Tying Knots in Particle Physics

Minoru Eto,[1,2,3] Yu Hamada,[4,2] and Muneto Nitta[5,2,3]
[1]Department of Physics, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan
[2]Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Yokohama, Kanagawa 223-8521, Japan
[3]International Institute for Sustainability with Knotted Chiral Meta Matter (WPI-SKCM[2]),
Hiroshima University, 1-3-2 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8511, Japan
[4]Deutsches Elektronen-Synchrotron DESY, Notkestraße 85, 22607 Hamburg, Germany
[5]Department of Physics, Keio University, 4-1-1 Hiyoshi, Kanagawa 223-8521, Japan

(Received 30 December 2024; revised 4 August 2025; accepted 8 August 2025; published 29 August 2025)

Knots emerge in various fields of mathematics and physics today. We show that knots indeed appear as stable solitons in a realistic extension of the standard model of particle physics that provides the QCD axion and right-handed neutrinos. This result suggests that, during the early Universe, a "knot dominated era" may have existed, where knots were a dominant component of the Universe, and this scenario can be tested through gravitational wave observations. Furthermore, we propose that the end of this era involves the collapse of the knots via quantum tunneling, leading to the generation of matter-antimatter asymmetry in the Universe.

DOI: 10.1103/s3vd-brsn

Introduction—Knots, mathematically defined as closed curves embedded into three-dimensional space, appear not only when tying a tie but also in various scientific fields today, as pioneered by Lord Kelvin. Although his hypothesis that atoms are knots of aether vortices [1] was eventually disproven, it stimulated the development of knot theory [2] and its applications across multiple fields of physics. The key characteristic of knots is their topological stability—knots cannot be undone without breaking or intersecting the strings, making them essential in persistent physical phenomena [3]. In fact, knots appear as vortex knots or topological solitons called Hopfions [4] in various physical systems [5–35].

The concept of knots has previously appeared in the context of particle physics. In particular, a knot invariant in mathematics arises from topological field theories known as the Chern-Simons theory [36], which exhibits a direct connection between theoretical physics and mathematics. In addition, a knot as a soliton is proposed [4,37–41] in a field-theoretic toy model called the Faddeev-Skyrme model, in which the stability of the knot soliton requires non-renormalizable interactions destroying the predictability of the model. Nevertheless, knots have never appeared in phenomenologically viable models in particle physics.

In this Letter, we present the first theoretical demonstration of stable knot solitons emerging within a particle physics framework. Specifically, we consider a simple model incorporating both global and local $U(1)$ symmetries, which can be identified with the Peccei-Quinn $U(1)$ symmetry [42,43] and the $B-L$ (baryon number minus lepton number) symmetry, respectively. The spontaneous breaking of these symmetries gives rise to two distinct types of stringlike topological defects: flux tubes (local strings) and superfluid vortices (global strings). We find that when these two types of strings are linked together, they form stable knot solitons.

Furthermore, we explore the potential cosmological implications of these knot solitons. Our analysis suggests that a "knot-dominated era" may have existed in the early Universe, where knot solitons played a significant role in cosmic evolution. This scenario can be tested through gravitational wave (GW) observations. Additionally, we propose a mechanism in which the quantum tunneling-induced collapse of knot solitons contributes to the generation of the observed matter-antimatter asymmetry.

The model—We start with a model that includes two complex scalar fields ϕ_1 and ϕ_2 and a $U(1)$ gauge field. The Lagrangian for the scalars is given as

$$\mathcal{L}_\phi = |D_\mu\phi_1|^2 + |D_\mu\phi_2|^2 - V, \tag{1}$$

with $V \equiv \lambda(|\phi_1|^2 + |\phi_2|^2 - m^2)^2 - \kappa|\phi_1|^2|\phi_2|^2$ the scalar potential and $\lambda, \kappa > 0$. Here V has an exchange symmetry between the two scalars for simplicity, which is, however, not necessary in the following argument. The Lagrangian is invariant under $U(1)$ phase rotations of the scalars ϕ_1 $[U(1)_1]$ and ϕ_2 $[U(1)_2]$. The covariant derivative $D_\mu\phi_{1(2)} \equiv (\partial_\mu - iq_{1(2)}gA_\mu)\phi_{1(2)}$ depends on the gauge field A_μ,

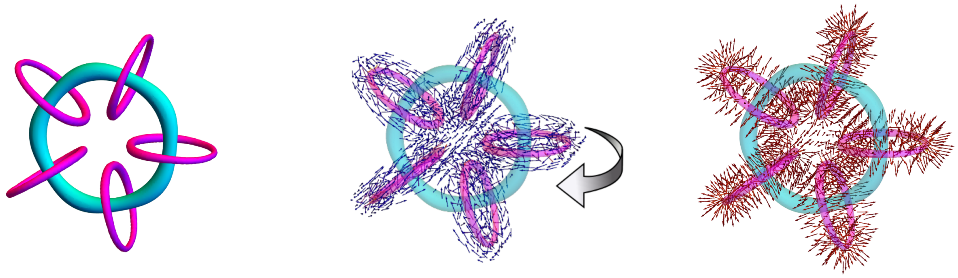

FIG. 1. 3D plots of the numerical solution for a knot soliton. The magenta and cyan regions represent the cores of the ϕ_1 and ϕ_2 strings ($|\phi_1|^2/v^2 < 0.1$ and $|\phi_2|^2/v^2 < 0.1$). In the middle and right panels, the arrows that are overlaid with transparent string cores indicate the magnetic flux $\vec{B}$ and the electric flux $\vec{E}$ of the massive gauge field A_μ, respectively, and their colors correspond to their strength $|\vec{B}|$ and $|\vec{E}|$. The magnetic flux is along the ϕ_1 strings, one of whose direction is indicated by the large gray arrow, while the electric one comes out from the strings, which means that the strings have net electric charges. The energy of the solution is $7.0 \times 10^3 v/g$.

