Solvable Nonunitary Fermionic Long-Range Model with Extended Symmetry

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We define and study a long-range version of the XX model, arising as the free-fermion point of the XXZ-type Haldane-Shastry (HS) chain. It has a description via nonunitary fermions, based on the free-fermion Temperley-Lieb algebra, and may also be viewed as an alternating $\mathfrak{gl}(1|1)$ spin chain. Even and odd lengths behave very differently; we focus on odd length. The model is integrable, and we explicitly identify two commuting Hamiltonians. While nonunitary, their spectrum is real by PT symmetry. One Hamiltonian is chiral and quadratic in fermions, while the other is parity invariant and quartic. Their one-particle spectra have two linear branches, realizing a massless relativistic dispersion on the lattice. The appropriate fermionic modes arise from "quasi-translation" symmetry, which replaces ordinary translation symmetry. The model exhibits exclusion statistics, like the isotropic HS chain, with even more "extended symmetry" and larger degeneracies.

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Introduction—Strongly interacting quantum many-body systems lie at the core of condensed-matter physics. In two dimensions such systems exhibit rich collective behaviors, e.g., fractional excitations and spin-charge separation. Several particularly interesting disorder-driven critical phenomena, like the plateau transition in the integer quantum Hall effect, and geometric problems, e.g., polymers or percolation, are inherently nonunitary [1–3]. The few tools available to treat such systems analytically are mostly based on super-spin chains, loop models, and the Heisenberg XXZ chain [4–6]. Despite their integrability, these nonunitary models remain challenging to analyze, as it is not yet well understood how their nonunitary infinite-dimensional symmetries are realized.

Models with *long-range* interactions constitute an important chapter of integrability. Prominent examples are Calogero-Sutherland systems [7] and the associated spin chains [8,9], which are deeply related to matrix models, exclusion statistics, and 2D CFT [10,11]. Long-range spin chains arise in AdS/CFT integrability too [12]. Rather than a Bethe ansatz, such models are tackled via symmetry-based algebraic methods. In particular, the trigonometric spin-Calogero-Sutherland system and the associated Haldane-Shastry (HS) chain have *extended* (Yangian) spin symmetry [13,14] rendering the spectrum very simple and degenerate [14,15]. Yet there are few, if any, examples of *nonunitary* spin chains with extended symmetry to serve as finite discretizations of the nonunitary CFTs with current-algebra symmetry expected for disordered critical systems [16].

Main result—We introduce a new integrable system that can be viewed as a long-range XX model, a long-range model of nonunitary fermions, or a long-range alternating $\mathfrak{gl}(1|1)$ superspin chain. It has (i) a family of conserved charges, (ii) extended symmetry, (iii) an extremely degenerate and simple spectrum.

This model arises as the free-fermion point of the HS chain. Underlying it is a "parent model" whose symmetries and spectrum are understood in detail: the XXZ-type HS chain [14,17,18]. It generalizes the HS chain by breaking the $\mathfrak{su}(2)$ spin symmetry to $\mathfrak{u}(1)$ while preserving its key properties. Crucially, the extended spin symmetry persists [14], where the Yangian is replaced by quantum-affine $\mathfrak{su}(2)$. A deformation parameter q plays the role of the anisotropy $\Delta = (q + q^{-1})/2$ of the Heisenberg XXZ chain, with $q = 1$ the isotropic case. For generic q, the parent model behaves like the HS chain, yet new features appear at roots of unity. Here we consider the important case $q = i$. For the Heisenberg XXZ chain this gives the XX model ($\Delta = 0$), equivalent to free fermions via the Jordan-Wigner transformation. The model introduced in this Letter is the point $q = i$ of the parent model, described from a new fermionic perspective. We combine general knowledge from the parent model with fermionic techniques capturing the special features at $q = i$.

