$O(\alpha_s^2)$ Timelike Wilson Coefficients for Parton-Fragmentation Functions in Mellin Space

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Abstract

We calculate the Mellin moments of the $O(\alpha_s^2)$ coefficient functions for the unpolarized and polarized fragmentation functions. They can be expressed in terms of multiple finite harmonic sums of maximal weight $w = 4$. Using algebraic and structural relations between harmonic sums one finds that besides the single harmonic sums only three basic sums and their derivatives w.r.t. the summation index contribute. The Mellin moments are analytically continued to complex values of the Mellin variable. This representation significantly reduces the large complexity being present in x -space calculations and allow very compact and fast numerical implementations.

1 Introduction

The production of hadrons in annihilation processes such as e^+e^- or quark-antiquark and gluon-gluon scattering is described by parton-hadron fragmentation in the final state of these processes. Due to the factorization theorems perturbative Quantum Chromodynamics describes the hard scattering sub-processes at the level of leading twist, while hadron formation requires non-perturbative functions, the fragmentation densities $D_p^H(z, M^2)$. At leading order in the coupling constant they describe the probability that a final state parton p involved in the hard scattering process produces a hadron H . Here z denotes the momentum fraction of the parton in the hadron and M^2 is the respective factorization scale. At higher orders the fragmentation process is described by fragmentation functions which are formed of Mellin convolutions of the fragmentation densities and the corresponding time-like Wilson coefficients.

The process of e^+e^- annihilation into hadrons provides the cleanest environment to measure the fragmentation functions and to extract the individual fragmentation densities :

$$e^+ + e^- \rightarrow \overline{H} + X . \quad (1)$$

By crossing from s - to t -channel this process transforms into deeply inelastic scattering of an electron off the hadron H :

$$e^- + H \rightarrow e^- + X . \quad (2)$$

Both processes are related by this transformation and the question arises for similarities or transformation properties of characteristic quantities contributing to each of the reactions. This question has been raised early [1] concerning both the non-perturbative and perturbative components. Crossing of the non-perturbative fragmentation- and structure functions is not expected to hold [2]. However, one may ask the question for the perturbative part order by order in the strong coupling constant $a_s = \alpha_s/(4\pi)$. Here it is not expected that the crossing holds for the Wilson coefficients and splitting functions defined in a specific factorization scheme, as e.g. the $\overline{\text{MS}}$ scheme, individually. However, physical quantities as physical evolution kernels K_{ij} obey the crossing relations. These functions describe the scale evolution of observables and are scheme-independent. To NLO they were derived in [3–6] for polarized and unpolarized targets (final state hadrons) referring to different observables as the structure (fragmentation) functions and their slope and combining the longitudinal and transverse projections F_2 and F_L . Recently the scheme-invariant evolution kernels were derived at NNLO in [7] for the space-like case¹. In Ref. [6] the conditions were derived up to the 2-loop Wilson coefficients. The crossing of the 2-loop splitting functions was studied [8] based on Refs. [9].

To be capable to construct physical evolution kernels in the space- and time-like cases, the complexity of the respective integrals requires to choose the Mellin space representation. For the whole set of the time-like Wilson coefficients so far only the x -space representation has been available [10–13]². In this paper we derive the Mellin-representation of the 2-loop Wilson coefficients. As observed in case of other single-scale quantities at the 2- and 3-loop level before [14–18] a considerable structural simplification for these quantities is obtained w.r.t. the number of contributing functions. In the Mellin-space representation the evolution equations can be solved analytically and fast and precise numerical implementations are obtained [19,20]. The 2-loop Wilson coefficients depend only on a few universal basic functions, for which representations in the complex Mellin-plane have to be derived.

¹The structure in the time-like case is analogous.

²In the present analysis typos contained in the print were accounted for.

The paper is organized as follows. In section 2 the differential cross sections for the process $e^+e^- \rightarrow H + X$ are summarized and expressed in terms of fragmentation densities and coefficient functions to $O(\alpha_s^2)$ in the unpolarized and the polarized case. In section 3 we derive the Mellin moments for the coefficient functions and discuss their structure w.r.t. the set of basic functions emerging. It turns out that this is a subset of the functions needed in the space-like and related cases [16, 18]. The analytic continuations for these functions to complex values of the Mellin variable N was given before in [21]. Section 4 contains the conclusions. In the Appendices A and B the explicit representations of the Wilson coefficients in the unpolarized and polarized case in Mellin space are presented.

2 Coefficient Functions

Let us consider the differential scattering cross section for e^+e^- -annihilation into a hadron H and the hadronic remainder part X

$$e^+ + e^- \rightarrow \gamma, Z \rightarrow H + 'X' . \quad (3)$$

This reaction is either studied for unpolarized or polarized leptons. If in the latter case the polarization of the produced hadron H is measured, one can form the polarization asymmetry. In the following we will consider both the case of unpolarized scattering and the final state polarization asymmetry in the polarized case.

2.1 Unpolarized Case

$$\frac{d^2\sigma^H}{dx d\cos\theta} = \frac{3}{8} (1 + \cos^2\theta) \frac{d\sigma_T^H}{dx} + \frac{3}{4} \sin^2\theta \frac{d\sigma_L^H}{dx} + \frac{3}{4} \cos\theta \frac{d\sigma_A^H}{dx} . \quad (4)$$

Here $x = 2p \cdot q / Q^2$ denotes the scaling variable with $Q^2 = q^2 > 0$ and θ is scattering angel of the produced hadron. p denotes the momentum of the produced hadron and q is the virtuality of the exchanged $\gamma(Z)$. x denotes the fraction of beam energy carried away by the produced hadron H . The differential scattering cross section (4) consists of three contributions $\sigma_{T,L,A}^H$ corresponding to the transverse (T), longitudinal (L) and the asymmetric (A) part due to $\gamma - Z$ interference and pure Z -boson exchange. The individual contributions to the differential cross sections are given by

$$\begin{aligned} \frac{d\sigma_k^H}{dx} = & \sigma_{\text{tot}}^{(0)}(Q^2) \left[D_S^H(x, M^2) \otimes \mathbb{C}_{k,q}^S \left(x, \frac{Q^2}{M^2} \right) + D_g^H(x, M^2) \otimes \mathbb{C}_{k,g}^S \left(x, \frac{Q^2}{M^2} \right) \right] \\ & + \sum_{l=1}^{N_f} \sigma_l^{(0)}(Q^2) D_{\text{NS},l}^H(x, M^2) \otimes \mathbb{C}_{k,q}^{\text{NS}} \left(x, \frac{Q^2}{M^2} \right) , \end{aligned} \quad (5)$$

with $k = T, L$. The asymmetric cross section is given by

$$\frac{d\sigma_A^H}{dx} = \sum_{l=1}^{N_f} A_l^{(0)}(Q^2) D_{A,l}^H(x, M^2) \otimes \mathbb{C}_{A,q}^{\text{NS}} \left(x, \frac{Q^2}{M^2} \right) . \quad (6)$$

N_f denotes the number of flavors. The Mellin convolution $\otimes$ is defined by

$$[A \otimes B](x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) . \quad (7)$$

In (5, 6) the flavor singlet and non-singlet combinations of the non-perturbative fragmentation densities $D_i(z, M^2)$ contribute, which are given by

$$D_S^H(z, M^2) = \frac{1}{N_f} \sum_{l=1}^{N_f} [D_l^H(z, M^2) + D_{\bar{l}}^H(z, M^2)] , \quad (8)$$

$$D_{NS,l}^H(z, M^2) = D_l^H(z, M^2) + D_{\bar{l}}^H(z, M^2) - D_S^H(z, M^2), \quad (9)$$

$$D_{A,l}^H(z, M^2) = D_l^H(z, M^2) - D_{\bar{l}}^H(z, M^2) , \quad (10)$$

with M^2 the factorization scale. The electro-weak pointlike cross sections are

$$\sigma_l^{(0)}(Q^2) = \frac{4\pi\alpha^2}{3Q^2} N_C \left[e_L^2 e_l^2 + \frac{2Q^2(Q^2 - M_Z^2)}{|Z(Q^2)|^2} e_L e_l C_{V,L} C_{V,l} + \frac{(Q^2)^2}{|Z(Q^2)|^2} (C_{V,L}^2 + C_{A,L}^2)(C_{V,l}^2 + C_{A,l}^2) \right], \quad (11)$$

$$\sigma_{\text{tot}}^{(0)}(Q^2) = \sum_{l=1} N_f \sigma_l^{(0)}(Q^2), \quad (12)$$

with

$$Z(Q^2) = Q^2 - M_Z^2 + iM_Z\Gamma_Z , \quad (13)$$

and M_Z, Γ_Z denote the mass and width of the Z -boson. The electric charges are $e_L = -1, e_u = 2/3, e_d = -1/3$ and $C_{V,(A),L}, C_{V,(A),l}$ are the electro-weak couplings of the charged lepton and the quarks, respectively. $N_C = 3$ denotes the number of colors. Correspondingly the asymmetry factor reads

$$A_l^{(0)} = \frac{4\pi\alpha^2}{3Q^2} N_C \left[\frac{2Q^2(Q^2 - M_Z^2)}{|Z(Q^2)|^2} e_L e_l C_{A,L} C_{A,l} + 4 \frac{(Q^2)^2}{|Z(Q^2)|^2} C_{A,L} C_{A,l} C_{V,L} C_{V,l} \right] . \quad (14)$$

The non-singlet and singlet coefficient functions are denoted by $\mathbb{C}_{l,q(g)}^S(x, Q^2/M^2)$ and $\mathbb{C}_{l,q(A)}^{\text{NS}}(x, Q^2/M^2)$. To $O(a_s^2)$ they are given by

$$\mathbb{C}_{L,q}^{\text{NS}} = a_s c_{L,q}^{(1)} + a_s^2 \left\{ \left[\frac{1}{2} P_{qq}^{(0)} \otimes c_{L,q}^{(1)} - \beta_0 c_{L,q}^{(1)} \right] \ln \left(\frac{Q^2}{M^2} \right) + c_{L,q}^{\text{NS},(2),nid} + c_{L,q}^{\text{NS},(2),id} \right\} , \quad (15)$$

$$\begin{aligned} \mathbb{C}_{T,q}^{\text{NS}} = & 1 + a_s \left[\frac{1}{2} P_{qq}^{(0)} \ln \left(\frac{Q^2}{M^2} \right) + c_{T,q}^{(1)} \right] + a_s^2 \left\{ \left[\frac{1}{8} P_{qq}^{(0)} \otimes P_{qq}^{(0)} - \frac{1}{4} \beta_0 P_{qq}^{(0)} \right] \ln^2 \left(\frac{Q^2}{M^2} \right) \right. \\ & + \left[\frac{1}{2} (P_{qq}^{(1),\text{NS}} + P_{q\bar{q}}^{(1),\text{NS}}) - \beta_0 c_{T,q}^{(1)} + \frac{1}{2} P_{qq}^{(0)} \otimes c_{T,q}^{(1)} \right] \ln \left(\frac{Q^2}{M^2} \right) \\ & \left. + c_{T,q}^{\text{NS},(2),nid} + c_{T,q}^{\text{NS},(2),id} \right\} , \end{aligned} \quad (16)$$

$$\mathbb{C}_{A,q}^{\text{NS}} = \mathbb{C}_{A,q}^{\text{NS},nid(2)} - \mathbb{C}_{A,q}^{\text{NS},id(2)} . \quad (17)$$

and

$$\mathbb{C}_{k,F}^S = \mathbb{C}_{k,F}^{\text{NS}} + \mathbb{C}_{k,F}^{\text{PS}}, \quad F = L, T \quad (18)$$

$$\mathbb{C}_{L,q}^{\text{PS}} = N_f a_s^2 \left[\left\{ \frac{1}{4} P_{qq}^{(0)} \otimes c_{L,q}^{(1)} \right\} \ln \left(\frac{Q^2}{M^2} \right) + c_{L,q}^{\text{PS},(2)} \right] \quad (19)$$

$$\begin{aligned} \mathbb{C}_{T,q}^{\text{PS}} = & N_f a_s^2 \left[\left\{ \frac{1}{8} P_{qq}^{(0)} \otimes P_{qq}^{(0)} \right\} \ln^2 \left(\frac{Q^2}{M^2} \right) + \left\{ \frac{1}{2} P_{qq}^{\text{PS},(1)} + \frac{1}{4} P_{qq}^{(0)} \otimes c_{T,g}^{(1)} \right\} \ln \left(\frac{Q^2}{M^2} \right) \right. \\ & \left. + c_{T,q}^{\text{PS},(2)} \right] \end{aligned} \quad (20)$$

$$\mathbb{C}_{L,g}^{\text{S}} = a_s c_{L,g}^{(1)} + a_s^2 \left[\left\{ -\beta_0 c_{L,g}^{(1)} + \frac{1}{2} P_{gg}^{(0)} \otimes c_{L,g}^{(1)} + P_{qq}^{(0)} \otimes c_{L,q}^{(1)} \right\} \ln \left(\frac{Q^2}{M^2} \right) + c_{L,g}^{(2)} \right] \quad (21)$$

$$\begin{aligned} \mathbb{C}_{T,g}^{\text{S}} = & a_s \left[P_{qq}^{(0)} \ln \left(\frac{Q^2}{M^2} \right) + c_{T,g}^{(0)} \right] \\ & + a_s^2 \left[\left\{ \frac{1}{4} P_{qq}^{(0)} \otimes (P_{gg}^{(0)} + P_{qq}^{(0)}) - \frac{1}{2} \beta_0 P_{qq}^{(0)} \right\} \ln^2 \left(\frac{Q^2}{M^2} \right) \right. \\ & \left. + \left\{ P_{qq}^{(1)} - \beta_0 c_{T,g}^{(1)} + \frac{1}{2} P_{gg}^{(0)} \otimes c_{T,g}^{(1)} + P_{qq}^{(0)} \otimes c_{T,q}^{(1)} \right\} \ln \left(\frac{Q^2}{M^2} \right) + c_{T,g}^{(2)} \right] \end{aligned} \quad (22)$$