coupling constant g, and $U(1)$ gauge charges q_1 and q_2 corresponding to ϕ_1 and ϕ_2, respectively. For simplicity, we take $q_1 = 1$ and $q_2 = 0$ for the moment, making $U(1)_1$ a gauge symmetry $[U(1)_{\text{local}}]$ and $U(1)_2$ a global symmetry $[U(1)_{\text{global}}]$. The Lagrangian for the kinetic term of A_μ is $\mathcal{L}_g = (-1/4)F_{\mu\nu}F^{\mu\nu}$, with $F_{\mu\nu}$ being the field strength. Thus, the full Lagrangian is given as $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_g$. The tachyonic mass $-\lambda m^2 < 0$ in V makes the two scalars condense and develop vacuum expectation values (VEVs) as $\langle\phi_1\rangle = \langle\phi_2\rangle = v/\sqrt{2}$, spontaneously breaking both of the $U(1)$ symmetries.

At the broken phase, quantum corrections can induce the so-called Chern-Simons coupling between the NG boson $a \equiv -i \arg \phi_2$ and A_μ [42,43],

$$\mathcal{L}_{\text{CS}} = C a F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ and C is a model-dependent dimensionless constant; for instance, it may be induced by one-loop effects of fermions [44]. To consider general setups, C is taken as a free parameter in this Letter.

The spontaneous breaking of the two symmetries $U(1)_{\text{local}}$ and $U(1)_{\text{global}}$ [or, $U(1)_1$ and $U(1)_2$] gives rise to two kinds of vortex strings as topological defects. We call these two strings the ϕ_1 and ϕ_2 strings, which are topologically protected with the phases winding around their defects. The ϕ_1 string is a local string and contains a magnetic flux of A_μ, $|\Phi_A| = 2\pi/g$ like superconductors while the ϕ_2 string is a global string without magnetic fluxes like superfluid vortices.

Knot soliton—The Chern-Simons coupling (2) does not affect the individual ϕ_1 or ϕ_2 string because the ϕ_1 (ϕ_2) string does not contain excitation of the NG boson a (the gauge field A_μ), resulting in (2) vanishing. Thus, loops of ϕ_1 or ϕ_2 strings alone are unstable and shrink due to their tension.

Now, make the ϕ_1 and ϕ_2 strings linked. Particularly, one simple example is a link made of a single ϕ_2 string and multiple ϕ_1 strings (see the left panel of Fig. 1). Here the coupling (2) plays a crucial role; it induces the electric flux of the A_μ field around the ϕ_1 string. In fact, we have $[\delta/(\delta A_0)]\int d^4x \mathcal{L}_{\text{CS}} = -C\vec{\nabla}a \cdot \vec{B}$, where $\vec{B}$ is the magnetic field of A_μ and we have used that all fields are static. This changes the Gauss law in the Maxwell equation for A_0; hence, $C\vec{\nabla}a \cdot \vec{B}$ behaves as a source of the electric field [45,46]. The total electric charge is estimated to be $C\int d^3x \vec{\nabla}a \cdot \vec{B} = 4\pi^2 C N_{\text{link}}/g$, where the strings are assumed to be sufficiently thin [47]. The integer N_{link} is the linking number of the ϕ_1 and the ϕ_2 strings [48]. From this, the ϕ_1 strings obtain the electric charge, which gives rise to the electric field $\vec{E}$ around them [50]. This electric field, though screened by ϕ_1's VEV, prevents the ϕ_1 string loops from shrinking and stabilizes them for large C [54].

The stability of the link requires another condition $\lambda \gg g^2, \kappa$, which ensures that the link of the ϕ_1 and ϕ_2 strings cannot be delinked, i.e., the strings cannot pass through each other. This is because in the limit of $\lambda \to \infty$, N_{link} is equivalent to the skyrmion number [57,58], which is characterized by the homotopy group $\pi_3(S^3) \simeq \mathbb{Z}$ defined by a map from S^3, obtained by a compactification of the real space $\mathbb{R}^3$, to S^3 defined by $|\phi_1|^2 + |\phi_2|^2 = \text{const}$, and hence is approximately a topological invariant. This is similar to the case of skyrmions in two-component Bose-Einstein condensates [59–61]. In practical cases with finite λ, the ϕ_1 and ϕ_2 strings feel an energy barrier to be delinked and are repelled from each other [11,62].

