DESY 06-036
DCPT/06/40, IPPP/06/20
April 2006

hep-ph/0604053

Next-to-Next-to-Leading Order Evolution of Non-Singlet Fragmentation Functions

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Abstract

We have investigated the next-to-next-to-leading order (NNLO) corrections to inclusive hadron production in e^+e^- annihilation and the related parton fragmentation distributions, the 'time-like' counterparts of the 'space-like' deep-inelastic structure functions and parton densities. We have re-derived the corresponding second-order coefficient functions in massless perturbative QCD, which so far had been calculated only by one group. Moreover we present, for the first time, the third-order splitting functions governing the NNLO evolution of flavour non-singlet fragmentation distributions. These results have been obtained by two independent methods relating time-like quantities to calculations performed in deep-inelastic scattering. We briefly illustrate the numerical size of the NNLO corrections, and make a prediction for the difference of the yet unknown time-like and space-like splitting functions at the fourth order in the strong coupling constant.

In this letter we address the evolution of the parton fragmentation distributions D^h and the corresponding fragmentation functions F_a^h in e^+e^- annihilation, $e^+e^- \to \gamma, Z \to h+X$ where

$$\frac{1}{\sigma_{\rm tot}} \frac{d^2\sigma}{dx\, d\cos\theta} \;=\; \frac{3}{8}(1+\cos^2\theta)\, F_T^h + \frac{3}{4}\sin^2\theta\, F_L^h + \frac{3}{4}\cos\theta\, F_A^h \; . \qquad (1)$$

Here θ represents the angle (in the center-of-mass frame) between the incoming electron beam and the hadron h observed with four-momentum p, and the scaling variable reads $x = 2pq/Q^2$ where q with $q^2 \equiv Q^2 > 0$ is the momentum of the virtual gauge boson. The transverse (T), longitudinal (L) and asymmetric (A) fragmentation functions in Eq. (1) have been measured especially at LEP, see Ref. [1] for a general overview. Disregarding corrections suppressed by inverse powers of Q^2, these observables are related to the universal fragmentation distributions D^h by

$$F_a^h(x,Q^2) \;=\; \sum_{\rm f=q,\,\bar{q},\,g} \int_x^1 \frac{dz}{z}\, c_{a,\rm f}\left(z,\alpha_{\rm s}(Q^2)\right) D_{\rm f}^h\left(\frac{x}{z},Q^2\right) . \qquad (2)$$

The coefficient functions $c_{a,\rm f}$ in Eq. (2) have been calculated by Rijken and van Neerven in Refs. [2–4] up to the next-to-next-to-leading order (NNLO) for Eq. (1), i.e., the second order in the strong coupling $a_{\rm s} \equiv \alpha_{\rm s}(Q^2)/(4\pi)$. Below we will present the results of a re-calculation of these functions by two approaches differing from that employed in Refs. [2–4].

Besides the second-order coefficient functions, a complete NNLO description also requires the third-order contributions to the splitting functions (so far calculated only up to the second order [5–7]) governing the scale dependence (evolution) of the parton fragmentation distributions. In a notation covering both the (time-like q, $\sigma = 1$) fragmentation distributions and the (space-like q, $Q^2 \equiv -q^2$, $\sigma = -1$) parton distributions, the flavour non-singlet evolution equations read

$$\frac{d}{d\ln Q^2}\, f_\sigma^{\rm ns}(x,Q^2) \;=\; \int_x^1 \frac{dz}{z}\, P_\sigma^{\rm ns}\left(z,\alpha_{\rm s}(Q^2)\right)\, f_\sigma^{\rm ns}\left(\frac{x}{z},Q^2\right) \qquad (3)$$

with

$$P_\sigma^{\rm ns}\left(x,\alpha_{\rm s}(Q^2)\right) \;=\; a_{\rm s}\, P^{(0)\,\rm ns}(x) + a_{\rm s}^2\, P_\sigma^{(1)\,\rm ns}(x) + a_{\rm s}^3 P_\sigma^{(2)\,\rm ns}(x) + \ldots \; . \qquad (4)$$

The superscript 'ns' in Eqs. (3) and (4) stands for any of the following three types of combinations of (parton or fragmentation) quark distributions,

$$f_{ik}^{\pm} \;=\; q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \; , \quad f^{\,\rm v} \;=\; \textstyle\sum_{r=1}^{n_f} (q_r - \bar{q}_r) \; , \qquad (5)$$

where n_f denotes the number of active (effectively massless) flavours. As detailed below, we have obtained the so far unknown time-like NNLO splitting functions $P_{\sigma=1}^{(2)\,\rm ns}(x)$ in Eq. (4).

As already indicated in Eq. (4), the space-like and time-like non-singlet splitting functions are identical at the leading order (LO) [8], a fact known as the Gribov-Lipatov relation. This relation does not hold beyond LO in the usual $\overline{\rm MS}$ scheme adopted also in this letter. However, the space-like and time-like cases are related by an analytic continuation in x, as shown in detailed diagrammatic analyses [5,9] at order $\alpha_{\rm s}^2$, see also Refs. [10,11]. Moreover, another approach relating the non-singlet splitting functions has been proposed in Ref. [12]. Hence it should be possible to derive time-like quantities from the space-like results computed to order $\alpha_{\rm s}^3$ in Refs. [13–15].

