

Einführung in Lagrange- und Hamilton-Formalismus.

Zunächst nichtrelativistische Theorie.

Dynamisches System: System von bewegten Massenpunkten unter dem Einfluss orts- und zeitabhängiger Kräfte.

Wenn die Kräfte $\vec{F}_i(\vec{r}_i, t)$ bekannt sind, lauten die Bewegungsgleichungen

$$m_i \ddot{\vec{r}}_i = \vec{F}_i$$

Generalisierte Koordinaten $q_k(t)$ $k=1, \dots, f$
Minimale Anzahl von beliebigen Koordinaten, die das dynam. System vollständig beschreiben.

$$f = N \cdot N_{\text{Dim}} - \lambda$$

f = Zahl der Freiheitsgrade

N = Anzahl der Teilchen

N_{Dim} : Raum-Dimensionen (1, 2, 3)

λ Anzahl der Restriktionen

Beispiele

1) harmon. Oszillator



$$N=1, N_{\text{Dim}}=1, q_1=q=x$$

$$f=1$$

2) Pendel



$$N=1, N_{\text{Dim}}=2, f=1 \quad (\lambda=1)$$

1 Restriktion: Fadenlänge ($x^2 + y^2 = l^2$)

$$q = \varphi$$

(2)

kinetische Energie

$$T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\vec{r}}_i^2 = T(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$$

potentielle Energie

$$U = - \sum_{i=1}^N \int \vec{F}_i \cdot d\vec{r}_i = U(q_1, \dots, q_f, t)$$

kinetische Energie enthält im allgemeinen q_k und $\dot{q}_k$.Beisp. Bewegung in Polarkoordinaten (r, φ)

$$T = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\varphi}^2) = T(r, \dot{r}, \dot{\varphi})$$

$$q_1 = r, \quad q_2 = \varphi$$

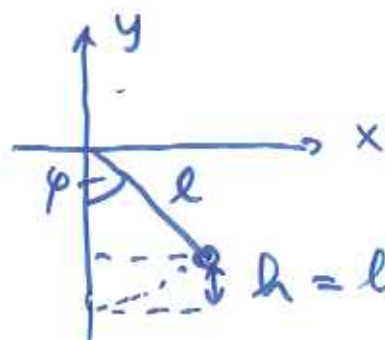
Lagrange Funktion

$$L(q_k, \dot{q}_k, t) = T(q_k, \dot{q}_k) - U(q_k, t)$$

① harmon. Oszillator:

$$T = \frac{m}{2} \dot{q}^2 \quad U = \frac{k}{2} q^2 \quad L = \frac{1}{2} (m \dot{q}^2 - k q^2)$$

② Pendel



$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) = \frac{m}{2} l^2 \dot{\varphi}^2$$

$$U = mgh = mgl(1 - \cos\varphi)$$

$$L(\varphi, \dot{\varphi}) = \frac{m}{2} l^2 \dot{\varphi}^2 - mgl(1 - \cos\varphi)$$

Lagrange - Gleichungen.

$$\boxed{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_h} \right) - \frac{\partial L}{\partial q_h} = 0} \quad h=1, \dots, f$$

① harm. Oszillator

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \frac{\partial L}{\partial q} = -kq$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \Rightarrow m\ddot{q} + kq = 0$$

② Pendel

$$\frac{\partial L}{\partial \dot{q}} = m\ell^2 \dot{\varphi} \quad \frac{\partial L}{\partial q} = -mg\ell \sin \varphi$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \Rightarrow \ddot{\varphi} + \frac{g}{\ell} \sin \varphi = 0$$

Man kann die Lagrange - Gleichungen aus dem Hamiltonschen Prinzip der kleinsten Wirkung herleiten.

Wirkungsintegral (action)

$$S = \int_{t_0}^{t_1} L(q_h, \dot{q}_h, t) dt$$

Randbedingungen sind festgelegt:

$$q_h(t_0) = q_{h0}, \quad q_h(t_1) = q_{h1}$$

Hier ist noch nicht festgelegt, auf welcher „Kurve“ sich $q_h(t)$ bewegt zwischen t_0 und t_1 .

Gegeben sei beliebige Sätze von Funktionen

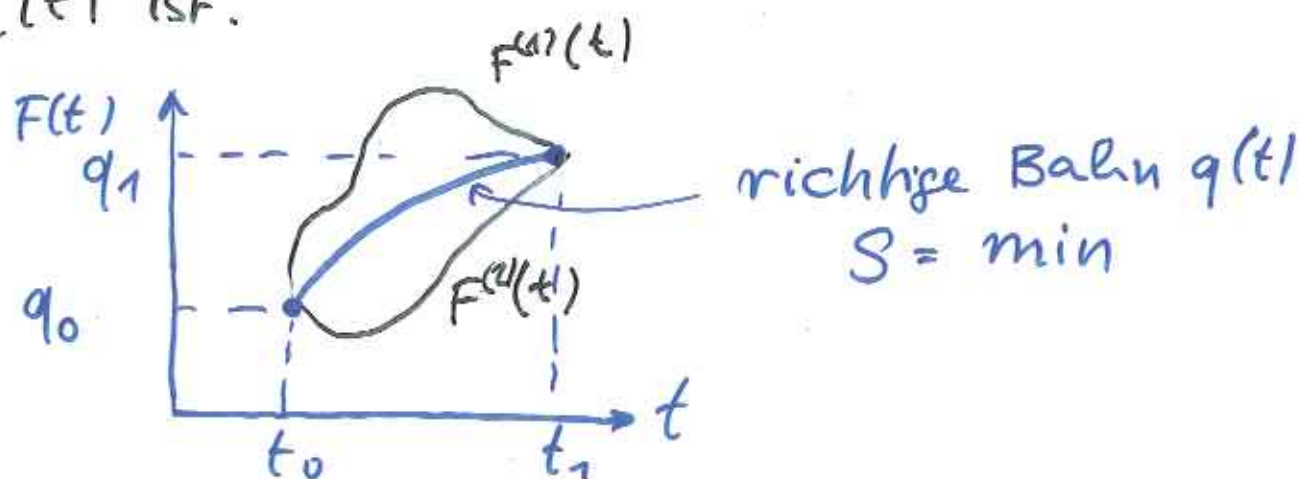
$$F_h(t) \quad \text{mit} \quad F_h(t_0) = q_{h0}$$

$$F_h(t_1) = q_{h1}$$

Die Wirkung t_1

$$S = \int_{t_0}^{t_1} L(F_h, \dot{F}_h, t) dt$$

nimmt ein Extremum (meist Minimum) an, wenn $F_h(t)$ gleich der „wirklichen“ Bahn $q_h(t)$ ist.



Beispiel (Übungsaufgabe).

harmon. Oszillator mit $m=1$, $k=1$

$$\ddot{q} + q = 0 \quad \text{Lösung: } q(t) = a \sin t$$

Wir sehen die Randbed. $q(0) = 0$, $q(1) = 1$

$$\Rightarrow a = \frac{1}{\sin 1}$$

Betrachte die Testfunktionen

$$F(\mu, t) = \frac{\sin \mu t}{\sin \mu} \quad ; \quad \text{Skizze für } \mu = 0, 1, 1, 2.5, 5.5$$

Zeige daß $S(\mu) = \min$ für $\mu=1$.

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Variation der Wirkung

$$\delta S = \int_{t_0}^{t_1} \left[\sum_k \frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt = 0$$

$$\delta \dot{q}_k = \frac{d}{dt} (\delta q_k) \quad , \text{ partiell integrieren:}$$

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt} (\delta q_k) dt = \underbrace{\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_0}^{t_1}}_0$$

$$- \int_{t_0}^{t_1} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k dt$$

$$\left[\delta q_k \right]_{t_0}^{t_1} = 0 \quad \text{wegen der festen Randbed. bei } t=t_0 \text{ und } t=t_1$$

$$\Rightarrow \int_{t_0}^{t_1} \sum \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k dt = 0$$

Die δq_k sind unabhängig voneinander beliebig wählbar

$$\Rightarrow \left[\quad \right] = 0 \quad , \text{ Lagrange-Gl.}$$

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Definition der generalisierten Impulse

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

harmon. Oszillator: $p = m\dot{q} = mv$ „normaler“ Impuls
 Pendel: $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = l(m\dot{\varphi})$
 $= \text{Drehimpuls}$

Definition der Hamiltonfunktion

$$H = \sum_{k=1}^f p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Hiernach scheint H eine Funktion der Variablen $q_k, \dot{q}_k, p_k, t$ zu sein. Man kann die $\dot{q}_k$ durch die q_k, p_k, t ausdrücken.

$$H = H(q_k, p_k, t)$$

Beispiele: 1) harmon. Oszillator

$$H = p \cdot \dot{q} - \frac{1}{2} m \dot{q}^2 + \frac{1}{2} k q^2 \quad m\dot{q} = p$$

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k q^2 = \text{totale Energie}$$

2) Pendel

$$H = p_\varphi \cdot \dot{\varphi} - \frac{m}{2} l^2 \dot{\varphi}^2 + mgl(1 - \cos \varphi)$$

$$q = \varphi, \quad \dot{q} = \dot{\varphi} = \frac{p}{ml^2} \quad (p \equiv p_\varphi)$$

$$H = \frac{p^2}{2ml^2} + mgl(1 - \cos q) = \text{totale Energie}$$

(p ist Drehimpuls!)

Hamiltonsche Gleichungen

$$H(q_k, p_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Wir bilden die totalen Differentiale der linken und rechten Seite

linke Seite

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt$$

rechte Seite

$$\sum_k \left[\dot{q}_k dp_k + p_k d\dot{q}_k - \underbrace{\frac{\partial L}{\partial q_k}}_{\dot{p}_k} dq_k - \underbrace{\frac{\partial L}{\partial \dot{q}_k}}_{p_k} d\dot{q}_k \right] - \frac{\partial L}{\partial t} dt$$

Die Terme mit $d\dot{q}_k$ heben sich weg.

Die dq_k, dp_k, dt sind unabhängige Differentiale.
Daraus folgt

$$\boxed{\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

Dividieren durch dt :

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_k \left(\frac{\partial H}{\partial q_k} \underbrace{\dot{q}_k}_{\frac{\partial H}{\partial p_k}} + \frac{\partial H}{\partial p_k} \underbrace{\dot{p}_k}_{-\frac{\partial H}{\partial q_k}} \right)$$

$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

Wenn H nicht explizit von der Zeit abhängt, hat es an jedem Ort der Bahn den gleichen Wert

(Gesamtenergie = const bei Oszillator, Pendel)

⑧

Kanonische Transformationen (J. Hagel)

Zur Vereinfachung der Schreibweise werden System mit einem Freiheitsgrad betrachtet ($f=1$).