Our model has several striking features. Its properties depend sensitively on the parity of the system size. The parent model's antiferromagnetic (ground)state is nondegenerate for even length and doubly degenerate for odd length. At $q = 1$, this yields two different CFT sectors in the scaling limit [15,19,20]. At $q = i$, the difference for

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finite size is even more dramatic, presumably due to the global symmetry discussed below. In this Letter we focus on an *odd* number of sites. We can give three conserved charges from (i) explicitly. One is a "quasi-translation," replacing the lattice translation since ordinary translation invariance is broken. The second charge is free fermionic, and parity odd (chiral). The third charge has quartic interactions and is parity even, thus naturally assuming the role of Hamiltonian. Like in [21], for $q \neq 1$ extended symmetry (ii) is incompatible with periodicity. Instead, the Hamiltonians are "quasi-periodic,", i.e., they commute with the quasi-translation. Although the model is nonunitary, the spectrum is real by PT invariance, cf. [22]. The reward for having complicated interactions is that this spectrum is extremely simple as in (iii): sums of quasiparticle energies with *linear* dispersions, comprising two branches associated with even and odd mode numbers. Linear dispersions also occur for the (antiperiodic) $1/r$ Hubbard model [23] and spin chains in $\mathrm{AdS}_3/\mathrm{CFT}_2$ integrability [24]. The interaction in the quartic Hamiltonian implements a statistical selection rule that excludes successive occupied mode numbers, originating in the parent model and matching the "motif" description of the HS chain [15]. This selection rule comes with high degeneracies caused by (ii). The extended symmetry, inherited from the parent model, contains $\mathfrak{gl}(1|1)$.

Significance—The HS chain captures the salient properties of the Heisenberg XXX chain, notably its description i n terms of spinons as quasiparticles with fractional statistics. This is particularly useful in the scaling limit in the antiferromagnetic regime, which is captured by the $\mathrm{SU}(2)_{k=1}$ Wess-Zumino-Witten CFT [13,19,20]. The present model, combining exclusion statistics with a fermionic realization, will play a similar role for nonunitary spin chains and their logarithmic CFT limits, which are notoriously hard to analyze. An explicit lattice regularization, with a new kind of realization for the symmetry algebra, is a great theoretical asset.

Methodology—Our results are obtained as follows [25].

Recall that the Heisenberg XXX chain has a hidden algebraic structure, the $\mathfrak{su}(2)$ Yangian, providing both commuting charges and their diagonalization via the algebraic Bethe ansatz. For XXZ the Yangian is q deformed to quantum-affine $\mathfrak{su}(2)$ [28]. At $q = i$ the Jordan-Wigner transformation provides a more direct treatment by fermionic methods. If periodicity is replaced by special boundary conditions, the $\mathfrak{su}(2)$ spin symmetry persists in the guise of $U_q\mathfrak{sl}(2)$, enabling techniques based on the Temperley-Lieb (TL) algebra, both for general q and at $q = i$ [6,21].

For the HS chain, the Yangian plays *another* role: its generators are different, and commute with the Hamiltonian [13], providing extended spin *symmetry* rather than the full spectrum. Instead, an algebraic formalism—using so-called Dunkl operators and "freezing"—allows one to construct the commuting charges, Yangian, and eigenvectors [14,29–31], cf. [32]. This lifts to the XXZ-type level [18], providing the commuting charges, *extended* (quantum-affine) spin symmetry, *explicit* spectrum, and eigenvectors of the parent model.

We leverage this knowledge of the parent model, reviewed in [25], supplemented by fermionic techniques. We extract the first few explicit commuting charges from the parent model by specializing to $q = i$. The TL algebra provides the bridge to Jordan-Wigner-like fermions. We exhibit discrete symmetries, and describe the extended symmetry, and exact spectrum inherited from the parent model. Exploiting the quasi-translation operator, we define fermions that can be Fourier transformed to bring the Hamiltonians to a simple form and describe part of the eigenstates in the corresponding Fock basis.

