C}_{kmn}) \quad (\text{D.16})$$

$$\omega_{5ab} = \frac{1}{2} N^n \hat{C}_{nab} - \frac{1}{2} (u_{ab} - u_{ba}) \quad (\text{D.17})$$

$$\omega_{ma\hat{5}} = -\frac{1}{2} N^{-1} N^k (\hat{C}_{mak} + \hat{C}_{amk}) + \frac{1}{2} N^{-1} (u_{am} + u_{ma}) \quad (\text{D.18})$$

$$\omega_{5a\hat{5}} = -\frac{1}{2} N^{-1} N^n N^k \hat{C}_{nak} + \frac{1}{2} N^{-1} N^k (u_{ka} + u_{ak}) + e_a^n \partial_n N. \quad (\text{D.19})$$

We see that in our gauge there are no $\partial_5 N_m$ in any of ω_{MAB} , which means that the spin connection coefficients are *non-singular*, i.e. contain no $\delta(z)$, in the orbifold picture!

D.4 Extrinsic curvature

Our (outward pointing) unit vector normal to the boundary $\partial\mathcal{M}$ is

$$n_M = (0_m, -N), \quad n^M = (N^{-1} N^m, -N^{-1}). \quad (\text{D.20})$$

This gives the following projector onto the boundary,

$$P^{MN} = \begin{pmatrix} \gamma^{mn} & 0 \\ 0 & 0 \end{pmatrix} \quad P_M{}^N = \begin{pmatrix} \delta_m^n & 0 \\ N^n & 0 \end{pmatrix} \quad P_{MN} = \begin{pmatrix} \gamma_{mn} & N_m \\ N_n & N^k N_{kl} \end{pmatrix}, \quad (\text{D.21})$$

and the extrinsic curvature,

$$K^{MN} = \begin{pmatrix} K^{mn} & 0 \\ 0 & 0 \end{pmatrix} \quad K_{MN} = \begin{pmatrix} K_{mn} & N^k K_{mk} \\ N^k K_{kn} & N^k N^l K_{kl} \end{pmatrix}, \quad (\text{D.22})$$

where

$$K_{mn} = -\Gamma_{mn}^5 n_5, \quad K^{mn} = \gamma^{mk} \gamma^{nl} K_{kl} . \quad (\text{D.23})$$

The trace of the extrinsic curvature is

$$K \equiv g_{MN} K^{MN} = \gamma_{mn} K^{mn} = \gamma^{mn} K_{mn} = -\gamma^{mn} \Gamma_{mn}^5 n_5 . \quad (\text{D.24})$$

Using the relation between the Christoffel symbols and the spin connection,

$$D_M e_N^A \equiv \partial_M e_N^A + \omega_{MC}^A e_N^C - \Gamma_{MN}^K e_K^A = 0 , \quad (\text{D.25})$$

we find

$$\begin{aligned} \Gamma_{mn}^5 &= \omega_{mn}^{\hat{5}} + e_A^5 \partial_m e_n^A = e_n^a \omega_{ma}^{\hat{5}} e_{\hat{5}}^5 \\ &\quad + e_n^a \omega_{ma}^b e_b^{\hat{5}} + e_n^{\hat{5}} \omega_{m\hat{5}}^b e_b^5 + e_a^5 \partial_m e_n^a + e_{\hat{5}}^5 \partial_m e_n^{\hat{5}} . \end{aligned} \quad (\text{D.26})$$

But all the terms in the second line vanish in our gauge, so

$$\Gamma_{mn}^5 = e_n^a \omega_{ma}^{\hat{5}} e_{\hat{5}}^5 . \quad (\text{D.27})$$

Therefore, we obtain

$$K_{ma} \equiv e_a^n K_{mn} = \omega_{ma\hat{5}}, \quad K = e^{ma} K_{ma} . \quad (\text{D.28})$$

This gives a geometrical meaning to the spin connection coefficient $\omega_{ma\hat{5}}$.

D.5 Advantages of the $e_m^{\hat{5}} = 0$ gauge

The following properties are unique to our gauge, $e_m^{\hat{5}} = 0$. (Another simple gauge, $e_5^a = 0$, frequently used in the Kaluza-Klein reductions,¹⁴ *does not* enjoy these properties.)

1. e_m^a is an induced vierbein on a slice $x^5 = \text{const}$.
2. ω_{mab} is a spin connection for e_m^a .
3. $\hat{D}_m e_n^a = \partial_m e_n^a + \omega_{mc}^a e_n^c - \Gamma_{mn}^k e_k^a = 0$, where ω_{mab} and Γ_{mn}^k are elements of ω_{MAB} and Γ_{MN}^K .
4. There is a simple relation between the extrinsic curvature and a spin connection coefficient: $K_{ma} = \omega_{ma\hat{5}}$.
5. There are no $\delta(z)$ -terms in any of the spin connection coefficients ω_{MAB} .
6. The compensating local Lorentz rotation (with the parameter $\omega^{a\hat{5}}$; see below) leaves the supersymmetry transformation of e_m^a unchanged.

