Einführung in Lagrange- und Hamilton-Formalismus.

Zunächst nichtrelativistische Theorie.

Dynamisches System: System von bewegten Massenpunkten unter dem Einfluß orts- und zeitabhängiger Kräfte.

Wenn die Kräfte $\vec{F}_i(\vec{r},t)$ bekannt sind, lauten die Bewegungsgleichungen

$$m_i \ddot{\vec{r}}_i = \vec{F}_i$$

Generalisierte Koordinaten $q_k(t)$ $k=1,..f$

Minimale Anzahl von beliebigen Koordinaten, die das dynam. System vollständig beschreiben.

$$f = N \cdot N_{Dim} - \lambda$$

f = Zahl der Freiheitsgrade

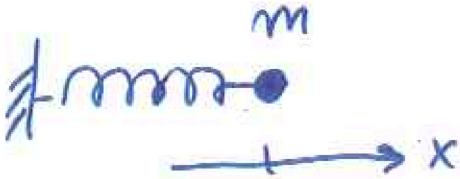
N = Anzahl der Teilchen

N_{Dim} : Raum-Dimensionen (1, 2, 3)

λ Anzahl der Restriktionen

Beispiele

1) harmon. Oszillator



$N=1$, $N_{Dim}=1$, $q_1=q=x$

$f=1$

2) Pendel

$N=1$, $N_{Dim}=2$, $f=1$ $(\lambda=1)$

1 Restriktion: Fadenlänge $(x^2+y^2=\ell^2)$

$q=\varphi$

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kinetische Energie

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = T(q_1, \dots q_f, \dot{q}_1 \dots \dot{q}_f)$$

potentielle Energie

$$U = -\sum_{i=1}^{N} \int \vec{F}_i \cdot d\vec{r}_i = U(q_1, \dots q_f, t)$$

kinetische Energie enthält im allgemeinen q_k und $\dot{q}_k$.

Beisp. Bewegung in Polarkoordinaten (r, φ)

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) = T(r, \dot{r}, \dot{\varphi})$$

$$q_1 = r, \quad q_2 = \varphi$$

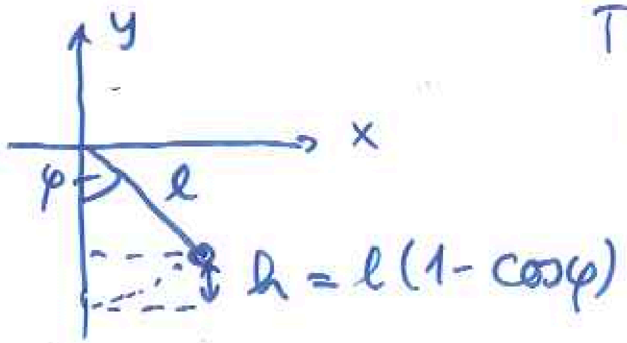
Lagrange Funktion

$$L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k) - U(q_k, t)$$

① harmon. Oszillator:

$$T = \frac{m}{2}\dot{q}^2 \qquad U = \frac{k}{2}q^2 \qquad L = \frac{1}{2}(m\dot{q}^2 - kq^2)$$

② Pendel



$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2} l^2 \dot{\varphi}^2$$

$$U = mgh = mgl(1 - \cos\varphi)$$

$$L(\varphi, \dot{\varphi}) = \frac{m}{2} l^2 \dot{\varphi}^2 - mgl(1 - \cos\varphi)$$

Lagrange-Gleichungen.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0} \qquad k = 1, .. f$$

① harm. Oszillator

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial L}{\partial q} = -kq$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \Rightarrow m\ddot{q} + kq = 0$$

② Pendel

$$\frac{\partial L}{\partial \dot{q}} = ml^2\dot{\varphi} \qquad \frac{\partial L}{\partial q} = -mgl\sin\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \Rightarrow \ddot{\varphi} + \frac{g}{l}\cdot\sin\varphi = 0$$

Man kann die Lagrange-Gleichungen aus dem Hamiltonschen Prinzip der kleinsten Wirkung herleiten.

Wirkungsintegral (action)

$$S = \int_{t_0}^{t_1} L(q_k, \dot{q}_k, t)\, dt$$

Randbedingungen sind festgelegt:

$$q_k(t_0) = q_{k0}, \quad q_k(t_1) = q_{k1}$$

Hier ist noch nicht festgelegt, auf welcher „Kurve" sich $q_k(t)$ bewegt zwischen t_0 und t_1.

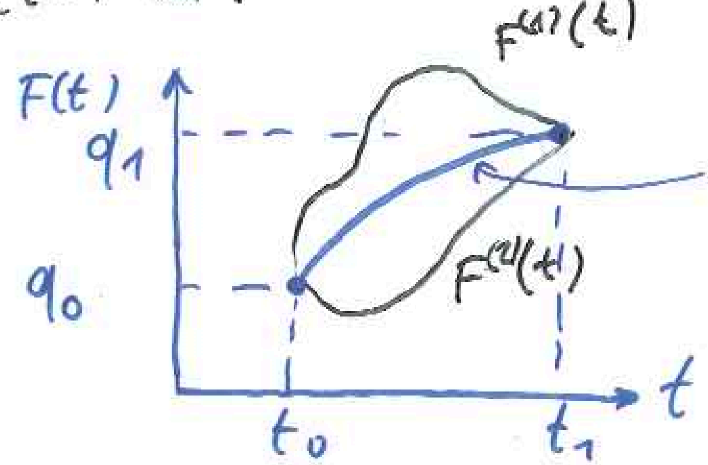
Gegeben sei beliebige Schar von Funktionen

$$F_k(t) \quad \text{mit} \quad F_k(t_0) = q_{k0}$$
$$F_k(t_1) = q_{k1}$$

Die Wirkung

$$S = \int_{t_0}^{t_1} L(F_k, \dot{F}_k, t)\, dt$$

nimmt ein Extremum (meist Minimum) an, wenn $F_k(t)$ gleich der „wirklichen" Bahn $q_k(t)$ ist.



Beispiel (Übungsaufgabe).
harmon. Oszillator mit $m=1$, $k=1$

$\ddot{q} + q = 0$ Lösung: $q(t) = a \sin t$

Wir sehen die Randbed. $q(0) = 0$, $q(1) = 1$

$$\Rightarrow a = \frac{1}{\sin 1}$$

Betrachte die Testfunktionen

$F(\mu, t) = \dfrac{\sin \mu t}{\sin \mu}$; Skizze für $\mu = 0{,}1,\ 1,\ 2{,}5,\ 5{,}5$

zeige daß $S(\mu) = \min$ für $\mu = 1$.

Variation der Wirkung

$$\delta S = \int_{t_0}^{t_1} \Big[\sum_k \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot q_k} \delta \dot q_k \Big] dt = 0$$

$$\delta \dot q_k = \frac{d}{dt} (\delta q_k)$$, partiell integrieren:

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot q_k} \frac{d}{dt} (\delta q_k) dt = \underbrace{\Big[\frac{\partial L}{\partial \dot q_k} \delta q_k \Big]_{t_0}^{t_1}}_{0}$$

$$- \int_{t_0}^{t_1} \frac{d}{dt} \Big(\frac{\partial L}{\partial \ddot q_k} \Big) \delta q_k \, dt$$

$\Big[\delta q_k \Big]_{t_0}^{t_1} = 0$ wegen der festen Randbed. bei $t = t_0$ und $t = t_1$

$$\Rightarrow \int_{t_0}^{t_1} \sum \Big[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \Big(\frac{\partial L}{\partial \dot q_k} \Big) \Big] \delta q_k \, dt = 0$$

Die δq_k sind unabhängig voneinander beliebig wählbar

$$\Rightarrow [\quad] = 0$$, Lagrange-Gl.

Definition der generalisierten Impulse

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

harmon. Oszillator: $p = m\dot{q} = mv$ „normaler" Impuls

Pendel : $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \ell\,(m\ell\dot{\varphi})$
$= $ Drehimpuls

Definition der Hamiltonfunktion

$$H = \sum_{k=1}^{f} p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Hiernach scheint H eine Funktion der Variablen $q_k, \dot{q}_k, p_k, t$ zu sein. Man kann die $\dot{q}_k$ durch die q_k, p_k, t ausdrücken.

$$H = H(q_k, p_k, t)$$

Beispiele: 1) harmon. Oszillator

$$H = p\cdot\dot{q} - \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}kq^2 \qquad m\dot{q} = p$$

$$H = \tfrac{1}{2m}p^2 + \tfrac{1}{2}kq^2 = \text{totale Energie}$$

2) Pendel

$$H = p_\varphi \cdot \dot{\varphi} - \frac{m}{2}\ell^2\dot{\varphi}^2 + mg\ell(1-\cos\varphi)$$

$$q = \varphi \;,\quad \dot{q} = \dot{\varphi} = \frac{p}{m\ell^2} \qquad (p \equiv p_\varphi)$$

$$H = \frac{p^2}{2m\ell^2} + mg\ell(1-\cos q) = \text{totale Energie}$$

(p ist Drehimpuls!)

Hamiltonsche Gleichungen

$$H(q_k, p_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Wir bilden die totalen Differentiale der linken und rechten Seite

linke Seite

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt$$

rechte Seite

$$\sum_k \left[\dot{q}_k dp_k + p_k d\dot{q}_k - \underbrace{\frac{\partial L}{\partial q_k}}_{\dot{p}_k} dq_k - \underbrace{\frac{\partial L}{\partial \dot{q}_k}}_{p_k} d\dot{q}_k \right] - \frac{\partial L}{\partial t} dt$$

Die Terme mit $d\dot{q}_k$ heben sich weg.
Die dq_k, dp_k, dt sind unabhängige Differentiale
Daraus folgt

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Dividieren durch dt:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_k \left(\frac{\partial H}{\partial q_k} \underbrace{\dot{q}_k}_{\frac{\partial H}{\partial p_k}} + \frac{\partial H}{\partial p_k} \underbrace{\dot{p}_k}_{-\frac{\partial H}{\partial q_k}} \right)$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

Wenn H nicht <u>explizit</u> von der Zeit abhängt, hat es an jedem Ort der Bahn den gleichen Wert

(Gesamtenergie = const bei: Oszillator, Pendel)

Kanonische Transformationen (J. Hagel)

Zur Vereinfachung der Schreibweise werden System mit einem Freiheitsgrad betrachtet ($f=1$).