We start the analytic continuation from the unrenormalized (and unfactorized) partonic transverse structure function $F_1^{\rm b}$ in deep-inelastic scattering, $\gamma^* q \rightarrow X$ (and correspondingly for F_L and $F_3 \rightarrow F_A$), calculated in dimensional regularization with $D = 4-2\varepsilon$ and the scale μ [13–15],

$$F_1^{\rm b}(a_{\rm s}^{\rm b},Q^2) \;=\; \delta(1-x) + \sum_{l=1}^{\infty} (a_{\rm s}^{\rm b})^l \left(\frac{Q^2}{\mu^2}\right)^{-l\varepsilon} F_{1,l}^{\rm b} \; . \tag{6}$$

The bare and renormalized coupling $\alpha_{\rm s}^{\rm b}$ and $\alpha_{\rm s}$ are related by (recall $a_{\rm s} \equiv \alpha_{\rm s}/(4\pi)$)

$$a_{\rm s}^{\rm b} \;=\; a_{\rm s} - \frac{\beta_0}{\varepsilon} a_{\rm s}^2 + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon}\right) a_{\rm s}^3 + \ldots \tag{7}$$

with $\beta_0 = 11/3\, C_A - 2/3\, n_f$ etc. The expansion coefficients in Eq. (6) are then decomposed into form-factor ($\mathcal{F}_l$) and real-emission ($\mathcal{R}_l$, defined analogous to Eq. (6)) contributions [16]

$$\begin{aligned} F_{1,1}^{\rm b} &= 2\mathcal{F}_1\,\delta(1-x) + \mathcal{R}_1 \\ F_{1,2}^{\rm b} &= 2\mathcal{F}_2\,\delta(1-x) + (\mathcal{F}_1)^2\,\delta(1-x) + 2\mathcal{F}_1\mathcal{R}_1 + \mathcal{R}_2 \\ F_{1,3}^{\rm b} &= 2\mathcal{F}_3\,\delta(1-x) + 2\mathcal{F}_1\mathcal{F}_2\,\delta(1-x) + (\,2\mathcal{F}_2 + (\mathcal{F}_1)^2\,)\,\mathcal{R}_1 + 2\mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_3\; . \end{aligned} \tag{8}$$

The analytic continuation of the form factor to the time-like case is known. The x-dependent functions $\mathcal{R}_l$ are continued from x to $1/x$ [5,9], taking into account the (complex) continuation of q^2 (see Eq. (4.1) of Ref. [16]) and the additional prefactor $x^{1-2\varepsilon}$ originating from the phase space of the detected parton in the time-like case [3]. Practically this continuation has been performed using routines for harmonic polylogarithms [17, 18] implemented in FORM [19]. The only subtle point in the analytic continuations is the treatment of logarithmic singularities for $x \rightarrow 1$, cf. Ref. [9], starting with

$$\ln(1-x) \;\rightarrow\; \ln(1-x) - \ln x + i\pi\; . \tag{9}$$

Finally the bare transverse fragmentation function $F_T^{\rm b}$ is re-assembled analogous to Eq. (8), keeping the real parts of the continued $\mathcal{R}_l$ only, and the time-like non-singlet splitting functions and coefficient functions can be read off iteratively from the non-singlet mass factorization formula

$$\begin{aligned} F_{T,1} &= -\varepsilon^{-1} P^{(0)} + c_T^{(1)} + \varepsilon\, a_T^{(1)} + \varepsilon^2\, b_T^{(1)} + \varepsilon^3\, d_T^{(1)} + \ldots \\ F_{T,2} &= \frac{1}{2\varepsilon^2} P^{(0)}(P^{(0)}+\beta_0) - \frac{1}{2\varepsilon}\left[P_{\sigma=1}^{(1)} + 2P^{(0)}c_T^{(1)}\right] + c_T^{(2)} - P^{(0)}a_T^{(1)} + \varepsilon\left[a_T^{(2)} - P^{(0)}b_T^{(1)}\right] + \ldots \\ F_{T,3} &= -\frac{1}{6\varepsilon^3} P^{(0)}(P^{(0)}+\beta_0)(P^{(0)}+2\beta_0) + \frac{1}{6\varepsilon^2}\Big[P_{\sigma=1}^{(1)}(3P^{(0)}+2\beta_0) + P^{(0)}(3P^{(0)}c_T^{(1)} \\ &\quad + 3\beta_0 c_T^{(1)} + 2\beta_1)\Big] - \frac{1}{6\varepsilon}\left[2P_{\sigma=1}^{(2)} + 3P_{\sigma=1}^{(1)}c_T^{(1)} + P^{(0)}(6c_T^{(2)} - 3P^{(0)}a_T^{(1)} - 3\beta_0 a_T^{(1)})\right] + \ldots \end{aligned} \tag{10}$$

where the expansion coefficients $F_{T,l}$ now refer to an expansion in the renormalized coupling at the scale $Q^2 \equiv q^2$. The products of the x-dependent (generalized) functions in Eq. (10) are to be read as Mellin convolutions or, more conveniently, as products in Mellin-N space, employing routines for harmonic sums and their inverse Mellin transform back to x-space [18–20].

Unlike the diagrammatic treatment of Refs. [5, 9] — which cannot be emulated at order α_s^3 using the only available space-like calculation based on the optical theorem [13–15] — the above procedure is not entirely rigorous, as Eq. (8) does not represent a full decomposition according to the number of emitted partons. This would be required, for instance, in a subtraction formalism for exclusive observables at NNLO [21] and beyond. In Eq. (8) the real functions $\mathcal{R}_{n\geq 2}$ do not only collect, e.g., n-gluon tree-level amplitudes, but also combinations of real emission and virtual corrections. Especially, starting at order α_s^3, the decomposition Eq. (8) includes overlapping divergences from triple unresolved configurations when two particles become soft and one is collinear. Thus one has to be prepared for some problem in the abelian (C_F^3) piece related to π^2 contributions originating from phase space integrations over unresolved regions.