¹⁴See, e.g., the paper by Chamseddine and Nicolai in Ref. [19].

E. Supersymmetry transformations

In our gauge and in the two-component spinor notation, the supersymmetry transformations of Eqs. (2.1–2.3) (dropping the 3-Fermi terms in the $\delta\Psi_{Mi}$) can be written as follows,

$$\begin{aligned}
\delta_{\mathcal{H}} e_m^a &= -i(\psi_{m1}\sigma^a\bar{\eta}_1 + \psi_{m2}\sigma^a\bar{\eta}_2) + h.c. \\
\delta_{\mathcal{H}} e_5^a &= -i(\psi_{51}\sigma^a\bar{\eta}_1 + \psi_{52}\sigma^a\bar{\eta}_2) + h.c. \\
\delta_{\mathcal{H}} e_m^{\hat{5}} &= -\psi_{m2}\eta_1 + \psi_{m1}\eta_2 + h.c. \\
\delta_{\mathcal{H}} e_5^{\hat{5}} &= -\psi_{52}\eta_1 + \psi_{51}\eta_2 + h.c. \\
\delta_{\mathcal{H}} B_m &= i\frac{\sqrt{6}}{2}(\psi_{m2}\eta_1 - \psi_{m1}\eta_2) + h.c. \\
\delta_{\mathcal{H}} B_5 &= i\frac{\sqrt{6}}{2}(\psi_{52}\eta_1 - \psi_{51}\eta_2) + h.c.
\end{aligned} \tag{E.1}$$

$$\begin{aligned}
\delta_{\mathcal{H}}\psi_{m1} &= 2\hat{D}_m\eta_1 + i\omega_{ma\hat{5}}\sigma^a\bar{\eta}_2 + i\lambda\sigma_m(q_{12}^*\bar{\eta}_1 + q_3\bar{\eta}_2) - i\sqrt{6}\lambda(q_3\eta_1 - q_{12}^*\eta_2)B_m \\
&\quad + \frac{1}{2\sqrt{6}}\left\{ -4i(\sigma_m{}^n + \delta_m^n)\eta_1(e_5^{\hat{5}}F_{n5} + e_5^kF_{nk}) + [i\epsilon_m{}^{nkl}\sigma_l + 4\delta_m^n\sigma^k]\bar{\eta}_2F_{nk} \right\} \\
\delta_{\mathcal{H}}\psi_{m2} &= 2\hat{D}_m\eta_2 - i\omega_{ma\hat{5}}\sigma^a\bar{\eta}_1 + i\lambda\sigma_m(q_3\bar{\eta}_1 - q_{12}\bar{\eta}_2) + i\sqrt{6}\lambda(q_3\eta_2 + q_{12}\eta_1)B_m \\
&\quad + \frac{1}{2\sqrt{6}}\left\{ -4i(\sigma_m{}^n + \delta_m^n)\eta_2(e_5^{\hat{5}}F_{n5} + e_5^kF_{nk}) - [i\epsilon_m{}^{nkl}\sigma_l + 4\delta_m^n\sigma^k]\bar{\eta}_1F_{nk} \right\} \\
\delta_{\mathcal{H}}\psi_{51} &= 2\hat{D}_5\eta_1 + i\omega_{5a\hat{5}}\sigma^a\bar{\eta}_2 + \lambda(e_5^{\hat{5}} - i\sqrt{6}B_5)(q_3\eta_1 - q_{12}^*\eta_2) \\
&\quad + \frac{1}{2\sqrt{6}}\left\{ -4(\sigma^n\bar{\eta}_2 - ie_5^n\eta_1)F_{n5} - 4ie_{5a}\sigma^{an}\eta_1(e_5^{\hat{5}}F_{n5} + e_5^kF_{nk}) \right. \\
&\quad \left. + (-2ie_5^{\hat{5}}\sigma^{nk}\eta_1 + ie_{5a}\epsilon^{ankl}\sigma_l\bar{\eta}_2)F_{nk} \right\} \\
\delta_{\mathcal{H}}\psi_{52} &= 2\hat{D}_5\eta_2 - i\omega_{5a\hat{5}}\sigma^a\bar{\eta}_1 - \lambda(e_5^{\hat{5}} - i\sqrt{6}B_5)(q_3\eta_2 + q_{12}\eta_1) \\
&\quad + \frac{1}{2\sqrt{6}}\left\{ +4(\sigma^n\bar{\eta}_1 + ie_5^n\eta_2)F_{n5} - 4ie_{5a}\sigma^{an}\eta_2(e_5^{\hat{5}}F_{n5} + e_5^kF_{nk}) \right. \\
&\quad \left. + (-2ie_5^{\hat{5}}\sigma^{nk}\eta_2 - ie_{5a}\epsilon^{ankl}\sigma_l\bar{\eta}_1)F_{nk} \right\}, \tag{E.2}
\end{aligned}$$

where

$$\hat{D}_M\eta = \partial_M\eta + \frac{1}{2}\omega_{Mab}\sigma^{ab}\eta. \tag{E.3}$$