Gegeben Hamiltonfunktion $H(q,p,t)$

$$\Rightarrow \quad \dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = - \frac{\partial H}{\partial q}$$

Dies sind gekoppelte Gln., die normalerweise nicht einfacher zu lösen sind als die Newton-Gl.

$$m\ddot{x} = F(x,t)$$

Häufig existiert eine Transformation $(q,p) \to (Q,P)$ die zu Gln für die neuen Variablen Q, P führt, die sich leichter lösen lassen.

Beispiel:

$$\dot{q} + q + q^2 = 0 \qquad \text{schwierig!}$$

Transformation $\quad Q = \frac{1}{q}$

$$-\frac{\dot{Q}}{Q^2} + \frac{1}{Q} + \frac{1}{Q^2} = 0$$

$$\Rightarrow \quad \dot{Q} - Q = 1 \qquad \text{leicht zu lösen}$$

$$Q(t) = C e^t - 1$$

$$q(t) = \frac{1}{Ce^t - 1}$$

(9) Eine Transformation $(q,p) \to (Q,P)$ heißt <u>kanonisch</u>, wenn die Gln. für Q, P aus einer neuen Hamiltonfunktion $\mathcal{H}(Q,P,t)$ hergeleitet werden können

$$\dot{q} = \frac{\partial H}{\partial p} \qquad Q = Q(q,p) \qquad \dot{Q} = \frac{\partial \mathcal{H}}{\partial P}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad P = P(q,p) \qquad \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

Wenn dies möglich ist, existieren auch die beiden Lagrange-Funktionen

$$L = p\dot{q} - H(q,p,t) \qquad \mathcal{L} = P\dot{Q} - \mathcal{H}(Q,P,t)$$

Außerdem gilt

$$\delta\left(\int_{t_0}^{t_1} L\,dt\right) = \delta\left(\int_{t_0}^{t_1} \mathcal{L}\,dt\right) = 0$$

Daraus folgt

$$L = \mathcal{L} + \frac{dF}{dt}$$

wobei F eine beliebige Funktion der Koordinaten und der Zeit ist.

Warum?

$$\delta\left(\int_{t_0}^{t_1} L\,dt\right) = \delta\left(\int_{t_0}^{t_1} \mathcal{L}\,dt\right) + \underbrace{\delta\left(\int_{t_0}^{t_1} \frac{dF}{dt}\,dt\right)}_{\underbrace{\delta[F(t_1) - F(t_0)]}_{0}}$$

(10)

F heißt <u>erzeugende Funktion</u> der kanon. Transf.

Es gibt 4 Typen.

<u>(1) $F = F_1(q, Q, t)$</u>

$$L = \mathcal{L} + \frac{dF_1}{dt}$$

$$p \cdot \dot{q} - H = P \cdot \dot{Q} - \mathcal{H} + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q} \cdot \dot{q} + \frac{\partial F_1}{\partial Q} \cdot \dot{Q}$$

$$\Rightarrow \quad p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}$$

$$\mathcal{H} = H + \frac{\partial F_1}{\partial t}$$

<u>(2) $F = F_2(q, P, t) - Q \cdot P$</u>

$$p \cdot \dot{q} - H = P \cdot \dot{Q} - \mathcal{H} + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q} \dot{q} + \frac{\partial F_2}{\partial P} \cdot \dot{P} - P \cdot \dot{Q} - Q \cdot \dot{P}$$

$$\Rightarrow \quad p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}$$

$$\mathcal{H} = H + \frac{\partial F_2}{\partial t}$$

<u>(3) $F = F_3(p, Q, t) + q \cdot p$</u>

$$\Rightarrow \quad q = -\frac{\partial F_3}{\partial p} \qquad P = -\frac{\partial F_3}{\partial Q}$$

$$\mathcal{H} = H + \frac{\partial F_3}{\partial t}$$

$$(4) \quad F = p \cdot q - P \cdot Q + F_4(p, P, t)$$

$$q = -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}, \quad \mathcal{H} = H + \frac{\partial F_4}{\partial t}$$

Umbenennung der unabh. Variablen

Üblicherweise ist die Zeit t die unabh. Variable

Hamilton-Gln. folgen aus Extremalprinzip

$$\delta\left(\int_{t_0}^{t_1} L\,dt\right) = \delta\left(\int_{t_0}^{t_1} \sum_{i=1}^{n} p_i dq_i - H dt\right) = 0 \quad (*)$$

Definiere neue Koord. und Impuls durch

$$q_0 = t, \quad p_0 = -H$$

Setze $s \equiv q_n$ als neue unabh. Variable an; $p_s \equiv p_n$ kann als Funktion der übrigen Variablen geschrieben werden.

$$-p_s = \mathcal{H}(q_0, q_1, \dots, q_n = s, p_0, \dots, p_{n-1})$$

$$\sum_{i=1}^{n} p_i dq_i - H dt = \sum_{j=0}^{n-1} p_j dq_j + \underbrace{p_n dq_n}_{-\mathcal{H} ds}$$

Aus $(*)$ folgt daher

$$\delta\left(\int_{s_0}^{s_1} \sum_{j=0}^{n-1} p_j dq_j - \mathcal{H} ds\right) = 0$$

Also gelten die Hamilton-Gleichungen

$$q_i' = \frac{dq_i}{ds} = \frac{\partial \mathcal{H}}{\partial p_i} \qquad p_i' = -\frac{\partial \mathcal{H}}{\partial q_i}$$

$$(i = 0, \dots n-1)$$

$\mathcal{H}$ ist keine Energie!

Lagrange and Hamilton functions for charged particles in electromagnetic fields

The electromagnetic field is expressed in terms of a scalar potential Φ and a vector potential **A**

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

We use Cartesian coordinates (x_1,x_2,x_3) and consider first nonrelativistic protons

$$q_i = x_i \quad , \quad \dot{q}_i = v_i$$

The kinetic energy is

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \sum_i \dot{q}_i^2$$

To obtain the Lorentz force

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

we need a velocity-dependent potential

$$U(q_i, \dot{q}_i) = e\Phi - e\vec{v}\cdot\vec{A}$$

(Here Φ and **A** are functions of the coordinates $q_i = x_i$)

The Lorentz force is given by

$$F_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right)$$

(tutorial)

The Lagrangian is

$$L(q_i, \dot{q}_i) = T(\dot{q}_i) - U(q_i, \dot{q}_i)$$

and the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)}_{m\dot{v}_i} \underbrace{- \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i}}_{-F_i} =$$

lead to Newton's equation of motion

$$m\dot{\vec{v}} = e(\vec{E} + \vec{v} \times \vec{B})$$

The canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial \dot{q}_i} = m\dot{q}_i + eA_i$$

$$\boxed{\vec{p} = m\vec{v} + e\vec{A}}$$

canonical momentum of proton in e.m. fiel

$m\vec{v}$ mechanical momentum

Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - L$$

$$= (m\vec{v} + e\vec{A}) \cdot \vec{v} - \frac{1}{2}m\vec{v}^2 - e\vec{A} \cdot \vec{v} + e\Phi$$

$$H = \underset{\substack{\uparrow \\ E_{kin}}}{\frac{1}{2}m\vec{v}^2} + \underset{\substack{\uparrow \\ E_{pot}}}{e\Phi}$$

sum of kinetic and potential energy

Result:

- H is the energy of the particle, i.e. the sum of kinetic and potential energy
- the vector potential has no influence on the energy

In the Hamiltonian formalism one has to write H as a function of the coordinates q_i and the canonical momenta p_i

$$H(q_i, p_i) = \frac{1}{2m}\left(\vec{p} - e\vec{A}(\vec{q})\right)^2 + e\Phi(\vec{q})$$

recipe: replace $m\vec{v}$ by $\vec{p} - e\vec{A}$

Relativistic case:

(a) without field (free protons)

$$E^2 = \vec{p}^{\,2}c^2 + m_0^2c^4$$

m_0 rest mass

$$H \triangleq E = \sqrt{\vec{p}^{\,2}c^2 + m_0^2c^4}$$

$\vec{p} = \underbrace{\gamma m_0}_{\text{"moving mass"}}\vec{v}$

(b) with field

$$\boxed{H = \sqrt{(\vec{p} - e\vec{A})^2c^2 + m_0^2c^4} + e\Phi}$$

Note: H is now the total energy of the particle, including

- the rest energy m_0c^2
- the kinetic energy $\gamma m_0c^2 - m_0c^2$
- the potential energy $e\Phi$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Hamiltonian treatment of transverse motion

(1) Hamiltonian in Cartesian coordinates

$$\vec{q} = (q_1, q_2, q_3) \equiv \vec{r} = (x_1, x_2, x_3)$$

The canonical momentum of a particle of charge e in an electromagnetic field is

$$\vec{p} = \gamma m_0 \vec{v} + e\vec{A} \qquad \gamma = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

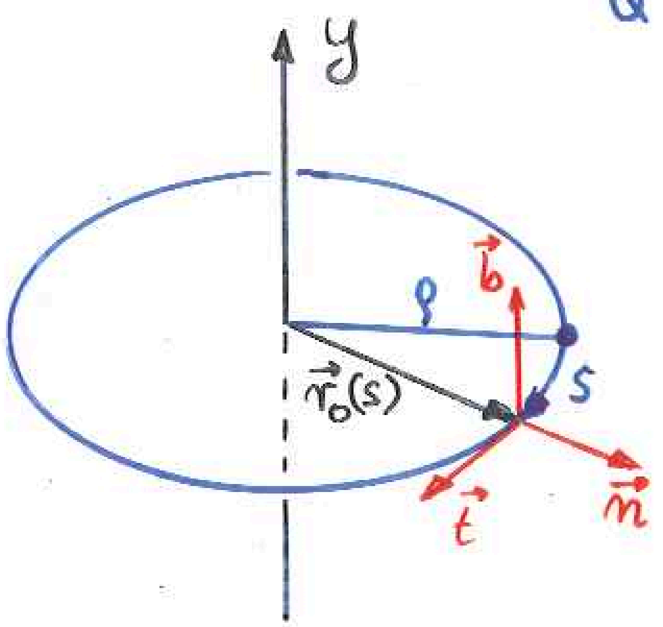
The relativistic Hamiltonian is

$$H_1(\vec{q}, \vec{p}, t) = c\left[(\vec{p} - e\vec{A})^2 + m_0^2 c^2\right]^{1/2} + e\Phi$$