Here $a_s = \alpha_s/(4\pi) = g_s^2/(4\pi)^2$ and g_s denotes the strong coupling constant. $P_{jk}^{(l)}$ are the $l+1$ -st order QCD splitting functions and $\beta_0 = 11C_A/3 - (4/3)T_R N_f$ with $C_A = 3, T_R = 1/2$ for $SU(3)_c$. The superscripts ‘*id*’ and ‘*nid*’ refer to identical quark contributions resp. the remainder part. The corresponding Wilson coefficients to $O(\alpha_s^2)$ were calculated in Refs. [10–12].

2.2 Polarized Case

The polarization asymmetry for the fragmentation process was calculated in [13] to $O(a_s^2)$. The hadronic tensor of the process is obtained from the hadronic tensor in polarized electro-weak deeply inelastic scattering [22] by crossing from t to s -channel. In general five fragmentation functions $g_i^H(x, Q^2)|_{i=1}^5$ contribute to the scattering cross section. If we limit the analysis to the level of twist-2 fragmentation functions the cross section is determined by $g_{1,4,5}^H$. Here $g_{4,5}^H$ are non-singlet fragmentation functions which contribute to the $\gamma - Z$ and $|Z|^2$ -exchange terms. The corresponding coefficient functions are the same as for the fragmentation functions F_2 and F_1 , respectively, in the unpolarized case due to the structure of the hadronic tensor, cf. [22]. In the following, we will therefore consider the case of pure photon exchange only. If the scattering cross section is integrated over the azimuthal angle ϕ of the produced hadron the differential cross section is given by

$$\frac{d\sigma^{H(\downarrow)}(\downarrow)}{dx d\cos\theta} - \frac{d\sigma^{H(\downarrow)}(\uparrow)}{dx d\cos\theta} = N_C \frac{\pi\alpha^2}{Q^2} \cos\theta \ g_1^H(x, Q^2) . \quad (23)$$

The fragmentation function g_1^H is given in terms of polarized fragmentation densities $\Delta_i^{H,S(\text{NS})}$ and Wilson coefficients $\Delta\mathbb{C}_{1,i}^{S(\text{NS})}$,

$$\begin{aligned} g_1^H(x, Q^2) = & \frac{1}{N_f} \sum_{k=1}^{N_f} e_k^2 \left[\Delta_q^{H,S}(x, M^2) \otimes \mathbb{C}_{1,q}^{\text{S}} \left(x, \frac{Q^2}{M^2} \right) + \Delta_g^{H,S}(x, M^2) \otimes \mathbb{C}_{T,g}^{\text{S}} \left(x, \frac{Q^2}{M^2} \right) \right. \\ & \left. + N_f \Delta_q^{H,\text{NS}}(x, M^2) \otimes \mathbb{C}_{1,q}^{\text{NS}} \left(x, \frac{Q^2}{M^2} \right) \right] . \end{aligned} \quad (24)$$

The polarized fragmentation densities are

$$\Delta_a^H(z, M^2) = D_{a\downarrow}^{H\downarrow}(z, M^2) - D_{a\downarrow}^{H\uparrow}(z, M^2), \quad a = q, \bar{q}, g . \quad (25)$$

The respective singlet and non-singlet combinations are given by

$$\Delta_a^{H,S}(z, M^2) = \sum_{k=1}^{N_f} \Delta_k^H(z, M^2) + \Delta_k^H(z, M^2) , \quad (26)$$

$$\Delta_a^{H,NS}(z, M^2) = \Delta_k^H(z, M^2) + \Delta_k^H(z, M^2) - \frac{1}{N_f} \Delta_q^{H,S}(z, M^2) . \quad (27)$$

The polarized non-singlet and singlet coefficient functions $\Delta\mathbb{C}_{1,a}^{NS}(x, Q^2/M^2)$ and $\Delta\mathbb{C}_{1,q(g)}^S(x, Q^2/M^2)$ to $O(a_s^2)$ are given by

$$\begin{aligned} \Delta\mathbb{C}_{1,q}^{NS} = & 1 + a_s \left[\frac{1}{2} \Delta P_{qq}^{(0)} \ln \left(\frac{Q^2}{M^2} \right) + \Delta c_{1,q}^{(1)} \right] \\ & + a_s^2 \left\{ \left[\frac{1}{8} \Delta P_{qq}^{(0)} \otimes \Delta P_{qq}^{(0)} - \frac{1}{4} \beta_0 \Delta P_{qq}^{(0)} \right] \ln^2 \left(\frac{Q^2}{M^2} \right) \right. \\ & + \left[\frac{1}{2} \left(\Delta P_{qq}^{(1),NS} + \Delta P_{q\bar{q}}^{(1),NS} \right) - \beta_0 c_{T,q}^{(1)} + \frac{1}{2} \Delta P_{qq}^{(0)} \otimes \Delta c_{1,q}^{(1)} \right] \ln \left(\frac{Q^2}{M^2} \right) \\ & \left. + \Delta c_{1,q}^{NS,(2),nid} + \Delta c_{T,q}^{NS,(2),id} \right\} , \end{aligned} \quad (28)$$

$$\begin{aligned} \Delta\mathbb{C}_{1,q}^{PS} = & N_f a_s^2 \left\{ \left[\frac{1}{8} \Delta P_{gq}^{(0)} \otimes \Delta P_{gq}^{(0)} \right] \ln^2 \left(\frac{Q^2}{M^2} \right) + \left[\frac{1}{2} \Delta P_{qq}^{(1),PS} + \frac{1}{4} \Delta P_{gq}^{(0)} \otimes \Delta c_{1,q}^{(1)} \right] \ln \left(\frac{Q^2}{M^2} \right) \right. \\ & \left. + \Delta c_{1,q}^{PS,(2)} \right\} , \end{aligned} \quad (29)$$

$$\begin{aligned} \Delta\mathbb{C}_{1,g}^S = & a_s \left[\Delta P_{gq}^{(0)} \ln \left(\frac{Q^2}{M^2} \right) + \Delta c_{1,g}^{(1)} \right] \\ & + a_s^2 \left\{ \left[\frac{1}{4} \Delta P_{gq}^{(0)} \otimes (\Delta P_{gg}^{(0)} + \Delta P_{qq}^{(0)}) - \frac{1}{2} \beta_0 \Delta P_{gq}^{(0)} \right] \ln^2 \left(\frac{Q^2}{M^2} \right) \right. \\ & \left. + \left[\Delta P_{gq}^{(1)} - \beta_0 c_{1,g}^{(1)} + \frac{1}{2} \Delta P_{gg}^{(0)} \otimes \Delta c_{1,g}^{(1)} + \Delta P_{gq}^{(0)} \otimes \Delta c_{1,q}^{(1)} \right] \ln \left(\frac{Q^2}{M^2} \right) + \Delta c_{1,g}^{(2)} \right\} , \end{aligned} \quad (30)$$

with

$$\Delta\mathbb{C}_{1,q}^S = \Delta\mathbb{C}_{1,q}^{NS} + \Delta\mathbb{C}_{1,q}^{PS} . \quad (31)$$

Here $\Delta P_{i,j}^{(k)}$ denote the polarized splitting functions. The non-singlet Wilson coefficients for the structure functions $g_{1,4,5}$ are the same as in the unpolarized case for the corresponding tensor structures, i.e.

$$\Delta\mathbb{C}_{1,q}^{NS} = \mathbb{C}_{1,q}^{NS} \quad (32)$$

$$\Delta\mathbb{C}_{4,q}^{NS} = \mathbb{C}_{2,q}^{NS} \quad (33)$$

$$\Delta\mathbb{C}_{5,q}^{NS} = \mathbb{C}_{1,q}^{NS} . \quad (34)$$

The structure functions $g_{4,5}$ are pure non-singlet.

3 Mellin Moments

The coefficient functions to $O(\alpha_s^2)$ in x -space can be expressed in terms of Nielsen integrals

$$S_{n,p}(y) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zy), \quad (35)$$

partly with y being a rational function of x [10–13]. Here a relatively large number of functions contributes. The **weight** $\mathbf{w}$ of these functions is defined by $\mathbf{w} = \mathbf{p} + \mathbf{n}$, where any power of a logarithm counts for $\mathbf{w} = 1$ as is the case for the denominators $1/x$, $1/(1+x)$ and $1/(1-x)$. In a product of functions the weights of the factors add. The Nielsen integrals cover the polylogarithms $S_{n-1,1}(y) = \text{Li}_n(y)$ and through $d\text{Li}_2(y)/d\ln(y) = -\ln(1-y)$ the logarithm.

For massless calculations it is suitable to consider the coefficient functions in Mellin space. The Mellin-transform of a function $F(x)$ is defined by

$$\mathbf{M}[F](N) = \int_0^1 dx x^{N-1} F(x). \quad (36)$$

In particular, for the Mellin convolution $[F_1 \otimes F_2](x)$ of two functions $F_1(x), F_2(x)$ one obtains

$$\mathbf{M}[F_1 \otimes F_2](N) = \mathbf{M}[F_1](N) \cdot \mathbf{M}[F_2](N). \quad (37)$$

In Mellin space fragmentation densities and the corresponding Wilson coefficients are connected multiplicatively. Mellin space representations have also the advantage that the QCD evolution equations can be solved analytically [19,20] which are numerically very precise and fast.

The Mellin transforms of single-scale quantities in QCD can be represented in terms of nested multiple harmonic sums [23,30] and rational functions of the Mellin variable N . In this way a standardization is obtained. Harmonic sums are described recursively by

$$S_{b,a_1,\dots,a_k}(N) = \sum_{l=1}^N \frac{\text{sign}(b)^l}{l^{|b|}} S_{a_1,\dots,a_k}(l), \quad (38)$$

where $b, a_1, \dots, a_k$ are positive or negative integers and N is a positive integer. $S_a(N)$ is called single harmonic sum.

The harmonic sums $S_{a_1,\dots,a_k}(N)$ form a Hopf algebra [24,25]. For each index set $\{a_1, \dots, a_k\}$ one can determine the basis elements through which all harmonic sums belonging to this set can be expressed algebraically. The number of these basic sums is given by the number of Lyndon words of this set and can be calculated by Witt-formulae [25,26].

For the application of the Mellin-space representation to experimental data one needs to derive analytic continuations of the harmonic sums from integer values of N , N - even or odd -, to complex values. While in the case of single harmonic sums these representations are known, they are more difficult to derive for multiple harmonic sums [17,21]. As finally the argument N will be complex one may consider further relations between harmonic sums for rational and real values of the argument N [27] beyond the algebraic relations. In this way the amount of basic sums is further reduced. In particular one may consider equivalence classes of functions, which are related by differentiation. If an analytic representation for a harmonic sum is found in the complex plane outside its singularities, any derivation is easily obtained. It is convenient to choose for the basic functions Mellin transforms of a suitable harmonic polylogarithm [28] or Nielsen integrals with argument x weighted by $1/(1 \mp x)$.