Gegeben Hamiltonfunktion $H(q, p, t)$

$$\Rightarrow \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Dies sind gekoppelte Gl., die normalerweise nicht einfacher zu lösen sind als die Newton-Gl.

$$m \ddot{x} = F(x, t)$$

Häufig existiert eine Transformation $(q, p) \rightarrow (Q, P)$ die zu Gl. für die neuen Variablen Q, P führt, die sich leichter lösen lassen.

Beispiel:

$$\dot{q} + q + q^2 = 0 \quad \text{schwierig!}$$

Transformation $Q = \frac{1}{q}$

$$-\frac{\dot{Q}}{Q^2} + \frac{1}{Q} + \frac{1}{Q^2} = 0$$

$$\Rightarrow \dot{Q} - Q = 1 \quad \text{leicht zu lösen}$$

$$Q(t) = C e^t - 1$$

$$q(t) = \frac{1}{C e^t - 1}$$

④ Eine Transformation $(q, p) \rightarrow (Q, P)$ heißt kanonisch, wenn die gln. für Q, P aus einer neuen Hamiltonfunktion $\mathcal{H}(Q, P, t)$ hergeleitet werden können

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$Q = Q(q, p)$$

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

$$P = P(q, p)$$

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}$$

Wenn dies möglich ist, existieren auch die beiden Lagrange-Funktionen

$$L = p\dot{q} - H(q, p, t)$$

$$\mathcal{L} = P\dot{Q} - \mathcal{H}(Q, P, t)$$

Außerdem gilt

$$\delta \left(\int_{t_0}^{t_1} L dt \right) = \delta \left(\int_{t_0}^{t_1} \mathcal{L} dt \right) = 0$$

Daraus folgt

$$L = \mathcal{L} + \frac{dF}{dt}$$

wobei F eine beliebige Funktion der Koordinaten und der Zeit ist.

Warum?

$$\delta \left(\int_{t_0}^{t_1} L dt \right) = \delta \left(\int_{t_0}^{t_1} \mathcal{L} dt \right) + \underbrace{\delta \left(\int_{t_0}^{t_1} \frac{dF}{dt} dt \right)}_{\delta [F(t_1) - F(t_0)]}$$

0

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F heißt erzeugende Funktion der kanon. Transf.

Es gibt 4 Typen.

(1) $F = F_1(q, Q, t)$

$$L = \mathcal{L} + \frac{dF_1}{dt}$$

$$p \cdot \dot{q} - H = P \cdot \dot{Q} - \mathcal{H} + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q} \cdot \dot{q} + \frac{\partial F_1}{\partial Q} \cdot \dot{Q}$$

$$\Rightarrow p = \frac{\partial F_1}{\partial q}, \quad P = - \frac{\partial F_1}{\partial Q}$$

$$\mathcal{H} = H + \frac{\partial F_1}{\partial t}$$

(2) $F = F_2(q, P, t) - Q \cdot P$

$$p \cdot \dot{q} - H = P \cdot \dot{Q} - \mathcal{H} + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q} \dot{q} + \frac{\partial F_2}{\partial P} \cdot \dot{P} \\ - P \cdot \dot{Q} - Q \cdot \dot{P}$$

$$\Rightarrow p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}$$

$$\mathcal{H} = H + \frac{\partial F_2}{\partial t}$$

(3) $F = F_3(p, Q, t) + q \cdot p$

$$\Rightarrow q = - \frac{\partial F_3}{\partial p}, \quad P = - \frac{\partial F_3}{\partial Q}$$

$$\mathcal{H} = H + \frac{\partial F_3}{\partial t}$$

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$$(4) \quad F = p \cdot q - P \cdot Q + F_4(p, P, t)$$

$$q = - \frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}, \quad \mathcal{H} = H + \frac{\partial F_4}{\partial t}$$

Umbenennung der unabh. Variablen

Üblicherweise ist die Zeit t die unabh. Variable
Hamilton-Gln. folgen aus Extremalprinzip

$$\delta \left(\int_{t_0}^{t_1} L dt \right) = \delta \left(\int_{t_0}^{t_1} \sum_{i=1}^n p_i dq_i - H dt \right) = 0 \quad (*)$$

Definiere neue Koord. und Impuls durch

$$q_0 = t, \quad p_0 = -H$$

Setze $s \equiv q_n$ als neue unabh. Variable an;
 $p_s \equiv p_n$ kann als Funktion der übrigen Variablen geschrieben werden.

$$-p_s = \mathcal{H}(q_0, q_1, \dots, q_n = s, p_0, \dots, p_{n-1})$$

$$\sum_{i=1}^n p_i dq_i - H dt = \sum_{j=0}^{n-1} p_j dq_j + \underbrace{p_n dq_n}_{-\mathcal{H} ds}$$

Aus (*) folgt daher

$$\delta \left(\int_{s_0}^{s_1} \sum_{j=0}^{n-1} p_j dq_j - \mathcal{H} ds \right) = 0$$

Also gelten die Hamilton-Gleichungen

$$q_i' = \frac{dq_i}{ds} = \frac{\partial \mathcal{H}}{\partial p_i} \quad p_i' = - \frac{\partial \mathcal{H}}{\partial q_i}$$

$\mathcal{H}$ ist keine Energie! ($i = 0, \dots, n-1$)

Lagrange and Hamilton functions for charged particles in electromagnetic fields

The electromagnetic field is expressed in terms of a scalar potential Φ and a vector potential $\mathbf{A}$

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

We use Cartesian coordinates (x_1, x_2, x_3) and consider first nonrelativistic protons

$$q_i = x_i, \quad \dot{q}_i = v_i$$

The kinetic energy is

$$T = \frac{1}{2} m \vec{v}^2 = \frac{1}{2} m \sum_i \dot{q}_i^2$$

To obtain the Lorentz force

$$\vec{F} = e (\vec{E} + \vec{v} \times \vec{B})$$

we need a velocity-dependent potential

$$U(q_i, \dot{q}_i) = e\Phi - e\vec{v} \cdot \vec{A}$$

(Here Φ and $\mathbf{A}$ are functions of the coordinates $q_i = x_i$)

The Lorentz force is given by

$$F_i = - \frac{\partial U}{\partial q_i} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_i} \right)$$

(tutorial)

The Lagrangian is

$$L(q_i, \dot{q}_i) = T(\dot{q}_i) - U(q_i, \dot{q}_i)$$

and the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \underbrace{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right)}_{m \dot{v}_i} - \underbrace{\frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_i} \right) + \frac{\partial U}{\partial q_i}}_{-F_i}$$

lead to Newton's equation of motion

$$m \dot{\vec{v}} = e (\vec{E} + \vec{v} \times \vec{B})$$

The canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial U}{\partial \dot{q}_i} = m \dot{q}_i + e A_i$$

$$\boxed{\vec{p} = m \vec{v} + e \vec{A}}$$

canonical momentum
of proton in e.m. field

$m \vec{v}$ mechanical momentum

Hamiltonian:

$$\begin{aligned} H &= \sum_i p_i \dot{q}_i - L \\ &= (m \vec{v} + e \vec{A}) \cdot \vec{v} - \frac{1}{2} m \vec{v}^2 - e \vec{A} \cdot \vec{v} + e \Phi \end{aligned}$$

$$H = \underbrace{\frac{1}{2} m \vec{v}^2}_{E_{\text{kin}}} + \underbrace{e \Phi}_{E_{\text{pot}}}$$

sum of kinetic and
potential energy

Result:

- H is the energy of the particle, i.e. the sum of kinetic and potential energy
- the vector potential has no influence on the energy

In the Hamiltonian formalism one has to write H as a function of the coordinates q_i and the canonical momenta p_i

$$H(q_i, p_i) = \frac{1}{2m} (\vec{p} - e\vec{A}(\vec{q}))^2 + e\Phi(\vec{q})$$

recipe : replace $m\vec{v}$ by $\vec{p} - e\vec{A}$

Relativistic case:

(a) without field (free protons)

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

m_0 rest mass

$$H \hat{=} E = \sqrt{\vec{p}^2 c^2 + m_0^2 c^4}$$

$\vec{p} = \underbrace{\gamma m_0}_{\text{"moving mass"}} \vec{v}$

(b) with field

$$H = \sqrt{(\vec{p} - e\vec{A})^2 c^2 + m_0^2 c^4} + e\Phi$$

Note: H is now the total energy of the particle, including

- the rest energy $m_0 c^2$
- the kinetic energy $\gamma m_0 c^2 - m_0 c^2$
- the potential energy $e\Phi$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Hamiltonian treatment of transverse motion

(1) Hamiltonian in Cartesian coordinates

$$\vec{q} = (q_1, q_2, q_3) \equiv \vec{r} = (x_1, x_2, x_3)$$

The canonical momentum of a particle of charge e in an electromagnetic field is

$$\vec{p} = \gamma m_0 \vec{v} + e \vec{A} \quad \gamma = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

The relativistic Hamiltonian is

$$H_1(\vec{q}, \vec{p}, t) = c [(\vec{p} - e \vec{A})^2 + m_0^2 c^2]^{1/2} + e \Phi$$

(In general, the scalar potential Φ and the vector potential $\mathbf{A}$ are functions of the coordinates q_i and the time t)

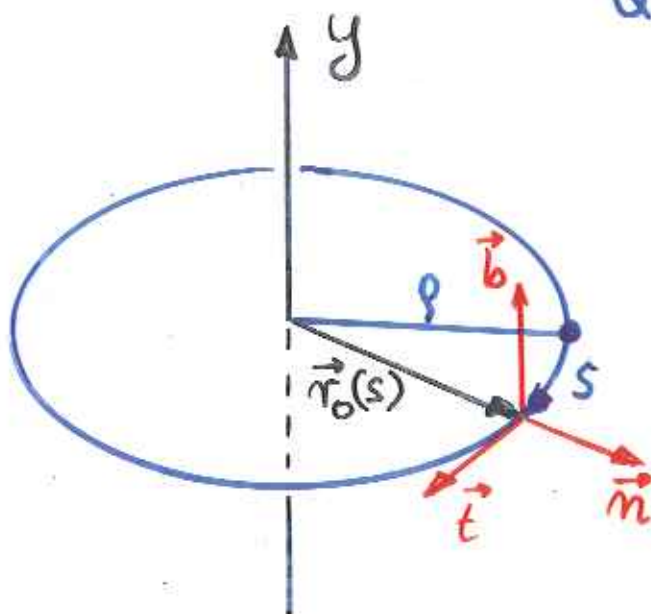
(2) Transformation to accelerator coordinates

$$\vec{Q} \equiv (Q_1, Q_2, Q_3) = (x, y, s)$$

position vector of particle

$$\vec{r}(s) = \vec{r}_0(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s)$$