(In general, the scalar potential Φ and the vector potential **A** are functions of the coordinates q_i and the time t)

(2) Transformation to accelerator coordinates

$$\vec{Q} \equiv (Q_1, Q_2, Q_3) = (x, y, s)$$



position vector of particle

$$\vec{r}(s) = \vec{r}_0(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s)$$

$(\vec{n}, \vec{b}, \vec{t})$ comoving unit vectors

Assumption: accelerator is circle of radius ρ in the plane y = 0 (no torsion)

$$\frac{d\vec{r}_0}{ds} = \vec{t} \qquad \frac{d\vec{t}}{ds} = -\frac{1}{\rho}\vec{n} \qquad \frac{d\vec{n}}{ds} = \frac{1}{\rho}\vec{t} \qquad \frac{d\vec{b}}{ds} = 0$$

Canonical transformation $(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$

using generating function

$$F_3(\vec{p}, \vec{Q}) = -\vec{r}\cdot\vec{p} = -(\vec{r}_0 + x\,\vec{n} + y\,\vec{b})\cdot\vec{p}$$

The new momenta are

$$P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$\left.\begin{aligned} P_x &= -\frac{\partial F_3}{\partial x} = \vec{p}\cdot\vec{n} = p_x \\ P_y &= -\frac{\partial F_3}{\partial y} = \vec{p}\cdot\vec{b} = p_y \end{aligned}\right\}$$ projection of $\vec{p}$ on unit vectors along x resp. y

but: $P_s = -\frac{\partial F_3}{\partial s} = \left(1 + \frac{x}{\rho}\right)\vec{p}\cdot\vec{t}$

Note that the momentum conjugate to the arc length s is not identical with the tangential component of the vector **p**

Since the generating function does not depend on time, H remains invariant

$$H_2 = H_1 + \frac{\partial F_3}{\partial t} = H_1$$

Write H_2 in terms of Q_i, P_i

$$H_2(\vec{Q},\vec{P},t) = c\left[(P_x-eA_x)^2+(P_y-eA_y)^2+\left(\frac{P_s-eA_s}{1+\frac{x}{\rho}}\right)^2+m_0^2c^2\right]^{1/2}+e\Phi$$

Note that H_2 is still the total energy of the particle

(3) Replace time t by orbit length s

This is basically a renaming of variables.
$(t, -H_2)$ can be considered as canonical variables like (s, P_s). The above equation is solved for $-P_s$ which is the new Hamiltonian:

$$\underbrace{-P_s}_{H_3} = -eA_s \mp \left(1+\frac{x}{\rho}\right)\left[\frac{1}{c^2}(H_2-e\Phi)^2-m_0^2c^2 - (P_x-eA_x)^2-(P_y-eA_y)^2\right]^{1/2}$$

$(H_2 - e\Phi) = \gamma m_0 c^2$ kinetic + rest energy

$$\frac{1}{c^2}(H_2-e\Phi)^2-m_0^2c^2 = (\gamma m_0 \vec{v})^2 = \vec{p}^{\,2}$$

here $\vec{p}$ mechanical momentum

For simplicity we restrict ourselves to transverse magnetic fields only

$B_s = 0 \Rightarrow$ choose $\vec{A}$ such that $A_x, A_y = 0$

Then the Hamiltonian reads

$$\boxed{H_3 = -eA_s - \left(1+\frac{x}{\rho}\right)\left[\vec{p}^{\,2}-p_x^2-p_y^2\right]^{1/2}}$$

Note that H_3 is no longer an energy but has the dimension of a momentum

Derivation of Hill's equation

We evaluate H_3 for a ring made of combined-function magnets with ideal dipole and quadrupole field components
Use cylindrical coordinate system (r,θ,y)

$r = \rho + x \quad , \quad \theta = s/\rho \quad$ (ring plane)
y = vertical coordinate

A has only a θ-component. To compute A_θ we evaluate $\vec{\nabla} \times \vec{A}$ in cylindrical coordinates

$$B_r \equiv B_x = -g y = -\frac{\partial A_\theta}{\partial y} \qquad (1)$$

$$B_y = -B_1 - g x = \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) \qquad (2)$$

From (1): $A_\theta(x,y) = +\frac{1}{2} g y^2 + f(x)$
Then (2) can be written

$$\frac{1}{1+\frac{x}{\rho}} \cdot \frac{\partial}{\partial x}\left(\underbrace{\left(1+\frac{x}{\rho}\right) f(x)}_{f_1(x)}\right) = -B_1 - g \cdot x$$

Use $B_1 = \frac{p_0}{e} \cdot \frac{1}{\rho}$, $k = \left(\frac{p_0}{e}\right)^{-1} \cdot g$

$$\Rightarrow f_1(x) = -\frac{p_0}{e}\left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} + k\right)\frac{x^2}{2} + \underbrace{\frac{g}{\rho}\frac{x^3}{3}}\right]$$

neglect 3rd order term
(x small)

$$A_s(x,y) = \left(1+\frac{x}{\rho}\right) A_\theta(x,y)$$
$$= f_1(x) + \frac{1}{2} g y^2 + \underbrace{\frac{1}{2}\frac{g}{\rho} y^2 x}_{\text{neglect 3rd order}}$$

$$-eA_s = p_0\left[\frac{x}{\rho} + \left(\frac{1}{\rho^2}+k\right)\frac{x^2}{2} - k\frac{y^2}{2}\right]$$

Now simplify second term in H_3

$$[\vec{p}^{\,2} - p_x^2 - p_y^2]^{1/2} \approx \underbrace{p}_{p_0+\Delta p} - \frac{p_x^2}{2p_0} - \frac{p_y^2}{2p_0}$$

assumption: $p_x, p_y \ll p_0$, $\Delta p \ll p_0$

Then H_3 becomes

$$H_3 \approx p_0\left[\frac{x}{\rho} + \left(\frac{1}{\rho^2}+k\right)\frac{x^2}{2} - k\frac{y^2}{2}\right]$$
$$-p - (p_0+\Delta p)\frac{x}{\rho} + \left(1+\frac{x}{\rho}\right)\left(\frac{p_x^2}{2p_0} + \frac{p_y^2}{2p_0}\right)$$

Keeping only terms up to 2^{nd} order in the small quantities x, y, p_x, p_y and neglecting the constant term $-p$ we obtain

$$\boxed{H_3 = -\Delta p\frac{x}{\rho} + p_0\left[\left(\frac{1}{\rho^2}+k\right)\frac{x^2}{2} - k\frac{y^2}{2}\right] + \frac{p_x^2}{2p_0} + \frac{p_y^2}{2p_0}}$$

Hamiltonian for transverse motion in the "linear optics" approximation

Note: H_3 is of 2^{nd} order in x, y; $\vec{B}$ is linear

Hamilton equations

$$x' = \frac{dx}{ds} = \frac{\partial H_3}{\partial p_x} = \frac{p_x}{p_0}$$

$$p_x' = -\frac{\partial H_3}{\partial x} = \frac{\Delta p}{\rho} - p_0\left(\frac{1}{\rho^2} + k\right)x$$

$$y' = \frac{\partial H_3}{\partial p_y} = \frac{p_y}{p_0} \quad , \quad p_y' = -\frac{\partial H_3}{\partial y} = +p_0 k y$$

$$\boxed{\begin{aligned} x'' + \left(\frac{1}{\rho^2} + k\right)x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\ y'' - ky &= 0 \end{aligned}}$$

(4) Normalized momenta

The Hamiltonian H_3 (which is not an energy but a momentum) and the transverse momenta are made dimensionless by dividing them by p_0 (nominal momentum)

$$H_4 = \frac{H_3}{p_0} \qquad \bar{p}_x = \frac{p_x}{p_0} \qquad \bar{p}_y = \frac{p_y}{p_0}$$

In the following we restrict ourselves to one plane of betatron oscillation, say x, and assume $\Delta p = 0$

$$\boxed{H_4(x, \bar{p}_x, s) = \frac{1}{2} K(s)\, x^2 + \frac{1}{2} \bar{p}_x^{\,2}}$$

$$x' = \frac{dx}{ds} = \frac{\partial H_4}{\partial \bar{p}_x} = \bar{p}_x \qquad \bar{p}_x' = -\frac{\partial H_4}{\partial x} = -Kx$$

The canonical momentum is identical with the slope of the trajectory

$$x'' + K(s)\, x = 0$$

$$x(s) = a\sqrt{\beta(s)}\, \cos(\phi(s) - \delta)$$

$$\bar{p}_x \equiv x'(s) = -a\,\beta^{-1/2}\left[\sin(\phi - \delta) - \frac{\beta'}{2}\cos(\phi - \delta)\right]$$

$$\phi'(s) = \frac{1}{\beta(s)}$$

Angle-action variables of harmonic oscillator

$$m\ddot{q} + kq = 0 \quad \Rightarrow \ddot{q} + \omega^2 q = 0 \ , \quad \omega = \sqrt{\frac{k}{m}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m}\Big(\underbrace{p^2}_{\sim \sin^2\varphi} + \underbrace{m^2\omega^2 q^2}_{\sim \cos^2\varphi}\Big)$$

Find angle variable φ with the property
$p^2 = C \cdot \sin^2\varphi$, $m^2\omega^2 q^2 = C \cdot \cos^2\varphi$ $(C = \text{const})$
$\Rightarrow$ new Hamiltonian independent of φ

Canonical transformation

$$(q, p) \longrightarrow (\varphi, J)$$
$$H \longrightarrow \mathcal{H}$$

with $F_1(q, \varphi) = -\frac{1}{2}m\omega q^2 \cdot \tan\varphi$

$$p = \frac{\partial F_1}{\partial q} = -m\omega q \cdot \tan\varphi$$

$$J = -\frac{\partial F_1}{\partial \varphi} = \frac{1}{2}m\omega q^2 \frac{1}{\cos^2\varphi}$$