At the second order, however, the above procedure works perfectly at least for the terms in the ε-expansion written down in the second line of Eq. (10), thus including the previously unknown ε-coefficients $a_a^{(2)}(x)$, $a = T, L, A$, which we will not write out here for brevity. We have verified this by comparing to a direct calculation, to be presented elsewhere, of $e^+e^- \to \gamma, Z \to h + X$ to this accuracy in ε using the approach of Ref. [22]. Especially, we also re-derive the $\mathcal{O}(a_s^2)$ coefficient functions $c_a^{(2)}(x)$ which so far were only calculated in Refs. [2–4]. The differences between the time-like non-singlet coefficient functions for Eq. (1) and the corresponding (by the structure of the respective hadronic tensors) quantities in deep-inelastic scattering read

$$\begin{aligned}
c^{(2)}_{T,\mathrm{ns}}(x) - c^{(2)}_{1,\mathrm{ns}}(x) \;=\;&\\
&+C_F^2\Big(72 - 8\zeta_2 + 176/3\,\mathrm{H}_0 - 48\mathrm{H}_{0,0} + 8\mathrm{H}_2 - 72\mathrm{H}_{1,0} + (1+x)\Big[-493/6 + 12\zeta_3\\
&\quad + 10\zeta_2 - 155/6\,\mathrm{H}_0 + 16\mathrm{H}_0\zeta_2 - 6\mathrm{H}_{0,0} + 12\mathrm{H}_{0,0,0} - 20\mathrm{H}_3 - 10\mathrm{H}_2 - 4\mathrm{H}_{2,0} - 29\mathrm{H}_1\\
&\quad + 42\mathrm{H}_{1,0} - 12\mathrm{H}_{1,1}\Big] + p_{\mathrm{qq}}(x)\Big[-84\zeta_3 - 106/3\,\zeta_2 + 389/6\,\mathrm{H}_0 - 44\mathrm{H}_0\zeta_2\\
&\quad + 196/3\,\mathrm{H}_{0,0} - 24\mathrm{H}_{0,0,0} + 36\mathrm{H}_3 + 62\mathrm{H}_2 - 20\mathrm{H}_{2,0} + 48\mathrm{H}_{2,1} - 40\mathrm{H}_1\zeta_2 + 134/3\,\mathrm{H}_{1,0}\\
&\quad + 32\mathrm{H}_{1,0,0} + 56\mathrm{H}_{1,2} + 40\mathrm{H}_{1,1,0}\Big] + \delta(1-x)\Big[608/3\,\zeta_2 - 24\zeta_2^2\Big]\Big)\\
&+C_F\,(C_A - 2C_F)\Big(-76/3\,x\mathrm{H}_0 + (1+x)\Big[-215/6 + 49/6\,\mathrm{H}_0 - 9\mathrm{H}_1\Big] + p_{\mathrm{qq}}(x)\Big[12\zeta_3\\
&\quad - 44/3\,\zeta_2 + 445/6\,\mathrm{H}_0 - 20\mathrm{H}_0\zeta_2 + 44/3\,\mathrm{H}_{0,0} + 24\mathrm{H}_{0,0,0} + 4\mathrm{H}_3 + 22\mathrm{H}_2 - 4\mathrm{H}_{2,0}\\
&\quad - 16\mathrm{H}_1\zeta_2 + 22/3\,\mathrm{H}_{1,0} + 8\mathrm{H}_{1,2} - 8\mathrm{H}_{1,1,0}\Big] + p_{\mathrm{qq}}(-x)\Big[32\mathrm{H}_{-1,0,0} + 8\mathrm{H}_0\zeta_2 + 16\mathrm{H}_{-2,0}\\
&\quad - 24\mathrm{H}_{0,0,0}\Big] + \delta(1-x)\Big[466/3\,\zeta_2 - 24\zeta_2^2\Big]\Big)\\
&+C_F n_f\Big(4/3\,x\mathrm{H}_0 + (1+x)\Big[19/3 + 1/3\,\mathrm{H}_0 + 2\mathrm{H}_1\Big] + p_{\mathrm{qq}}(x)\Big[8/3\,\zeta_2 - 35/3\,\mathrm{H}_0\\
&\quad - 8/3\,\mathrm{H}_{0,0} - 4\mathrm{H}_2 - 4/3\,\mathrm{H}_{1,0}\Big] + \delta(1-x)\Big[-76/3\,\zeta_2\Big]\Big)\,, \qquad (11)
\end{aligned}$$

$$\begin{aligned}
c^{(2)}_{L,\mathrm{ns}}(x)|_{e^+e^-} - c^{(2)}_{L,\mathrm{ns}}(x)|_{ep} \;=\;&\\
&+C_F^2\Big(583/9 - 860/9\,x - (14/3 + 152/3\,x)\mathrm{H}_0 - (12+16x)\mathrm{H}_{0,0} + (74/3 - 112/3\,x)\mathrm{H}_1\\
&\quad - 16(1+x)\mathrm{H}_{1,0} + 8(1-2x)\mathrm{H}_{1,1} + 4(1+6x)\Big[\zeta_2 - \mathrm{H}_2\Big]\Big)\\
&+C_F\,(C_A - 2C_F)\Big(2317/45 + 8/5\,x^{-1} - 3752/45\,x + 32/5\,x^2 + 32x\zeta_3 + (40 + 8/5x^{-2})\mathrm{H}_{-1,0}
\end{aligned}$$

$$- 1/5(98/3+8x^{-1}+728/3\,x-32x^2)\mathrm{H}_0+16\mathrm{H}_{0,-1,0}+32(x-1/5\,x^3)\Big[\zeta_2+\mathrm{H}_{-1,0}-\mathrm{H}_{0,0}\Big]$$
$$- 8(1+2x)\Big[\mathrm{H}_{-1}\zeta_2+2\mathrm{H}_{-1,-1,0}-\mathrm{H}_{-1,0,0}\Big]+(1-2x)\Big[46/3\,\mathrm{H}_1-8\mathrm{H}_1\zeta_2+8\mathrm{H}_{1,0,0}\Big]\Big)$$
$$+C_F n_f\Big(-74/9+112/9\,x+4/3\,(1+4x)\mathrm{H}_0-4/3\,(1-2x)\mathrm{H}_1\Big)\;, \tag{12}$$