$(\vec{n}, \vec{b}, \vec{t})$ comoving unit vectors



Assumption: accelerator is circle of radius ρ in the plane $y = 0$ (no torsion)

$$\frac{d\vec{r}_0}{ds} = \vec{t} \quad \frac{d\vec{t}}{ds} = -\frac{1}{\rho} \vec{n} \quad \frac{d\vec{n}}{ds} = \frac{1}{\rho} \vec{t} \quad \frac{d\vec{b}}{ds} = 0$$

Canonical transformation $(\vec{q}, \vec{p}) \rightarrow (\vec{Q}, \vec{P})$

using generating function

$$F_3(\vec{p}, \vec{Q}) = -\vec{r} \cdot \vec{p} = -(\vec{r}_0 + x \vec{n} + y \vec{b}) \cdot \vec{p}$$

The new momenta are

$$P_i = - \frac{\partial F_3}{\partial Q_i}$$

$$\left. \begin{aligned} P_x &= - \frac{\partial F_3}{\partial x} = \vec{p} \cdot \vec{n} = p_x \\ P_y &= - \frac{\partial F_3}{\partial y} = \vec{p} \cdot \vec{b} = p_y \end{aligned} \right\} \begin{array}{l} \text{projection of } \vec{p} \\ \text{on unit vectors} \\ \text{along } x \text{ resp. } y \end{array}$$

$$\text{but: } P_s = - \frac{\partial F_3}{\partial s} = \left(1 + \frac{x}{\rho}\right) \vec{p} \cdot \vec{t}$$

Note that the momentum conjugate to the arc length s is not identical with the tangential component of the vector $\vec{p}$

Since the generating function does not depend on time, H remains invariant

$$H_2 = H_1 + \frac{\partial F_3}{\partial t} = H_1$$

Write H_2 in terms of Q_i, P_i

$$H_2(\vec{Q}, \vec{P}, t) = c \left[(P_x - eA_x)^2 + (P_y - eA_y)^2 + \left(\frac{P_s - eA_s}{1 + \frac{x}{\rho}} \right)^2 + m_0^2 c^2 \right]^{1/2} + e\Phi$$

Note that H_2 is still the total energy of the particle

(3) Replace time t by orbit length s

This is basically a renaming of variables.

$(t, -H_2)$ can be considered as canonical variables like (s, P_s) . The above equation is solved for $-P_s$ which is the new Hamiltonian:

$$\underbrace{-P_s}_{H_3} = -eA_s - \left(1 + \frac{x}{\rho}\right) \left[\frac{1}{c^2} (H_2 - e\Phi)^2 - m_0^2 c^2 - (P_x - eA_x)^2 - (P_y - eA_y)^2 \right]^{1/2}$$

$$(H_2 - e\Phi) = \gamma m_0 c^2 \quad \text{kinetic + rest energy}$$

$$\frac{1}{c^2} (H_2 - e\Phi)^2 - m_0^2 c^2 = (\gamma m_0 \vec{v})^2 = \vec{p}^2$$

here $\vec{p}$ mechanical momentum

For simplicity we restrict ourselves to transverse magnetic fields only

$$\vec{B}_s = 0 \Rightarrow \text{choose } \vec{A} \text{ such that } A_x, A_y = 0$$

Then the Hamiltonian reads

$$H_3 = -eA_s - \left(1 + \frac{x}{\rho}\right) \left[\vec{p}^2 - p_x^2 - p_y^2 \right]^{1/2}$$

Note that H_3 is no longer an energy but has the dimension of a momentum

Derivation of Hill's equation

We evaluate H_3 for a ring made of combined-function magnets with ideal dipole and quadrupole field components

Use cylindrical coordinate system (r, θ, y)

$$r = \rho + x, \quad \theta = s/\rho \quad (\text{ring plane})$$

y = vertical coordinate

$\mathbf{A}$ has only a θ -component. To compute A_θ we evaluate $\vec{\nabla} \times \vec{A}$ in cylindrical coordinates

$$B_r \equiv B_x = -g y = -\frac{\partial A_\theta}{\partial y} \quad (1)$$

$$B_y = -B_1 - g x = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \quad (2)$$

From (1): $A_\theta(x, y) = +\frac{1}{2} g y^2 + f(x)$

Then (2) can be written

$$\frac{1}{1 + \frac{x}{\rho}} \cdot \frac{\partial}{\partial x} \left(\underbrace{\left(1 + \frac{x}{\rho}\right) f(x)}_{f_1(x)} \right) = -B_1 - g \cdot x$$

Use $B_1 = \frac{p_0}{e} \cdot \frac{1}{\rho}, \quad k = \left(\frac{p_0}{e}\right)^{-1} \cdot g$

$$\Rightarrow f_1(x) = -\frac{p_0}{e} \left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} + k\right) \frac{x^2}{2} + \frac{g}{\rho} \frac{x^3}{3} \right]$$

neglect 3rd order term
(x small)

$$A_s(x, y) = \left(1 + \frac{x}{\rho}\right) A_\theta(x, y)$$

$$= f_1(x) + \frac{1}{2} g y^2 + \underbrace{\frac{1}{2} \frac{g}{\rho} y^2 x}_{\text{neglect 3rd order}}$$

$$-e A_s = p_0 \left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} + k \right) \frac{x^2}{2} - k \frac{y^2}{2} \right]$$

Now simplify second term in H_3

$$[\vec{p}^2 - p_x^2 - p_y^2]^{1/2} \approx \underbrace{p}_{p_0 + \Delta p} - \frac{p_x^2}{2p_0} - \frac{p_y^2}{2p_0}$$

assumption: $p_x, p_y \ll p_0$, $\Delta p \ll p_0$

Then H_3 becomes

$$H_3 \approx p_0 \left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} + k \right) \frac{x^2}{2} - k \frac{y^2}{2} \right]$$

$$- p - (p_0 + \Delta p) \frac{x}{\rho} + \left(1 + \frac{x}{\rho}\right) \left(\frac{p_x^2}{2p_0} + \frac{p_y^2}{2p_0} \right)$$

Keeping only terms up to 2nd order in the small quantities x, y, p_x, p_y and neglecting the constant term $-p$ we obtain

$$H_3 = -\Delta p \frac{x}{\rho} + p_0 \left[\left(\frac{1}{\rho^2} + k \right) \frac{x^2}{2} - k \frac{y^2}{2} \right] + \frac{p_x^2}{2p_0} + \frac{p_y^2}{2p_0}$$

Hamiltonian for transverse motion in the "linear optics" approximation

Note: H_3 is of 2nd order in x, y ; $\vec{B}$ is linear

Hamilton equations

$$x' = \frac{dx}{ds} = \frac{\partial H_3}{\partial p_x} = \frac{p_x}{p_0}$$

$$p_x' = -\frac{\partial H_3}{\partial x} = \frac{\Delta p}{\rho} - p_0 \left(\frac{1}{\rho^2} + k \right) x$$

$$y' = \frac{\partial H_3}{\partial p_y} = \frac{p_y}{p_0}, \quad p_y' = -\frac{\partial H_3}{\partial y} = +p_0 k y$$

$ \begin{aligned} x'' + \left(\frac{1}{\rho^2} + k \right) x &= \frac{1}{\rho} \frac{\Delta p}{p_0} \\ y'' - k y &= 0 \end{aligned} $

(4) Normalized momenta

The Hamiltonian H_3 (which is not an energy but a momentum) and the transverse momenta are made dimensionless by dividing them by p_0 (nominal momentum)

$$H_4 = \frac{H_3}{p_0} \quad \bar{p}_x = \frac{p_x}{p_0} \quad \bar{p}_y = \frac{p_y}{p_0}$$

In the following we restrict ourselves to one plane of betatron oscillation, say x , and assume $\Delta p = 0$

$$H_4(x, \bar{p}_x, s) = \frac{1}{2} K(s) x^2 + \frac{1}{2} \bar{p}_x^2$$

$$x' = \frac{dx}{ds} = \frac{\partial H_4}{\partial \bar{p}_x} = \bar{p}_x \quad \bar{p}_x' = - \frac{\partial H_4}{\partial x} = -Kx$$

The canonical momentum is identical with the slope of the trajectory

$$x'' + K(s)x = 0$$

$$x(s) = a \sqrt{\beta(s)} \cos(\phi(s) - \delta)$$

$$\bar{p}_x \equiv x'(s) = -a \beta^{-1/2} [\sin(\phi - \delta) - \frac{\beta'}{2} \cos(\phi - \delta)]$$

$$\phi'(s) = \frac{1}{\beta(s)}$$

Angle-action variables of harmonic oscillator

$$m\ddot{q} + kq = 0 \Rightarrow \ddot{q} + \omega^2 q = 0, \quad \omega = \sqrt{\frac{k}{m}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m} \left(\underbrace{p^2}_{\sim \sin^2 \psi} + \underbrace{m^2 \omega^2 q^2}_{\sim \cos^2 \psi} \right)$$

Find angle variable ψ with the property
 $p^2 = C \cdot \sin^2 \psi$, $m^2 \omega^2 q^2 = C \cdot \cos^2 \psi$ ($C = \text{const}$)
 $\Rightarrow$ new Hamiltonian independent of ψ

Canonical transformation

$$(q, p) \rightarrow (\psi, J)$$

$$H \rightarrow \mathcal{H}$$

with $F_1(q, \psi) = -\frac{1}{2} m \omega q^2 \cdot \tan \psi$

$$p = \frac{\partial F_1}{\partial q} = -m \omega q \cdot \tan \psi$$

$$J = -\frac{\partial F_1}{\partial \psi} = \frac{1}{2} m \omega q^2 \frac{1}{\cos^2 \psi}$$