It follows: $q = \sqrt{\frac{2J}{m\omega}} \cos\varphi$

$$p = -\sqrt{2Jm\omega}\,\sin\varphi$$

$$\mathcal{H} = H + \frac{\partial F_1}{\partial t} = J \cdot \omega\,(\sin^2\varphi + \cos^2\varphi) = J \cdot \omega$$

$$\boxed{\mathcal{H} = J \cdot \omega}$$

$$\dot{J} = -\frac{\partial \mathcal{H}}{\partial \varphi} = 0 \quad \Rightarrow \quad \text{action } J = \text{const} \ , \ J = \frac{E}{\omega}$$

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial J} = \omega \qquad \varphi = \omega t + \alpha$$

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t + \alpha)$$

(5) Transformation to angle-action variables

$$(x, x') \rightarrow (\psi, J) \quad \text{with} \quad \psi = \phi(s) - \delta$$

Generating function

$$F_1(x, \psi, s) = -\frac{x^2}{2\beta(s)}\left[\tan\psi - \tfrac{1}{2}\beta'(s)\right]$$

$$x' = \frac{\partial F_1}{\partial x} = -\frac{x}{\beta}\left[\tan\psi - \frac{\beta'}{2}\right]$$

insert $x = a\sqrt{\beta}\cos\psi$

$$\Rightarrow x' = -\frac{a}{\sqrt{\beta}}\left[\sin\psi - \frac{\beta'}{2}\cos\psi\right]$$

The new momentum, called 'action', is

$$J = -\frac{\partial F_1}{\partial \psi} = \frac{x^2}{2\beta}\cdot\frac{1}{\cos^2\psi} = \frac{a^2\beta\cos^2\psi}{2\beta\cos^2\psi} = \frac{a^2}{2}$$

From Basic Optics course, formula (4.32)

$$\frac{1}{\beta}\left(x^2 + \left(\beta x' - \tfrac{1}{2}\beta' x\right)^2\right) = a^2 = \text{const}$$

Hence 2J is identical with the Courant-Snyder invariant and therefore a constant of motion

New Hamiltonian

$$H_5 = H_4 + \frac{\partial F_1}{\partial s}$$

$$= \frac{a^2}{2}\left[K\beta\cos^2\psi + \frac{1}{\beta}\left(\sin\psi - \frac{\beta'}{2}\cos\psi\right)^2\right]$$

$$-\frac{\partial}{\partial s}\left\{\frac{x^2}{2\beta(s)}\left(\tan\psi - \tfrac{1}{2}\beta'(s)\right)\right\}$$

Second term:

$$+\left\{\frac{x^2\beta'}{2\beta^2}\left(\tan\psi-\frac{\beta'}{2}\right)+\frac{x^2}{4\beta}\beta''\right\}$$

Insertion of $x = a\beta^{1/2}\cos\psi$ yields

$$+\frac{a^2}{2}\left\{\sin\psi\cos\psi\cdot\frac{\beta'}{\beta}-\cos^2\psi\cdot\frac{\beta'^2}{2\beta}+\cos^2\psi\,\frac{\beta''}{2}\right\}$$

Then H_5 becomes

$$H_5=\frac{a^2}{2}\left[\cos^2\psi\left\{K\beta-\frac{\beta'^2}{4\beta}+\frac{1}{2}\beta''\right\}+\frac{1}{\beta}\sin^2\psi\right]$$

The β function fulfills the equation (4.21)

$$\frac{1}{2}\beta\beta''-\frac{1}{4}\beta'^2+K\beta^2=1 \quad\Rightarrow\{\ \}=\frac{1}{\beta}$$

$$\boxed{H_5(\psi, J, s) = \frac{J}{\beta(s)}}$$

The new Hamiltonian does not depend on the angle variable ψ. The conjugate momentum, the action J, is thus a constant of motion

$$J' = -\frac{\partial H_5}{\partial\psi} = 0 \qquad J = \text{const}$$

We know this already, $2J$ = Courant-Snyder invariant

$$\psi' = \frac{d\psi}{ds} = \frac{\partial H_5}{\partial J} = \frac{1}{\beta(s)}$$

$$\psi(s) = \psi(0) + \int_0^s \frac{ds'}{\beta(s')}$$

The Hamiltonian H_5 is still s-dependent

(6) Transformation to angle variable that is linear in s

$$(\psi, J) \rightarrow (\psi_1, J_1)$$

Generating function

$$F_2(\psi, J_1, s) = \psi \cdot J_1 + \left(\frac{Q}{R} \cdot s - \int_0^s \frac{ds'}{\beta(s')} \right) \cdot J_1$$

$2\pi R$ circumference of machine

Q = number of betatron oscillations per turn

$$\psi_1 = \frac{\partial F_2}{\partial J_1} = \psi(s) + \frac{Q}{R} s - \int_0^s \frac{ds'}{\beta(s')}$$

$$\psi_1(s) = \psi(0) + \frac{Q}{R} \cdot s$$

phase ψ_1 is linear in s

$$J = \frac{\partial F_2}{\partial \psi} = J_1$$

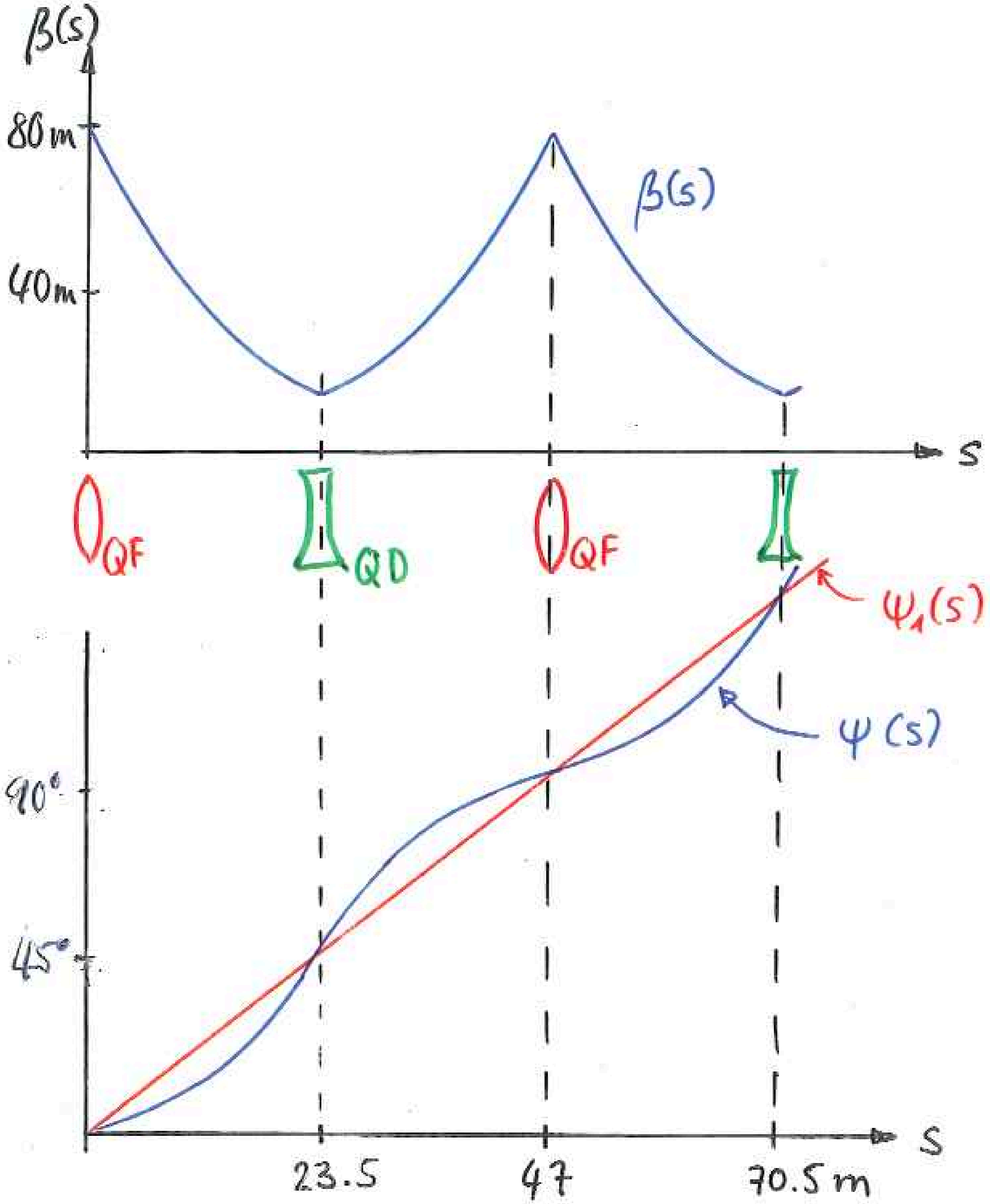
$$H_6 = H_5 + \frac{\partial F_2}{\partial s} = \frac{J}{\beta(s)} + \frac{Q}{R} J_1 - \frac{J_1}{\beta(s)}$$

$$\boxed{H_6 = \frac{Q}{R} J = \omega J}$$

same as for harmonic oscillator

$J = \text{const} \Rightarrow H_6 = \text{const}$

HERA proton ring (horizontal plane x)



$$\psi_1(s) = \frac{Q}{R} \cdot s \qquad \psi(s) = \int_0^s \frac{ds'}{\beta(s')} \qquad (\psi(0) = 0)$$

$$= \phi(s) - \delta$$

Application: gradient error

We use normalized momentum and start from Hamiltonian of type H_4.
Unperturbed Hamiltonian is called H_0 from here on.

$$H_0(x,p,s) = K(s)\frac{x^2}{2} + \frac{p^2}{2} \qquad p = \frac{p_x}{p_0}$$

perturbation $\Delta H = \Delta K(s)\frac{x^2}{2}$

Transformation to angle-action variables (type H_5)

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H} = \frac{J}{\beta(s)} + \frac{1}{2}\Delta K(s) x^2$$

For x we insert the unperturbed solution

$$x = a\beta^{1/2}\cos\varphi$$

$$x^2 = 2J\beta\cos^2\varphi$$

$$\Rightarrow \mathcal{H}(\varphi, J, s) = \frac{J}{\beta(s)} + J\cdot\beta(s)\Delta K(s)\cos^2\varphi$$

$$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$$

Question: what is the Q-value?