We numerically solve the classical equations of motion (EOMs) to confirm the existence of the stable knot soliton [63]. Figure 1 shows the cores of the strings of the obtained solution. We here take $\lambda/g^2 = 10^3$, $\kappa/g^2 = 0.0008$, and $C = 400$. The magnetic and electric fields are also shown by arrows. In this figure, the single ϕ_2 string loop is linked with five ϕ_1 string loops (with the same linking direction), so that the solution has the linking number 5. One can also see that the magnetic flux is along the ϕ_1 strings, one of whose direction is indicated by the large gray arrow. On the

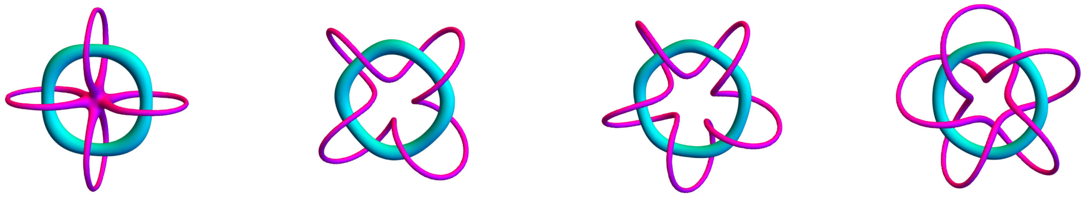

FIG. 2. 3D plots of numerical solutions for other knot solitons. The colors and the parameters are the same as those in Fig. 1. The two left plots have the same linking number 4, and the other plots have the linking number 5. The energy of the solutions is $6.0 \times 10^3 v/g$, $6.3 \times 10^3 v/g$, $7.3 \times 10^3 v/g$, and $7.5 \times 10^3 v/g$ (from left to right).

other hand, the electric one comes out from the strings, which means that the strings have net electric charges consistent with the argument above [99].

Note that a similar solution is obtained by acting parity transformation on the knot soliton shown here. This solution has the opposite linking direction of the ϕ_1 and ϕ_2 strings compared to the knot soliton, so that we call this the antiknot soliton. The antiknot one has the $U(1)_{\mathrm{local}}$ electric charge (and Skyrme number) with the opposite signs to those of the knot one.

We can also obtain other knot solitons, which are shown in Fig. 2. The leftmost panel shows a single loop of the ϕ_2 string linking with four ϕ_1 string loops. In the other panels, single ϕ_1 and ϕ_2 loops make a higher linking number by linking multiple times. For both types, the energy grows as the linking number gets higher because the electric charge contained in the ϕ_1 string becomes larger. The energy of all configurations is almost dominated by the ϕ_1 strings and the electric flux. Within the same linking number, the energy gets larger as the length of the ϕ_1 string gets longer. Therefore, the leftmost solution in Fig. 2 and that in Fig. 1 have the least energy with the linking number 4 and 5, respectively. We did not find any other knot solitons with our numerical code, which would imply that the knot soliton with the linking number smaller than 4 cannot be stable due to the smaller electric charges and that it necessitates a larger simulation box size than that we used to show the knot solitons with the linking number larger than 5. However, more detailed studies are necessary to conclude.

Even when the (anti)knot solitons are classically stable, they can decay due to quantum tunneling, whose lifetime is denoted by τ_{decay}. One possible decay process is the delinking process, i.e., the ϕ_1 and ϕ_2 strings pass through each other by quantum tunneling. The calculation of the decay rate is complicated and beyond the scope of this Letter. Instead, we will later relate the decay rate to the cosmological history of the knot solitons.

Particle phenomenology—In spite of the great success of the SM, there are still several problems that the SM is not able to answer, such as dark matter in the Universe, tiny neutrino masses, strong *CP* problem, and baryon asymmetry in the Universe. (For a review, see, e.g., Ref. [101].) It is believed that (some of) these problems should be resolved by new physics beyond the SM.

One way to extend the SM is to introduce new symmetries. The strong *CP* problem and dark matter are simultaneously explained by introducing a global $U(1)$ symmetry called the Peccei-Quinn symmetry [42,43] (denoted by $U(1)_{\mathrm{PQ}}$), which spontaneously breaks and provides a NG boson, the QCD axion [102,103]. In addition, it is well motivated to introduce the gauge $B-L$ (baryon number minus lepton number) symmetry, $U(1)_{B-L}$, because it requires the existence of the right-handed (RH) neutrinos and its spontaneous breaking gives them Majorana masses, which naturally explain the tiny neutrino masses in terms of the type-I seesaw mechanism [104–107]. The RH neutrinos are also expected to play key roles to generate baryon asymmetry in the Universe. Furthermore, the $U(1)_{B-L}$ gauge symmetry naturally arises from grand unified theories (GUTs).

Now we apply our Letter on the knot soliton to particle physics models and discuss its influence on cosmology. One realistic and promising setup is obtained by identifying the $U(1)_{\mathrm{local}}$ and $U(1)_{\mathrm{global}}$ symmetries in (1) as the $U(1)_{B-L}$ and $U(1)_{\mathrm{PQ}}$ symmetries, respectively. To couple ϕ_1 with RH neutrinos, its $B-L$ charge is set to $q_1 = 2$ while keeping $q_2 = 0$. Here the QCD axion is identified with a. For this to solve the strong *CP* problem, we need additional fermionic fields inducing the Chern-Simons coupling between a and gluon, which may be realized by introducing Kim-Shifman-Vainshtein-Zakharov (KSVZ)like [108,109] heavy quarks. How a couples to the $B-L$ and SM gauge bosons depends on the charge assignment of the fermionic sectors, which is kept as general as possible hereafter [110]. From the astrophysical constraints on the QCD axion, we take $v \gtrsim 10^8$ GeV [119–122]. We emphasize that this setup is well motivated because it can elegantly resolve some of the mysteries in particle physics as stated above. It is natural to have coexistence of the two symmetries with symmetry breaking scales of the same order of magnitude since neither symmetry alone can fully address the shortcomings of the SM, and a comparable symmetry breaking scale avoids introducing a new hierarchy problem [123].