It follows: $q = \sqrt{\frac{2J}{m\omega}} \cos \psi$

$$p = -\sqrt{2Jm\omega} \sin \psi$$

$$\mathcal{H} = H + \frac{\partial F_1}{\partial t} = J \cdot \omega (\sin^2 \psi + \cos^2 \psi) = J \cdot \omega$$

$$\boxed{\mathcal{H} = J \cdot \omega}$$

$$\dot{J} = -\frac{\partial \mathcal{H}}{\partial \psi} = 0 \Rightarrow \text{action } J = \text{const}, \quad J = \frac{E}{\omega}$$

$$\dot{\psi} = \frac{\partial \mathcal{H}}{\partial J} = \omega \quad \psi = \omega t + \alpha$$

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t + \alpha)$$

(5) Transformation to angle-action variables

$$(x, x') \rightarrow (\psi, J) \text{ with } \psi = \phi(s) - \delta$$

Generating function

$$F_1(x, \psi, s) = -\frac{x^2}{2\beta(s)} \left[\tan \psi - \frac{1}{2} \beta'(s) \right]$$

$$x' = \frac{\partial F_1}{\partial x} = -\frac{x}{\beta} \left[\tan \psi - \frac{\beta'}{2} \right]$$

insert $x = a \sqrt{\beta} \cos \psi$

$$\Rightarrow x' = -\frac{a}{\sqrt{\beta}} \left[\sin \psi - \frac{\beta'}{2} \cos \psi \right]$$

The new momentum, called 'action', is

$$J = -\frac{\partial F_1}{\partial \psi} = \frac{x^2}{2\beta} \cdot \frac{1}{\cos^2 \psi} = \frac{a^2 \beta \cos^2 \psi}{2\beta \cos^2 \psi} = \frac{a^2}{2}$$

From Basic Optics course, formula (4.32)

$$\frac{1}{\beta} (x^2 + (\beta x' - \frac{1}{2} \beta' x)^2) = a^2 = \text{const}$$

Hence $2J$ is identical with the Courant-Snyder invariant and therefore a constant of motion

New Hamiltonian

$$\begin{aligned} H_5 &= H_4 + \frac{\partial F_1}{\partial s} \\ &= \frac{a^2}{2} \left[K\beta \cos^2 \psi + \frac{1}{\beta} \left(\sin \psi - \frac{\beta'}{2} \cos \psi \right)^2 \right] \\ &\quad - \frac{\partial}{\partial s} \left\{ \frac{x^2}{2\beta(s)} \left(\tan \psi - \frac{1}{2} \beta'(s) \right) \right\} \end{aligned}$$

Second term:

$$+ \left\{ \frac{x^2 \beta'}{2 \beta^2} (\tan \psi - \frac{\beta'}{2}) + \frac{x^2}{4 \beta} \beta'' \right\}$$

Insertion of $x = a \beta^{1/2} \cos \psi$ yields

$$+ \frac{a^2}{2} \left\{ \sin \psi \cos \psi \cdot \frac{\beta'}{\beta} - \cos^2 \psi \cdot \frac{\beta'^2}{2 \beta} + \cos^2 \psi \frac{\beta''}{2} \right\}$$

Then H_5 becomes

$$H_5 = \frac{a^2}{2} \left[\cos^2 \psi \left\{ K \beta - \frac{\beta'^2}{4 \beta} + \frac{1}{2} \beta'' \right\} + \frac{1}{\beta} \sin^2 \psi \right]$$

The β function fulfills the equation (4.21)

$$\frac{1}{2} \beta \beta'' - \frac{1}{4} \beta'^2 + K \beta^2 = 1 \Rightarrow \{ \} = \frac{1}{\beta}$$

$$H_5(\psi, J, s) = \frac{J}{\beta(s)}$$

The new Hamiltonian does not depend on the angle variable ψ . The conjugate momentum, the action J , is thus a constant of motion

$$J' = - \frac{\partial H_5}{\partial \psi} = 0 \quad J = \text{const}$$

We know this already, $2J = \text{Courant-Snyder invariant}$

$$\psi' = \frac{d\psi}{ds} = \frac{\partial H_5}{\partial J} = \frac{1}{\beta(s)}$$

$$\psi(s) = \psi(0) + \int_0^s \frac{ds'}{\beta(s')}$$

The Hamiltonian H_5 is still s -dependent

(6) Transformation to angle variable that is linear in s

$$(\psi, J) \rightarrow (\psi_1, J_1)$$

Generating function

$$F_2(\psi, J_1, s) = \psi \cdot J_1 + \left(\frac{Q}{R} \cdot s - \int_0^s \frac{ds'}{\beta(s')} \right) \cdot J_1$$

$2\pi R$ circumference of machine

Q = number of betatron oscillations per turn

$$\psi_1 = \frac{\partial F_2}{\partial J_1} = \psi(s) + \frac{Q}{R} s - \int_0^s \frac{ds'}{\beta(s')}$$

$$\psi_1(s) = \psi(0) + \frac{Q}{R} \cdot s$$

phase ψ_1 is linear in s

$$J = \frac{\partial F_2}{\partial \psi} = J_1$$

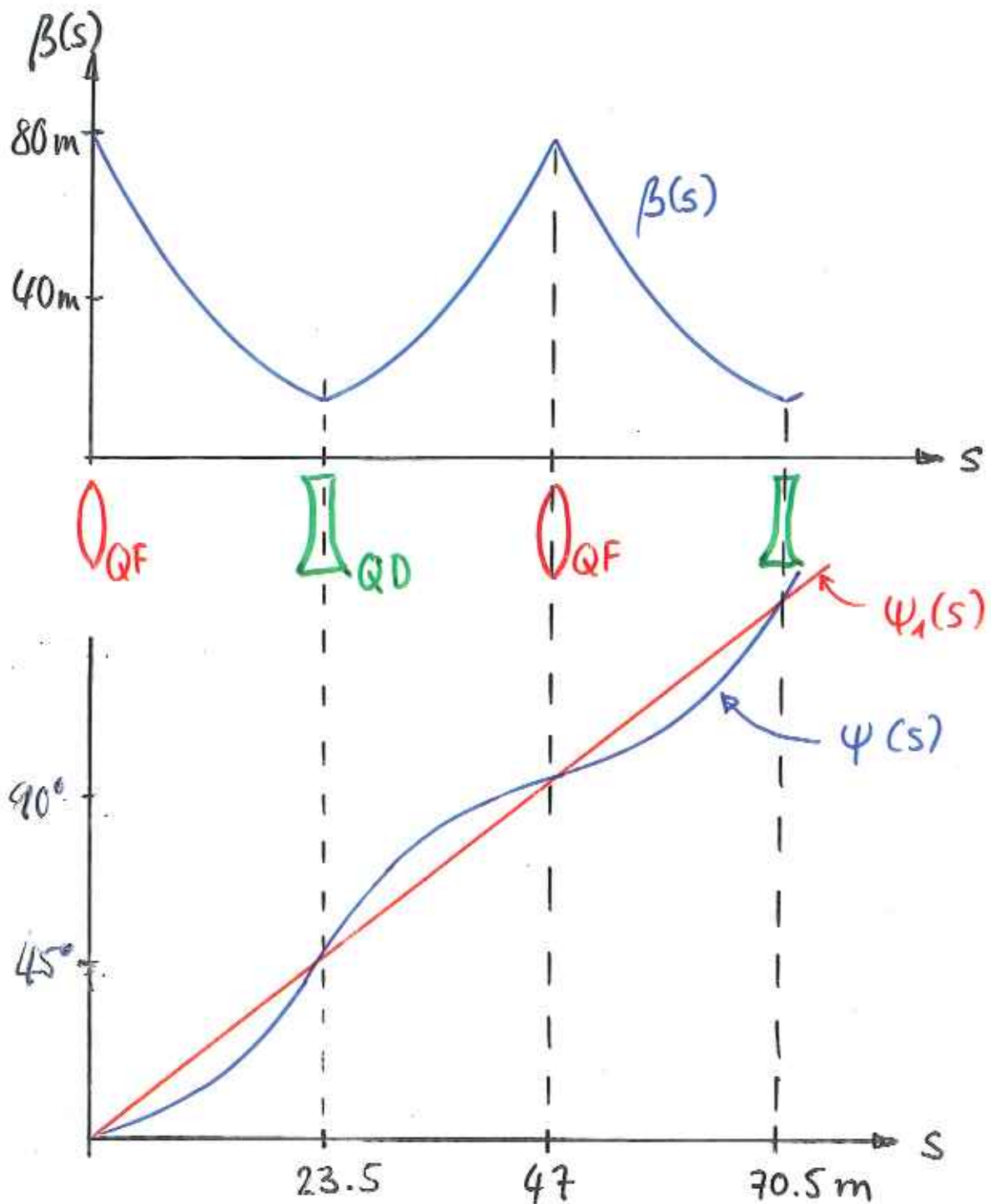
$$H_6 = H_5 + \frac{\partial F_2}{\partial s} = \frac{J}{\beta(s)} + \frac{Q}{R} J_1 - \frac{J_1}{\beta(s)}$$

$$\boxed{H_6 = \frac{Q}{R} J = \omega J}$$

same as for harmonic oscillator

$$J = \text{const} \Rightarrow H_6 = \text{const}$$

HERA proton ring (horizontal plane x)



$$\psi_1(s) = \frac{Q}{R} \cdot s$$

$$\psi(s) = \int_0^s \frac{ds'}{\beta(s')} \quad (\psi(0)=0)$$

$$= \phi(s) - \phi$$

P. Schmüser

Application: gradient error

We use normalized momentum and start from Hamiltonian of type H_4 .