$$Q = \frac{1}{2\pi}\oint \frac{d\varphi}{ds}\,ds$$

$$\frac{d\varphi}{ds} = \frac{\partial\mathcal{H}}{\partial J} = \frac{1}{\beta(s)} + \frac{1}{2}\beta(s)\Delta K(s)(1+\cos 2\varphi)$$

Averaged over many turns: $\cos 2\varphi$ cancels

$$Q = \underbrace{\frac{1}{2\pi}\oint \frac{ds}{\beta(s)}}_{Q_0} + \underbrace{\frac{1}{4\pi}\oint \beta(s)\,\Delta k(s)\,ds}_{\text{tune shift } \Delta Q}$$

(Basic Optics, Eq. (5.14))

Higher-order multipoles

(Here uniformly distributed around ring)

Sextupole

$$H = H_0 + S \cdot x^3$$

$$\mathcal{H} = \frac{J}{\beta(s)} + S\,(2J)^{3/2} \cos^3\varphi$$

$$\cos^3\varphi = \frac{1}{4}\left(\cos 3\varphi + 3\cos\varphi\right)$$

both terms average to zero upon many revolutions

Conclusion: a sextupole does not lead to a tune shift (in first-order perturbation theory)

Octupole

$$\Delta\mathcal{H} \propto x^4 = J^2 \cos^4\varphi$$

$$\cos^4\varphi = \frac{1}{8}\left(\cos 4\varphi + 4\cos 2\varphi + 3\right)$$

does not average out

$\Rightarrow$ octupole makes Q shift

$\Delta Q \propto J \propto$ square of amplitude

Canonical Perturbation Theory

In the linear-optics approximation the magnetic fields are linear in x and y while the Hamiltonian (type H_4) is of second order. Transformation to angle-action variables

$$(x, x') \rightarrow (\psi_1, J)$$

leads to a Hamiltonian (type H_6) which is independent of the angular variable ψ_1 and also of the orbit length s (our 'quasi-time'). Instead of s one can also use the azimuthal angle θ.

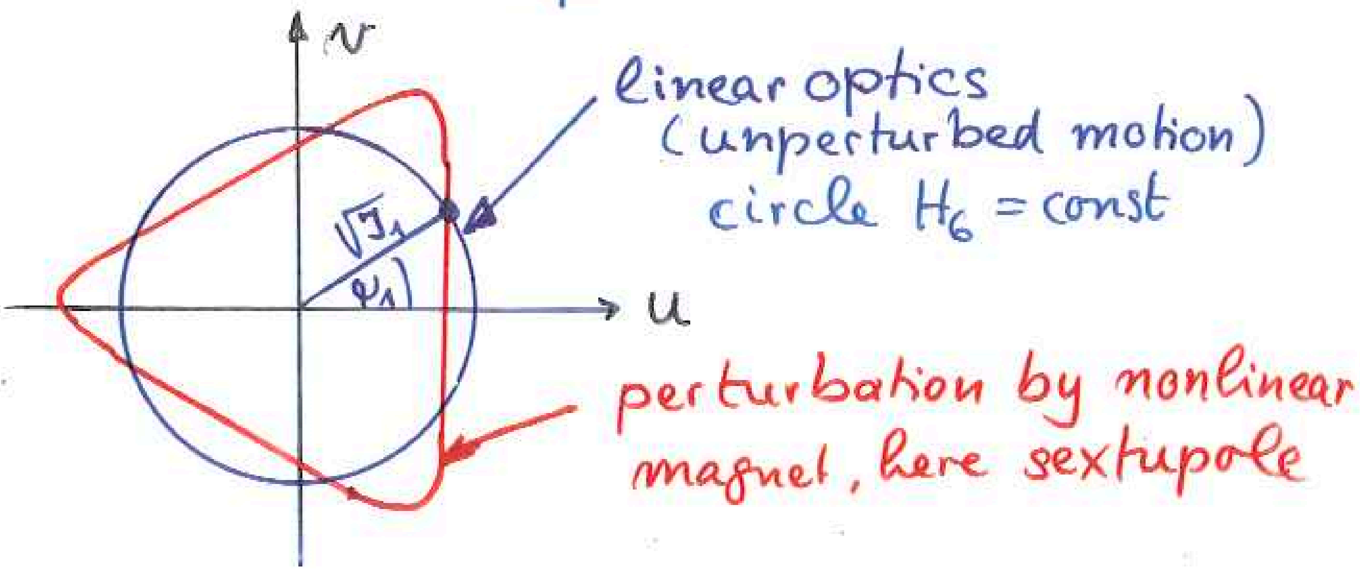
$$H_6 = \frac{Q}{R} J = \text{const}$$

$$s \rightarrow \Theta = \frac{s}{R} : \quad H_7 = R \cdot H_6 = Q \cdot J_1$$

Normalized phase space diagram

plot $\quad u = \frac{1}{\sqrt{2\beta}} x = \sqrt{J_1} \cos\psi_1$

$$v = \frac{-1}{\sqrt{2\beta}} \left(\beta x' - \frac{\beta'}{2} x\right) = + \sqrt{J_1} \sin\psi_1$$



Perturbation by nonlinear fields:
terms of order >2 in x and y in the Hamiltonian. The sextupoles, octupoles etc are usually localized in the ring, hence the perturbation depends on s resp. θ.

Start with Hamiltonian of type H_7 which is called $\mathcal{H}_0$ in the following

$$\mathcal{H}_0 = Q_0 \cdot J_1 \quad \text{(independent of } \psi_1 \text{ and } \theta\text{)}$$

Perturbed Hamiltonian

$$\mathcal{H}_1(\psi_1, J_1, \theta) = \underbrace{\mathcal{H}_0(J_1)}_{Q_0 \cdot J_1} + \underbrace{U(\psi_1, J_1, \theta)}_{\text{small perturbation}}$$

Q_0 unperturbed tune

The perturbation U is periodic in the betatron phase ψ_1 and the azimuthal angle θ. It can be expanded in a double Fourier series

$$U(\psi_1, J_1, \theta) = \sum_{m,n} U_{mn}(J_1)\, e^{i(m\psi_1 - n\theta)}$$

Change of angles per revolution:

$$\theta \rightarrow \theta + 2\pi$$

$$\psi_1 \rightarrow \psi_1 + Q_0 \cdot 2\pi$$

$Q \approx Q_0$ since perturbation is "small"

Behaviour near a resonance

Let $Q_0 \approx n/m$ and $U_{mn} \neq 0$. Then the phase factor

$$\exp\left(i\left(m\psi_1 - n\theta\right)\right)$$

is almost constant from turn to turn since

$$m\psi_1 - n\theta \rightarrow m\left(\psi_1 + Q_0 2\pi\right) - n\left(\theta + 2\pi\right)$$
$$\approx m\psi_1 - n\theta$$

Hence the perturbation adds up coherently from turn to turn.

Other terms $U_{m'n'}$: phases vary from turn to turn $\Rightarrow$ perturbation averages out

Close to a resonance we can restrict ourselves to the terms U_{00} and U_{mn}.

$$\mathcal{H}_1 \approx Q_0 \cdot J_1 + U_{00}(J_1) + 2U_{mn}(J_1)\cos\left(m\psi_1 - n\theta\right)$$
$$\text{for } Q_0 \approx \frac{n}{m}$$

If a single resonance term is present the Hamiltonian can be made independent of θ by transformation into a rotating coordinate system

Canonical transformation into rotating coordinate system: use generating function of type $F_2(q,P,t)$

$$F_2(\varphi_1, J_2, \theta) = \left(\varphi_1 - \frac{n}{m}\theta\right)\cdot J_2$$

$$J_1 = \frac{\partial F_2}{\partial \varphi_1} = J_2 \qquad \varphi_2 = \frac{\partial F_2}{\partial J_2} = \varphi_1 - \frac{n}{m}\theta$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial F_2}{\partial \theta}$$

$$= Q_0 \cdot J_1 + U_{00}(J_1) + 2\,U_{mn}(J_1)\cos(m\varphi_2) - \frac{n}{m} J_2$$

$$\boxed{\mathcal{H}_2(\varphi_2, J_2) = \delta \cdot J_2 + U_{00}(J_2) + 2U_{mn}(J_2)\cos m\varphi_2}$$

$\delta = Q_0 - \frac{n}{m}$ distance to resonance

The new Hamiltonian is indeed independent of the 'quasi time' θ.

Simplified notation: omit index "2"

$$\mathcal{H}(\varphi, J) = \delta \cdot J + \alpha(J) + f(J)\cos m\varphi$$

Third-integer resonance

Sextupole magnet yields term

$$S(\theta)\cdot x^3 = S(\theta)\,(2J\beta)^{3/2}\cos^3\psi$$

sextupole strength as a function of θ

$$\cos^3\psi = \frac{1}{4}(\cos 3\psi + 3\cos\psi)$$

The Fourier series of U contains the terms m = 1 and m = 3. Since Q_0 must be far away from an integer to avoid integer resonances we look only at the term m = 3.