Cosmology, baryogenesis, and gravitational wave—Now let us discuss the cosmological fate of the knot solitons. When the temperature T of the thermal bath in the Universe is larger than v, the $U(1)_{B-L}$ and $U(1)_{\mathrm{PQ}}$ symmetries remain unbroken. At $T \simeq v$, they get spontaneously broken

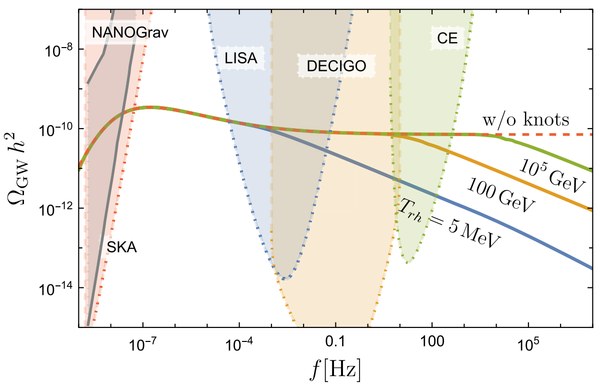

FIG. 3. Stochastic GW spectrum from the network of the ϕ_1 string with the tension $\mu \simeq v_1^2 \simeq 10^{27}$ GeV^2. The red dashed line corresponds to the cosmological evolution without the knot solitons while the solid ones include the knot dominated era with different reheating temperatures due to the knot decay, $T_{rh} = 5$ MeV (blue), 100 GeV (orange), and 10^5 GeV (green).

simultaneously producing the ϕ_1 and ϕ_2 strings via the Kibble-Zurek mechanism [127,128]. The produced strings have random configurations, in which some of them are linked with finite probability. After the oscillation and the random motion are relaxed, long ϕ_1 and ϕ_2 strings form conventional networks whose typical length scale is the Hubble scale. Besides them, small loops with the linking configurations become the knot solitons that we have studied above. The number density of the produced (anti)knot solitons is estimated as $n_{\rm knot}(T \simeq v) \sim (0.04)^4 \xi^{-3}$ [63].

After the production, the knot solitons behave as heavy long-lived particles, whose energy density scales as $\rho_{\rm knot} \propto R(t)^{-3}$, with $R(t)$ the scale factor. Therefore, their energy density eventually overcomes that of the radiation and dominates the total energy density of the Universe. We call this epoch "knot dominated era." Afterward, at $t \sim \tau_{\rm decay}$, they decay by the quantum tunneling and release their energy into light particles, leading to the radiation dominated epoch again via a secondary reheating. Assuming instantaneous decays, the temperature of the reheated thermal bath is given by [129] $T_{rh} \simeq \{[30/(\pi^2 g_*)][(3M_{\rm pl}^2)/8\pi]\Gamma^2\}^{1/4}$ with g_* the number of the relativistic degrees of freedom, $g_* = \mathcal{O}(10^2)$, $\Gamma \equiv 1/\tau_{\rm decay}$ the decay rate, and $M_{\rm pl}$ the Planck mass. Note that Γ is typically exponentially suppressed because of nonperturbative processes. In order to realize the successful big bang nucleosynthesis (BBN), T_{rh} must be larger than $\mathcal{O}(1)$ MeV [130–132]. Therefore, the Universe experiences the knot dominated era sandwiched by two radiation dominated eras before BBN.

When T_{rh} is larger than the electroweak (EW) scale $\mathcal{O}(10^2)$ GeV, this end of the knot domination can produce baryon asymmetry in the Universe in a similar way to nonthermal leptogenesis via inflaton decay [133–137]. The observed baryon asymmetry is given by $Y_B|_{\rm obs} \equiv (n_B - n_{\bar B})/s \simeq 0.8 \times 10^{-10}$ [138] where $n_{B(\bar B)}$ and s are the number density of (anti)baryon and the entropy density, respectively.

In this scenario, the RH neutrinos ν_R play key roles. They get Majorana masses from the VEV of ϕ_1 with mass eigenvalues M_{Ri} and eigenstates N_i. The two lightest ones are assumed to be nearly degenerated $M_{R1} \lesssim M_{R2} (< M_{R3})$ like resonant leptogenesis [139,140]. Through the couplings with ϕ_1 and A_μ, the decay of a knot soliton produces N_i. We parameterize an energy fraction of the produced N_1 compared to the knot static energy as $f_{N_1} \in [0, 1]$, i.e., the produced number of N_1 is given as $f_{N_1} M/M_{R1}$, in which $M \simeq \mathcal{O}(10^3) v/g$ is the knot soliton mass and f_{N_1} is not expected to be small when $M_{R1} < gv$. The produced N_1 immediately decay into the SM Higgs and charged leptons, providing the lepton asymmetry $n_L - n_{\bar L}$ [63]. Through the EW sphaleron process, some of the lepton asymmetry is converted into the baryon asymmetry, resulting in

$$Y_B \simeq 0.8 \times 10^{-10} f_{N_1} \left(\frac{T_{rh}}{10^2\ \text{GeV}}\right)\left(\frac{10^{12}\ \text{GeV}}{M_{R1}}\right), \quad (3)$$

reproducing $Y_B|_{\rm obs}$ by taking appropriate parameters. There can be an alternative scenario of baryogenesis even when T_{rh} is smaller than the electroweak scale. This is complementary to the scenario presented here [63].