Unperturbed Hamiltonian is called H_0 from here on.

$$H_0(x, p, s) = K(s) \frac{x^2}{2} + \frac{p^2}{2} \quad p = \frac{p_x}{p_0}$$

perturbation $\Delta H = \Delta K(s) \frac{x^2}{2}$

Transformation to angle-action variables (type H_5)

$$\mathcal{H} = \mathcal{H}_0 + \Delta \mathcal{H} = \frac{J}{\beta(s)} + \frac{1}{2} \Delta K(s) x^2$$

For x we insert the unperturbed solution

$$x = a \beta^{1/2} \cos \psi$$

$$x^2 = 2J \beta \cos^2 \psi$$

$$\Rightarrow \mathcal{H}(\psi, J, s) = \frac{J}{\beta(s)} + J \cdot \beta(s) \Delta K(s) \cos^2 \psi$$

$$\cos^2 \psi = \frac{1}{2}(1 + \cos 2\psi)$$

Question: what is the Q-value?

$$Q = \frac{1}{2\pi} \oint \frac{d\psi}{ds} ds$$

$$\frac{d\psi}{ds} = \frac{\partial \mathcal{H}}{\partial J} = \frac{1}{\beta(s)} + \frac{1}{2} \beta(s) \Delta K(s) (1 + \cos 2\psi)$$

Averaged over many turns: $\cos 2\psi$ cancels

$$Q = \underbrace{\frac{1}{2\pi} \oint \frac{ds}{\beta(s)}}_{Q_0} + \underbrace{\frac{1}{4\pi} \oint \beta(s) \Delta k(s) ds}_{\text{tune shift } \Delta Q}$$

(Basic Optics, Eq. (5.14))

Higher-order multipoles

(Here uniformly distributed around ring)

Sextupole

$$H = H_0 + S \cdot x^3$$

$$\mathcal{H} = \frac{J}{\beta(s)} + S (2J)^{3/2} \cos^3 \psi$$

$$\cos^3 \psi = \frac{1}{4} (\cos 3\psi + 3\cos \psi)$$

both terms average to zero upon many revolutions

Conclusion: a sextupole does not lead to a tune shift (in first-order perturbation theory)

Octupole

$$\Delta \mathcal{H} \propto x^4 = J^2 \cos^4 \psi$$

$$\cos^4 \psi = \frac{1}{8} (\cos 4\psi + 4\cos 2\psi + 3)$$

does not average out

⇒ octupole makes Q shift

$\Delta Q \propto J \propto \text{square of amplitude}$

Canonical Perturbation Theory

In the linear-optics approximation the magnetic fields are linear in x and y while the Hamiltonian (type H_4) is of second order. Transformation to angle-action variables

$$(x, x') \rightarrow (\psi_1, J)$$

leads to a Hamiltonian (type H_6) which is independent of the angular variable ψ_1 and also of the orbit length s (our 'quasi-time'). Instead of s one can also use the azimuthal angle θ .

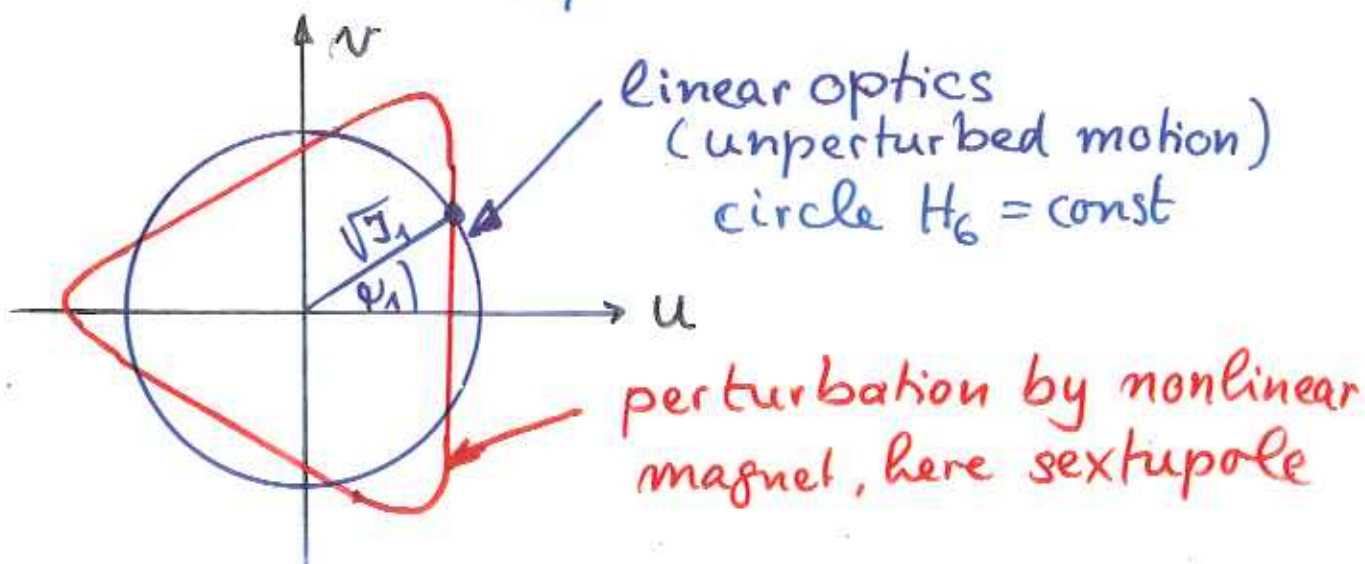
$$H_6 = \frac{Q}{R} J = \text{const}$$

$$s \rightarrow \theta = \frac{s}{R} : H_7 = R \cdot H_6 = Q \cdot J_1$$

Normalized phase space diagram

plot $u = \frac{1}{\sqrt{2}\beta} x = \sqrt{J_1} \cos \psi_1$

$$v = \frac{-1}{\sqrt{2}\beta} (\beta x' - \frac{\beta'}{2} x) = + \sqrt{J_1} \sin \psi_1$$



Perturbation by nonlinear fields:

terms of order >2 in x and y in the Hamiltonian. The sextupoles, octupoles etc are usually localized in the ring, hence the perturbation depends on s resp. θ .

Start with Hamiltonian of type H_7 which is called $\mathcal{H}_0$ in the following

$$\mathcal{H}_0 = Q_0 \cdot J_1 \quad (\text{independent of } \psi_1 \text{ and } \theta)$$

Perturbed Hamiltonian

$$\mathcal{H}_1(\psi_1, J_1, \theta) = \underbrace{\mathcal{H}_0(J_1)}_{Q_0 \cdot J_1} + \underbrace{U(\psi_1, J_1, \theta)}_{\text{small perturbation}}$$

Q_0 unperturbed tune

The perturbation U is periodic in the betatron phase ψ_1 and the azimuthal angle θ . It can be expanded in a double Fourier series.

$$U(\psi_1, J_1, \theta) = \sum_{m,n} U_{mn}(J_1) e^{i(m\psi_1 - n\theta)}$$

Change of angles per revolution:

$$\theta \rightarrow \theta + 2\pi$$

$$\psi_1 \rightarrow \psi_1 + Q_0 \cdot 2\pi$$

$\uparrow Q \approx Q_0$ since perturbation is "small"

Behaviour near a resonance

Let $Q_0 \approx n/m$ and $U_{mn} \neq 0$. Then the phase factor

$$\exp(i(m\psi_1 - n\theta))$$

is almost constant from turn to turn since

$$m\psi_1 - n\theta \rightarrow m(\psi_1 + Q_0 2\pi) - n(\theta + 2\pi) \\ \approx m\psi_1 - n\theta$$

Hence the perturbation adds up coherently from turn to turn.

Other terms $U_{m'n'}$: phases vary from turn to turn $\Rightarrow$ perturbation averages out

Close to a resonance we can restrict ourselves to the terms U_{00} and U_{mn} .

$$\mathcal{H}_1 \approx Q_0 \cdot J_1 + U_{00}(J_1) + 2U_{mn}(J_1) \cos(m\psi_1 - n\theta) \\ \text{for } Q_0 \approx \frac{n}{m}$$

If a single resonance term is present the Hamiltonian can be made independent of θ by transformation into a rotating coordinate system

Canonical transformation into rotating coordinate system: use generating function of type $F_2(q, P, t)$

$$F_2(\psi_1, J_2, \theta) = \left(\psi_1 - \frac{n}{m} \theta\right) \cdot J_2$$

$$J_1 = \frac{\partial F_2}{\partial \psi_1} = J_2 \quad \psi_2 = \frac{\partial F_2}{\partial J_2} = \psi_1 - \frac{n}{m} \theta$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial F_2}{\partial \theta}$$

$$= Q_0 \cdot J_1 + U_{00}(J_1) + 2 U_{mn}(J_1) \cos(m\psi_2) - \frac{n}{m} J_2$$

$$\mathcal{H}_2(\psi_2, J_2) = \delta \cdot J_2 + U_{00}(J_2) + 2 U_{mn}(J_2) \cos m\psi_2$$

$$\delta = Q_0 - \frac{n}{m} \quad \text{distance to resonance}$$

The new Hamiltonian is indeed independent of the 'quasi time' θ .

Simplified notation: omit index "2"

$$\mathcal{H}(\psi, J) = \delta \cdot J + \alpha(J) + f(J) \cos m\psi$$

Third-integer resonance

Sextupole magnet yields term

$$S(\theta) \cdot x^3 = S(\theta) (2J\beta)^{3/2} \cos^3 \psi$$

↑ sextupole strength as a function of θ
↑ $\frac{1}{4} (\cos 3\psi + 3 \cos \psi)$

The Fourier series of U contains the terms $m = 1$ and $m = 3$. Since Q_0 must be far away from an integer to avoid integer resonances we look only at the term $m = 3$.

Third-integer resonance: $Q_0 - \frac{n}{3} = \delta$ small

$$\mathcal{H}(\psi, J) = \delta \cdot J + A \cdot J^{3/2} \cos 3\psi$$

Phase space representation: for constant Hamiltonian plot $\sqrt{J}$ against angle ψ in a polar diagram