Third-integer resonance: $Q_0 - \frac{n}{3} = \delta$ small

$$\mathcal{H}(\psi, J) = \delta\cdot J + A\cdot J^{3/2}\cos 3\psi$$

Phase space representation: for constant Hamiltonian plot $\sqrt{J}$ against angle ψ in a polar diagram

Fixed points: points in phase space which do not move as a function of θ ($\hat{=}$ time)

$$\frac{d\psi}{d\theta} = \frac{\partial\mathcal{H}}{\partial J} = 0 \quad \Rightarrow \quad \delta + \frac{3}{2}AJ^{\frac{1}{2}}\cos 3\psi = 0$$

$$\frac{dJ}{d\theta} = -\frac{\partial\mathcal{H}}{\partial\psi} = 0 \quad \Rightarrow \quad \sin 3\psi = 0$$

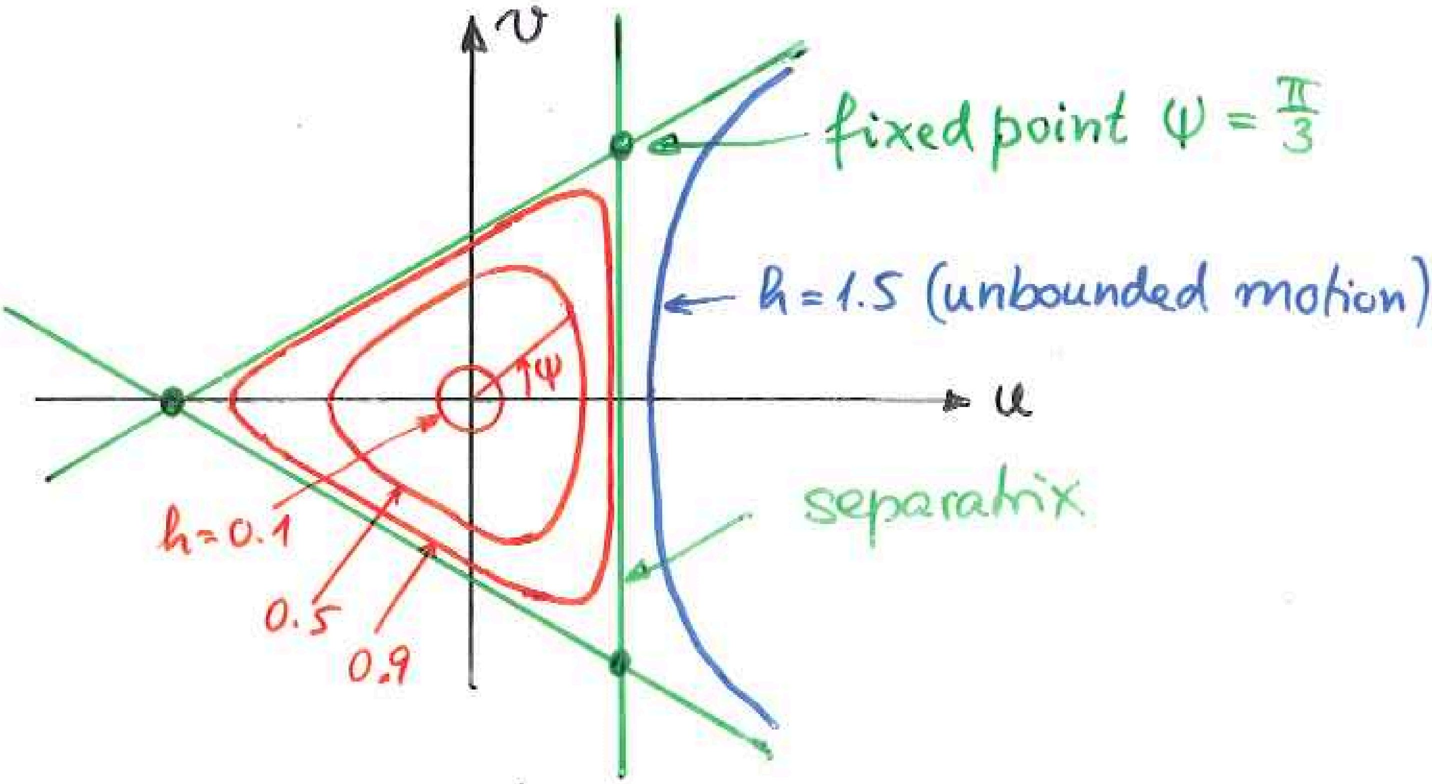
$$\Rightarrow \psi = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Take $\delta, A > 0$

$$\underbrace{\delta}_{>0} + \underbrace{\tfrac{3}{2} A J^{\frac{1}{2}}}_{>0} \cos 3\psi = 0$$

$\Rightarrow$ fixed points at $\psi = \frac{\pi}{3}, \pi, \frac{5}{3}\pi$

$$J_{F.P.} = \left(\frac{2\delta}{3A}\right)^2 \qquad \mathcal{H}_{F.P.} = \frac{4}{27} \cdot \frac{\delta^3}{A^2}$$



equation of a phase space trajectory:

$$\cos 3\psi = \frac{h - 3j}{2 j^{3/2}} \qquad h = \frac{\mathcal{H}}{\mathcal{H}_{F.P.}} \quad j = \frac{J}{J_{F.P.}}$$

The separatrix ($h = 1$) separates regions of bounded motion and unbounded motion

Fourth-integer resonance

Octupole magnet yields

$$\mathcal{O}(\theta)\cdot x^4 = \mathcal{O}(\theta)\cdot(2J\beta)^2 \cos^4\psi$$

$$\frac{1}{8}(\cos 4\psi + 4\cos 2\psi + 3)$$

The term cos 2ψ can be ignored for Q0 ≈ n/4
however the constant term must be kept

Close to a fourth-integer resonance $\delta = Q_0 - \frac{n}{4}$ small

$$\mathcal{H}(\psi, J) = \delta\cdot J + A\cdot J^2\,(3+\cos 4\psi)$$

The constant A contains the strength and the sign of the octupole
The Hamilton equations read (remember θ is our "time" variable)

$$\frac{d\psi}{d\theta} = \frac{\partial\mathcal{H}}{\partial J} = \delta + 2AJ\,(3+\cos 4\psi) \qquad (1)$$

$$\frac{dJ}{d\theta} = -\frac{\partial\mathcal{H}}{\partial\psi} = 4AJ^2\sin 4\psi \qquad (2)$$

The conditions for fixed points are

$$\frac{dJ}{d\theta} = 0 \quad\Rightarrow\quad \psi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \ldots$$

$\frac{d\psi}{d\theta} = 0$: can only be fulfilled if δ and A are of opposite sign

Let $Q_0 > n/4$, i.e. $\delta > 0$. Then $A < 0$ needed to obtain fixed points.

There are two kinds of fixed points:

(a) Stable (or "elliptic") fixed points

$$\psi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Around a stable fixed point is an island of stable motion

(b) Instable ("hyperbolic") fixed points

$$\psi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

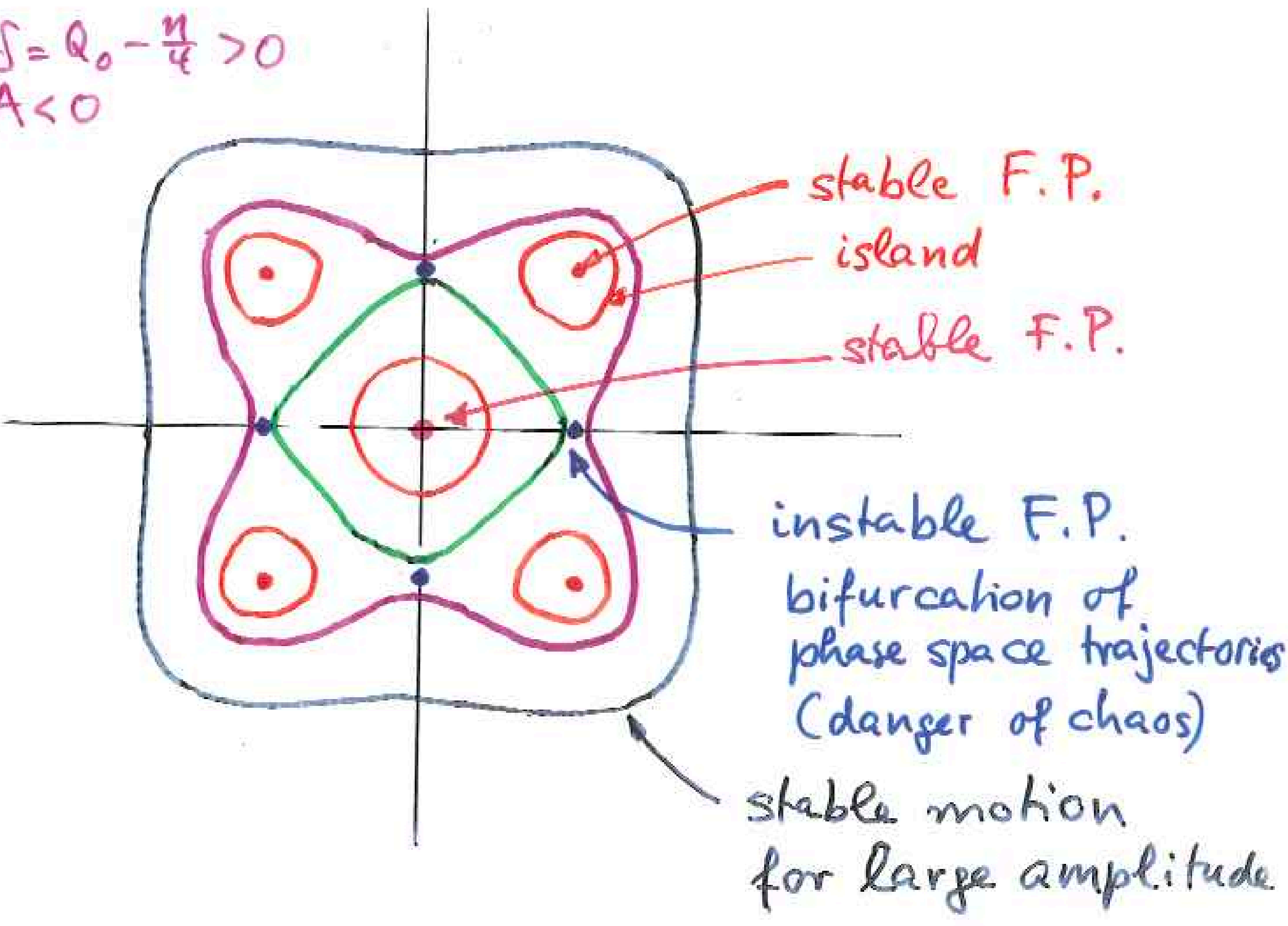
Bifurcation of trajectories occurs, chaos may happen

Phase space trajectories: iterative solution of (1), (2)

$$\begin{pmatrix} \psi \\ J \end{pmatrix}_{i+1} = \begin{pmatrix} \psi \\ J \end{pmatrix}_i + \begin{pmatrix} \delta + 2AJ_i(3+\cos 4\psi_i) \\ 4AJ_i^2 \sin 4\psi_i \end{pmatrix} \cdot \Delta\Theta$$

$\delta = Q_0 - \frac{n}{4} > 0$

$A < 0$



What happens if δ and A have the same sign?
No fixed points are present, the phase space trajectories are squares with rounded corners.