In former analyses of the two-loop anomalous dimensions, the two-loop Wilson coefficients for unpolarized and polarized deeply inelastic structure functions [18], the unpolarized and polarized Drell-Yan process and scalar and pseudoscalar Higgs boson production [16], $O(\alpha_s^3)$ heavy flavor contributions to $F_L(x, Q^2)$ [29] and the unpolarized $O(\alpha_s^3)$ anomalous dimensions [17] the following minimal sets basic functions w.r.t the equivalence classes defined above were found. We list the newly emerging basic functions.

i) 1 loop anomalous dimensions and coefficient functions :³

$$\mathbf{M} \left[\frac{1}{1-x} \right] (N) \quad (39)$$

ii) 2 loop anomalous dimensions :

$$A_3(N) = \mathbf{M} \left[\frac{\text{Li}_2(x)}{1+x} \right] (N) \quad (40)$$

iii) 2 loop coefficient functions :

$$\begin{aligned} A_{18}(N) &= \mathbf{M} \left[\frac{\text{Li}_2(x)}{1-x} \right] (N) & A_6(N) &= \mathbf{M} \left[\frac{\text{Li}_3(x)}{1+x} \right] (N) \\ A_{21}(N) &= \mathbf{M} \left[\frac{S_{1,2}(x)}{1-x} \right] (N) & A_8(N) &= \mathbf{M} \left[\frac{S_{1,2}(x)}{1+x} \right] (N) . \end{aligned} \quad (41)$$

In the case of the 3-loop anomalous dimensions further eight basic functions contribute [17]. The notation above follows Ref. [21]. Since

$$\frac{\partial^k}{\partial N^k} \mathbf{M}[f(x)](N) = \mathbf{M} [\ln^k(x) f(x)] (N) , \quad (42)$$

one obtains

$$A_5(N) = \frac{\partial}{\partial N} A_3(N) \quad (43)$$

$$A_{22}(N) = \frac{\partial}{\partial N} A_{18}(N) . \quad (44)$$

$A_5(N)$ and $A_{22}(N)$ are represented by $A_3(N)$ and $A_{18}(N)$ in the sense of equivalence classes as outlined above.

The Mellin transforms of the unpolarized and polarized time-like coefficient functions to $O(\alpha_s^2)$ are given in Appendix A and B. Here we discuss their principal structure. Unlike the case for the $O(\alpha_s^2)$ coefficient functions for deeply inelastic scattering, the process related through crossing from the time-like case considered in this paper to the space like-region [6], besides the single harmonic sums only three other basic function contribute. For the deep-inelastic Wilson coefficients [18] and the polarized and unpolarized Drell-Yan process and scalar and pseudoscalar Higgs-boson production cross sections in the heavy mass limit [16] five additional functions occurred. As the corrections to the latter hadronic processes are initial-state corrections they belong to the same class as space-like processes. Furthermore it is interesting to see, whether

³In case of functions which diverge as $x \rightarrow 1$ the $(...)_+$ prescription is understood.

any of the harmonic sums contributing contains an index $\{-1\}$ since those sums were generally absent in all cases analyzed before. Alternating single harmonic sums are given by

$$S_{-1}(N) = (-1)^N \beta(N+1) - \ln(2) \quad (45)$$

$$S_{-k}(N) = (-1)^{(N+1+k)} \frac{1}{(k-1)!} \beta^{(k-1)}(N+1) - \left(1 - \frac{1}{2^{k-1}}\right) \zeta(k), \quad k > 1, \quad (46)$$

with

$$\beta(x) = \frac{1}{2} \left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) \right] \quad (47)$$

and $\psi(x) = d \ln(\Gamma(x))/dx$. In Appendix A and B the β -function indeed occurs. However, it emerges always together with the function $A_3(N)$:

$$A_3(N) = \mathbf{M} \left[\frac{\text{Li}_2(x)}{1+x} \right] (N) = (-1)^{N+1} \left[S_{-2,1}(N) - \zeta(2) S_{-1}(N) + \frac{5}{8} \zeta(3) - \zeta(2) \ln(2) \right] \quad (48)$$

in such a way, that it cancels. Therefore, all contributing harmonic sums do not contain the index $\{-1\}$ also in the present cases.

The analytic continuations of the Wilson coefficients from integer values to complex values of N is now easily performed. The alternating single sums are expressed in terms of Eqs. (45,46). The non-alternating single sums obey the representation

$$S_1(N) = \psi(N+1) + \gamma_E \quad (49)$$

$$S_k(N) = (-1)^{(1+k)} \frac{1}{(k-1)!} \psi^{(k-1)}(N+1) + \zeta(k). \quad (50)$$

The analytic continuations for the remaining functions $A_3(N)$, $A_{18}(N)$ and $A_{21}(N)$ were performed in [21] and $A_5(N)$ and $A_{22}(N)$ are obtained as derivatives (43,44).

In the case of the longitudinal fragmentation functions a transformation to Mellin space was given in [31] recently using the transformation tables of one of the authors [30], see also [32]. In Appendix A and B we present the Mellin transform in such a way, that factors of $(-1)^N$ still present in [31], Eq. (A5), do not occur, and use further compactifications compared to earlier results. As well known, the analytic continuation of the Mellin moments is carried out either from the even or odd integer values of N determined by the crossing relations $q \rightarrow -q$, $p \rightarrow p$ of the respective quantity, cf. [22]. Thus, factors of $(-1)^N$ are not present in the analytic continuation.

A FORTRAN-programme of the 2-loop Wilson coefficients for the representation of the fragmentation functions for complex values of the Mellin variable suitable to extend analyses of the experimental fragmentation data [31,33] can be obtained by the authors.

4 Conclusion

The mathematical structure of the 2-loop Wilson coefficients for the polarized and unpolarized time-like fragmentation functions simplifies considerably in Mellin space expressing the result in terms of nested harmonic sums. Using algebraic and structural identities for the harmonic sums and their continuations to rational, real and finally complex values of the Mellin variable. The physical evolution kernels for deeply inelastic scattering and parton fragmentation into hadrons are related by crossing from t - to s -channel order by order in perturbation theory.

Since these quantities are rational functions of the respective Wilson coefficients and anomalous dimensions in Mellin space one expects a familiarity of the mathematical structure of the space- and time-like Wilson coefficients. While in the space-like case five non-trivial basic functions emerge besides the single harmonic sums, in the time-like case only three of them contribute to $O(\alpha_s^2)$. In x -space many more functions are required to express the Wilson coefficients. As observed for various other processes, harmonic sums with index $\{-1\}$ do not contribute. The Mellin-space expressions allow to use the time-like Wilson coefficients in fast evolution codes for the experimental analysis of fragmentation function data.

Note added. The time-like Wilson coefficients in the unpolarized case have been newly calculated in [34] very recently.

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A Unpolarized Time-like Coefficient Functions

In this appendix, we present the Mellin moments of unpolarized time-like coefficient functions to $O(\alpha_s^2)$:

$$\mathbb{C}_{T,L,a}^F = C_{T,L,a}^{F,(0)} + a_s C_{T,L,a}^{F,(1)} + a_s^2 C_{T,L,a}^{F,(2)} \quad (\text{A.1})$$

with $F = \text{NS, PS, S}$.

The individual contributions are

$$C_{T,q}^{\text{NS}(0)} = 1 \quad (\text{A.2})$$

$$\begin{aligned} C_{T,q}^{\text{NS}(1)} = C_F & \left[\left(-\frac{2}{(N+1)} - \frac{2}{N} + 3 - 4S_1(N-1) \right) \log \left(\frac{Q^2}{\mu^2} \right) - 9 + \frac{4}{N^2} - \frac{3}{N+1} \right. \\ & + \frac{3}{N} + \frac{4}{(N+1)^2} + 2S_1^2(N-1) + 2\frac{S_1(N+1)}{(N+1)} + 2\frac{S_1(N)}{N} + 3S_1(N-1) \\ & \left. + 10S_2(N-1) \right] \quad (\text{A.3}) \end{aligned}$$

$$\begin{aligned} C_{T,g}^{\text{S}(1)} = C_F & \left[\left(\frac{4}{(N+1)} - \frac{8}{N} + \frac{8}{(N-1)} \right) \log \left(\frac{Q^2}{\mu^2} \right) + \frac{16}{N^2} - \frac{16}{(-1+N)^2} + \frac{8}{N} \right. \\ & \left. - \frac{8}{(N+1)^2} - 8\frac{S_1(-1+N)}{(-1+N)} - 4\frac{S_1(N+1)}{(N+1)} + 8\frac{S_1(N)}{N} - \frac{8}{(-1+N)} \right] \quad (\text{A.4}) \end{aligned}$$

$$\begin{aligned} C_{T,q}^{\text{NS}(2)} = C_F & \left(C_F - \frac{C_A}{2} \right) \left[\frac{36}{N^4} + \frac{24}{N^3} - \frac{36}{(N+1)^4} + \frac{16}{(N+1)^3} + \frac{82}{5(N+1)^2} \right. \\ & - \frac{46}{5(N+1)} - \frac{16}{(N+2)^3} - \frac{24}{5(N+2)^2} + \frac{24}{5(N+2)} + \frac{24}{5(N+3)^3} \\ & - \frac{24}{5(N-1)} - \frac{24}{5(N-1)^2} - 16\beta^{(1)}(N) - \frac{28}{3}\beta^{(3)}(N) - 48A_5(N-1) \\ & + 16\frac{\zeta_2}{(N+1)^2} + 4\frac{\zeta_2}{(N+1)} + 8\frac{\zeta_2}{(N+2)} - \frac{12}{5}\frac{\zeta_2}{(N+3)} - 8\frac{\zeta_2}{(N-1)} \\ & + \frac{12}{5}\frac{\zeta_2}{(N-2)} - 48S_2(N-1)\beta^{(1)}(N) + \frac{24}{5}\frac{\beta^{(1)}(N-1)}{(N-2)} - 16\frac{\beta^{(1)}(N+3)}{(N+2)} \\ & + \frac{24}{5}\frac{\beta^{(1)}(4+N)}{(N+3)} + 72\beta^{(1)}(N)\zeta_2 - 16\frac{\beta^{(1)}(N)}{(N-1)} - 32\beta^{(1)}(N)S_1^2(N-1) \\ & + 32\frac{\beta^{(2)}(N+2)}{(N+1)} + 32\frac{A_3(N+1)}{(N+1)} - 32\frac{\beta(N+2)\zeta_2}{(N+1)} + \left(32S_1(N-1)\beta^{(1)}(N) \right. \\ & - 8\beta^{(2)}(N) + \frac{8}{(N+1)^3} - \frac{8}{(N+1)^2} + 16\frac{\beta^{(1)}(N+2)}{(N+1)} + 32\beta(N)\zeta_2 \\ & - \frac{16}{(N+1)} - 32A_3(N-1) + \frac{16-16\beta^{(1)}(N+1)}{N} - \frac{8}{N^2} - \frac{8}{N^3} \Big) \log \left(\frac{Q^2}{\mu^2} \right) \\ & + \left(-32\beta(N)\zeta_2 + 56\beta^{(2)}(N) + 32A_3(N-1) \right) S_1(N-1) + \left(-4\zeta_2 - 24\beta^{(2)}(N+1) \right. \\ & \left. + \frac{46}{5} \right) \frac{1}{N} + \left(\frac{8}{(N+1)^2} - \frac{8}{(N+1)^3} - 48\frac{\beta^{(1)}(N+2)}{(N+1)} + \frac{16}{(N+1)} \right) S_1(N+1) \end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{158}{5} + 16\beta^{(1)}(N+1) - 8\zeta_2 \right) \frac{1}{N^2} + \left(\frac{-16 + 16\beta^{(1)}(N+1)}{N} \right. \\
& \left. + \frac{8}{N^2} + \frac{8}{N^3} \right) S_1(N) \Big] \\
& + N_F C_F T_F \left[\left(-\frac{8}{3} S_1(N-1) - \frac{4}{3(N+1)} - \frac{4}{3N} + 2 \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(\frac{8}{3} \frac{S_1(N+1)}{(N+1)} \right. \right. \\
& + \frac{28}{9N} + \frac{8}{3} \frac{S_1(N)}{N} + 8 S_2(N-1) + \frac{8}{3(N+1)^2} + \frac{116}{9} S_1(N-1) + \frac{52}{9(N+1)} \\
& + \frac{8}{3} S_1^2(N-1) + \left. \frac{8}{3N^2} - \frac{38}{3} \right) \log \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{494}{27} + \frac{16}{3} \zeta_2 - \frac{8}{3} S_2(N-1) \right) S_1(N-1) \\
& + \left(-\frac{8}{3(N+1)^2} - \frac{52}{9(N+1)} \right) S_1(N+1) + \left(-\frac{8}{3N^2} - \frac{28}{9N} \right) S_1(N) \\
& - \frac{58}{9} S_1^2(N-1) - \frac{4}{3} \frac{S_1^2(N+1)}{(N+1)} - \frac{4}{3} \frac{S_1^2(N)}{N} - \frac{8}{9} S_1^3(N-1) - \frac{250}{9} S_2(N-1) \\
& - \frac{4}{3} \frac{S_2(N+1)}{(N+1)} - \frac{4}{3} \frac{S_2(N)}{N} - \frac{40}{9} S_3(N-1) - 4\zeta_2 + \frac{16}{3} \zeta_3 + \frac{457}{18} - \frac{34}{27(N+1)} \\
& + \frac{8}{3} \frac{\zeta_2}{(N+1)} - \frac{4}{3(N+1)^3} - \frac{12}{(N+1)^2} + \frac{16}{3} A_{18}(N-1) + \left(\frac{8}{3} \zeta_2 - \frac{118}{27} \right) \frac{1}{N} \\
& \left. - \frac{20}{3N^2} - \frac{4}{3N^3} \right] \\
& + C_A C_F \left[\left(\left(8\zeta_2 - \frac{367}{9} \right) S_1(N-1) - \frac{22}{3} \frac{S_1(N+1)}{(N+1)} - \frac{22}{3} \frac{S_1(N)}{N} - \frac{22}{3} S_1^2(N-1) \right. \right. \\
& - 22 S_2(N-1) - 8 S_3(N-1) + \frac{215}{6} - \frac{34}{3(N+1)^2} - \frac{4}{(N+1)^3} + 4 \frac{\zeta_2}{(N+1)} \\
& - 4\zeta_3 - \frac{275}{9(N+1)} + \left(4\zeta_2 + \frac{7}{9} \right) \frac{1}{N} - \frac{34}{3N^2} - \frac{4}{N^3} \Big) \log \left(\frac{Q^2}{\mu^2} \right) + 3\zeta_2 \\
& + \frac{18}{N^4} + \frac{47}{3N^3} + \frac{18}{(N+1)^4} + \frac{47}{3(N+1)^3} + \frac{184}{5(N+1)^2} + \frac{3587}{270(N+1)} \\
& - \frac{8}{(N+2)^3} + \frac{12}{5(N+2)^2} - \frac{12}{5(N+2)} - \frac{12}{5(N+3)^3} + \frac{12}{5(N-1)} \\
& + \frac{12}{5(N-1)^2} - 8 \frac{\zeta_2}{(N+1)^2} - \frac{22}{3} \frac{\zeta_2}{(N+1)} + 4 \frac{\zeta_2}{(N+2)} + \frac{6}{5} \frac{\zeta_2}{(N+3)} \\
& - 4 \frac{\zeta_2}{(N-1)} - \frac{6}{5} \frac{\zeta_2}{(N-2)} - \frac{12}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} - 8 \frac{\beta^{(1)}(N+3)}{(N+2)} - \frac{12}{5} \frac{\beta^{(1)}(4+N)}{(N+3)} \\
& - 8 \frac{\beta^{(1)}(N)}{(N-1)} + \frac{118}{3} \zeta_3 + \frac{11}{3} \frac{S_1^2(N+1)}{(N+1)} + \frac{11}{3} \frac{S_1^2(N)}{N} + \frac{110}{9} S_3(N-1) \\
& - \frac{44}{3} A_{18}(N-1) + \frac{22}{9} S_1^3(N-1) - 34 \frac{\zeta_3}{(N+1)} + 8 \frac{S_3(N+1)}{(N+1)} + 16 \frac{\beta^{(1)}(N+2)}{(N+1)^2} \\
& + 4 \frac{\beta^{(1)}(N+2)}{(N+1)} - 8 \frac{A_{18}(N+1)}{(N+1)} - 12 (\beta^{(1)}(N))^2 + 24 A_{21}(N-1) + 44 S_4(N-1) \\
& + 8 S_2^2(N-1) - \frac{97}{5} \zeta_2^2 - \frac{5465}{72} + \left(-4 S_2(N-1) + \frac{367}{18} \right) S_1^2(N-1)
\end{aligned}$$