Fixed points: points in phase space which do not move as a function of θ ($\hat{=}$ time)

$$\frac{d\psi}{d\theta} = \frac{\partial \mathcal{H}}{\partial J} = 0 \quad \Rightarrow \quad \delta + \frac{3}{2} A J^{1/2} \cos 3\psi = 0$$

$$\frac{dJ}{d\theta} = -\frac{\partial \mathcal{H}}{\partial \psi} = 0 \quad \Rightarrow \quad \sin 3\psi = 0$$

$$\Rightarrow \psi = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

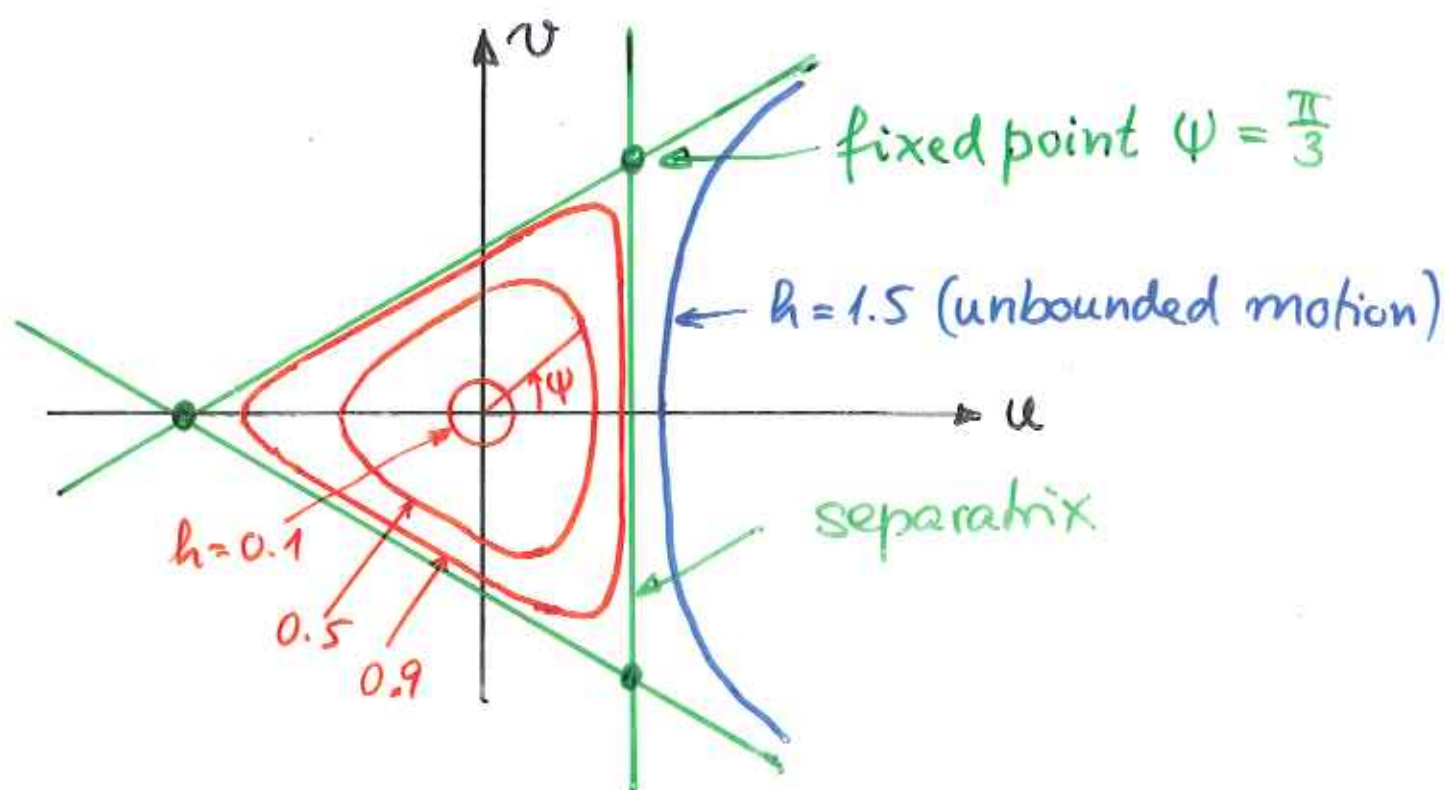
Take $\delta, A > 0$

$$\underbrace{\delta}_{>0} + \underbrace{\frac{3}{2} A J^{\frac{1}{2}}}_{>0} \cos 3\psi = 0$$

$\Rightarrow$ fixed points at $\psi = \frac{\pi}{3}, \pi, \frac{5}{3}\pi$

$$J_{\text{F.P.}} = \left(\frac{2\delta}{3A}\right)^2$$

$$\mathcal{H}_{\text{F.P.}} = \frac{4}{27} \cdot \frac{\delta^3}{A^2}$$



equation of a phase space trajectory:

$$\cos 3\psi = \frac{h - 3j}{2j^{3/2}} \quad h = \frac{\mathcal{H}}{\mathcal{H}_{\text{F.P.}}} \quad j = \frac{J}{J_{\text{F.P.}}}$$

The separatrix ($h=1$) separates regions of bounded motion and unbounded motion

Fourth-integer resonance

Octupole magnet yields

$$O(\theta) \cdot x^4 = O(\theta) \cdot (2J\beta)^2 \cos^4 \psi$$

$$\frac{1}{8} (\cos 4\psi + 4 \cos 2\psi + 3)$$

The term $\cos 2\psi$ can be ignored for $Q_0 \approx n/4$
however the constant term must be kept

Close to a fourth-integer resonance $\delta = Q_0 - \frac{n}{4}$ small

$$\mathcal{H}(\psi, J) = \delta \cdot J + A \cdot J^2 (3 + \cos 4\psi)$$

The constant A contains the strength and the sign of the octupole

The Hamilton equations read (remember θ is our "time" variable)

$$\frac{d\psi}{d\theta} = \frac{\partial \mathcal{H}}{\partial J} = \delta + 2A J (3 + \cos 4\psi) \quad (1)$$

$$\frac{dJ}{d\theta} = -\frac{\partial \mathcal{H}}{\partial \psi} = 4A J^2 \sin 4\psi \quad (2)$$

The conditions for fixed points are

$$\frac{dJ}{d\theta} = 0 \Rightarrow \psi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \dots$$

$\frac{d\psi}{d\theta} = 0$: can only be fulfilled if δ and A are of opposite sign

Let $Q_0 > n/4$, i.e. $\delta > 0$. Then $A < 0$ needed to obtain fixed points.

There are two kinds of fixed points:

(a) Stable (or "elliptic") fixed points

$$\psi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Around a stable fixed point is an island of stable motion

(b) Instable ("hyperbolic") fixed points

$$\psi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

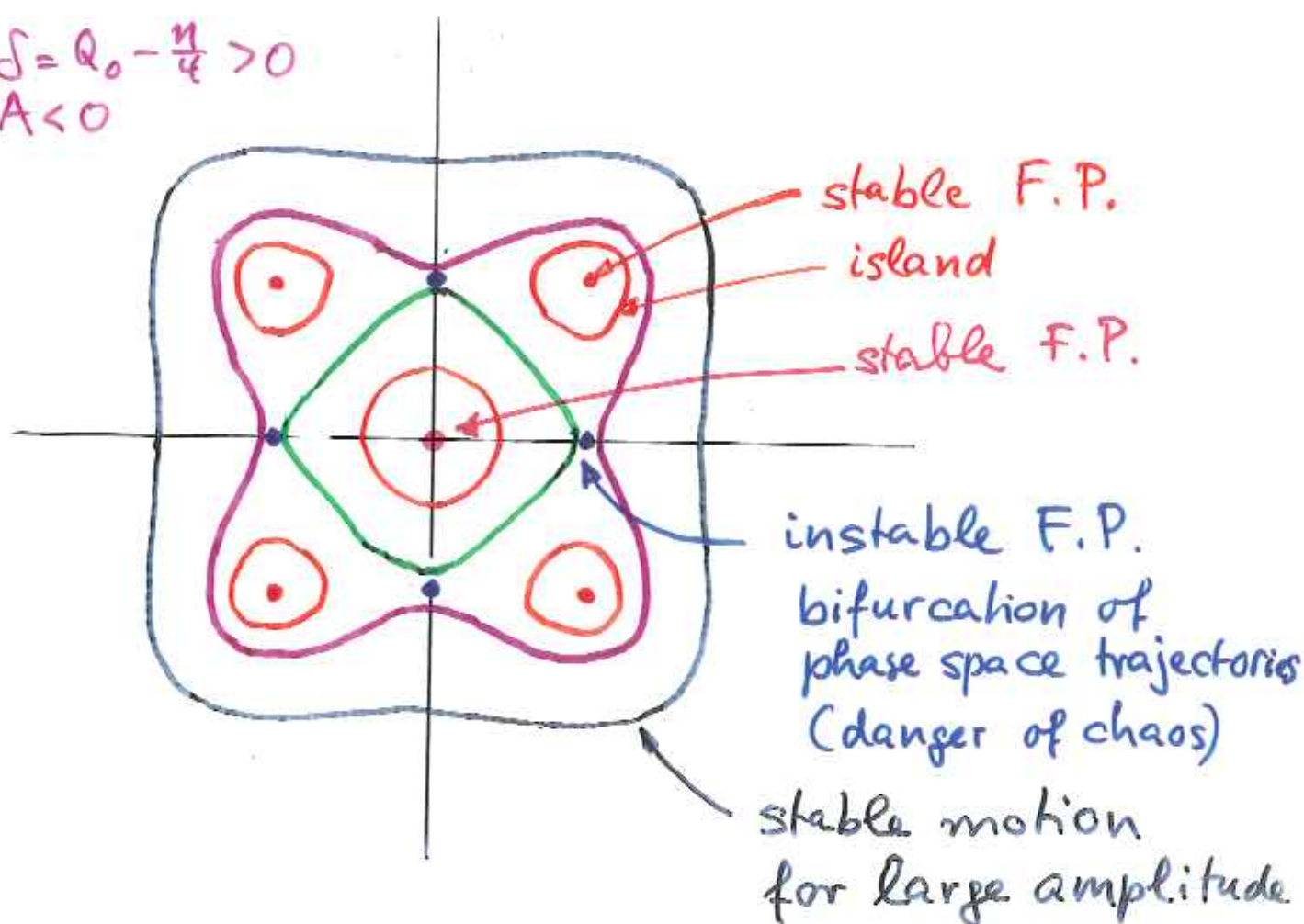
Bifurcation of trajectories occurs, chaos may happen

Phase space trajectories: iterative solution of (1), (2)

$$\begin{pmatrix} \psi \\ J \end{pmatrix}_{i+1} = \begin{pmatrix} \psi \\ J \end{pmatrix}_i + \begin{pmatrix} \delta + 2A J_i (3 + \cos 4\psi_i) \\ 4A J_i^2 \sin 4\psi_i \end{pmatrix} \cdot \Delta\theta$$

$$\delta = Q_0 - \frac{n}{4} > 0$$

$$A < 0$$



What happens if δ and A have the same sign?
 No fixed points are present, the phase space trajectories are squares with rounded corners.