$$c^{(2)}_{A,\mathrm{ns}}(x)-c^{(2)}_{3,\mathrm{ns}}(x) \;=\; c^{(2)}_{T,\mathrm{ns}}(x)-c^{(2)}_{1,\mathrm{ns}}(x)$$
$$+C_F^2\Big((1-x)\Big[-16\zeta_2+46\mathrm{H}_0+20\mathrm{H}_{0,0}+16\mathrm{H}_2+24\mathrm{H}_{1,0}\Big]\Big)+C_F n_f\Big(-4(1-x)\mathrm{H}_0\Big)$$
$$+C_F\,(C_A-2C_F)\Big(p_{\mathrm{qq}}(-x)\Big[-64\mathrm{H}_{-1,0,0}-16\mathrm{H}_0\zeta_2-32\mathrm{H}_{-2,0}+48\mathrm{H}_{0,0,0}\Big]$$
$$+54(1-x)\mathrm{H}_0+32(1+x)\mathrm{H}_{0,0}\Big)\;. \tag{13}$$

Here our notation for the harmonic polylogarithms $H_{m_1,\ldots,m_w}(x)$, $m_j=0,\pm1$ follows Ref. [17]. Furthermore we have employed the short-hand notation

$$H_{\underbrace{0,\ldots,0}_{m},\pm1,\underbrace{0,\ldots,0}_{n},\pm1,\ldots}(x) \;=\; H_{\pm(m+1),\pm(n+1),\ldots}(x)\;, \tag{14}$$

suppressed the argument of the polylogarithms, and used the function $p_{\mathrm{qq}}(x)=2\,(1-x)^{-1}-1-x$. The divergences for $x\to1$ in Eq. (11) are to be read as plus-distributions.

Eqs. (11) - (13) agree with the results in Refs. [3,4] up to a few (presumably typographical) errors in those articles. Specifically, for $c_T^{(2)}(z)$ the term $-3C_F^2\,(1+z)\ln^2 z$ in Eq. (A.6) of Ref. [3] has to be replaced by $-3C_F^2\,(1+z)\ln^3 z$, and the contribution $24\,(C_F^2-C_AC_F/2)\,\ln z/(5z^2)$ in Eq. (A.8) by $24\,(C_F^2-C_AC_F/2)\,\ln z/(5z)$. The argument of $S_{1,2}$ should read $-z$ instead of $1-z$ in the first term of Eq. (A.15) for $c_L^{(2)}(z)$. Finally, in Eq. (17) of Ref. [4] for $c_A^{(2)}(z)$ the term $24C_F^2\ln z/(5z^2)$ has to be replaced by $24C_F^2\ln z/(5z)$. We have also re-calculated the second-order gluon and pure-singlet coefficient functions for c_T and c_L, finding complete agreement with Ref. [4].

We now turn to the corresponding NNLO (third-order) splitting functions $P^{(2)\,\mathrm{ns}}_{\sigma=1}$ in Eq. (4). Using the same notation as above these functions are given by

$$\delta P^{(2)+}(x) \;\equiv\; P^{(2)+}_{\sigma=1}(x)-P^{(2)+}_{\sigma=-1}(x) \;=$$
$$+16C_F^3\Big(p_{\mathrm{qq}}(x)\Big[311/24\,\mathrm{H}_0+4/3\,\mathrm{H}_0\zeta_2-169/9\,\mathrm{H}_{0,0}+8\mathrm{H}_{0,0}\zeta_2-22\mathrm{H}_{0,0,0}$$
$$-268/9\,\mathrm{H}_{1,0}+8\mathrm{H}_{1,0}\zeta_2-44/3\,\mathrm{H}_{1,0,0}-268/9\,\mathrm{H}_2+8\mathrm{H}_2\zeta_2-44/3\,\mathrm{H}_{2,0}-44/3\,\mathrm{H}_3\Big]$$
$$+(1+x)\Big[-4\mathrm{H}_{0,0}\zeta_2+25/2\,\mathrm{H}_{0,0,0}+\mathrm{H}_{2,0}+2\mathrm{H}_3\Big]-(1-x)\Big[325/18\,\mathrm{H}_0+50/3\,\mathrm{H}_{1,0}$$
$$+50/3\,\mathrm{H}_2\Big]+(3-5x)\mathrm{H}_0\zeta_2-(173/18-691/18\,x)\mathrm{H}_{0,0}\Big)$$
$$+16C_F^2\,(C_A-2C_F)\Big(p_{\mathrm{qq}}(x)\Big[151/24\,\mathrm{H}_0+\mathrm{H}_0\zeta_3+13/6\,\mathrm{H}_0\zeta_2-169/18\,\mathrm{H}_{0,0}+8\mathrm{H}_{0,0}\zeta_2$$
$$-13/2\,\mathrm{H}_{0,0,0}-8\mathrm{H}_{0,0,0,0}-134/9\,\mathrm{H}_{1,0}+4\mathrm{H}_{1,0}\zeta_2-22/3\,\mathrm{H}_{1,0,0}-6\mathrm{H}_{1,0,0,0}-134/9\,\mathrm{H}_2$$
$$+4\mathrm{H}_2\zeta_2-22/3\,\mathrm{H}_{2,0}-2\mathrm{H}_{2,0,0}-22/3\,\mathrm{H}_3-2\mathrm{H}_{3,0}-6\mathrm{H}_4\Big]+p_{\mathrm{qq}}(-x)\Big[-8\mathrm{H}_{-3,0}$$
$$+8\mathrm{H}_{-2}\zeta_2+8\mathrm{H}_{-2,-1,0}+3\mathrm{H}_{-2,0}-14\mathrm{H}_{-2,0,0}-4\mathrm{H}_{-2,2}+8\mathrm{H}_{-1,-2,0}+16\mathrm{H}_{-1,-1,0,0}$$
$$+8\mathrm{H}_{-1,0}\zeta_2+6\mathrm{H}_{-1,0,0}-18\mathrm{H}_{-1,0,0,0}-4\mathrm{H}_{-1,2,0}-8\mathrm{H}_{-1,3}-7\mathrm{H}_0\zeta_3+3/2\,\mathrm{H}_0\zeta_2$$