$$\begin{aligned}
& + \left(8 S_3(N-1) + \frac{22}{3} S_2(N-1) - 48 \zeta_3 + \frac{3155}{54} - 16 A_{18}(N-1) \right. \\
& - \frac{44}{3} \zeta_2 \left. \right) S_1(N-1) + \left(\frac{1603}{18} - 20 \zeta_2 \right) S_2(N-1) + \left(\frac{4}{(N+1)^2} \right. \\
& + \frac{11}{3(N+1)} \left. \right) S_2(N+1) + \left(-4 \frac{S_2(N+1)}{(N+1)} + \frac{34}{3(N+1)^2} + \frac{383}{9(N+1)} \right. \\
& - 4 \frac{\zeta_2}{(N+1)} \left. \right) S_1(N+1) + \left(\frac{11}{3N} + \frac{4}{N^2} \right) S_2(N) + \left(\frac{2513}{270} - 14 \zeta_3 - \frac{22}{3} \zeta_2 \right. \\
& + 4 \beta^{(1)}(N+1) - 8 A_{18}(N) \left. \right) \frac{1}{N} + \left(\frac{122}{15} - 12 \zeta_2 + 8 \beta^{(1)}(N+1) \right) \frac{1}{N^2} + \left(\frac{11}{3N} \right. \\
& - \frac{11}{2} + \frac{22}{3} S_1(N-1) + \frac{11}{3(N+1)} \left. \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(-4 \frac{S_2(N)}{N} + \left(4 \zeta_2 - \frac{97}{9} \right) \frac{1}{N} \right. \\
& + \frac{34}{3N^2} \left. \right) S_1(N) \Big] \\
& + C_F^2 \left[67 \zeta_2 - \frac{102}{N^4} - \frac{90}{N^3} - \frac{102}{(N+1)^4} - \frac{46}{(N+1)^3} - \frac{248}{5(N+1)^2} \right. \\
& - \frac{187}{10(N+1)} + \frac{16}{(N+2)^3} - \frac{24}{5(N+2)^2} + \frac{24}{5(N+2)} + \frac{24}{5(N+3)^3} \\
& - \frac{24}{5(N-1)} - \frac{24}{5(N-1)^2} + \left(-8 \frac{\zeta_2}{(N+1)} - \frac{4}{(N+1)^2} + 36 \frac{S_2(N+1)}{(N+1)} \right. \\
& - \frac{16}{(N+1)^3} - \frac{67}{(N+1)} \left. \right) S_1(N+1) + 12 \frac{\zeta_2}{(N+1)^2} - 12 \frac{\zeta_2}{(N+1)} - 8 \frac{\zeta_2}{(N+2)} \\
& - \frac{12}{5} \frac{\zeta_2}{(N+3)} + 8 \frac{\zeta_2}{(N-1)} + \frac{12}{5} \frac{\zeta_2}{(N-2)} + \frac{24}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} + 16 \frac{\beta^{(1)}(N+3)}{(N+2)} \\
& + \frac{24}{5} \frac{\beta^{(1)}(4+N)}{(N+3)} + 16 \frac{\beta^{(1)}(N)}{(N-1)} - 36 \zeta_3 - 30 S_3(N-1) + 12 A_{18}(N-1) \\
& + 6 S_1^3(N-1) + 20 \frac{\zeta_3}{(N+1)} + 4 \frac{S_3(N+1)}{(N+1)} - 32 \frac{\beta^{(1)}(N+2)}{(N+1)^2} - 8 \frac{\beta^{(1)}(N+2)}{(N+1)} \\
& + 8 \frac{A_{18}(N+1)}{(N+1)} + 24 (\beta^{(1)}(N))^2 - 24 A_{21}(N-1) - 92 S_4(N-1) + 54 S_2^2(N-1) \\
& + \frac{38}{5} \zeta_2^2 + 8 A_{22}(N-1) + 20 \frac{S_3(N)}{N} + 4 \frac{S_1^3(N+1)}{(N+1)} + 4 \frac{S_1^3(N)}{N} + 2 S_1^4(N-1) \\
& + \frac{331}{8} + \left(\frac{8}{(N+1)^2} + \frac{4}{(N+1)} \right) S_1^2(N+1) + \left(-24 \zeta_2 - \frac{51}{2} + 42 S_2(N-1) \right. \\
& + 24 S_3(N-1) + 16 A_{18}(N-1) \left. \right) S_1(N-1) + \left(36 \frac{S_2(N)}{N} + \frac{-24 \zeta_2 + 29}{N} - \frac{20}{N^2} \right. \\
& - \frac{16}{N^3} \left. \right) S_1(N) + \left(\frac{8}{N} + \frac{8}{N^2} \right) S_1^2(N) + \left(-16 \zeta_2 - \frac{27}{2} + 36 S_2(N-1) \right) S_1^2(N-1) \\
& + \left(8 S_1^2(N-1) - \frac{6}{N^2} + \frac{9}{2} + 8 \frac{S_1(N+1)}{(N+1)} - 12 S_1(N-1) + 8 \frac{S_1(N)}{N} - \frac{10}{N} \right. \\
& - \frac{6}{(N+1)^2} - \frac{2}{(N+1)} \left. \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{239}{2} + 24 \zeta_2 \right) S_2(N-1) + \left(\frac{36}{N} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{44}{N^2} \Big) S_2(N) + \left(\frac{44}{(N+1)^2} - \frac{24}{(N+1)} \right) S_2(N+1) + \left(\frac{187}{10} - 20\zeta_3 - 12\zeta_2 \right. \\
& + 8A_{18}(N) - 8\beta^{(1)}(N+1) \Big) \frac{1}{N} + \left(-\frac{218}{5} + 20\zeta_2 - 16\beta^{(1)}(N+1) \right) \frac{1}{N^2} \\
& + \left(\left(-56S_2(N-1) + 45 + 16\zeta_2 \right) S_1(N-1) + \left(\frac{4}{(N+1)} - \frac{12}{(N+1)^2} \right) S_1(N+1) \right. \\
& + \left(-\frac{12}{N^2} - \frac{4}{N} \right) S_1(N) - 6S_1^2(N-1) - 12\frac{S_1^2(N+1)}{(N+1)} - 12\frac{S_1^2(N)}{N} - 8S_1^3(N-1) \\
& + 42S_2(N-1) - 28\frac{S_2(N+1)}{(N+1)} - 28\frac{S_2(N)}{N} + 16S_3(N-1) - 24\zeta_2 + 8\zeta_3 \\
& + \frac{31}{(N+1)} + \frac{44}{(N+1)^3} - \frac{51}{2} + 8\frac{\zeta_2}{(N+1)} + \frac{20}{(N+1)^2} + \frac{8\zeta_2 + 5}{N} \\
& \left. + \frac{52}{N^2} + \frac{44}{N^3} \right) \log \left(\frac{Q^2}{\mu^2} \right) \Big] \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
C_{T,q}^{\text{PS}(2)} = & N_F C_F T_F \left[-\frac{88}{N^4} - \frac{28}{N^3} - \frac{88}{(N+1)^4} - \frac{28}{(N+1)^3} + \frac{208}{3(N+1)^2} \right. \\
& + \frac{140}{3(N+1)} + \frac{32}{3(N+2)^3} + \frac{256}{9(N+2)^2} + \frac{1024}{27(N+2)} - \frac{160}{27(N-1)} \\
& + \frac{32}{3(N-1)^2} + 16\frac{\zeta_2}{(N+1)^2} - 8\frac{\zeta_2}{(N+1)} + 16\frac{\zeta_2}{(N+2)} - 16\frac{\zeta_2}{(N-1)} \\
& + \frac{32}{3}\frac{\beta^{(1)}(N+3)}{(N+2)} + \frac{32}{3}\frac{\beta^{(1)}(N)}{(N-1)} + 32\frac{\beta^{(1)}(N+2)}{(N+1)} + \frac{128}{3(N-1)^3} + \frac{16}{3}\frac{S_2(N-1)}{(N-1)} \\
& - \frac{16}{3}\frac{S_2(N+2)}{(N+2)} + \frac{16}{3}\frac{S_1^2(N-1)}{(N-1)} - \frac{16}{3}\frac{S_1^2(N+2)}{(N+2)} + \left(-\frac{32}{3}\frac{S_1(N-1)}{(N-1)} \right. \\
& + \left(\frac{16}{(N+1)^2} + \frac{8}{(N+1)} \right) S_1(N+1) + \left(\frac{16}{N^2} - \frac{8}{N} \right) S_1(N) + \frac{136}{3(N+1)} \\
& - \frac{64}{3(N-1)^2} + \frac{8}{N^2} - \frac{184}{3N} - \frac{16}{3(N-1)} + \frac{64}{3(N+2)} \\
& + \frac{48}{(N+1)^3} + \frac{48}{N^3} + \frac{32}{3}\frac{S_1(N+2)}{(N+2)} + \frac{40}{(N+1)^2} + \frac{32}{3(N+2)^2} \Big) \log \left(\frac{Q^2}{\mu^2} \right) \\
& + \left(-\frac{8}{N^2} + \frac{4}{N} \right) S_1^2(N) + \left(\frac{16}{3(N-1)} + \frac{64}{3(N-1)^2} \right) S_1(N-1) \\
& + \left(-\frac{48}{(N+1)^3} - \frac{40}{(N+1)^2} - \frac{136}{3(N+1)} \right) S_1(N+1) + \left(-\frac{48}{N^3} \right. \\
& + \frac{184}{3N} - \frac{8}{N^2} \Big) S_1(N) + \left(-\frac{64}{3(N+2)} - \frac{32}{3(N+2)^2} \right) S_1(N+2) + \left(-\frac{8}{N^2} \right. \\
& + \frac{4}{N} \Big) S_2(N) + \left(-\frac{16}{3(N+2)} + \frac{16}{3(N-1)} - \frac{8}{(N+1)^2} + \frac{4}{N} - \frac{4}{(N+1)} \right. \\
& - \frac{8}{N^2} \Big) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{4}{(N+1)} - \frac{8}{(N+1)^2} \right) S_2(N+1) + \left(-\frac{4}{(N+1)} \right. \\
& \left. - \frac{8}{(N+1)^2} \right) S_1^2(N+1) + \left(-\frac{236}{3} + 8\zeta_2 + 32\beta^{(1)}(N+1) \right) \frac{1}{N}
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{400}{3} + 16 \zeta_2 \right) \frac{1}{N^2} \Big] \tag{A.6} \\
C_{T,g}^{S(2)} = & C_A C_F \Bigg[\frac{176}{N^4} - \frac{32}{N^3} + \frac{248}{(N+1)^4} + \frac{4}{(N+1)^3} - \frac{172}{3(N+1)^2} \\
& - \frac{106}{(N+1)} - \frac{32}{3(N+2)^3} - \frac{256}{9(N+2)^2} - \frac{928}{27(N+2)} + \frac{4438}{27(N-1)} \\
& - \frac{496}{3(N-1)^2} - 64 \frac{\zeta_2}{(N+1)^2} + 20 \frac{\zeta_2}{(N+1)} - 16 \frac{\zeta_2}{(N+2)} + 96 \frac{\zeta_2}{(N-1)} \\
& - \frac{32}{3} \frac{\beta^{(1)}(N+3)}{(N+2)} - \frac{80}{3} \frac{\beta^{(1)}(N)}{(N-1)} - 16 \frac{\beta^{(2)}(N+2)}{(N+1)} + 32 \frac{A_3(N+1)}{(N+1)} \\
& - 32 \frac{\beta(N+2)\zeta_2}{(N+1)} + 60 \frac{\zeta_3}{(N+1)} - \frac{200}{3} \frac{S_3(N+1)}{(N+1)} + 32 \frac{\beta^{(1)}(N+2)}{(N+1)^2} - 56 \frac{\beta^{(1)}(N+2)}{(N+1)} \\
& - 16 \frac{A_{18}(N+1)}{(N+1)} + \frac{400}{3} \frac{S_3(N)}{N} - \frac{4}{3} \frac{S_1^3(N+1)}{(N+1)} + \frac{8}{3} \frac{S_1^3(N)}{N} - \frac{464}{3(N-1)^3} \\
& + \frac{16}{3} \frac{S_2(N+2)}{(N+2)} + \frac{16}{3} \frac{S_1^2(N+2)}{(N+2)} + \frac{320}{(N-1)^4} - 96 \frac{\zeta_2}{(N-1)^2} + 80 \frac{\zeta_3}{(N-1)} \\
& - \frac{352}{3} \frac{S_3(N-1)}{(N-1)} + 64 \frac{\beta^{(1)}(N)}{(N-1)^2} - 40 \frac{\beta^{(2)}(N)}{(N-1)} + 32 \frac{A_3(N-1)}{(N-1)} - 32 \frac{A_{18}(N-1)}{(N-1)} \\
& - \frac{8}{3} \frac{S_1^3(N-1)}{(N-1)} - 32 \frac{\beta(N)\zeta_2}{(N-1)} + \left(-\frac{96}{N^3} - \frac{64}{N^2} - \frac{112}{(N+1)^3} - \frac{40}{(N+1)^2} \right. \\
& \left. - \frac{100}{3(N+1)} - \frac{32}{3(N+2)^2} - \frac{32}{3(N+2)} + \frac{188}{3(N-1)} + \frac{400}{3(N-1)^2} \right. \\
& \left. + 8 \frac{\zeta_2}{(N+1)} + 16 \frac{\zeta_2}{(N-1)} - 32 \frac{\beta^{(1)}(N)}{(N-1)} + 40 \frac{S_2(N+1)}{(N+1)} - 80 \frac{S_2(N)}{N} + 8 \frac{S_1^2(N+1)}{(N+1)} \right. \\
& \left. - 16 \frac{S_1^2(N)}{N} - 16 \frac{\beta^{(1)}(N+2)}{(N+1)} - \frac{128}{(N-1)^3} + \frac{344}{3} \frac{S_1(N-1)}{(N-1)} - \frac{32}{3} \frac{S_1(N+2)}{(N+2)} \right. \\
& \left. + 80 \frac{S_2(N-1)}{(N-1)} + 16 \frac{S_1^2(N-1)}{(N-1)} + \left(-\frac{32}{3} - 16 \zeta_2 - 32 \beta^{(1)}(N+1) \right) \frac{1}{N} \right. \\
& \left. + \left(-\frac{16}{(N+1)^2} + 8 \frac{1}{(N+1)} \right) S_1(N+1) + \left(-\frac{64}{N^2} - 96 \frac{1}{N} \right) S_1(N) \right) \log \left(\frac{Q^2}{\mu^2} \right) \\
& + \left(64 \frac{\zeta_2}{(N-1)} - 72 \frac{S_2(N-1)}{(N-1)} - \frac{496}{3(N-1)^2} + \frac{96}{(N-1)^3} \right. \\
& \left. - \frac{356}{3(N-1)} \right) S_1(N-1) + \left(\frac{120}{N^2} + \frac{112}{N} \right) S_2(N) + \left(-16 \frac{S_1(N-1)}{(N-1)} \right. \\
& \left. - 8 \frac{S_1(N+1)}{(N+1)} + \frac{16}{3(N+2)} + \frac{16}{(N-1)^2} + \frac{32}{N} - \frac{124}{3(N-1)} + 16 \frac{S_1(N)}{N} \right. \\
& \left. + \frac{16}{(N+1)^2} + \frac{4}{(N+1)} + \frac{16}{N^2} \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(-36 \frac{S_2(N+1)}{(N+1)} - 16 \frac{\beta^{(1)}(N+2)}{(N+1)} \right. \\
& \left. + \frac{16}{(N+1)^2} + \frac{4}{3(N+1)} + \frac{96}{(N+1)^3} + 40 \frac{\zeta_2}{(N+1)} \right) S_1(N+1) + \left(72 \frac{S_2(N)}{N} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{236}{3} - 32 \beta^{(1)}(N+1) - 80 \zeta_2 \right) \frac{1}{N} + \frac{96}{N^2} + \frac{128}{N^3} \Big) S_1(N) + \left(\frac{32}{3(N+2)^2} \right. \\
& + \left. \frac{32}{3(N+2)} \right) S_1(N+2) + \left(\frac{8}{(N-1)^2} - \frac{172}{3(N-1)} \right) S_1^2(N-1) \\
& + \frac{1}{N} \left(-120 \zeta_3 - 104 \zeta_2 - 32 \beta^{(2)}(N+1) - 36 + 64 A_3(N) + 32 A_{18}(N) \right. \\
& - 64 \beta(N+1) \zeta_2 - 80 \beta^{(1)}(N+1) \Big) + \left(-\frac{772}{3} + 64 \beta^{(1)}(N+1) + 32 \zeta_2 \right) \frac{1}{N^2} + \left(\frac{24}{N^2} \right. \\
& + \left. \frac{48}{N} \right) S_1^2(N) + \left(-\frac{8}{(N+1)} + \frac{12}{(N+1)^2} \right) S_1^2(N+1) + \left(-\frac{364}{3(N-1)} \right. \\
& - \left. \frac{88}{(N-1)^2} \right) S_2(N-1) + \left(-\frac{16}{(N+1)} - \frac{36}{(N+1)^2} \right) S_2(N+1) \Big] \\
& + C_F^2 \left[-\frac{88}{N^4} - \frac{64}{N^3} + \frac{44}{(N+1)^4} + \frac{166}{(N+1)^3} + \frac{262}{5(N+1)^2} + \frac{154}{5(N+1)} \right. \\
& + \frac{16}{5(N+2)^2} - \frac{16}{5(N+2)} - \frac{16}{5(N+3)^3} + \frac{316}{5(N-1)} - \frac{144}{5(N-1)^2} \\
& + 24 \frac{\zeta_2}{(N+1)^2} - 12 \frac{\zeta_2}{(N+1)} + \frac{8}{5} \frac{\zeta_2}{(N+3)} + 96 \frac{\zeta_2}{(N-1)} + \frac{32}{5} \frac{\zeta_2}{(N-2)} \\
& + \frac{64}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} - \frac{16}{5} \frac{\beta^{(1)}(4+N)}{(N+3)} - 16 \frac{\beta^{(2)}(N+2)}{(N+1)} - 64 \frac{A_3(N+1)}{(N+1)} \\
& + 64 \frac{\beta(N+2) \zeta_2}{(N+1)} - 32 \frac{\zeta_3}{(N+1)} + \frac{128}{3} \frac{S_3(N+1)}{(N+1)} + 96 \frac{\beta^{(1)}(N+2)}{(N+1)} + 16 \frac{A_{18}(N+1)}{(N+1)} \\
& - \frac{256}{3} \frac{S_3(N)}{N} - \frac{20}{3} \frac{S_1^3(N+1)}{(N+1)} + \frac{40}{3} \frac{S_1^3(N)}{N} + 64 \frac{\zeta_2}{(N-1)^2} + 16 \frac{\zeta_3}{(N-1)} \\
& + \frac{160}{3} \frac{S_3(N-1)}{(N-1)} - 16 \frac{\beta^{(2)}(N)}{(N-1)} - 64 \frac{A_3(N-1)}{(N-1)} + 32 \frac{A_{18}(N-1)}{(N-1)} - \frac{40}{3} \frac{S_1^3(N-1)}{(N-1)} \\
& + 64 \frac{\beta(N) \zeta_2}{(N-1)} + \left(-\frac{48}{(N-1)} - \frac{48}{(N-1)^2} \right) S_1^2(N-1) + \left(-\frac{20}{(N+1)^2} \right. \\
& - \left. \frac{14}{(N+1)} \right) S_1^2(N+1) + \left(\frac{40}{N^2} + \frac{56}{N} \right) S_1^2(N) + \left(-\frac{8}{N^2} + \frac{8}{N} \right. \\
& - 16 \frac{S_1(N-1)}{(N-1)} + 16 \frac{S_1(N)}{N} + \frac{4}{(N+1)^2} - 8 \frac{S_1(N+1)}{(N+1)} - \frac{2}{(N+1)} \Big) \log^2 \left(\frac{Q^2}{\mu^2} \right) \\
& + \left(\left(\frac{48}{(N-1)} + \frac{64}{(N-1)^2} \right) S_1(N-1) + \left(\frac{24}{(N+1)^2} + \frac{12}{(N+1)} \right) S_1(N+1) \right. \\
& + \left(-\frac{64}{N} - \frac{48}{N^2} \right) S_1(N) + 32 \frac{S_1^2(N-1)}{(N-1)} + 16 \frac{S_1^2(N+1)}{(N+1)} - 32 \frac{S_1^2(N)}{N} \\
& + 32 \frac{S_2(N-1)}{(N-1)} + 16 \frac{S_2(N+1)}{(N+1)} - 32 \frac{S_2(N)}{N} - \frac{24}{(N+1)^3} + \frac{8}{(N+1)} \\
& - \frac{20}{(N+1)^2} + \frac{8}{(N-1)} - 64 \frac{\zeta_2}{(N-1)} - 32 \frac{\zeta_2}{(N+1)} + \frac{-20+64\zeta_2}{N} + \frac{16}{N^2} \\
& + \frac{48}{N^3} \Big) \log \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{604}{5} - 80 \zeta_2 + 64 \zeta_3 - 128 A_3(N) + 128 \beta(N+1) \zeta_2 \right.
\end{aligned}$$