Remarkably, the existence of the knot dominated era can be observed in terms of GWs when there exists a sufficient GW source. Fortunately, this model already contains a possible source of GW: network of ϕ_1 strings. The knot dominated era leaves imprints on the GW spectrum from ϕ_1 strings [141–143], as shown in Fig. 3 with three benchmarks for T_{rh} [63]. Here $\Omega_{\rm GW} h^2 = [d\rho_{\rm GW}(f)/d\log f] h^2/\rho_c$, with h being the dimensionless Hubble constant, $\rho_{\rm GW}(f)$ the GW energy density for a frequency f observed today, and ρ_c the critical energy density. We also show in Fig. 3 the observed signals of NANOGrav 15-year dataset [144] as the gray region and power-law integrated sensitivity curves of future GW observatories (taken from Ref. [145]): DECIGO [146] (orange), Cosmic Explorer (CE) [147] (green), LISA [148] (blue), and Square Kilometer Array (SKA) [149] (red). Because of the existence of the knot dominated era, solid lines are deviated from the conventional spectrum without the knot domination (red dashed line). Especially, taking $M_{R1} = 10^{12}$ GeV and $f_{N_1} \sim \mathcal{O}(1)$ in Eq. (3), one can predict T_{rh} to be $\mathcal{O}(10^2)$ GeV, which is the lower limit to realize leptogenesis, and hence the deviation in the GW spectrum from the dashed line appears within the range of CE (the orange line in Fig. 3). Therefore, this scenario is expected to be tested. Note that this imprint is the byproduct of the knot-based baryogenesis. In other words, if future GW detectors do not observe the deviation of the stochastic GW spectrum from the conventional flat GW spectrum (red dashed line), our scenario based on knot domination would be ruled out within certain parameter space. This is a significant advantage of our scenario as standard thermal leptogenesis suffers from lack of testability.

To summarize, within reasonable setups, the decay of the knot solitons can produce the correct baryon asymmetry. Furthermore, knot domination can be probed by stochastic GW backgrounds in the near future. While Kelvin originally conjectured knots as fundamental building blocks of matter, our findings provide, for the first time, a realistic particle physics model in which knots may play a crucial role in the origin of matter.

For further studies in the phenomenological perspective, we need concrete model building. Particularly, it is worth considering embedding this setup into GUTs such as $SO(10)$ GUTs. Although we have taken C, the coefficient of the Chern-Simons coupling, as a free parameter, it may be explained as a one-loop effect of matter fermions charged under the $U(1)_{B-L}$ and $U(1)_{\rm PQ}$ symmetries. To conclude, the knot soliton in particle physics has many potential applications for understanding fundamental aspects of nature.

Acknowledgments—The authors would like to thank Simone Blasi, Asuka Ito, Ryusuke Jinno, Kohei Kamada, Yuka Kotorii, Shunya Mizoguchi, Nobuyuki Matsumoto, Hiroto Shibuya, Fumio Uchida, Wen Yin, and Ryo Yokokura for useful comments. The numerical calculations were carried out on Yukawa-21 at YITP in Kyoto University. This work is supported in part by JSPS Grant-in-Aid for Scientific Research KAKENHI Grants No. JP22H01221, No. JP23K22492 (M. E. and M. N.), and No. JP22KJ3123 (Y. H.) and by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy—EXC 2121 Quantum Universe—390833306. The work of M. E. and M. N. is supported in part by the World Premier International Research Center Initiative program "Sustainability with Knotted Chiral Meta Matter (WPI – SKCM^2)" at Hiroshima University.

Data availability—The data that support the findings of this Letter are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