$$
\begin{aligned}
&-8\mathrm{H}_{0,0}\zeta_2 - 9/2\,\mathrm{H}_{0,0,0} + 8\mathrm{H}_{0,0,0,0} + 2\mathrm{H}_{3,0} + 6\mathrm{H}_4\Big] - (1+x)\Big[4\mathrm{H}_{-2,0} + 8\mathrm{H}_{-1,0,0}\Big] \\
&+ (1-x)\Big[4\mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,0,0} - 88/9\,\mathrm{H}_0 + 3\mathrm{H}_0\zeta_3 - 28/3\,\mathrm{H}_{1,0} - 28/3\,\mathrm{H}_2\Big] - 4x\mathrm{H}_0\zeta_2 \\
&- (50/9 - 184/9\,x)\mathrm{H}_{0,0} - 4x\mathrm{H}_{0,0}\zeta_2 + (11/2 + 35/2\,x)\mathrm{H}_{0,0,0} + 8x\mathrm{H}_{0,0,0,0}\Big) \\
&+ 16 C_F^2 n_f \Big(p_{\mathrm{qq}}(x)\Big[-11/12\,\mathrm{H}_0 - 2/3\mathrm{H}_0\zeta_2 + 11/9\,\mathrm{H}_{0,0} + 2\mathrm{H}_{0,0,0} + 20/9\,\mathrm{H}_{1,0} \\
&+ 4/3\,\mathrm{H}_{1,0,0} + 20/9\,\mathrm{H}_2 + 4/3\,\mathrm{H}_{2,0} + 4/3\,\mathrm{H}_3\Big] - (1+x)\,\mathrm{H}_{0,0,0} + (1-x)\Big[13/9\,\mathrm{H}_0 \\
&+ 4/3\,\mathrm{H}_{1,0} + 4/3\,\mathrm{H}_2\Big] + (8/9 - 28/9\,x)\mathrm{H}_{0,0}\Big)\;, \qquad (15)
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta P^{(2)\xi}(x) - \delta P^{(2)+}(x) \;\equiv \\
&+ 16 C_F^2 (C_A - 2C_F)\Big(p_{\mathrm{qq}}(-x)\Big[16\mathrm{H}_{-3,0} - 16\mathrm{H}_{-2}\zeta_2 - 16\mathrm{H}_{-2,-1,0} - 6\mathrm{H}_{-2,0} + 28\mathrm{H}_{-2,0,0} \\
&+ 8\mathrm{H}_{-2,2} - 16\mathrm{H}_{-1,-2,0} - 32\mathrm{H}_{-1,-1,0,0} - 16\mathrm{H}_{-1,0}\zeta_2 - 12\mathrm{H}_{-1,0,0} + 36\mathrm{H}_{-1,0,0,0} + 8\mathrm{H}_{-1,2,0} \\
&+ 16\mathrm{H}_{-1,3} + 14\mathrm{H}_0\zeta_3 - 3\mathrm{H}_0\zeta_2 + 16\mathrm{H}_{0,0}\zeta_2 + 9\mathrm{H}_{0,0,0} - 16\mathrm{H}_{0,0,0,0} - 4\mathrm{H}_{3,0} - 12\mathrm{H}_4\Big] \\
&+ (1+x)\Big[8\mathrm{H}_{-2,0} + 16\mathrm{H}_{-1,0,0} + 8\mathrm{H}_0\zeta_2 - 4\mathrm{H}_{2,0} - 8\mathrm{H}_3\Big] - (1-x)\Big[8\mathrm{H}_{-3,0} + 8\mathrm{H}_{-2,0,0} + 10\mathrm{H}_0 \\
&+ 6\mathrm{H}_0\zeta_3 + 4\mathrm{H}_{0,0}\zeta_2 - 8\mathrm{H}_{0,0,0,0} + 8\mathrm{H}_{1,0} + 8\mathrm{H}_2\Big] - (10 - 6x)\mathrm{H}_{0,0} - (12 + 24x)\mathrm{H}_{0,0,0}\Big) \qquad (16)
\end{aligned}
$$

for both $\xi = -$ and $\xi = \mathrm{v}$. Eq. (15) is, in fact, not quite the result of the analytic continuation as described above, which returns a different coefficient for the term $C_F^3\, p_{\mathrm{qq}}(x)\,\mathrm{H}_{0,0}\,\zeta_2$ in the first line. We have corrected this term by imposing the correct (vanishing) first moment of $P^{(2)-}_{\sigma=1}$. As it is conceivable that the present form of the analytic continuation leads to other problems not affecting the first moment — recall the discussion in the paragraph below Eq. (10) — we obviously need a second, independent confirmation of our new results (15) and (16).