$$\begin{aligned}
& -32\beta^{(2)}(N+1) + 64\beta^{(1)}(N+1) - 32A_{18}(N) \Big) \frac{1}{N} + \left(-128\beta^{(1)}(N+1) - \frac{418}{5} \right. \\
& \left. -112\zeta_2 \right) \frac{1}{N^2} + \left(-\frac{48}{(N-1)} - \frac{112}{(N-1)^2} \right) S_2(N-1) + \left(-\frac{8}{(N-1)} \right. \\
& \left. +64\frac{\beta^{(1)}(N)}{(N-1)} - \frac{64}{(N-1)^3} - 56\frac{S_2(N-1)}{(N-1)} - \frac{96}{(N-1)^2} \right) S_1(N-1) \\
& + \left(-\frac{52}{(N+1)^2} + \frac{2}{(N+1)} \right) S_2(N+1) + \left(-\frac{12}{(N+1)^2} + \frac{24}{(N+1)} \right. \\
& \left. +64\frac{\beta^{(1)}(N+2)}{(N+1)} - 28\frac{S_2(N+1)}{(N+1)} - 16\frac{\zeta_2}{(N+1)} - \frac{8}{(N+1)^3} \right) S_1(N+1) \\
& + \left(56\frac{S_2(N)}{N} + \frac{32\zeta_2 + 128\beta^{(1)}(N+1)}{N} + \frac{80}{N^2} + \frac{16}{N^3} \right) S_1(N) \\
& + \left(\frac{56}{N} + \frac{104}{N^2} \right) S_2(N) \Big] \tag{A.7}
\end{aligned}$$