Important: one observes a strong amplitude dependence of betatron tune

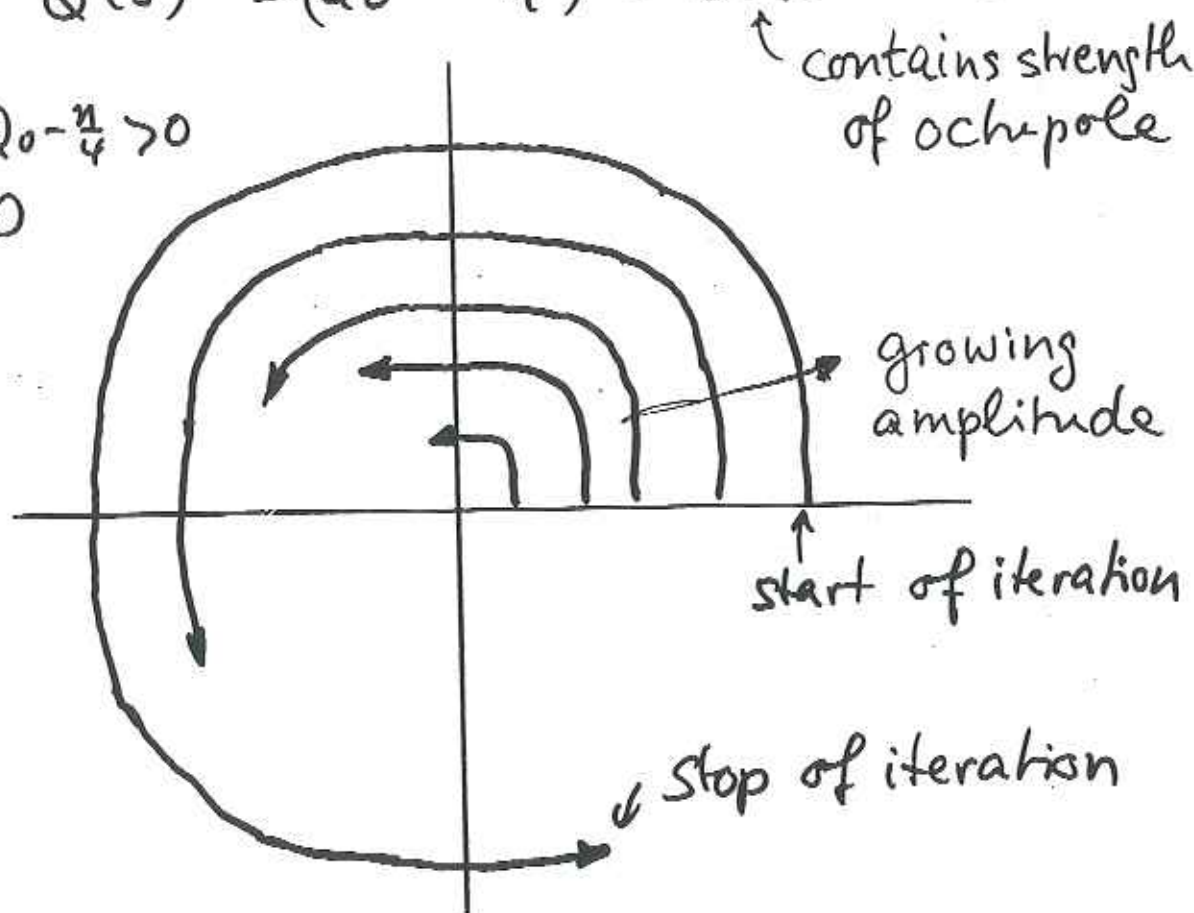
$$Q = \frac{\partial \mathcal{H}}{\partial J} = \left(Q_0 - \frac{n}{4}\right) + 2A \cdot J (3 + \cos 4\psi)$$

↑
averages to zero

$$Q(J) = \left(Q_0 - \frac{n}{4}\right) + 6A \cdot J$$

$$\delta = Q_0 - \frac{n}{4} > 0$$

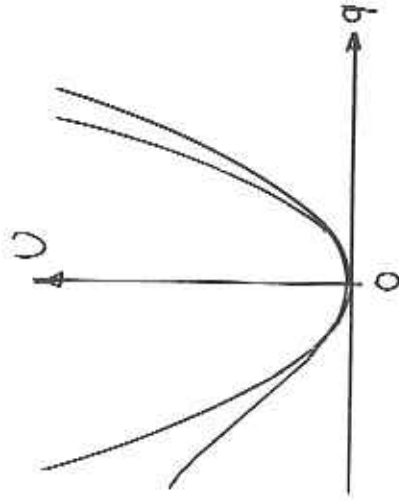
$$A > 0$$



$A > 0$: Q increases with increasing J
 $A < 0$: Q decreases

Betatron motion in an accelerator with only octupoles is always bounded.

Anharmonic oscillator: q^3 -term



$$U(q) = \frac{1}{2} k q^2 + \alpha q^3$$

$$H(q, p) = \underbrace{\frac{p^2}{2m} + \frac{1}{2} k q^2}_{H_0} + \underbrace{\alpha q^3}_{\Delta H}$$

$$\mathcal{H}(\psi, J) = \omega_0 J + \beta J^{3/2} \underbrace{\cos^3 \psi}_{\frac{1}{4}(\cos 3\psi + 3\cos \psi)}$$

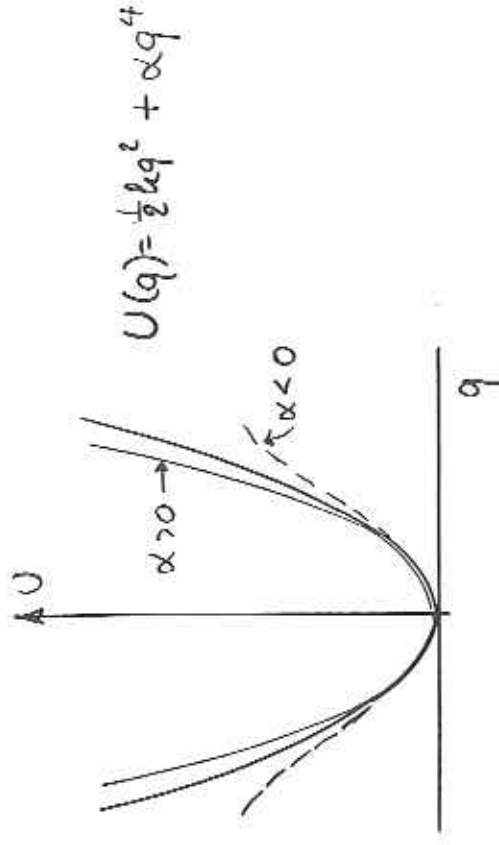
averaged over one period

$$\langle \mathcal{H} \rangle = \omega_0 J$$

$$\omega = \frac{\partial \langle \mathcal{H} \rangle}{\partial J} = \omega_0$$

Third-order term makes no frequency shift (in lowest-order perturbation theory)

anharmonic oscillator: q^4 -term



$$U(q) = \frac{1}{2} k q^2 + \alpha q^4$$

$$\mathcal{H}(\psi, J) = \omega_0 J + \beta J^2 \underbrace{\cos^4 \psi}_{\frac{1}{8}(\cos^4 \psi + 4\cos^2 \psi + 3)}$$

$$\frac{1}{8}(\cos^4 \psi + 4\cos^2 \psi + 3)$$

The $\cos^4 \psi$ term does not average out.

$$\langle \mathcal{H} \rangle = \omega_0 J + \frac{3}{8} \beta J^2$$

$$\omega = \frac{\partial \langle \mathcal{H} \rangle}{\partial J} = \omega_0 + \frac{3}{4} \beta J$$

We get a frequency shift which depends on the square of the oscillation amplitude

Coupling resonances

Coupling of horizontal and vertical betatron oscillations may be caused by

skew quadrupole $A_s(x, y, s) = f(s) \cdot x \cdot y$

sextupole normal $f(s) \cdot (x^3 - 3xy^2)$
 skew " $(3x^2y - y^3)$

octupole normal " $(x^4 - 6x^2y^2 + y^4)$
 skew " $(4x^3y - 4xy^3)$

These terms have to be added to the Hamiltonian (type H_4) which will be called H_0 here

$$H = \underbrace{\frac{1}{2} K(s) (y^2 - x^2)}_{H_0} + \frac{1}{2} (p_x^2 + p_y^2) + f(s) xy$$

Canonical transformation to angle-action variables

$$(x, p_x) \rightarrow (\psi_1, J_1), \quad (y, p_y) \rightarrow (\psi_2, J_2)$$

$$\mathcal{H}_1 = \underbrace{Q_1 J_1 + Q_2 J_2}_{\mathcal{H}_0 \text{ (type } H_2)} + U(\psi_1, \psi_2, J_1, J_2, \theta)$$

The perturbation U is periodic in ψ_1, ψ_2 and θ and can be expanded in a triple Fouries series

$$U(\psi_1, \psi_2, J_1, J_2, \theta) = \sum_{m_1, m_2, n} U_{m_1 m_2 n} e^{i(m_1 \psi_1 + m_2 \psi_2 - n \theta)}$$

$U_{m_1 m_2 n}$ is function of J_1, J_2

Change of phase per turn:

$$\psi_i \rightarrow \psi_i + 2\pi Q_i, \quad \theta \rightarrow \theta + 2\pi$$

$$m_1\psi_1 + m_2\psi_2 - n\theta \rightarrow m_1\psi_1 + m_2\psi_2 - n\theta + 2\pi(m_1Q_1 + m_2Q_2 - n)$$

Resonance condition:

$$\boxed{m_1Q_1 + m_2Q_2 - n \approx 0.}$$

Close to such a resonance

$$\mathcal{H}_1 \approx Q_1 \cdot J_1 + Q_2 \cdot J_2 + f(J_1, J_2) \cos(m_1\psi_1 + m_2\psi_2 - n\theta)$$

$$(f(J_1, J_2) = 2U_{m_1 m_2 n}(J_1, J_2); \quad U_{000} \text{ neglected})$$

$$\delta = m_1Q_1 + m_2Q_2 - n \quad \text{distance to resonance}$$

Canonical transformation into rotating system

$$(\psi_i, J_i) \rightarrow (\phi_i, K_i) \quad i = 1, 2 \quad (x, y)$$

$$F_2(\psi_1, K_1, \psi_2, K_2, \theta) = (m_1\psi_1 + m_2\psi_2 - n\theta) \cdot K_1 + \psi_2 \cdot K_2$$

$$\phi_1 = \frac{\partial F_2}{\partial K_1} = m_1\psi_1 + m_2\psi_2 - n\theta$$

$$\phi_2 = \frac{\partial F_2}{\partial K_2} = \psi_2$$

$$J_1 = \frac{\partial F_2}{\partial \psi_1} = m_1 K_1 \quad J_2 = \frac{\partial F_2}{\partial \psi_2} = m_2 K_1 + K_2$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial F_2}{\partial \theta} = Q_1 J_1 + Q_2 J_2 + f(J_1, J_2) \cos \phi_1 - n K_1$$