For this purpose we adopt the approach of Dokshitzer, Marchesini and Salam [12] (see also Appendix B1 of Ref. [23]), where Eq. (3) is rewritten as[1]

$$
\frac{d}{d\ln Q^2}\, f_\sigma^{\mathrm{ns}}(x,Q^2) \;=\; \int_x^1 \frac{dz}{z}\, P^{\mathrm{ns}}_{\mathrm{univ}}\left(z,\alpha_{\mathrm{s}}(Q^2)\right)\, f_\sigma^{\mathrm{ns}}\left(\frac{x}{z}, z^\sigma Q^2\right)\,, \qquad (17)
$$

and the modified splitting functions $P^{\mathrm{ns}}_{\mathrm{univ}}$ are postulated to be identical for the time-like and space-like cases. Working out the perturbative expansion of the integrand along the lines of Ref. [12] one arrives at a successful 'postdiction' for the NLO difference $P^{(1)\,\mathrm{ns}}_{\sigma=1}(x) - P^{(1)\,\mathrm{ns}}_{\sigma=-1}(x)$ of Refs. [5, 7] (see Eq. (4) of Ref. [12]) and the new NNLO prediction ($\xi = +, -, \mathrm{v}$, recall Eq. (5))

$$
\delta P^{(2)\xi}(x) \;=\; 2\left\{\left[\ln x \cdot \widetilde{P}^{(1)\xi}\right] \otimes P^{(0)} + \left[\ln x \cdot P^{(0)}\right] \otimes \widetilde{P}^{(1)\xi}\right\} \qquad (18)
$$

with $\otimes$ denoting the Mellin convolution (cf. Eqs. (2) and (3)) and

$$
2\widetilde{P}^{(n)\xi}(x) \;=\; P^{(n)\xi}_{\sigma=1}(x) + P^{(n)\xi}_{\sigma=-1}(x)\;. \qquad (19)
$$

[1]Ref. [12] also includes a shift in the argument of α_{s} in Eq. (17) which is irrelevant for our purpose of relating the time-like and space-like results. Note also that the notation for the α_{s} expansion in Ref. [12] differs from Eq. (4).

The evaluation of Eq. (18), again performing the convolutions via a transformation to Mellin-N space, yields exactly Eqs. (15) and (16), thus providing both the desired confirmation of these results and further evidence supporting the ansatz (17).

Consequently, it is possible to make even a prediction for the fourth-order (N^{3}LO) difference $\delta P^{(3)\xi}$ of the (both unknown) time-like and space-like non-singlet splitting functions on this basis. Using the notation of Eq. (18) together with $A^{\otimes 2} \equiv A \otimes A$ etc, this prediction reads

$$\begin{aligned}\delta P^{(3)\xi}(x) \;=\; & 2\Big\{[\ln x \cdot \widetilde{P}^{(2)\xi}] \otimes P^{(0)} + [\ln x \cdot P^{(0)}] \otimes \widetilde{P}^{(2)\xi} + [\ln x \cdot \widetilde{P}^{(1)\xi}] \otimes \widetilde{P}^{(1)\xi}\Big\} \\ & - 2\,P^{(0)} \otimes [\ln x \cdot P^{(0)}]^{\otimes 3} - 4\,[P^{(0)}]^{\otimes 2} \otimes [\ln x \cdot P^{(0)}] \otimes [\ln^2 x \cdot P^{(0)}] \\ & - 2/3\;[P^{(0)}]^{\otimes 3} \otimes [\ln^3 x \cdot P^{(0)}]\;. \end{aligned} \tag{20}$$

For brevity we refrain from writing out the resulting explicit expressions which are, of course, much more lengthy than Eq. (15). We expect that a first check of Eq. (20), or rather its first line (which dominates the large-x behaviour, cf. Ref. [12]), will be obtained via next-to-leading order calculations in the large-n_f expansion, generalizing the leading-n_f result of Ref. [24]. The corresponding contribution to Eq. (20) is identical for all three non-singlet cases and reads

$$\begin{aligned}&\delta P^{(3),\mathrm{ns}}(x)\,\big|_{n_f^2} \;\equiv\; P^{(3),\mathrm{ns}}_{\sigma=1}(x) - P^{(3),\mathrm{ns}}_{\sigma=-1}(x) \;= \\ &16/81\, C_F^2\, n_f^2\, \Big(p_{\mathrm{qq}}(x)\,[\,-(159/4 - 120\zeta_2 + 36\zeta_3)\mathrm{H}_0 + (23 + 72\zeta_2)\mathrm{H}_{0,0} - 279\mathrm{H}_{0,0,0} \\ &\quad - 216\mathrm{H}_{0,0,0,0} - 76(\mathrm{H}_{1,0} + \mathrm{H}_2) - 240(\mathrm{H}_{1,0,0} + \mathrm{H}_{2,0} + \mathrm{H}_3) - 108(\mathrm{H}_{1,0,0,0} + \mathrm{H}_{2,0,0} \\ &\quad + \mathrm{H}_{3,0} + \mathrm{H}_4)\,] + (1-x)\,[\,-(260 - 72\zeta_2)\mathrm{H}_0 - 276(\mathrm{H}_{1,0} + \mathrm{H}_2) - 144(\mathrm{H}_{1,0,0} + \mathrm{H}_{2,0} \\ &\quad + \mathrm{H}_3)\,] + (1+x)\,108\mathrm{H}_{0,0,0,0} - (466 - 398x)\mathrm{H}_{0,0} - (90 - 450x)\mathrm{H}_{0,0,0}\Big)\;. \end{aligned} \tag{21}$$

The $a_s^n C_F^2 n_f^{n-2}$ contributions dominating $\delta P^{(n-1),\mathrm{ns}}$ in the large-n_f limit are given to all higher orders n by a straightforward generalization of Eq. (18) and the first line of Eq. (20).