$$\begin{aligned}
C_{L,q}^{(2),nid} = & C_F^2 \left[-\frac{20}{N^3} + \frac{16}{(N+1)^3} - \frac{32}{5(N+1)^2} + \frac{86}{5(N+1)} + \frac{8}{(N+2)^3} \right. \\
& - \frac{16}{5(N+2)^2} + \frac{16}{5(N+2)} + \frac{16}{5(N+3)^3} + \frac{24}{5(N-1)} + \frac{24}{5(N-1)^2} \\
& - 8\frac{\zeta_2}{(N+1)} - 4\frac{\zeta_2}{(N+2)} - \frac{8}{5}\frac{\zeta_2}{(N+3)} - 8\frac{\zeta_2}{(N-1)} - \frac{12}{5}\frac{\zeta_2}{(N-2)} - 4\frac{S_1(N+1)}{(N+1)} \\
& - \frac{24}{5}\frac{\beta^{(1)}(N-1)}{(N-2)} + 8\frac{\beta^{(1)}(N+3)}{(N+2)} + \frac{16}{5}\frac{\beta^{(1)}(4+N)}{(N+3)} - 16\frac{\beta^{(1)}(N)}{(N-1)} + 20\frac{S_2(N)}{N} \\
& + 4\frac{S_1^2(N)}{N} + 16\frac{\beta^{(1)}(N+2)}{(N+1)} - 16\frac{S_3(N)}{N} + \left(\frac{4}{(N+1)} - 8\frac{S_1(N)}{N} + \frac{2}{N} \right. \\
& \left. + \frac{4}{N^2} \right) \log\left(\frac{Q^2}{\mu^2}\right) + \left(\frac{16\zeta_2 - 14}{N} + \frac{4}{N^2} \right) S_1(N) + \left(-\frac{211}{5} + 40\zeta_3 + 8\zeta_2 \right) \frac{1}{N} \\
& \left. + \left(\frac{78}{5} - 8\zeta_2 - 16\beta^{(1)}(N+1) \right) \frac{1}{N^2} \right] \\
& + C_A C_F \left[\frac{4}{N^3} - \frac{8}{(N+1)^3} - \frac{4}{5(N+1)^2} - \frac{254}{15(N+1)} - \frac{4}{(N+2)^3} + \frac{8}{5(N+2)^2} \right. \\
& - \frac{8}{5(N+2)} - \frac{8}{5(N+3)^3} - \frac{12}{5(N-1)} - \frac{12}{5(N-1)^2} - \frac{22}{3N} \log\left(\frac{Q^2}{\mu^2}\right) \\
& + 4\frac{\zeta_2}{(N+1)} + 2\frac{\zeta_2}{(N+2)} + \frac{4}{5}\frac{\zeta_2}{(N+3)} + 4\frac{\zeta_2}{(N-1)} + \frac{6}{5}\frac{\zeta_2}{(N-2)} + \frac{12}{5}\frac{\beta^{(1)}(N-1)}{(N-2)} \\
& - 4\frac{\beta^{(1)}(N+3)}{(N+2)} - \frac{8}{5}\frac{\beta^{(1)}(4+N)}{(N+3)} + 8\frac{\beta^{(1)}(N)}{(N-1)} - 8\frac{\beta^{(1)}(N+2)}{(N+1)} + 8\frac{S_3(N)}{N} \\
& + \left(\frac{46}{3} - 8\zeta_2 \right) S_1(N) \frac{1}{N} + \left(-\frac{22}{15} + 4\zeta_2 + 8\beta^{(1)}(N+1) \right) \frac{1}{N^2} + \left(\frac{2017}{45} \right. \\
& \left. - 20\zeta_3 - 4\zeta_2 \right) \frac{1}{N} \Big] \\
& + N_F C_F T_F \left[-\frac{8}{3N^2} + \frac{8}{3N} \log\left(\frac{Q^2}{\mu^2}\right) - \frac{100}{9N} + \frac{8}{3(N+1)} - \frac{8}{3} \frac{S_1(N)}{N} \right] \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
C_{L,q}^{(2),id} = & C_F^2 \left[32 \frac{S_1(N)\beta^{(1)}(N+1)}{N} - \frac{104}{5(N+1)} + 8 \frac{\zeta_2}{(N-1)} - \frac{12}{5} \frac{\zeta_2}{(N-2)} \right. \\
& + 16 \frac{\beta^{(1)}(N+2)}{(N+1)} - 8 \frac{\beta^{(1)}(N+3)}{(N+2)} + \frac{16}{5} \frac{\beta^{(1)}(4+N)}{(N+3)} - \frac{24}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} + 4 \frac{\zeta_2}{(N+2)} \\
& - 8 \frac{\zeta_2}{(N+1)} - \frac{8}{5} \frac{\zeta_2}{(N+3)} - \frac{8}{(N+2)^3} - \frac{16}{5(N+2)^2} + \frac{16}{5(N+2)} \\
& + \frac{16}{5(N+3)^3} + \frac{24}{5(N-1)} + \frac{24}{5(N-1)^2} + \frac{16}{(N+1)^3} + \frac{8}{5(N+1)^2} \\
& + 16 \frac{\beta^{(1)}(N)}{(N-1)} + \left(32 \beta(N+1)\zeta_2 + 16 \beta^{(1)}(N+1) + \frac{64}{5} - 32 A_3(N) \right. \\
& \left. - 8 \beta^{(2)}(N+1) \right) \frac{1}{N} + \left(-8 \zeta_2 - \frac{112}{5} - 16 \beta^{(1)}(N+1) \right) \frac{1}{N^2} + \frac{8}{N^3} \Big] \\
& + C_A C_F \left[-16 \frac{S_1(N)\beta^{(1)}(N+1)}{N} + \frac{12}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} + \frac{6}{5} \frac{\zeta_2}{(N-2)} - 4 \frac{\zeta_2}{(N-1)} \right. \\
& - 8 \frac{\beta^{(1)}(N)}{(N-1)} + 4 \frac{\beta^{(1)}(N+3)}{(N+2)} + 4 \frac{\zeta_2}{(N+1)} + \frac{4}{5} \frac{\zeta_2}{(N+3)} - \frac{12}{5} \frac{1}{(N-1)^2} \\
& - \frac{8}{5(N+3)^3} - \frac{12}{5(N-1)} - \frac{4}{5(N+1)^2} + \frac{8}{5(N+2)^2} - \frac{8}{5(N+2)} \\
& + \frac{4}{(N+2)^3} + \frac{52}{5(N+1)} - \frac{8}{(N+1)^3} - 8 \frac{\beta^{(1)}(N+2)}{(N+1)} - \frac{8}{5} \frac{\beta^{(1)}(4+N)}{(N+3)} \\
& - 2 \frac{\zeta_2}{(N+2)} + \left(-\frac{32}{5} + 16 A_3(N) - 8 \beta^{(1)}(N+1) - 16 \beta(N+1)\zeta_2 \right. \\
& \left. + 4 \beta^{(2)}(N+1) \right) \frac{1}{N} + \left(\frac{56}{5} + 8 \beta^{(1)}(N+1) + 4 \zeta_2 \right) \frac{1}{N^2} - \frac{4}{N^3} \Big] \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
C_{L,q}^{\text{PS}(2)} = & N_F C_F T_F \left[\left(-\frac{16}{(N+1)} - \frac{16}{N^2} + \frac{32}{3(N-1)} + \frac{16}{3(N+2)} \right) \log \left(\frac{Q^2}{\mu^2} \right) \right. \\
& + \frac{48}{N^3} + 16 \frac{S_1(N+1)}{(N+1)} + \frac{32}{N^2} - \frac{112}{3N} + \frac{32}{(N+1)^2} - \frac{16}{(N+2)} \\
& + \frac{208}{3(N+1)} - \frac{16}{3(N+2)^2} - \frac{16}{(N-1)} - \frac{32}{3} \frac{S_1(N-1)}{(N-1)} - \frac{64}{3(N-1)^2} \\
& \left. - \frac{16}{3} \frac{S_1(N+2)}{(N+2)} + 16 \frac{S_1(N)}{N^2} \right] \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
C_{L,g}^{S(2)} = & C_A C_F \left[\left(\frac{32}{N^2} + 32 \frac{S_1(N)}{N} - \frac{272}{3} \frac{1}{(N-1)} + \frac{32}{(N-1)^2} - \frac{16}{3(N+2)} \right. \right. \\
& + \frac{16}{(N+1)} + \frac{80}{N} - 32 \frac{S_1(N-1)}{(N-1)} \Big) \log \left(\frac{Q^2}{\mu^2} \right) + \frac{464}{3} \frac{S_1(N-1)}{(N-1)} - 16 \frac{S_1(N+1)}{(N+1)} \\
& + \left(-\frac{64}{N^2} - \frac{144}{N} \right) S_1(N) + \frac{16}{3} \frac{S_1(N+2)}{(N+2)} + 16 \frac{S_1^2(N-1)}{(N-1)} - 16 \frac{S_1^2(N)}{N} \\
& + 80 \frac{S_2(N-1)}{(N-1)} - 80 \frac{S_2(N)}{N} + \frac{352}{3(N-1)^2} + \frac{16}{3(N+2)^2} - \frac{128}{(N-1)^3} \\
& \left. + \frac{448}{3(N-1)} - 32 \frac{\beta^{(1)}(N)}{(N-1)} + \frac{32}{3(N+2)} + 16 \frac{\zeta_2}{(N-1)} - \frac{160}{3(N+1)} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{32}{(N+1)^2} + \left(-32\beta^{(1)}(N+1) - \frac{320}{3} - 16\zeta_2 \right) \frac{1}{N} - \frac{112}{N^2} - \frac{96}{N^3} \Big] \\
& + C_F^2 \left[\left(-\frac{8}{(N+1)} - \frac{8}{N} - \frac{16}{N^2} + \frac{16}{(N-1)} \right) \log \left(\frac{Q^2}{\mu^2} \right) - 32 \frac{S_1(N-1)}{(N-1)} \right. \\
& + 8 \frac{S_1(N+1)}{(N+1)} + \left(\frac{24}{N} + \frac{16}{N^2} \right) S_1(N) - \frac{64}{5} \frac{\beta^{(1)}(N-1)}{(N-2)} - \frac{96}{5(N-1)^2} \\
& + \frac{224}{15(N+1)^2} + \frac{248}{15(N+1)} + \frac{32}{15(N+2)^2} - \frac{32}{15(N+2)} - \frac{32}{15(N+3)^3} \\
& - \frac{32}{5} \frac{\zeta_2}{(N-2)} - \frac{32}{15} \frac{\beta^{(1)}(4+N)}{(N+3)} - \frac{96}{5(N-1)} + \frac{16}{15} \frac{\zeta_2}{(N+3)} + \left(\frac{24}{5} + \frac{16}{3} \zeta_2 \right. \\
& \left. \left. + \frac{32}{3} \beta^{(1)}(N+1) \right) \frac{1}{N} + \frac{8}{5N^2} + \frac{48}{N^3} \right] \tag{A.11}
\end{aligned}$$