Important: one observes a strong amplitude dependence of betatron tune

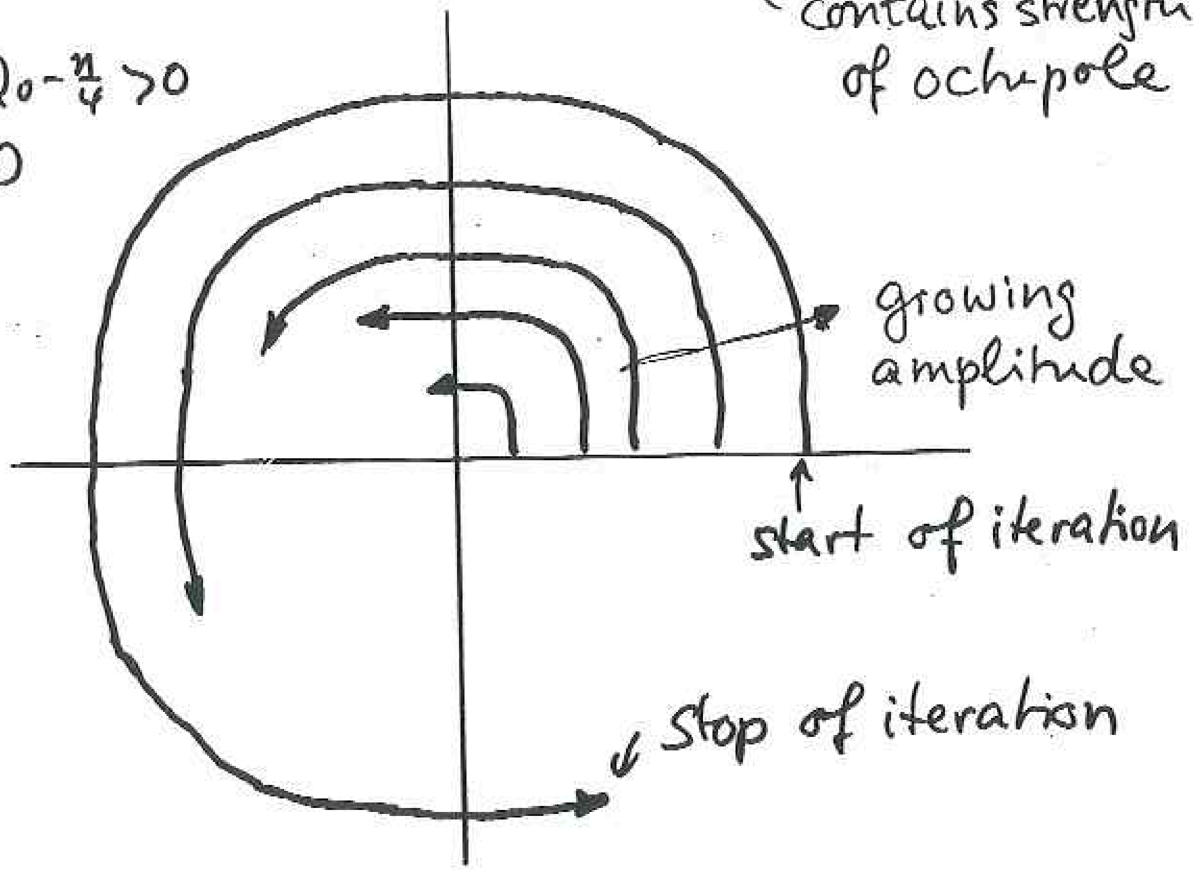
$$Q = \frac{\partial \mathcal{H}}{\partial J} = \left(Q_0 - \frac{n}{4}\right) + 2A\cdot J\,(3 + \cos 4\psi)$$

averages to zero

$$Q(J) = \left(Q_0 - \frac{n}{4}\right) + 6\,A\cdot J$$

contains strength of octupole

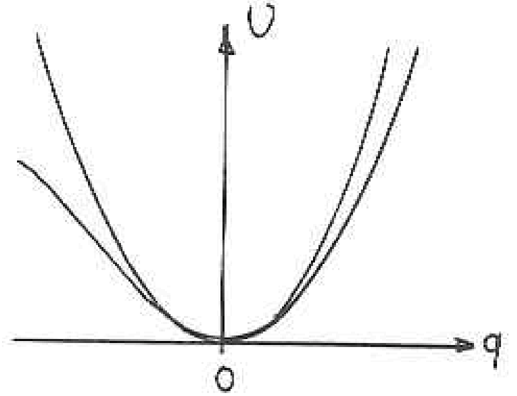
$\delta = Q_0 - \frac{n}{4} > 0$
$A > 0$



$A > 0$: Q increases with increasing J
$A < 0$: Q decreases

Betatron motion in an accelerator with only octupoles is always bounded.

<u>Anharmonic oscillator</u> : q^3-term



$$U(q) = \tfrac{1}{2} k q^2 + \alpha q^3$$

$$H(q,p) = \underbrace{\frac{p^2}{2m} + \frac{1}{2} k q^2}_{H_0} + \underbrace{\alpha q^3}_{\Delta H}$$

$$\mathcal{H}(\psi, J) = \omega_0 \cdot J + \beta J^{3/2} \underbrace{\cos^3\psi}_{\frac{1}{4}(\cos 3\psi + 3\cos\psi)}$$

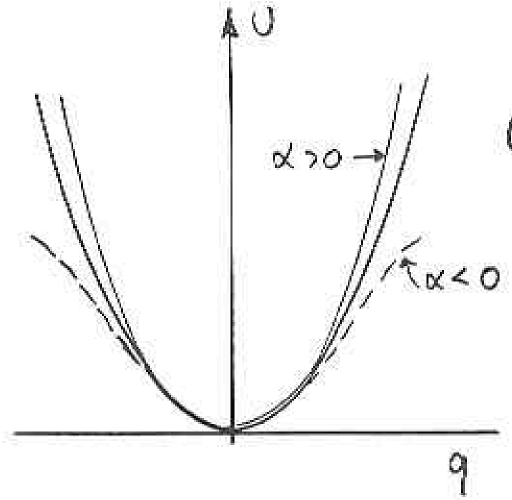
averaged over one period

$$\langle \mathcal{H} \rangle = \omega_0 J$$

$$\omega = \frac{\partial \langle \mathcal{H} \rangle}{\partial J} = \omega_0$$

Third-order term makes no frequency shift (in lowest-order perturbation theory)

<u>anharmonic oscillator</u>: q^4-term



$$U(q) = \tfrac{1}{2} k q^2 + \alpha q^4$$

$$\mathcal{H}(\psi, J) = \omega_0 \cdot J + \beta \cdot J^2 \underbrace{\cos^4\psi}_{\frac{1}{8}(\cos 4\psi + 4\cos 2\psi + 3)}$$

The $\cos^4\psi$ term does not average out:

$$\langle \mathcal{H} \rangle = \omega_0 J + \tfrac{3}{8} \beta J^2$$

$$\omega = \frac{\partial \langle \mathcal{H} \rangle}{\partial J} = \omega_0 + \tfrac{3}{4} \beta \cdot J$$

We get a frequency shift which depends on the square of the oscillation amplitude

Coupling resonances

Coupling of horizontal and vertical betatron oscillations may be caused by

skew quadrupole		$A_s(x,y,s) = f(s)\cdot x\cdot y$
sextupole	normal	$f(s)\cdot(x^3-3xy^2)$
	skew	" $(3x^2y-y^3)$
octupole	normal	" $(x^4-6x^2y^2+y^4)$
	skew	" $(4x^3y-4xy^3)$

These terms have to be added to the Hamiltonian (type H_4) which will be called H_0 here

$$H = \underbrace{\tfrac{1}{2}K(s)(y^2-x^2) + \tfrac{1}{2}(p_x^2+p_y^2)}_{H_0} + f(s)\,xy$$

Canonical transformation to angle-action variables

$$(x,p_x)\to(\varphi_1,J_1)\ ,\ (y,p_y)\to(\varphi_2,J_2)$$

$$\mathcal{H}_1 = \underbrace{Q_1\cdot J_1 + Q_2\cdot J_2}_{\mathcal{H}_0\ (\text{type } H_7)} + U(\varphi_1,\varphi_2,J_1,J_2,\theta)$$

The perturbation U is periodic in ψ_1, ψ_2 and θ and can be expanded in a triple Fouries series

$$U(\varphi_1,\varphi_2,J_1,J_2,\theta) = \sum_{m_1,m_2,n} U_{m_1m_2n}\, e^{i(m_1\varphi_1+m_2\varphi_2-n\theta)}$$

$U_{m_1m_2n}$ is function of J_1, J_2

Change of phase per turn:

$$\psi_i \to \psi_i + 2\pi Q_i \quad , \quad \theta \to \theta + 2\pi$$

$$m_1\psi_1 + m_2\psi_2 - n\theta \to m_1\psi_1 + m_2\psi_2 - n\theta$$
$$+ 2\pi(m_1 Q_1 + m_2 Q_2 - n)$$

Resonance condition:

$$\boxed{m_1 Q_1 + m_2 Q_2 - n \approx 0.}$$

Close to such a resonance

$$\mathcal{H}_1 \approx Q_1 \cdot J_1 + Q_2 \cdot J_2 + f(J_1, J_2)\cos(m_1\psi_1 + m_2\psi_2 - n\theta)$$

$(f(J_1, J_2) = 2U_{m_1 m_2 n}(J_1, J_2);\ U_{000}$ neglected$)$

$\delta = m_1 Q_1 + m_2 Q_2 - n$ distance to resonance

Canonical transformation into rotating system

$$(\psi_i, J_i) \to (\phi_i, K_i) \qquad i = 1, 2 \ (x, y)$$

$$F_2(\psi_1, K_1, \psi_2, K_2, \theta) = (m_1\psi_1 + m_2\psi_2 - n\theta) \cdot K_1 + \psi_2 \cdot K_2$$

$$\phi_1 = \frac{\partial F_2}{\partial K_1} = m_1\psi_1 + m_2\psi_2 - n\theta$$

$$\phi_2 = \frac{\partial F_2}{\partial K_2} = \psi_2$$

$$J_1 = \frac{\partial F_2}{\partial \psi_1} = m_1 K_1 \qquad J_2 = \frac{\partial F_2}{\partial \psi_2} = m_2 K_1 + K_2$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial F_2}{\partial \theta} = Q_1 J_1 + Q_2 J_2 + f(J_1, J_2)\cos\phi_1 - nK_1$$