Returning to the NNLO coefficient-function and splitting-function differences (11) – (16), we note that these functions include harmonic polylogarithms up to the same weights as the corresponding space-like results [13–15,18], with the interesting exception that weight-4 functions enter Eq. (15) only for the SU(n_c) group-factor combination $C_A - 2C_F$ suppressed as $1/n_c$ in the limit of a large number of colours n_c. Except for the longitudinal coefficient function $c_{L,\mathrm{ns}}$, the differences between the time-like and space-like quantities are parametrically suppressed in the large-x limit, since the logarithmically enhanced soft-emission contributions to $\mathcal{R}_l$ in Eq. (8) are, as they have to be, invariant under the analytic continuation. Specifically, the (identical) leading large-x contributions for $c^{(2)}_{T,\mathrm{ns}} - c^{(2)}_{1,\mathrm{ns}}$ and $c^{(2)}_{A,\mathrm{ns}} - c^{(2)}_{3,\mathrm{ns}}$ contain plus-distributions only up to $[(1-x)^{-1}\ln(1-x)]_+$ (all proportional to π^2-terms arising from the analytic continuation of the form factor), and those for the splitting-function differences read,

$$\delta P^{(2),\mathrm{ns}}(x \to 1) \;=\; -4\,A_{\mathrm{q}}^{(1)} A_{\mathrm{q}}^{(2)} \ln(1-x) \,+\, \mathcal{O}(1)\;, \tag{22}$$

as predicted in Ref. [12], where $A_{\mathrm{q}}^{(n)}$ are the coefficients of $a_s^n\,[1-x]_+^{-1}$ in Eq. (4), cf. Ref. [13]. The leading small-x terms, on the other hand, differ between the space-like and time-like cases.

The numerical impact of the second-order contributions to the time-like coefficient functions $c_a^{(2)}(x)$, $a = T, L, A$, has been discussed in some detail already in Refs. [2–4]. Here we confine ourselves to the transverse fragmentation function F_T, the largest contribution to the right-hand-side of Eq. (1), see Ref. [1]. In Fig. 1 we compare the corresponding non-singlet coefficient function $c_{T,\text{ns}}$ to its counterpart $c_{1,\text{ns}}$ in deep-inelastic scattering. In order to facilitate a direct comparison, the same schematic shape has been used for the non-singlet fragmentation distributions and parton distributions. As obvious from the figure, the higher-order corrections for c_T are in general considerably larger than those for c_1. Nevertheless the second-order term changes the NLO results, under the conditions of Fig. 1, by 5% or less from $x \simeq 0.55$ down to very small values of x.

The pattern is quite different for the non-singlet splitting functions illustrated, in a similar manner but at a lower scale, in Fig. 2. In N-space, for example, the ratio $\delta P^{\text{ns}}/(P^{\text{ns}}_{\sigma=-1} - P^{\text{ns}}_{\text{LO}})$ quickly decreases with increasing N, at NNLO from about 1/2 at $N = 2$ to about 1/6 at $N = 8$. Consequently the total time-like splitting functions $P^{\text{ns}}_{\sigma=1}$ is only mildly enhanced, e.g., by 8% and 2% for these two values of N and $\alpha_s = 0.2$, with respect to their space-like counterparts $P^{\text{ns}}_{\sigma=-1}$ discussed in detail in Ref. [13]. As shown in the right part of Fig. 2, the small-x scaling violations of the non-singlet fragmentation distributions are weaker than those of the parton distributions. For the chosen input distribution this reduction increases, in a perturbatively stable manner, from about 10% at $x = 10^{-2}$ to about 30% at $x = 10^{-4}$.

To summarize, we have re-derived the $\mathcal{O}(\alpha_s^2)$ coefficient functions [2–4] for the inclusive production of single hadrons in e^+e^- annihilation [1] and obtained, for the first time, the corresponding third-order splitting functions for the flavour non-singlet fragmentation distributions. Our derivation of the latter quantities rests on relations between the time-like and space-like cases, see especially Refs. [5, 7, 9, 12], and the third-order calculation of deep-inelastic scattering of Ref. [13–15]. We expect that a further study of these relations, backed up by fixed Mellin-N calculations along the lines of Ref. [22], will facilitate an extension of our derivation to the NNLO flavour-singlet splitting functions and, at least for F_L, the $\mathcal{O}(\alpha_s^3)$ coefficient functions.

Once this step has been taken, the way is open for full NNLO analyses, e.g., along the lines of Ref. [25], of high-precision data on $e^+e^- \to h + X$ from LEP and a future International Linear Collider. Due to the universality of the splitting functions, our results also represents a first step towards NNLO analyses of high-p_T hadron production in ep and pp collisions, where very large NLO corrections strongly suggest sizeable higher-order contributions, see Refs. [26–28] and [29–31], respectively. Another application concerns the b-quark spectrum in top decays (see Ref. [32]) where, after the calculations of Refs. [33, 34], the third-order time-like splitting functions will facilitate a complete NNLO treatment in the framework of perturbative fragmentation.

FORM and FORTRAN files of our results can be obtained from `http://arXiv.org` by downloading the source of this article. Furthermore they are available from the authors upon request.

Acknowledgments: We are grateful to L. Dixon, P. Uwer and W. Vogelsang for useful discussions. Our numerical results have been computed using the FORTRAN package of Ref. [35]. A.M. acknowledges support by the Alexander von Humboldt Foundation. The work of S.M. has been supported in part by the Helmholtz Gemeinschaft under contract VH-NG-105.