[1] W. H. Thomson, Trans. R. Soc. Edinburgh **25**, 217 (1869).
[2] P. G. Tait, Sci. Papers **1**, 273 (1898).
[3] L. H. Kauffman, *Knots and Physics*, Series on Knots & Everything (World Scientific Publishing Co Pte Ltd, Singapore, 1991).
[4] L. D. Faddeev and A. J. Niemi, Nature (London) **387**, 58 (1997).
[5] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[6] R. L. Ricca and M. A. Berger, Phys. Today **49**, No. 12, 28 (1996).
[7] R. L. Ricca, Nuovo Cimento Soc. Ital. Fis. **32C**, 185 (2009).
[8] D. Kleckner and W. T. M. Irvine, Nat. Phys. **9**, 253 (2013).
[9] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics* (Springer, Cham, 2021).
[10] E. Babaev, L. D. Faddeev, and A. J. Niemi, Phys. Rev. B **65**, 100512(R) (2002).
[11] F. N. Rybakov, J. Garaud, and E. Babaev, Phys. Rev. B **100**, 094515 (2019).
[12] D. Proment, M. Onorato, and C. F. Barenghi, Phys. Rev. E **85**, 036306 (2012).
[13] D. Kleckner, L. H. Kauffman, and W. T. M. Irvine, Nat. Phys. **12**, 650 (2016).
[14] Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. **100**, 180403 (2008); **101**, 029902(E) (2008).
[15] D. Hall, M. Ray, K. Tiurev *et al.*, Nat. Phys. **12**, 478 (2016).
[16] T. Ollikainen, A. Blinova, M. Möttönen, and D. S. Hall, Phys. Rev. Lett. **123**, 163003 (2019).
[17] G. Volovik and V. P. Mineev, J. Exp. Theor. Phys. **46**, 401 (1977).
[18] G. E. Volovik, *The Universe in a Helium Droplet*, International Series of Monographs on Physics, Oxford Scholarship Online (Oxford University Press, Oxford, England, 2009).
[19] BryanGin-ge Chen, P. J. Ackerman, G. P. Alexander, R. D. Kamien, and I. I. Smalyukh, Phys. Rev. Lett. **110**, 237801 (2013).
[20] T. Machon and G. P. Alexander, Phys. Rev. Lett. **113**, 027801 (2014).
[21] P. Ackerman, J. van de Lagemaat, and I. Smalyukh, Nat. Commun. **6**, 6012 (2015).
[22] P. Ackerman and I. Smalyukh, Nat. Mater. **16**, 426 (2017).
[23] P. J. Ackerman and I. I. Smalyukh, Phys. Rev. X **7**, 011006 (2017).
[24] J.-S. Tai, P. Ackerman, and I. Smalyukh, Proc. Natl. Acad. Sci. U.S.A. **115**, 921 (2018).
[25] J.-S. B. Tai and I. I. Smalyukh, Science **365**, 1449 (2019).
[26] G. P. Alexander, BryanGin-ge Chen, E. A. Matsumoto, and R. D. Kamien, Rev. Mod. Phys. **84**, 497 (2012).
[27] I. I. Smalyukh, Rep. Prog. Phys. **83**, 106601 (2020).
[28] J.-S. Wu and I. I. Smalyukh, *Hopfions, Heliknotons, Skyrmions, Torons and Both Abelian and Non-Abelian Vortices in Chiral Liquid Crystals* (Taylor & Francis, London, 2022).
[29] N. Kent *et al.*, Nat. Commun. **12**, 1562 (2021).
[30] M. R. Dennis, R. P. King, B. Jack, K. O'Holleran, and M. J. Padgett, Nat. Phys. **6**, 118 (2010).
[31] A. Trautman, Int. J. Theor. Phys. **16**, 561 (1977).
[32] A. F. Ranada, Lett. Math. Phys. **18**, 97 (1989).
[33] H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas, and W. T. M. Irvine, Phys. Rev. Lett. **111**, 150404 (2013).
[34] M. Arrayás, D. Bouwmeester, and J. L. Trueba, Phys. Rep. **667**, 1 (2017).
[35] S. Shankar, A. Souslov, M. J. Bowick, M. C. Marchetti, and V. Vitelli, Nat. Rev. Phys. **4**, 380 (2022).
[36] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[37] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998).
[38] N. S. Manton and P. Sutcliffe, *Topological Solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2004).
[39] E. Radu and M. S. Volkov, Phys. Rep. **468**, 101 (2008).
[40] M. Kobayashi and M. Nitta, Phys. Lett. B **728**, 314 (2014).
[41] Y. M. Shnir, *Topological and Non-Topological Solitons in Scalar Field Theories* (Cambridge University Press, Cambridge, England, 2018).
[42] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38**, 1440 (1977).
[43] R. D. Peccei and H. R. Quinn, Phys. Rev. D **16**, 1791 (1977).
[44] If we assume the existence of a UV theory, C is quantized as $C = Ng^2/32\pi^2$ with an integer N.
[45] P. A. Horvathy, arXiv:0704.3220.
[46] P. A. Horvathy and P. Zhang, Phys. Rep. **481**, 83 (2009).
[47] N. Yokoi, Geophys. Astrophys. Fluid Dyn. **107**, 114 (2013).
[48] Formally this is the same as cross helicity in magnetohydro dynamics when we regard $\vec{\nabla}a$ as the velocity of fluid [47,49].