The Hamiltonian in the rotating frame is

$$\boxed{\mathcal{H}_2 = \delta \cdot K_1 + Q_2 \cdot K_2 + \tilde{f}(K_1, K_2)\cos\phi_1}$$

The Hamiltonian is independent of the quasi time θ, hence the phase space trajectories are curves of constant Hamiltonian

$$\frac{d\mathcal{H}_2}{d\theta} = \frac{\partial \mathcal{H}_2}{\partial \theta} = 0$$

Furthermore $\mathcal{H}_2$ is independent of ϕ_2, hence K_2 = const

$$\frac{\partial \mathcal{H}_2}{\partial \phi_2} = 0 \quad \Rightarrow \quad \boxed{m_2 J_1 - m_1 J_2 = const}$$

In the following we consider a skew quadrupole. Choose $m_1 = +1$, then m_2 maybe +1 or -1

Sum resonance

For $m_2 = +1$ the difference of horizontal and vertical action

$$J_1 - J_2 = const$$

must stay constant. However, J_1 and J_2 can both grow indefinitely
A sum resonance

$$Q_1 + Q_2 = n$$

is unstable and leads to beam loss

Difference resonance

$$m_1 = +1, m_2 = -1 \quad \Rightarrow \quad J_1 + J_2 = \text{const}$$

The action variables are squares of amplitudes and hence always positive. Therefore the motion remains bounded

A difference resonance

$$Q_1 - Q_2 = n$$

is stable

However there is energy exchange between the horizontal and vertical betatron oscillations which is undesirable in electron-positron storage rings

Time development of coupled motion

$$\dot{K}_2 \equiv \frac{dK_2}{d\theta} = -\frac{\partial \mathcal{H}_2}{\partial \phi_2} = 0 \quad , \quad K_2 = \text{const}$$

$$\dot{\phi}_1 = \frac{\partial \mathcal{H}_2}{\partial K_1} = \delta + \frac{\partial \tilde{f}}{\partial K_1} \cos\phi_1$$

$$\dot{K}_1 = -\frac{\partial \mathcal{H}_2}{\partial \phi_1} = \tilde{f} \sin\phi_1$$

$$\ddot{K}_1 = \left(\frac{\partial \tilde{f}}{\partial K_1} \dot{K}_1 + \frac{\partial \tilde{f}}{\partial K_2} \underset{0}{\dot{K}_2} \right) \sin\phi_1 + \tilde{f} \cos\phi_1 \cdot \dot{\phi}_1$$

$$\ddot{K}_1 = \tilde{f} \frac{\partial \tilde{f}}{\partial K_1} \underbrace{(\sin^2\phi_1 + \cos^2\phi_1)}_{1} + \delta \underbrace{\tilde{f} \cos\phi_1}_{A - \delta \cdot K_1}$$

with $A = \mathcal{H}_2 - Q_2 K_2 = \text{const}$

$$\ddot{K}_1 = \tilde{f}\,\frac{\partial \tilde{f}}{\partial K_1} + \delta\,(A - \delta \cdot K_1)$$

Special case skew quadrupole

$$f(J_1, J_2) = B \cdot \sqrt{J_1 \cdot J_2}$$ $B \propto$ strength of quad.

$$\tilde{f}(K_1, K_2) = B\,\sqrt{K_1 \cdot (m_2 K_1 + K_2)}$$ $(m_1 = +1)$

$$\tilde{f}\,\frac{\partial \tilde{f}}{\partial K_1} = \frac{B^2}{2}\,(K_2 + 2 m_2 K_1)$$

The differential equation for K_1 becomes

$$\ddot{K}_1 + (\delta^2 - m_2 B^2) K_1 = \underbrace{\delta \cdot A + \frac{B^2}{2} K_2}_{\text{const}}$$

<u>Sum resonance</u> $(m_2 = +1)$

$$\ddot{K}_1 - \alpha^2 K_1 = \text{const}$$

$$\alpha^2 = B^2 - \delta^2$$

$\delta = Q_1 + Q_2 - n$ distance to resonance

$B \simeq$ strength of quad

Except for very weak skew quad we get

$\alpha^2 > 0$

$\Longrightarrow$ exponential growth of K_1.

Difference resonance ($m_2 = -1$)

$$\ddot{K}_1 + \omega^2 K_1 = \text{const}$$

$$\omega^2 = \delta^2 + B^2 > 0$$

Periodic oscillation, stability

Energy exchange in difference resonance

Initial condition $\theta = 0$:

$$J_1(0) = J_0 \quad , \quad J_2(0) = 0$$

$$K_1(0) = J_1(0) = J_0$$

$$K_2(0) = J_2(0) - \underset{\substack{\uparrow \\ -1}}{m_2} K_1(0) = J_0$$

$$\mathcal{H}_2(0) = (\delta + Q_2) J_0 + \underbrace{B\sqrt{J_0(J_0 - J_0)}\cos\phi_1}_{0}$$

$$A = \mathcal{H}_2 - Q_2 K_2 = \delta \cdot J_0$$

$$\ddot{K}_1 + \omega^2 K_1 = \left(\delta^2 + \frac{B^2}{2}\right) J_0$$

$$K_1(\theta) = c_0 + c_1 \cos(\omega\theta)$$

$$c_0 = \frac{\delta^2 + B^2/2}{\delta^2 + B^2} \cdot J_0 \quad , \quad c_1 = \frac{B^2/2}{\delta^2 + B^2} \cdot J_0$$

$$J_1(\theta) = \frac{J_0}{\omega^2}\left(\delta^2 + B^2 \cos^2\left(\frac{\omega}{2}\theta\right)\right)$$

$$J_2(\theta) = \frac{J_0}{\omega^2} B^2 \sin^2\left(\frac{\omega}{2}\theta\right)$$

(G. Ripken, F. Willeke, DESY 88-114)

Second-order perturbation theory

In the following we take a small perturbation but assume that the tune Q_0 is far away from a resonance condition

Start with Hamiltonian of type H_7

$$\mathcal{H}_1(\psi_1, J_1, \theta) = \underbrace{\mathcal{H}_0(J_1)}_{Q_0 \cdot J_1} + \underbrace{U(\psi_1, J_1, \theta)}_{\text{small perturbation}}$$

A canonical transformation is performed which differs only little from an identity transformation

$$(\psi_1, J_1) \longrightarrow (\psi_2, J_2)$$

$$F_2(\psi_1, J_2, \theta) = \underbrace{\psi_1 \cdot J_2}_{\text{identity transf.}} + \underbrace{\chi(\psi_1, J_2, \theta)}_{\text{"small"}}$$

$$\psi_2 = \frac{\partial F_2}{\partial J_2} = \psi_1 + \frac{\partial \chi}{\partial J_2} \qquad J_1 = \frac{\partial F_2}{\partial \psi_1} = J_2 + \frac{\partial \chi}{\partial \psi_1}$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial \chi}{\partial \theta}$$

First trick: replace in $\mathcal{H}_1$ the action J_1 by

$$J_1 = J_2 + \frac{\partial \chi}{\partial \psi_1}$$

$$\Rightarrow \mathcal{H}_2 = \mathcal{H}_0\left(J_2 + \frac{\partial \chi}{\partial \psi_1}\right) + U\left(\psi_1, J_2 + \frac{\partial \chi}{\partial \psi_1}, \theta\right) + \frac{\partial \chi}{\partial \theta}$$

second trick: add a term and subtract it again

$$\mathcal{H}_2 = Q_0 \cdot J_2 + \left\{ U\left(\psi_1, J_2 + \frac{\partial \chi}{\partial \psi_1}, \theta\right) - \underline{U(\psi_1, J_2, \theta)} \right\}$$
$$+ Q_0 \cdot \frac{\partial \chi}{\partial \psi_1} + \underline{U(\psi_1, J_2, \theta)} + \frac{\partial \chi}{\partial \theta}$$

$$\Rightarrow \{ \; \} = \left[\frac{\partial U}{\partial J_1} \cdot \frac{\partial \chi}{\partial \psi_1} \right]_{J_1 = J_2}$$ small of 2nd order

Now the function χ is determined such that

$$\boxed{Q_0 \frac{\partial \chi}{\partial \psi_1} + U(\psi_1, J_2, \theta) + \frac{\partial \chi}{\partial \theta} = 0}$$

Then the perturbation vanishes in first order and appears only in second order

Computation of perturbed Q-value

One is interested in the average tune shift, hence the perturbation term is averaged over the phase Ψ_1

$$\mathcal{H}_2 = Q_0 \cdot J_2 + \underbrace{\left[\frac{\partial U}{\partial J_1} \cdot \frac{\partial \chi}{\partial \psi_1} \right]_{J_1 = J_2}}_{V(\psi_1, J_2, \theta)}$$

$$\langle \mathcal{H}_2 \rangle = Q_0 \cdot J_2 + \langle V \rangle$$ averaged over ψ_1

If <V> is known one can compute the action-dependent tune shift

$$\frac{d\varphi_2}{d\theta} \approx \frac{\partial \mathcal{H}_2}{\partial J_2} = Q_0 + \frac{\partial \langle V \rangle}{\partial J_2}$$

$$Q(J_2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi_2}{d\theta} d\theta = Q_0 + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \langle V \rangle}{\partial J_2} d\theta$$

Example : sextupole

$$H_4 = \tfrac{1}{2} K(s)\, x^2 + \tfrac{1}{2} p^2 + S(s) \cdot x^3$$

sextupole at s_0

$$\mathcal{H}_2 = Q_0 \cdot J_2 + V(\varphi_1, J_2, \theta)$$

What is the J_2 dependence of V ?

$$x^3 = (2 J_1 \beta)^{3/2} \cos^3 \varphi \quad \Rightarrow U \sim J_2^{3/2} \qquad (J_2 \approx J_1)$$

From the determining equation of X we get

$$X \sim J_2^{3/2}$$

Therefore $\quad \dfrac{\partial U}{\partial J_2} \cdot \dfrac{\partial X}{\partial \varphi_1} \sim J_2^2$

$$\Rightarrow \langle V \rangle = f(\theta) \cdot J_2^2$$

$$Q(J_2) = Q_0 + \frac{1}{2\pi} \int_0^{2\pi} 2 f(\theta) J_2 \, d\theta$$

$$= Q_0 + A \cdot J_2$$

In second-order perturbation theory a sextupole makes a tune-shift $\sim$ action J (like octupole in first order)