The Hamiltonian in the rotating frame is

$$\mathcal{H}_2 = \delta \cdot K_1 + Q_2 \cdot K_2 + \tilde{f}(K_1, K_2) \cos \phi_1$$

The Hamiltonian is independent of the quasi time θ , hence the phase space trajectories are curves of constant Hamiltonian

$$\frac{d\mathcal{H}_2}{d\theta} = \frac{\partial \mathcal{H}_2}{\partial \theta} = 0$$

Furthermore $\mathcal{H}_2$ is independent of ϕ_2 , hence $K_2 = \text{const}$

$$\frac{\partial \mathcal{H}_2}{\partial \phi_2} = 0 \Rightarrow m_2 J_1 - m_1 J_2 = \text{const}$$

In the following we consider a skew quadrupole. Choose $m_1 = +1$, then m_2 maybe $+1$ or -1

Sum resonance

For $m_2 = +1$ the difference of horizontal and vertical action

$$J_1 - J_2 = \text{const}$$

must stay constant. However, J_1 and J_2 can both grow indefinitely

A sum resonance

$$Q_1 + Q_2 = n$$

is unstable and leads to beam loss

Difference resonance

$$m_1 = +1, m_2 = -1 \rightarrow J_1 + J_2 = \text{const}$$

The action variables are squares of amplitudes and hence always positive. Therefore the motion remains bounded

A difference resonance

$$Q_1 - Q_2 = n$$

is stable

However there is energy exchange between the horizontal and vertical betatron oscillations which is undesirable in electron-positron storage rings

Time development of coupled motion

$$\dot{K}_2 \equiv \frac{dK_2}{d\theta} = - \frac{\partial \mathcal{H}_2}{\partial \phi_2} = 0, \quad K_2 = \text{const}$$

$$\dot{\phi}_1 = \frac{\partial \mathcal{H}_2}{\partial K_1} = \delta + \frac{\partial \tilde{f}}{\partial K_1} \cos \phi_1$$

$$\dot{K}_1 = - \frac{\partial \mathcal{H}_2}{\partial \phi_1} = \tilde{f} \sin \phi_1$$

$$\ddot{K}_1 = \left(\frac{\partial \tilde{f}}{\partial K_1} \dot{K}_1 + \frac{\partial \tilde{f}}{\partial K_2} \dot{K}_2 \right) \sin \phi_1 + \tilde{f} \cos \phi_1 \cdot \dot{\phi}_1$$

$\uparrow$
 0

$$\ddot{K}_1 = \tilde{f} \frac{\partial \tilde{f}}{\partial K_1} \underbrace{(\sin^2 \phi_1 + \cos^2 \phi_1)}_1 + \delta \underbrace{\tilde{f} \cos \phi_1}_{A - \delta \cdot K_1}$$

$$\text{with } A = \mathcal{H}_2 - Q_2 K_2 = \text{const}$$

$$\ddot{K}_1 = \tilde{f} \frac{\partial \tilde{f}}{\partial K_1} + \delta(A - \delta \cdot K_1)$$

Special case skew quadrupole

$$f(J_1, J_2) = B \cdot \sqrt{J_1 J_2} \quad B \propto \text{strength of quad.}$$

$$\tilde{f}(K_1, K_2) = B \sqrt{K_1 (m_2 K_1 + K_2)} \quad (m_1 = +1)$$

$$\tilde{f} \frac{\partial \tilde{f}}{\partial K_1} = \frac{B^2}{2} (K_2 + 2m_2 K_1)$$

The differential equation for K_1 becomes

$$\ddot{K}_1 + (\delta^2 - m_2 B^2) K_1 = \underbrace{\delta \cdot A + \frac{B^2}{2} K_2}_{\text{const}}$$

Sum resonance ($m_2 = +1$)

$$\ddot{K}_1 - \alpha^2 K_1 = \text{const}$$

$$\alpha^2 = B^2 - \delta^2$$

$\delta = Q_1 + Q_2 - n$ distance to resonance

$B \approx$ strength of quad

Except for very weak skew quad we get

$$\alpha^2 > 0$$

$\Rightarrow$ exponential growth of K_1 .

Difference resonance ($m_2 = -1$)

$$\ddot{K}_1 + \omega^2 K_1 = \text{const}$$

$$\omega^2 = \delta^2 + B^2 > 0$$

Periodic oscillation, stability

Energy exchange in difference resonance

Initial condition $\theta = 0$:

$$J_1(0) = J_0, \quad J_2(0) = 0$$

$$K_1(0) = J_1(0) = J_0$$

$$K_2(0) = J_2(0) - \underset{\substack{\uparrow \\ -1}}{m_2} K_1(0) = J_0$$

$$\mathcal{H}_2(0) = (\delta + Q_2) J_0 + \underbrace{B \sqrt{J_0(J_0 - J_0)}}_0 \cos \phi_1$$

$$A = \mathcal{H}_2 - Q_2 K_2 = \delta \cdot J_0$$

$$\ddot{K}_1 + \omega^2 K_1 = (\delta^2 + \frac{B^2}{2}) J_0$$

$$K_1(\theta) = C_0 + C_1 \cos(\omega \theta)$$

$$C_0 = \frac{\delta^2 + B^2/2}{\delta^2 + B^2} \cdot J_0, \quad C_1 = \frac{B^2/2}{\delta^2 + B^2} \cdot J_0$$

$$J_1(\theta) = \frac{J_0}{\omega^2} (\delta^2 + B^2 \cos^2(\frac{\omega}{2} \theta))$$

$$J_2(\theta) = \frac{J_0}{\omega^2} B^2 \sin^2(\frac{\omega}{2} \theta)$$

(G. Ripken, F. Willeke, DESY 88-114)

Second-order perturbation theory

In the following we take a small perturbation but assume that the tune Q_0 is far away from a resonance condition

Start with Hamiltonian of type H_7

$$\mathcal{H}_1(\psi_1, J_1, \theta) = \underbrace{\mathcal{H}_0(J_1)}_{Q_0 \cdot J_1} + \underbrace{U(\psi_1, J_1, \theta)}_{\text{small perturbation}}$$

A canonical transformation is performed which differs only little from an identity transformation

$$(\psi_1, J_1) \rightarrow (\psi_2, J_2)$$

$$F_2(\psi_1, J_2, \theta) = \underbrace{\psi_1 \cdot J_2}_{\text{identity transf.}} + \underbrace{\chi(\psi_1, J_2, \theta)}_{\text{"small"}}$$

$$\psi_2 = \frac{\partial F_2}{\partial J_2} = \psi_1 + \frac{\partial \chi}{\partial J_2} \quad J_1 = \frac{\partial F_2}{\partial \psi_1} = J_2 + \frac{\partial \chi}{\partial \psi_1}$$

$$\mathcal{H}_2 = \mathcal{H}_1 + \frac{\partial \chi}{\partial \theta}$$

First trick: replace in $\mathcal{H}_1$ the action J_1 by

$$J_1 = J_2 + \frac{\partial \chi}{\partial \psi_1}$$

$$\Rightarrow \mathcal{H}_2 = \mathcal{H}_0\left(J_2 + \frac{\partial \chi}{\partial \psi_1}\right) + U\left(\psi_1, J_2 + \frac{\partial \chi}{\partial \psi_1}, \theta\right) + \frac{\partial \chi}{\partial \theta}$$

second trick: add a term and subtract it again

$$\mathcal{H}_2 = Q_0 \cdot J_2 + \left\{ U(\psi_1, J_2 + \frac{\partial \chi}{\partial \psi_1}, \theta) - \underline{U(\psi_1, J_2, \theta)} \right\} \\ + Q_0 \cdot \frac{\partial \chi}{\partial \psi_1} + \underline{U(\psi_1, J_2, \theta)} + \frac{\partial \chi}{\partial \theta}$$

$$\Rightarrow \{ \} = \left[\frac{\partial U}{\partial J_1} \cdot \frac{\partial \chi}{\partial \psi_1} \right]_{J_1=J_2} \quad \text{small of 2nd order}$$

Now the function χ is determined such that

$$Q_0 \frac{\partial \chi}{\partial \psi_1} + U(\psi_1, J_2, \theta) + \frac{\partial \chi}{\partial \theta} = 0$$

Then the perturbation vanishes in first order and appears only in second order

Computation of perturbed Q-value

One is interested in the average tune shift, hence the perturbation term is averaged over the phase ψ_1

$$\mathcal{H}_2 = Q_0 \cdot J_2 + \underbrace{\left[\frac{\partial U}{\partial J_1} \cdot \frac{\partial \chi}{\partial \psi_1} \right]_{J_1=J_2}}_{V(\psi_1, J_2, \theta)}$$

$$\langle \mathcal{H}_2 \rangle = Q_0 \cdot J_2 + \langle V \rangle \quad \text{averaged over } \psi_1$$

If $\langle V \rangle$ is known one can compute the action-dependent tune shift

$$\frac{d\psi_2}{d\theta} = \frac{\partial \mathcal{H}_2}{\partial J_2} = Q_0 + \frac{\partial \langle V \rangle}{\partial J_2}$$

$$Q(J_2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi_2}{d\theta} d\theta = Q_0 + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \langle V \rangle}{\partial J_2} d\theta$$

Example : sextupole

$$H_4 = \frac{1}{2} K(s) x^2 + \frac{1}{2} p^2 + \underset{\substack{\uparrow \\ \text{sextupole at } s_0}}{S(s)} \cdot x^3$$

$$\mathcal{H}_2 = Q_0 \cdot J_2 + V(\psi_1, J_2, \theta)$$

What is the J_2 dependence of V ?

$$x^3 = (2J_1/\beta)^{3/2} \cos^3 \psi \Rightarrow U \sim J_2^{3/2} \quad (J_2 \approx J_1)$$

From the determining equation of X we get

$$X \sim J_2^{3/2}$$

$$\text{Therefore } \frac{\partial U}{\partial J_2} \cdot \frac{\partial X}{\partial \psi_1} \sim J_2^2$$

$$\Rightarrow \langle V \rangle = f(\theta) \cdot J_2^2$$

$$\begin{aligned} Q(J_2) &= Q_0 + \frac{1}{2\pi} \int_0^{2\pi} 2 f(\theta) J_2 d\theta \\ &= Q_0 + A \cdot J_2 \end{aligned}$$

In second-order perturbation theory a sextupole makes a tune-shift $\sim$ action J (like octupole in first order)