[49] H. Nastase and J. Sonnenschein, J. High Energy Phys. 12 (2022) 144.
[50] Knots are also discussed in similar systems without the Chern-Simons coupling [51–53].
[51] E. Babaev, A. Sudbo, and N. W. Ashcroft, Nature (London) **431**, 666 (2004).
[52] J. Smiseth, E. Smorgrav, E. Babaev, and A. Sudbo, Phys. Rev. B **71**, 214509 (2005).
[53] E. Babaev, Phys. Rev. D **70**, 043001 (2004).
[54] A similar vortex loop with the charge/current is known as the vorton [55,56]. Our knot soliton considered here is different from those as it does not contain internal degrees of freedom on the loop.
[55] R. L. Davis and E. P. S. Shellard, Nucl. Phys. **B323**, 209 (1989).
[56] R. L. Davis and E. P. S. Shellard, Phys. Lett. B **209**, 485 (1988).
[57] S. B. Gudnason and M. Nitta, Phys. Rev. D **101**, 065011 (2020).
[58] S. B. Gudnason and M. Nitta, Phys. Rev. D **102**, 045022 (2020).
[59] J. Ruostekoski and J. R. Anglin, Phys. Rev. Lett. **86**, 3934 (2001).
[60] R. A. Battye, N. R. Cooper, and P. M. Sutcliffe, Phys. Rev. Lett. **88**, 080401 (2002).
[61] M. Nitta, K. Kasamatsu, M. Tsubota, and H. Takeuchi, Phys. Rev. A **85**, 053639 (2012).
[62] M. Eto, K. Kasamatsu, M. Nitta, H. Takeuchi, and M. Tsubota, Phys. Rev. A **83**, 063603 (2011).
[63] See Supplemental Material at http://link.aps.org/supplemental/10.1103/s3vd-brsn for the details of the numerical calculation, which includes Ref. [64]; for the derivation of the number density, which includes Refs. [65–69]; for the details of the alternaive baryogenesis scenario, which includes Refs. [70–84]; for the calculation of the lepton asymmetry, which includes Refs. [85–88]; and for the calculation of the GW spectrum, which includes Refs. [89–98].
[64] M. R. Hestenes, J. Optim. Theory Appl. **4**, 303 (1969).
[65] N. Turok, Phys. Rev. Lett. **63**, 2625 (1989).
[66] J. Borrill, E. J. Copeland, and A. R. Liddle, Phys. Lett. B **258**, 310 (1991).
[67] R. A. Leese and T. Prokopec, Phys. Rev. D **44**, 3749 (1991).
[68] A. Vilenkin and E. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 2000).
[69] T. Vachaspati and G. B. Field, Phys. Rev. Lett. **73**, 373 (1994).
[70] J. M. Hyde, A. J. Long, and T. Vachaspati, Phys. Rev. D **89**, 065031 (2014).
[71] K. Kamada and A. J. Long, Phys. Rev. D **94**, 063501 (2016).
[72] K. Kamada and A. J. Long, Phys. Rev. D **94**, 123509 (2016).
[73] I. Affleck and M. Dine, Nucl. Phys. **B249**, 361 (1985).
[74] R. T. Co and K. Harigaya, Phys. Rev. Lett. **124**, 111602 (2020).
[75] M. S. Turner and L. M. Widrow, Phys. Rev. D **37**, 2743 (1988).
[76] W. D. Garretson, G. B. Field, and S. M. Carroll, Phys. Rev. D **46**, 5346 (1992).
[77] M. M. Anber and L. Sorbo, J. Cosmol. Astropart. Phys. 10 (2006) 018.
[78] K. Kamada, Phys. Rev. D **97**, 103506 (2018).
[79] V. Domcke, K. Kamada, K. Mukaida, K. Schmitz, and M. Yamada, Phys. Rev. Lett. **126**, 201802 (2021).
[80] M. Yoshimura, Phys. Rev. Lett. **41**, 281 (1978); **42**, 746(E) (1979).
[81] A. Y. Ignatiev, N. V. Krasnikov, V. A. Kuzmin, and A. N. Tavkhelidze, Phys. Lett. **76B**, 436 (1978).
[82] S. Weinberg, Phys. Rev. Lett. **42**, 850 (1979).
[83] A. Maleknejad, Phys. Rev. D **104**, 083518 (2021).
[84] Y. Akamatsu and N. Yamamoto, Phys. Rev. Lett. **111**, 052002 (2013).
[85] M. Flanz, E. A. Paschos, and U. Sarkar, Phys. Lett. B **345**, 248 (1995); **384**, 487(E) (1996); **382**, 447(E) (1996).
[86] L. Covi, E. Roulet, and F. Vissani, Phys. Lett. B **384**, 169 (1996).
[87] W. Buchmuller and M. Plumacher, Phys. Lett. B **431**, 354 (1998).
[88] S. Davidson and A. Ibarra, Phys. Lett. B **535**, 25 (2002).
[89] C. J. A. P. Martins and E. P. S. Shellard, Phys. Rev. D **53**, R575 (1996).
[90] C. J. A. P. Martins and E. P. S. Shellard, Phys. Rev. D **54**, 2535 (1996).
[91] C. J. A. P. Martins and E. P. S. Shellard, Phys. Rev. D **65**, 043514 (2002).
[92] A. Vilenkin, Phys. Lett. **107B**, 47 (1981).
[93] N. Turok, Nucl. Phys. **B242**, 520 (1984).
[94] J. M. Quashnock and D. N. Spergel, Phys. Rev. D **42**, 2505 (1990).
[95] J. J. Blanco-Pillado, K. D. Olum, and B. Shlaer, Phys. Rev. D **89**, 023512 (2014).
[96] J. J. Blanco-Pillado and K. D. Olum, Phys. Rev. D **96**, 104046 (2017).