$$\begin{aligned}
& \mathbb{C}_{A,q}^{\text{NS},nid,(2)} - \mathbb{C}_{T,q}^{\text{NS},nid,(2)} = \\
& C_A C_F \left(\log \left(\frac{Q^2}{\mu^2} \right) \left(\frac{22}{3N} - \frac{22}{3(N+1)} \right) + \left(-\frac{8\zeta_2 S_1(N)}{N} + \frac{8\zeta_2 S_1(N+1)}{(N+1)} \right. \right. \\
& + \frac{4\zeta_2}{N^2} - \frac{4\zeta_2}{(N+1)^2} - \frac{2\zeta_2}{(N+2)} - \frac{6\zeta_2}{5(N+3)} + \frac{2\zeta_2}{(N-1)} + \frac{6\zeta_2}{5(N-2)} - \frac{20\zeta_3}{N} \\
& + \frac{20\zeta_3}{(N+1)} + \frac{50S_1(N)}{3N} + \frac{8S_3(N)}{N} - \frac{50S_1(N+1)}{3(N+1)} - \frac{8S_3(N+1)}{(N+1)} \\
& + \frac{12\beta^{(1)}(N-1)}{5(N-2)} + \frac{4\beta^{(1)}(N)}{(N-1)} + \frac{8\beta^{(1)}(N+1)}{N^2} - \frac{8\beta^{(1)}(N+1)}{N} \\
& - \frac{8\beta^{(1)}(N+2)}{(N+1)^2} - \frac{8\beta^{(1)}(N+2)}{(N+1)} + \frac{4\beta^{(1)}(N+3)}{(N+2)} + \frac{12\beta^{(1)}(N+4)}{5(N+3)} \\
& - \frac{4}{N^3} - \frac{182}{15N^2} - \frac{823}{45N} - \frac{4}{(N+1)^3} + \frac{398}{15(N+1)^2} + \frac{823}{45(N+1)} + \frac{4}{(N+2)^3} \\
& \left. \left. - \frac{12}{5(N+2)^2} + \frac{12}{5(N+2)} + \frac{12}{5(N+3)^3} - \frac{12}{5(N-1)^2} - \frac{12}{5(N-1)} \right) \right) \\
& + N_F C_F T_F \left(\log \left(\frac{Q^2}{\mu^2} \right) \left(\left(-\frac{8}{3N} + \frac{8}{3(N+1)} \right) + \left(\frac{8S_1(N)}{3N} - \frac{8S_1(N+1)}{3(N+1)} \right) \right. \right. \\
& + \frac{8}{3N^2} + \frac{76}{9N} - \frac{8}{3(N+1)^2} - \frac{76}{9(N+1)} \Big) \Big) + C_F^2 \left(\log \left(\frac{Q^2}{\mu^2} \right) \left(\frac{8S_1(N)}{N} \right. \right. \\
& - \frac{8S_1(N+1)}{(N+1)} - \frac{4}{N^2} + \frac{2}{N} + \frac{4}{(N+1)^2} - \frac{2}{(N+1)} \Big) + \left(\frac{16\zeta_2 S_1(N)}{N} \right. \\
& - \frac{16\zeta_2 S_1(N+1)}{(N+1)} - \frac{8\zeta_2}{N^2} + \frac{8\zeta_2}{(N+1)^2} + \frac{4\zeta_2}{(N+2)} + \frac{12\zeta_2}{5(N+3)} - \frac{4\zeta_2}{(N-1)} \\
& - \frac{12\zeta_2}{5(N-2)} + \frac{40\zeta_3}{N} - \frac{40\zeta_3}{(N+1)} - \frac{4S_1(N)}{N^2} - \frac{50S_1(N)}{N} - \frac{4S_1^2(N)}{N} \\
& - \frac{20S_2(N)}{N} - \frac{16S_3(N)}{N} + \frac{4S_1(N+1)}{(N+1)^2} + \frac{50S_1(N+1)}{(N+1)} + \frac{4S_1^2(N+1)}{(N+1)} \\
& \left. + \frac{20S_2(N+1)}{(N+1)} + \frac{16S_3(N+1)}{(N+1)} - \frac{24\beta^{(1)}(N-1)}{5(N-2)} - \frac{8\beta^{(1)}(N)}{(N-1)} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{16\beta^{(1)}(N+1)}{N^2} + \frac{16\beta^{(1)}(N+1)}{N} + \frac{16\beta^{(1)}(N+2)}{(N+1)^2} + \frac{16\beta^{(1)}(N+2)}{(N+1)} \\
& -\frac{8\beta^{(1)}(N+3)}{(N+2)} - \frac{24\beta^{(1)}(N+4)}{5(N+3)} + \frac{20}{N^3} - \frac{2}{5N^2} + \frac{9}{5N} - \frac{4}{(N+1)^3} \\
& -\frac{202}{5(N+1)^2} - \frac{9}{5(N+1)} - \frac{8}{(N+2)^3} + \frac{24}{5(N+2)^2} - \frac{24}{5(N+2)} \\
& -\frac{24}{5(N+3)^3} + \frac{24}{5(N-1)} + \frac{24}{5(N-1)^2} \Big) \Big) \tag{A.12}
\end{aligned}$$

$$\begin{aligned}
\mathbb{C}_{A,q}^{\text{NS},id,(2)} - \mathbb{C}_{T,q}^{\text{NS},id,(2)} = & \left(C_F^2 - \frac{1}{2} C_A C_F \right) \left(\frac{32\zeta_2\beta(N+1)}{N} + \frac{32\zeta_2\beta(N+2)}{(N+1)} - \frac{8\zeta_2}{N^2} + \frac{8\zeta_2}{N} - \frac{8\zeta_2}{(N+1)^2} \right. \\
& -\frac{8\zeta_2}{(N+1)} - \frac{4\zeta_2}{(N+2)} + \frac{12\zeta_2}{5(N+3)} + \frac{4\zeta_2}{(N-1)} - \frac{12\zeta_2}{5(N-2)} + \frac{32S_1(N)\beta^{(1)}(N+1)}{N} \\
& + \frac{32S_1(N+1)\beta^{(1)}(N+2)}{(N+1)} - \frac{32A_3(N)}{N} - \frac{32A_3(N+1)}{(N+1)} - \frac{24\beta^{(1)}(N-1)}{5(N-2)} \\
& + \frac{8\beta^{(1)}(N)}{(N-1)} - \frac{16\beta^{(1)}(N+1)}{N^2} - \frac{8\beta^{(2)}(N+1)}{N} - \frac{16\beta^{(1)}(N+2)}{(N+1)^2} \\
& - \frac{8\beta^{(2)}(N+2)}{(N+1)} + \frac{8\beta^{(1)}(N+3)}{(N+2)} - \frac{24\beta^{(1)}(N+4)}{5(N+3)} - \frac{8}{N^3} - \frac{72}{5N^2} \\
& + \frac{104}{5N} + \frac{8}{(N+1)^3} - \frac{72}{5(N+1)^2} - \frac{104}{5(N+1)} + \frac{8}{(N+2)^3} + \frac{24}{5(N+2)^2} \\
& \left. - \frac{24}{5(N+2)} - \frac{24}{5(N+3)^3} + \frac{24}{5(N-1)^2} + \frac{24}{5(N-1)} \right) \tag{A.13}
\end{aligned}$$

The functions $\mathbb{C}_{T,q}^{\text{NS},nid(id),(2)}(N)$ were given in (A.5) summing over the color factors.

B Polarized Time-like Coefficient Functions

In this appendix, we present the Mellin moments of the polarized time-like coefficient functions to $O(\alpha_s^2)$.

$$\Delta\mathbb{C}_{T,a} = \Delta C_{T,a}^{(0)} + a_s \Delta C_{T,a}^{(1)} + a_s^2 \Delta C_{T,a}^{(2)} \quad (\text{B.1})$$

The individual contributions are

$$\begin{aligned} \Delta C_{1,q}^{\text{PS}(2)} = & N_F C_F T_F \left[\left(\frac{20}{N} - \frac{20}{(N+1)} - \frac{8}{N^2} - \frac{8}{(N+1)^2} \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(\frac{40}{(N+1)} \right. \right. \\ & + \frac{16}{(N+1)^2} S_1(N+1) + \left(-\frac{40}{N} + \frac{16}{N^2} \right) S_1(N) + \frac{136}{(N+1)} + \frac{88}{(N+1)^2} \\ & - \frac{136}{N} + \frac{48}{N^3} + \frac{48}{(N+1)^3} - \frac{8}{N^2} \log \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{88}{(N+1)^2} \right. \\ & - \frac{136}{(N+1)} - \frac{48}{(N+1)^3} \Big) S_1(N+1) + \left(\frac{136}{N} - \frac{48}{N^3} + \frac{8}{N^2} \right) S_1(N) \\ & + \left(-\frac{20}{(N+1)} - \frac{8}{(N+1)^2} \right) S_1^2(N+1) + \left(\frac{20}{N} - \frac{8}{N^2} \right) S_1^2(N) + \left(-\frac{20}{(N+1)} \right. \\ & - \frac{8}{(N+1)^2} \Big) S_2(N+1) + \left(\frac{20}{N} - \frac{8}{N^2} \right) S_2(N) - \frac{284}{3(N+1)^2} + 56 \frac{\zeta_2}{(N+1)} \\ & - \frac{16}{3} \frac{\zeta_2}{(N+2)} - \frac{424}{3(N+1)} + \frac{32}{3} \frac{\beta^{(1)}(N+3)}{(N+2)} + \frac{32}{3} \frac{\beta^{(1)}(N)}{(N-1)} - \frac{100}{(N+1)^3} \\ & + \frac{16}{3} \frac{\zeta_2}{(N-1)} + \frac{32}{3(N+2)^3} - \frac{88}{(N+1)^4} + 16 \frac{\zeta_2}{(N+1)^2} + 32 \frac{\beta^{(1)}(N+2)}{(N+1)} \\ & \left. + \left(\frac{424}{3} - 56 \zeta_2 + 32 \beta^{(1)}(N+1) \right) \frac{1}{N} + \left(\frac{340}{3} + 16 \zeta_2 \right) \frac{1}{N^2} - \frac{44}{N^3} - \frac{88}{N^4} \right] \quad (\text{B.2}) \end{aligned}$$

$$\begin{aligned} \Delta C_{1,g}^{(2)} = & C_A C_F \left[\frac{496}{N^4} + \frac{144}{N^3} + \frac{88}{(N+1)^4} + \frac{76}{(N+1)^3} + \frac{128}{3(N+1)^2} \right. \\ & + \frac{76}{3(N+1)} - \frac{32}{3(N+2)^3} + 16 \frac{\zeta_2}{(N+1)^2} - 112 \frac{\zeta_2}{(N+1)} + \frac{16}{3} \frac{\zeta_2}{(N+2)} \\ & - \frac{88}{3} \frac{\zeta_2}{(N-1)} - \frac{32}{3} \frac{\beta^{(1)}(N+3)}{(N+2)} - \frac{176}{3} \frac{\beta^{(1)}(N)}{(N-1)} - 16 \frac{\beta^{(2)}(N+2)}{(N+1)} + 32 \frac{A_3(N+1)}{(N+1)} \\ & - 32 \frac{\beta(N+2)\zeta_2}{(N+1)} - 60 \frac{\zeta_3}{(N+1)} + \frac{200}{3} \frac{S_3(N+1)}{(N+1)} + 32 \frac{\beta^{(1)}(N+2)}{(N+1)^2} \\ & - 32 \frac{\beta^{(1)}(N+2)}{(N+1)} + 16 \frac{A_{18}(N+1)}{(N+1)} + \left(-16 \frac{\beta^{(1)}(N+2)}{(N+1)} + \frac{64}{(N+1)^3} \right. \\ & + \frac{136}{(N+1)^2} + 36 \frac{S_2(N+1)}{(N+1)} - 40 \frac{\zeta_2}{(N+1)} + \frac{240}{(N+1)} \Big) S_1(N+1) \\ & \left. + \left(\frac{12}{(N+1)^2} + \frac{48}{(N+1)} \right) S_1^2(N+1) - \frac{400}{3} \frac{S_3(N)}{N} + \frac{4}{3} \frac{S_1^3(N+1)}{(N+1)} - \frac{8}{3} \frac{S_1^3(N)}{N} \right] \end{aligned}$$