The model—Consider fermions hopping on a 1D lattice with an *odd* number N sites. The simplest formulation of our model uses *nonunitary* fermionic operators with anticommutation relations [6]

$$\{f_i, f_j^+\} = (-1)^i \delta_{ij}, \qquad \{f_i, f_j\} = \{f_i^+, f_j^+\} = 0. \tag{1}$$

They are related to canonical Jordan-Wigner fermions as $f_j = (-i)^j c_j$, $f_j^+ = (-i)^j c_j^\dagger$. The fs will avoid a proliferation of factors of i and make the symmetries more transparent. From the two-site fermionic operators

$$g_i \equiv f_i + f_{i+1}, \quad g_i^+ = f_i^+ + f_{i+1}^+, \quad 1 \leqslant i < N, \tag{2}$$

we construct the quadratic combinations

$$e_i \equiv g_i^+ g_i, \qquad 1 \leqslant i < N, \tag{3}$$

which obey the free-fermion TL algebra relations

$$e_i^2 = 0, \quad e_i e_{i\pm 1} e_i = e_i, \quad [e_i, e_j] = 0 \ \text{ if } |i-j| > 1. \tag{4}$$

Further define the nested TL commutators [25] (§C)

$$\begin{aligned} e_{[i,j]} &\equiv [[\cdots[e_i, e_{i+1}], \cdots], e_j] \\ &= s_{ij}(g_j^+ g_i + (-1)^{i-j} g_i^+ g_j), \qquad i \neq j, \end{aligned} \tag{5}$$

where $s_{ij} \equiv (-1)^{(i-j)(i+j-1)/2}$, and we set $e_{[i,i]} \equiv e_i$. Note that (5) is bilinear in the fermions (1). Finally, set

$$t_k \equiv \tan\frac{\pi k}{N}, \qquad t_{k,l} \equiv \prod_{i=k}^{l-1} t_i \ (k < l), \qquad t_{k,k} \equiv 1. \tag{6}$$

Then the chiral Hamiltonian reads

$$\mathrm{H}^{\mathrm{L}}=\frac{i}{2}\sum_{1\leqslant i\leqslant j<N}h^{\mathrm{L}}_{ij}e_{[i,j]},$$
$$h^{\mathrm{L}}_{ij}\equiv\sum_{n=j+1}^{N}t_{n-j,n-i}(1-(-1)^{i}t^{2}_{n-i,n}). \tag{7}$$

We do not imply any periodicity of the sites; (7) is not translation invariant. Instead, it commutes with the *quasi-translation* operator

$$\mathrm{G}=(1+t_{N-1}e_{N-1})\cdots(1+t_1e_1),\qquad \mathrm{G}^{N}=1, \tag{8}$$

which takes over the role of the usual lattice translation.

The next charge is a linear combination of anticommutators of the nested commutators (5),

$$\mathrm{H}=-\frac{1}{4N}\sum_{i\leqslant j<k\leqslant l}^{N-1}(h^{\mathrm{L}}_{ij;kl}+h^{\mathrm{R}}_{ij;kl})\{e_{[i,j]},e_{[k,l]}\},$$
$$h^{\mathrm{L}}_{ij;kl}\equiv(-1)^{k-j}\sum_{n(>l)}^{N}t_{n-l,n-j}t_{n-k,n-i}(1-(-1)^{i}t^{2}_{n-i,n}),$$
$$h^{\mathrm{R}}_{ij;kl}\equiv(-1)^{l-j+k-i}h^{\mathrm{L}}_{N-l,N-k;N-j,N-i}. \tag{9}$$

Stemming from the parent model, these quantities, and higher charges that we do not give here, commute [25],

$$[\mathrm{G},\mathrm{H}^{\mathrm{L}}]=[\mathrm{G},\mathrm{H}]=[\mathrm{H}^{\mathrm{L}},\mathrm{H}]=0. \tag{10}$$

Symmetries—The commuting charges (7)–(9) have several transformation properties and symmetries.

Parity—Parity reverses the lattice sites, $\mathrm{P}(f_i)=f_{N+1-i}$. This preserves (1) since N is odd. The TL generators transform as $\mathrm{P}(e_i)=e_{N-i}$. We have

$$\mathrm{P}(\mathrm{H}^{\mathrm{L}})=-\mathrm{H}^{\mathrm{L}},\qquad \mathrm{P}(\mathrm{H})=\mathrm{H},\qquad \mathrm{P}(\mathrm{G})=\mathrm{G}^{-1}. \tag{11}$$

The first of these relations is highly nontrivial [25].