Note that *all* e_5^a and e_5^m have been explicitly separated out (thus, $\sigma^m = e_a^m\sigma^a$ and so on).

However, we have to modify the supersymmetry transformations by a compensating Lorentz transformation ($\omega_{AB} = -\omega_{BA}$),

$$\delta_{\omega}e_M^A = e_M^B\omega_B^A, \quad \delta_{\omega}\Psi_{Mi} = \frac{1}{4}\omega_{AB}\Gamma^{AB}\Psi_{Mi}, \tag{E.4}$$

in order to stay in our $e_m^{\hat{5}} = 0$ gauge. Using only off-diagonal coefficient $\omega^{a\hat{5}}$,

$$\delta_\omega e_m^a = -e_{m\hat{5}}\omega^{a\hat{5}}, \quad \delta_\omega e_5^a = -e_{5\hat{5}}\omega^{a\hat{5}}, \quad \delta_\omega e_m^{\hat{5}} = e_{mc}\omega^{c\hat{5}}, \quad \delta_\omega e_5^{\hat{5}} = e_{5c}\omega^{c\hat{5}}. \quad (\text{E.5})$$

We define the modified supersymmetry transformations by

$$\delta'_\mathcal{H} e_M^A = \delta_\mathcal{H} e_M^A + \delta_\omega e_M^A. \quad (\text{E.6})$$

We want the supersymmetry variation to preserve $e_m^{\hat{5}} = 0$,

$$\delta'_\mathcal{H} e_m^{\hat{5}} = \delta_\mathcal{H} e_m^{\hat{5}} + e_{mc}\omega^{c\hat{5}} = 0, \quad (\text{E.7})$$

which fixes

$$\omega^{a\hat{5}} = -e^{ma}\delta_\mathcal{H} e_m^{\hat{5}} = e^{ma}(\psi_{m2}\eta_1 - \psi_{m1}\eta_2) + h.c. \quad (\text{E.8})$$

The modified supersymmetry transformations, therefore, are

$$\delta'_\mathcal{H} e_m^a = \delta_\mathcal{H} e_m^a \quad (\text{E.9})$$

$$\delta'_\mathcal{H} e_5^a = \delta_\mathcal{H} e_5^a + e_{5\hat{5}}e^{ma}\delta_\mathcal{H} e_m^{\hat{5}} \quad (\text{E.10})$$

$$\delta'_\mathcal{H} e_m^{\hat{5}} = 0 \quad (\text{E.11})$$

$$\delta'_\mathcal{H} e_5^{\hat{5}} = \delta_\mathcal{H} e_5^{\hat{5}} - e_5^a e_a^m \delta_\mathcal{H} e_m^{\hat{5}}. \quad (\text{E.12})$$

Note that the supersymmetry variation of the e_m^a stays the same, which is one of the advantages of our gauge.

The gravitino supersymmetry transformations also get modified, but only in the 3-Fermi terms (since $\omega^{a\hat{5}}$ is 2-Fermi) which we omit.

F. Bulk Lagrangian

The fermionic part of the bulk supergravity Lagrangian is

$$\begin{aligned} \mathcal{L}_{5F} = & \frac{i}{2}\tilde{\Psi}_M^i \Gamma^{MNK} D_N \Psi_{Ki} - i\frac{\sqrt{6}}{8}F^{MN}\tilde{\Psi}_M^i \Psi_{Ni} - i\frac{\sqrt{6}}{16}F_{MN}\tilde{\Psi}_P^i \Gamma^{MNPQ}\Psi_{Qi} \\ & + i\frac{3}{4}\lambda Q_i^j \tilde{\Psi}_M^i \Gamma^{MN} \Psi_{Nj} - i\frac{\sqrt{6}}{4}\lambda Q_i^j \tilde{\Psi}_M^i \Gamma^{MNK} \Psi_{Kj} B_N. \end{aligned} \quad (\text{F.1})$$