$$\begin{aligned}
& +16 \frac{\zeta_2}{(N-1)^2} - 40 \frac{\zeta_3}{(N-1)} + 16 \frac{S_3(N-1)}{(N-1)} + 32 \frac{\beta^{(1)}(N)}{(N-1)^2} + 8 \frac{\beta^{(2)}(N)}{(N-1)} \\
& + 32 \frac{A_3(N-1)}{(N-1)} - 32 \frac{\beta(N)\zeta_2}{(N-1)} + \left(\frac{24}{N^2} - \frac{56}{N} \right) S_1^2(N) + \left(\frac{8}{(N+1)^2} + \frac{32}{N^2} \right. \\
& + \frac{48}{(N+1)} - 16 \frac{S_1(N)}{N} - \frac{48}{N} + 8 \frac{S_1(N+1)}{(N+1)} \left. \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{124}{3} \right. \\
& - 32 \beta^{(2)}(N+1) + 128 \zeta_2 + 120 \zeta_3 - 64 \beta^{(1)}(N+1) - 64 \beta(N+1) \zeta_2 + 64 A_3(N) \\
& \left. - 32 A_{18}(N) \right) \frac{1}{N} + \left(-\frac{844}{3} - 128 \zeta_2 + 64 \beta^{(1)}(N+1) \right) \frac{1}{N^2} + \left(-72 \frac{S_2(N)}{N} \right. \\
& + \frac{1}{N} \left(80 \zeta_2 - 276 - 32 \beta^{(1)}(N+1) \right) - \frac{48}{N^2} + \frac{192}{N^3} \left. \right) S_1(N) + \left(\left(-\frac{112}{(N+1)} \right. \right. \\
& \left. \left. - \frac{32}{(N+1)^2} \right) S_1(N+1) + \left(-\frac{32}{N^2} + \frac{128}{N} \right) S_1(N) - 8 \frac{S_1^2(N+1)}{(N+1)} + 16 \frac{S_1^2(N)}{N} \right. \\
& \left. - 40 \frac{S_2(N+1)}{(N+1)} + 80 \frac{S_2(N)}{N} - 16 \frac{\beta^{(1)}(N+2)}{(N+1)} - \frac{224}{(N+1)} - 8 \frac{\zeta_2}{(N+1)} \right. \\
& \left. - \frac{136}{(N+1)^2} - \frac{48}{(N+1)^3} + \frac{232 - 32 \beta^{(1)}(N+1) + 16 \zeta_2}{N} + \frac{32}{N^2} \right. \\
& \left. - \frac{224}{N^3} \right) \log \left(\frac{Q^2}{\mu^2} \right) + \left(\frac{128}{(N+1)} + \frac{60}{(N+1)^2} \right) S_2(N+1) + \left(\frac{8}{(N-1)} \right. \\
& \left. - 16 \frac{\zeta_2}{(N-1)} - 32 \frac{\beta^{(1)}(N)}{(N-1)} \right) S_1(N-1) + \left(-\frac{72}{N^2} - \frac{168}{N} \right) S_2(N) \Big] \\
& + C_F^2 \left[\frac{88}{N^4} + \frac{256}{N^3} - \frac{44}{(N+1)^4} - \frac{86}{(N+1)^3} - \frac{196}{3(N+1)^2} - \frac{110}{3(N+1)} \right. \\
& + \frac{64}{3(N+2)^3} - 56 \frac{\zeta_2}{(N+1)^2} - 92 \frac{\zeta_2}{(N+1)} - \frac{32}{3} \frac{\zeta_2}{(N+2)} + \frac{104}{3} \frac{\zeta_2}{(N-1)} \\
& + \frac{64}{3} \frac{\beta^{(1)}(N+3)}{(N+2)} + \frac{208}{3} \frac{\beta^{(1)}(N)}{(N-1)} - 16 \frac{\beta^{(2)}(N+2)}{(N+1)} - 64 \frac{A_3(N+1)}{(N+1)} \\
& + 64 \frac{\beta(N+2)\zeta_2}{(N+1)} + 32 \frac{\zeta_3}{(N+1)} - \frac{128}{3} \frac{S_3(N+1)}{(N+1)} - 64 \frac{\beta^{(1)}(N+2)}{(N+1)^2} \\
& - 16 \frac{A_{18}(N+1)}{(N+1)} + \frac{256}{3} \frac{S_3(N)}{N} + \frac{20}{3} \frac{S_1^3(N+1)}{(N+1)} - \frac{40}{3} \frac{S_1^3(N)}{N} - 32 \frac{\zeta_2}{(N-1)^2} \\
& + 80 \frac{\zeta_3}{(N-1)} - 32 \frac{S_3(N-1)}{(N-1)} - 64 \frac{\beta^{(1)}(N)}{(N-1)^2} - 16 \frac{\beta^{(2)}(N)}{(N-1)} - 64 \frac{A_3(N-1)}{(N-1)} \\
& + 64 \frac{\beta(N)\zeta_2}{(N-1)} + \left(\frac{98}{3} + 96 \zeta_2 - 64 \zeta_3 - 32 \beta^{(2)}(N+1) + 64 \beta^{(1)}(N+1) \right. \\
& \left. - 128 A_3(N) + 32 A_{18}(N) + 128 \beta(N+1) \zeta_2 \right) \frac{1}{N} + \left(-\frac{94}{3} + 48 \zeta_2 \right) \frac{1}{N^2} \\
& + \left(\frac{52}{(N+1)^2} + \frac{86}{(N+1)} \right) S_2(N+1) + \left(\left(-\frac{68}{(N+1)} \right. \right. \\
& \left. \left. - \frac{24}{(N+1)^2} \right) S_1(N+1) + \left(\frac{80}{N} + \frac{48}{N^2} \right) S_1(N) - 16 \frac{S_1^2(N+1)}{(N+1)} + 32 \frac{S_1^2(N)}{N} \right.
\end{aligned}$$

$$\begin{aligned}
& -16 \frac{S_2(N+1)}{(N+1)} + 32 \frac{S_2(N)}{N} + \frac{28}{(N+1)^2} + 32 \frac{\zeta_2}{(N+1)} - \frac{52}{(N+1)} \\
& + \frac{24}{(N+1)^3} + \frac{-64\zeta_2 + 36}{N} - \frac{64}{N^2} - \frac{48}{N^3} \log \left(\frac{Q^2}{\mu^2} \right) + \left(-\frac{104}{N^2} \right. \\
& - \frac{104}{N} \Big) S_2(N) + \left(\frac{20}{(N+1)^2} + \frac{54}{(N+1)} \right) S_1^2(N+1) + \left(-\frac{40}{N^2} \right. \\
& - \frac{72}{N} \Big) S_1^2(N) + \left(32 \frac{\zeta_2}{(N-1)} + 64 \frac{\beta^{(1)}(N)}{(N-1)} - \frac{16}{(N-1)} \right) S_1(N-1) \\
& + \left(\frac{6}{(N+1)} - \frac{4}{(N+1)^2} - 16 \frac{S_1(N)}{N} + 8 \frac{S_1(N+1)}{(N+1)} + \frac{8}{N^2} \right) \log^2 \left(\frac{Q^2}{\mu^2} \right) \\
& + \left(-56 \frac{S_2(N)}{N} + \frac{128\beta^{(1)}(N+1) - 32\zeta_2 - 88}{N} - \frac{64}{N^2} - \frac{16}{N^3} \right) S_1(N) \\
& + \left(16 \frac{\zeta_2}{(N+1)} + \frac{52}{(N+1)^2} + \frac{108}{(N+1)} + 64 \frac{\beta^{(1)}(N+2)}{(N+1)} + 28 \frac{S_2(N+1)}{(N+1)} \right. \\
& \left. + \frac{8}{(N+1)^3} \right) S_1(N+1) \Big] \tag{B.3}
\end{aligned}$$

References

- [1] S. D. Drell, D. J. Levy and T. M. Yan, Phys. Rev. **187** (1969) 2159;
S. D. Drell, D. J. Levy and T. M. Yan, Phys. Rev. D **1** (1970) 1617.
- [2] A. Suri, Phys. Rev. D **4** (1971) 570;
R. Gatto and G. Preparata, Nucl. Phys. B **47** (1972) 313;
H. D. Dahmen and F. Steiner, Phys. Lett. B **43** (1973) 217;
P. Menotti, in: Proceedings of the Informal Meeting on Electromagnetic Interactions, Frascati, May 2–3, 1972, Pisa Preprint SNS 3/72.
- [3] W. Furmanski and R. Petronzio, Z. Phys. C **11** (1982) 293.
- [4] G. Grunberg, Phys. Rev. D **29** (1984) 2315.
- [5] S. Catani, Z. Phys. C **75** (1997) 665 [arXiv:hep-ph/9609263].
- [6] J. Blümlein, V. Ravindran and W. L. van Neerven, Acta Phys. Polon. B **29** (1998) 2581 [arXiv:hep-ph/9806355];
J. Blümlein, V. Ravindran and W. L. van Neerven, Nucl. Phys. B **586** (2000) 349 [arXiv:hep-ph/0004172].
- [7] J. Blümlein and A. Guffanti, arXiv:hep-ph/0411110.
- [8] M. Stratmann and W. Vogelsang, Nucl. Phys. B **496** (1997) 41 [arXiv:hep-ph/9612250].
- [9] G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;
E. G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417.
- [10] P. J. Rijken and W. L. van Neerven, Nucl. Phys. B **487** (1997) 233 [arXiv:hep-ph/9609377].
- [11] P. J. Rijken and W. L. van Neerven, Phys. Lett. B **392** (1997) 207 [arXiv:hep-ph/9609379].
- [12] P. J. Rijken and W. L. van Neerven, Phys. Lett. B **386** (1996) 422 [arXiv:hep-ph/9604436].
- [13] P. J. Rijken and W. L. van Neerven, Nucl. Phys. B **523** (1998) 245 [arXiv:hep-ph/9711335].
- [14] J. Blümlein, Nucl. Phys. Proc. Suppl. **135** (2004) 225 [arXiv:hep-ph/0407044].
- [15] J. Blümlein and V. Ravindran, Nucl. Phys. Proc. Suppl. **135** (2004) 24 [arXiv:hep-ph/0407045].
- [16] J. Blümlein and V. Ravindran, Nucl. Phys. B **716** (2005) 128 [arXiv:hep-ph/0501178].
- [17] J. Blümlein and S. O. Moch, Phys. Lett. B **614** (2005) 53 [arXiv:hep-ph/0503188].
- [18] J. Blümlein and S.-O. Moch, DESY 05–008.
- [19] M. Glück, E. Reya and A. Vogt, Phys. Rev. D **45** (1992) 3986;
- [20] J. Blümlein and A. Vogt, Phys. Rev. D **58** (1998) 014020 [arXiv:hep-ph/9712546].
- [21] J. Blümlein, Comput. Phys. Commun. **133** (2000) 76 [arXiv:hep-ph/0003100].

- [22] J. Blümlein and N. Kochelev, Nucl. Phys. B **498** (1997) 285 [arXiv:hep-ph/9612318];
J. Blümlein and A. Tkabladze, Nucl. Phys. B **553** (1999) 427 [arXiv:hep-ph/9812478].
- [23] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].
- [24] S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [arXiv:hep-ph/0110083].
- [25] J. Blümlein, Comput. Phys. Commun. **159** (2004) 19 [arXiv:hep-ph/0311046].
- [26] E. Witt, Journ. Reine und Angewandte Mathematik, **177** (1937) 152; Math. Zeitschr. **64** (1956) 195.
- [27] J. Blümlein, in preparation.
- [28] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237].
- [29] J. Blümlein, A. De Freitas, S. Klein, S.-O. Moch , and W.L. van Neerven, in preparation.
- [30] J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241].
- [31] S. Albino, B. A. Kniehl and G. Kramer, Nucl. Phys. B **725** (2005) 181 [arXiv:hep-ph/0502188].
- [32] J. Blümlein and S. Kurth, arXiv:hep-ph/9708388.
- [33] J. Binnewies, B. A. Kniehl and G. Kramer, Z. Phys. C **65** (1995) 471 [arXiv:hep-ph/9407347];
J. Binnewies, B. A. Kniehl and G. Kramer, Phys. Rev. D **52** (1995) 4947 [arXiv:hep-ph/9503464];
B. A. Kniehl, G. Kramer and B. Pötter, Nucl. Phys. B **597** (2001) 337 [arXiv:hep-ph/0011155],
S. Kretzer, Phys. Rev. D **62** (2000) 054001 [arXiv:hep-ph/0003177];
L. Bourhis, M. Fontannaz, J. P. Guillet and M. Werlen, Eur. Phys. J. C **19** (2001) 89 [arXiv:hep-ph/0009101].
- [34] A. Mitov and S.-O. Moch, to appear.