Time reversal—We define time reversal as complex conjugation of coefficients with respect to the Fock basis $f^{+}_{i_1}\cdots f^{+}_{i_M}|\varnothing\rangle$. By using the fs, $\mathrm{T}(e_i)=e_i$ and (7)–(9) have real matrix elements, except for the prefactor in (7). Thus,

$$\mathrm{T}(\mathrm{H}^{\mathrm{L}})=-\mathrm{H}^{\mathrm{L}},\qquad \mathrm{T}(\mathrm{H})=\mathrm{H},\qquad \mathrm{T}(\mathrm{G})=\mathrm{G}. \tag{12}$$

Since the Hamiltonians (and their eigenstates) are PT invariant, their spectrum is real [33–36]. The same is true for the "quasi-momentum" $p=-i\log\mathrm{G}$.

Charge conjugation—The particle-hole transformation

$$\mathrm{C}'(f_i)=f^{+}_i,\qquad \mathrm{C}'(f^{+}_i)=f_i, \tag{13}$$

gives $\mathrm{C}'(e_i)=-e_i$, preserving (4). Including a suitable antilinear transformation U, see [25], gives charge conjugation $\mathrm{C}=\mathrm{C}'\mathrm{U}$. It acts on the conserved charges by

$$\mathrm{C}(\mathrm{H}^{\mathrm{L}})=-\mathrm{H}^{\mathrm{L}},\qquad \mathrm{C}(\mathrm{H})=\mathrm{H},\qquad \mathrm{C}(\mathrm{G})=\mathrm{G}. \tag{14}$$

Global symmetry—The model can be seen as a long-range spin chain with alternating $\mathfrak{gl}(1|1)$ representations. Recall that $\mathfrak{gl}(1|1)$ has two bosonic and two fermionic generators, which we denote by N, E, and $\mathrm{F}_1,\mathrm{F}^{+}_1$, respectively. The nontrivial (anti)commutation relations are

$$[\mathrm{N},\mathrm{F}_1]=-\mathrm{F}_1,\qquad [\mathrm{N},\mathrm{F}^{+}_1]=\mathrm{F}^{+}_1,\qquad \{\mathrm{F}_1,\mathrm{F}^{+}_1\}=\mathrm{E}, \tag{15}$$

and E is central. This is just a fermionic version of the usual spin algebra. Each site i carries a $\mathfrak{gl}(1|1)$ representation generated by f_i, f^{+}_i, the number operator $(-1)^i f^{+}_i f_i$ and central charge $(-1)^i$. From this perspective, our model is a long-range $\mathfrak{gl}(1|1)$ superspin chain. It has a global $\mathfrak{gl}(1|1)$ symmetry generated by

$$\mathrm{F}_1=\sum_{i=1}^{N}f_i,\qquad \mathrm{F}^{+}_1=\sum_{i=1}^{N}f^{+}_i,$$
$$\mathrm{N}=\sum_{i=1}^{N}(-1)^i f^{+}_i f_i,\qquad \mathrm{E}=\sum_{i=1}^{N}(-1)^i=-1, \tag{16}$$

Indeed, these operators commute with all e_i, and thus with the conserved charges (7)–(9).

Since $\mathrm{F}_1^2=(\mathrm{F}^{+}_1)^2=0$, $\mathfrak{gl}(1|1)$ produces fewer descendants than $\mathfrak{su}(2)$ does at $q=1$. This is compensated by additional bosonic generators

$$\mathrm{F}_2=\sum_{i<j}^{N}f_i f_j,\qquad \mathrm{F}^{+}_2=\sum_{i<j}^{N}f^{+}_i f^{+}_j, \tag{17}$$

which commute with the e_i, hence with (7)–(9). Together, (16) and (17) generate the full global-symmetry algebra [5,37]. It is the $U_q\mathfrak{sl}(2)|_{q=i}$ symmetry from the parent model in fermionic language, cf. [6] (Sec. 2.3).

Extended symmetry—The parent model has quantum-affine $\mathfrak{sl}(2)$ symmetry, which underpins its large degeneracies. Its specialization to $q=i$ is tricky and seems absent in the mathematical literature. A detailed study will be performed elsewhere. The extended symmetry is visible in the highly degenerate spectrum.