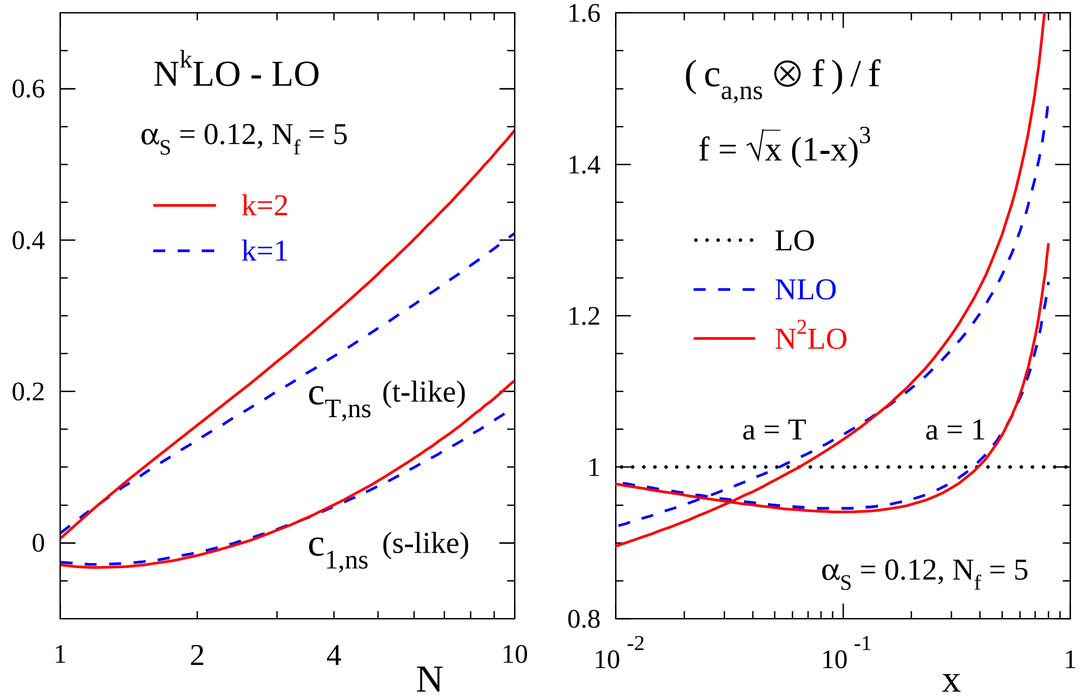

Figure 1: Comparison of the coefficient function $c_{T,\mathrm{ns}}$ for the (time-like) process $e^+e^- \rightarrow h+X$ with its counterpart $c_{1,\mathrm{ns}}$ in (space-like) deep-inelastic scattering for $Q^2 \simeq M_Z^2$. Left plot: Mellin moments, right plot: convolutions (2) with a schematic input shape denoted by f.

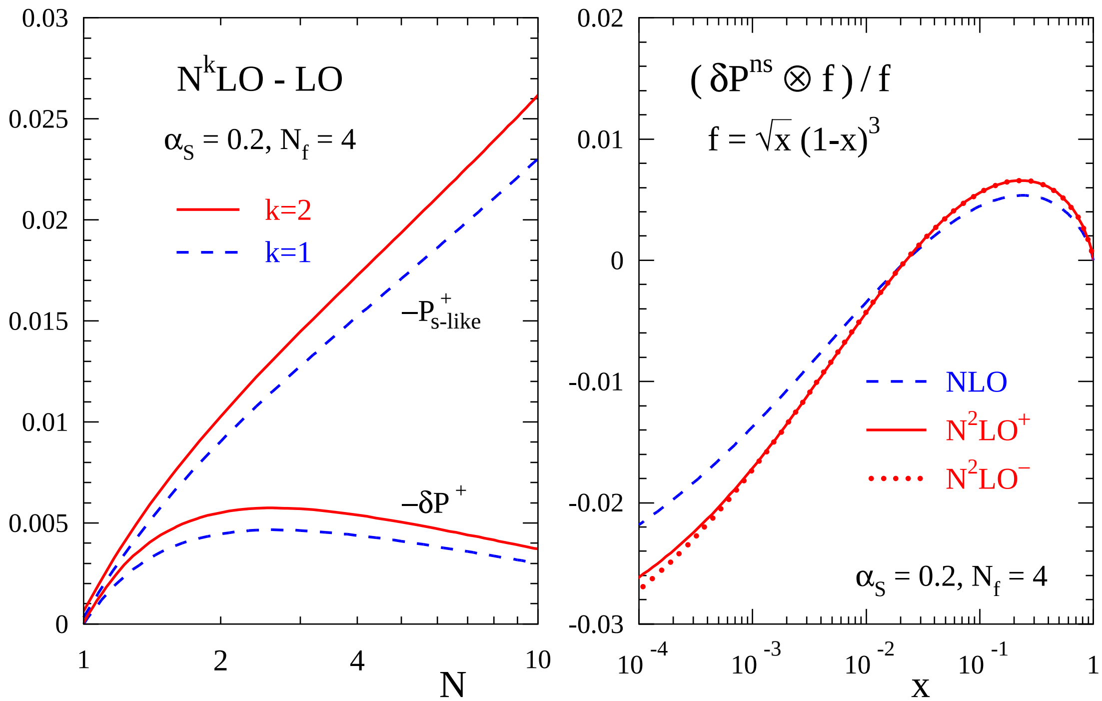

Figure 2: The differences $\delta P^{\mathrm{ns}} = P^{\mathrm{ns}}_{\sigma=1} - P^{\mathrm{ns}}_{\sigma=-1}$ between the time-like ($\sigma = 1$) and space-like ($\sigma = -1$) non-singlet splitting functions at a 'low' scale characterized by the (order-independent) value $\alpha_s = 0.2$ of the strong coupling constant. Left: moment-space comparison with the higher-order corrections in the space-like case. Right: convolutions with a schematic input shape.

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