In the two-component spinor notation it becomes

$$\begin{aligned}
\mathcal{L}_{5F} = & \frac{1}{2}\epsilon^{mnkl}(\bar{\psi}_{m2}\bar{\sigma}_l D_n \psi_{k2} + \bar{\psi}_{m1}\bar{\sigma}_l D_n \psi_{k1}) + (\psi_{m1}\sigma^{mn} D_{\hat{5}} \psi_{n2} - \psi_{m2}\sigma^{mn} D_{\hat{5}} \psi_{n1}) \\
& + (\psi_{m2}\sigma^{mn} D_n \psi_{\hat{5}1} - \psi_{\hat{5}1}\sigma^{mn} D_m \psi_{n2}) + (\psi_{\hat{5}2}\sigma^{mn} D_m \psi_{n1} - \psi_{m1}\sigma^{mn} D_n \psi_{\hat{5}2}) \\
& - i\frac{\sqrt{6}}{4}\left[\gamma^{mk}\gamma^{nl}F_{kl}(\psi_{m2}\psi_{n1}) + \gamma^{mk}F_{k\hat{5}}(\psi_{m2}\psi_{\hat{5}1} - \psi_{m1}\psi_{\hat{5}2})\right] \\
& - \frac{\sqrt{6}}{8}\epsilon^{mnpq}\left[F_{pq}(\psi_{m2}\psi_{n1}) - iF_{pq}(\psi_{m2}\sigma_n\bar{\psi}_{\hat{5}2} + \psi_{m1}\sigma_n\bar{\psi}_{\hat{5}1})\right. \\
& \quad \left.- iF_{m\hat{5}}(\psi_{p2}\sigma_n\bar{\psi}_{q2} + \psi_{p1}\sigma_n\bar{\psi}_{q1})\right] \\
& - \frac{3}{2}\lambda\left\{q_3[2\psi_{m1}\sigma^{mn}\psi_{n2} + i(\psi_{m2}\sigma^m\bar{\psi}_{\hat{5}2} - \psi_{m1}\sigma^m\bar{\psi}_{\hat{5}1})]\right. \\
& \quad \left.+ q_{12}[(\psi_{m1}\sigma^{mn}\psi_{n1} - \bar{\psi}_{m2}\bar{\sigma}^{mn}\bar{\psi}_{n2}) + i(\psi_{m1}\sigma^m\bar{\psi}_{\hat{5}2} + \bar{\psi}_{m2}\bar{\sigma}^m\psi_{\hat{5}1})]\right\} \\
& + \frac{\sqrt{6}}{4}\lambda\left\{q_3[iB_n\epsilon^{mnkl}(\psi_{m2}\sigma_l\bar{\psi}_{k2} - \psi_{m1}\sigma_l\bar{\psi}_{k1}) + 4iB_{\hat{5}}(\psi_{m1}\sigma^{mn}\psi_{n2})\right. \\
& \quad \left.- 4iB_n(\psi_{m2}\sigma^{mn}\psi_{\hat{5}1} + \psi_{m1}\sigma^{mn}\psi_{\hat{5}2})]\right. \\
& \quad \left.+ q_{12}[2iB_n\epsilon^{mnkl}(\psi_{m1}\sigma_l\bar{\psi}_{k2}) + 2iB_{\hat{5}}(\psi_{m1}\sigma^{mn}\psi_{n1} + \bar{\psi}_{m2}\bar{\sigma}^{mn}\bar{\psi}_{n2})\right. \\
& \quad \left.- 4iB_n(\psi_{m1}\sigma^{mn}\psi_{\hat{5}1} + \bar{\psi}_{m2}\bar{\sigma}^{mn}\bar{\psi}_{\hat{5}2})]\right\} \\
& + h.c.
\end{aligned} \tag{F.2}$$

Note that e_5^g does not appear at all, whereas e_5^m appears *only* in the following combinations,

$$\begin{aligned}
B_{\hat{5}} &= e_5^5 B_5 + e_5^m B_m, & F_{m\hat{5}} &= e_5^5 F_{m5} + e_5^n F_{mn} \\
D_{\hat{5}} &= e_5^5 D_5 + e_5^m D_m, & \psi_{\hat{5}1,2} &= e_5^5 \psi_{51,2} + e_5^m \psi_{m1,2}.
\end{aligned} \tag{F.3}$$

In particular, $\sigma^m = e_a^m \sigma^a$ and $\epsilon^{mnkl} = e_a^m e_b^n e_c^k e_d^l \epsilon^{abcd}$. The derivatives can be further decomposed as follows,

$$\begin{aligned}
D_M \psi_1 &= \hat{D}_M \psi_1 + \frac{i}{2}\omega_{Ma\hat{5}}\bar{\psi}_2 \\
D_M \psi_2 &= \hat{D}_M \psi_2 - \frac{i}{2}\omega_{Ma\hat{5}}\bar{\psi}_1.
\end{aligned} \tag{F.4}$$

Note that $D_M e_N^A = 0$, but $\hat{D}_M e_N^A \neq 0$.

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