Exact spectrum—The spectrum and degeneracies of the parent model are known explicitly [14,17,18]. Like for the HS chain, the quantum numbers are "motifs" [15] $\{\mu_m\}$, consisting of integers $1\leqslant\mu_m<N$ increasing as

$$\mu_{m+1}>\mu_m+1,\qquad 1\leqslant m<M. \tag{18}$$

Such a motif labels an M-fermion state with quasimomentum $p=(2\pi/N)\sum_m\mu_m \bmod 2\pi$ setting the eigenvalue e^{ip} of G. Its energy is additive:

$$E^{\mathrm{L}}_{\{\mu_m\}}=\sum_{m=1}^{M}\varepsilon^{\mathrm{L}}_{\mu_m},\qquad E_{\{\mu_m\}}=\sum_{m=1}^{M}\varepsilon_{\mu_m}, \tag{19}$$

with dispersions having two linear branches (Fig. 1):

$$\varepsilon_n^{\mathrm{L}} = \begin{cases} n, & n \text{ even}, \\ n-N, & n \text{ odd}, \end{cases} \qquad \varepsilon_n = |\varepsilon_n^{\mathrm{L}}|. \tag{20}$$

This state has (often many) descendants due to the extended symmetry. Its multiplicity is [13,18] $N+1$ for the empty motif (at $M=0$), and otherwise

$$\mu_1(N-\mu_M)\prod_{m=1}^{M-1}(\mu_{m+1}-\mu_m-1). \tag{21}$$

Given the extremely simple dispersion, further ("accidental") degeneracies between different motifs occur much more often than even for the HS chain.

Fermionic approach—Lacking periodicity, we cannot simply Fourier transform the fs as usual. The key to defining a good basis of fermions is to start at one end of the lattice and use the quasi-translation operator:

$$\Phi_i \equiv \mathrm{G}^{1-i} f_1 \mathrm{G}^{i-1}, \qquad \Phi_i^+ \equiv \mathrm{G}^{1-i} f_1^+ \mathrm{G}^{i-1}. \tag{22}$$

These dressed fermions, reminiscent of [38], *are* periodic,

$$\Phi_{i+N} = \Phi_i, \qquad \Phi_{i+N}^+ = \Phi_i^+, \tag{23}$$

and obey nonlocal anticommutation relations [25]

$$\{\Phi_i, \Phi_j^+\} = -(1+t_{j-i}), \qquad \{\Phi_i, \Phi_j\} = \{\Phi_i^+, \Phi_j^+\} = 0. \tag{24}$$

The nontrivial relation only depends on the distance. Because of (23) we may Fourier transform (22) as usual. Set $a_0 \equiv i$ and $a_n \equiv i^{n+1/2}$ else. The *rescaled* Fourier modes

$$\tilde{\Psi}_n \equiv \frac{a_n}{N}\sum_{j=1}^{N} \mathrm{e}^{-2i\pi nj/N}\Phi_j, \qquad \tilde{\Psi}_n^+ \equiv \frac{a_n}{N}\sum_{j=1}^{N} \mathrm{e}^{2i\pi nj/N}\Phi_j^+, \tag{25}$$

obey canonical anticommutation relations

$$\{\tilde{\Psi}_n, \tilde{\Psi}_m^+\} = \delta_{nm}, \qquad \{\tilde{\Psi}_n, \tilde{\Psi}_m\} = \{\tilde{\Psi}_n^+, \tilde{\Psi}_m^+\} = 0. \tag{26}$$

They are covariant under quasi-translations in the sense

$$\mathrm{G}\tilde{\Psi}_n \mathrm{G}^{-1} = \mathrm{e}^{-2i\pi n/N}\tilde{\Psi}_n, \quad \mathrm{G}\tilde{\Psi}_n^+ \mathrm{G}^{-1} = \mathrm{e}^{2i\pi n/N}\tilde{\Psi}_n^+. \tag{27}$$

The relation to the original fermions is strikingly simple. The zero modes commute with the Hamiltonians: they are just the fermionic $\mathfrak{gl}(1|1)$ generators from (16) [25],