[97] S. Blasi, V. Brdar, and K. Schmitz, Phys. Rev. Res. **2**, 043321 (2020).
[98] S. Blasi, V. Brdar, and K. Schmitz, Phys. Rev. Lett. **126**, 041305 (2021).
[99] The electric flux may cause an instability for a and A_i through the Chern-Simons coupling [100]. Even though our setup has some differences from theirs, this effect still exists in our case and causes oscillating profiles of a and A_i inside the ϕ_1 strings. However, the inner profiles are not crucial for the stability because the electric charge is fixed by the linking number unless the strings do not have overlap.
[100] H. Ooguri and M. Oshikawa, Phys. Rev. Lett. **108**, 161803 (2012).
[101] R. L. Workman *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2022**, 083C01 (2022).
[102] S. Weinberg, Phys. Rev. Lett. **40**, 223 (1978).
[103] F. Wilczek, Phys. Rev. Lett. **40**, 279 (1978).
[104] P. Minkowski, Phys. Lett. **67B**, 421 (1977).
[105] M. Gell-Mann, P. Ramond, and R. Slansky, Conf. Proc. C **790927**, 315 (1979).
[106] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).
[107] T. Yanagida, Conf. Proc. C **7902131**, 95 (1979).
[108] J. E. Kim, Phys. Rev. Lett. **43**, 103 (1979).
[109] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B166**, 493 (1980).
[110] Because of the couplings of the knot soliton with fermions, they could cause instability on the knot soliton such as Schwinger pair production to screen the electric field [111,112]. However, when the typical size of the knot solitons are smaller than the typical length scale of the fermions, the knot soliton behaves like a point particle, and hence the electric field inside it, which produces the repulsive force among the ϕ_1 strings, is hard to be screened by them. In addition, axion strings can be superconducting due to current/charge carriers [113–118]. This, however, does not affect the stability when the size of the knot solitons is much smaller than localization scale of the carriers. The investigation of the parameter space of the fermion couplings for the stability is beyond the scope of this Letter.
[111] P. Agrawal, A. Hook, J. Huang, and G. Marques-Tavares, J. High Energy Phys. 01 (2022) 103.
[112] K. Harigaya, X. Niu, W. Xue, and F. Yang, J. High Energy Phys. 03 (2025) 063.
[113] G. Lazarides and Q. Shafi, Phys. Lett. **151B**, 123 (1985).
[114] A. Iwazaki, Phys. Rev. Lett. **79**, 2927 (1997).
[115] N. Ganoulis and G. Lazarides, Nucl. Phys. **B316**, 443 (1989).
[116] G. Lazarides, C. Panagiotakopoulos, and Q. Shafi, Nucl. Phys. **B296**, 657 (1988).
[117] H. Fukuda, A. V. Manohar, H. Murayama, and O. Telem, J. High Energy Phys. 06 (2021) 052.
[118] Y. Abe, Y. Hamada, and K. Yoshioka, J. High Energy Phys. 06 (2021) 172.
[119] I. G. Irastorza and J. Redondo, Prog. Part. Nucl. Phys. **102**, 89 (2018).
[120] J. H. Chang, R. Essig, and S. D. McDermott, J. High Energy Phys. 09 (2018) 051.
[121] K. Hamaguchi, N. Nagata, K. Yanagi, and J. Zheng, Phys. Rev. D **98**, 103015 (2018).
[122] A. Caputo and G. Raffelt, Proc. Sci. COSMICWISPers (2024) 041 [arXiv:2401.13728].
[123] Giving ϕ_2 a small nonzero charge under $U(1)_{B-L}$, the setup may additionally solve the axion quality problem [124–126].
[124] H. Fukuda, M. Ibe, M. Suzuki, and T. T. Yanagida, Phys. Lett. B **771**, 327 (2017).
[125] M. Ibe, M. Suzuki, and T. T. Yanagida, J. High Energy Phys. 08 (2018) 049.
[126] Y.-C. Qiu, J.-W. Wang, and T. T. Yanagida, Phys. Rev. Lett. **131**, 071802 (2023).
[127] T. W. B. Kibble, Phys. Rep. **67**, 183 (1980).
[128] W. H. Zurek, Nature (London) **317**, 505 (1985).
[129] E. W. Kolb and M. S. Turner, *The Early Universe* (CRC Press in Boca Raton, Florida, 1990), Vol. 69.
[130] M. Kawasaki, K. Kohri, and N. Sugiyama, Phys. Rev. Lett. **82**, 4168 (1999).
[131] M. Kawasaki, K. Kohri, and N. Sugiyama, Phys. Rev. D **62**, 023506 (2000).
[132] S. Hannestad, Phys. Rev. D **70**, 043506 (2004).
[133] T. Asaka, K. Hamaguchi, M. Kawasaki, and T. Yanagida, Phys. Lett. B **464**, 12 (1999).
[134] G. F. Giudice, M. Peloso, A. Riotto, and I. Tkachev, J. High Energy Phys. 08 (1999) 014.
[135] T. Asaka, K. Hamaguchi, M. Kawasaki, and T. Yanagida, Phys. Rev. D **61**, 083512 (2000).
[136] G. Lazarides and Q. Shafi, Phys. Lett. B **258**, 305 (1991).
[137] K. Kumekawa, T. Moroi, and T. Yanagida, Prog. Theor. Phys. **92**, 437 (1994).
[138] N. Aghanim *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A6 (2020); **652**, C4(E) (2021).
[139] A. Pilaftsis, Phys. Rev. D **56**, 5431 (1997).
[140] A. Pilaftsis and T. E. J. Underwood, Nucl. Phys. **B692**, 303 (2004).
[141] Y. Cui, M. Lewicki, D. E. Morrissey, and J. D. Wells, Phys. Rev. D **97**, 123505 (2018).
[142] Y. Cui, M. Lewicki, D. E. Morrissey, and J. D. Wells, J. High Energy Phys. 01 (2019) 081.
[143] Y. Gouttenoire, G. Servant, and P. Simakachorn, J. Cosmol. Astropart. Phys. 07 (2020) 032.
[144] G. Agazie *et al.* (NANOGrav Collaboration), Astrophys. J. Lett. **951**, L8 (2023).
[145] K. Schmitz, J. High Energy Phys. 01 (2021) 097.
[146] S. Kawamura *et al.*, Prog. Theor. Exp. Phys. **2021**, 05A105 (2021).
[147] D. Reitze *et al.*, Bull. Am. Astron. Soc. **51**, 035 (2019).
[148] N. Bartolo *et al.*, J. Cosmol. Astropart. Phys. 12 (2016) 026.
[149] G. Janssen *et al.*, Proc. Sci. AASKA14 (**2015**) 037 [arXiv:1501.00127].