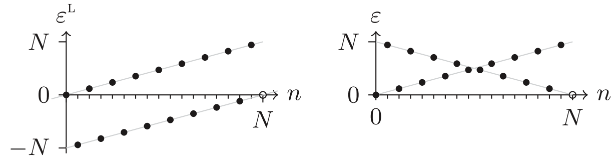

FIG. 1. The dispersion relations (20) alternate between two linear branches, realizing chiral and "full" (up to a shift) massless relativistic dispersions on the lattice.

$$\frac{1}{a_0}\tilde{\Psi}_0 = \sum_{i=1}^{N} f_i = \mathrm{F}_1, \qquad \frac{1}{a_0}\tilde{\Psi}_0^+ = \sum_{i=1}^{N} f_i^+ = \mathrm{F}_1^+. \tag{28}$$

The other modes are explicit *linear* combinations of the two-site fermions (2), with coefficients given in [25], Sec. C2. In these terms, H^{L} is diagonal:

$$\mathrm{H}^{\mathrm{L}} = \sum_{n=1}^{N-1} \varepsilon_n^{\mathrm{L}} \tilde{\Psi}_n^+ \tilde{\Psi}_n. \tag{29}$$

Numerics for low N confirms the equality with (7). If $|\varnothing\rangle$ is the fermionic vacuum, then by (26) the Fock states

$$|n_1, \ldots, n_M\rangle \equiv \tilde{\Psi}_{n_1}^+ \ldots \tilde{\Psi}_{n_M}^+ |\varnothing\rangle, \tag{30}$$

form an H^{L} eigenbasis labeled by all 2^N fermionic mode numbers $0 \leqslant n_1 < \ldots < n_M < N$. The quasi-momentum is $p = (2\pi/N)\sum_m n_m \bmod 2\pi$, and the chiral energy $E_{\{n_m\}}^{\mathrm{L}} = \sum_m \varepsilon_{n_m}^{\mathrm{L}}$ matches (19)–(20) when $\{n_m\}$ is a motif.

Next, (9) takes the quartic form [25]

$$\mathrm{H} = \sum_{n=1}^{N-1} \varepsilon_n \tilde{\Psi}_n^+ \tilde{\Psi}_n + \sum_{\substack{1\leqslant m<n<N \\ 1\leqslant r<s<N}} \tilde{V}_{mn;rs} \tilde{\Psi}_m^+ \tilde{\Psi}_n^+ \tilde{\Psi}_r \tilde{\Psi}_s. \tag{31}$$

The commutation (10) only allows $\tilde{V}_{mn;rs} \neq 0$ if [25] the quasi-momentum and chiral energy are conserved:

$$m+n = r+s \mod N, \qquad \varepsilon_m^{\mathrm{L}} + \varepsilon_n^{\mathrm{L}} = \varepsilon_r^{\mathrm{L}} + \varepsilon_s^{\mathrm{L}}. \tag{32}$$

Numerics for odd $N \leqslant 9$ suggest the stronger selection rule that $m+n = r+s$ be odd, with nonzero values $\tilde{V} = \pm 4$ determined by $\tilde{V}_{mn;rs} = \tilde{V}_{rs;mn}$ and

$$\tilde{V}_{mn;m+k,n-k} = (-1)^{k+1} 4\delta_{m\,\mathrm{odd}}, \qquad 0 \leqslant 2k < n-m. \tag{33}$$

For one-particle states $|n\rangle$ only the quadratic part of (31) contributes, reproducing the nonchiral dispersion (20). The quartic part implements the statistical repulsion rule (18): H is genuinely interacting. Thus, (30) are generally *not* eigenstates of H. In [25] Sec. D we illustrate this for the two-fermion spectrum, showing how $\tilde{V}$ "squeezes" adjacent modes to extended-symmetry descendants. E.g., for $N=5$ we know from the parent model that $|1,2\rangle$ is an extended-symmetry descendant of the motif $\{3\}$. This matches $\varepsilon_n^{\mathrm{L}} + \varepsilon_{n+1}^{\mathrm{L}} = \varepsilon_{2n+1 \bmod N}^{\mathrm{L}}$ (cf. Fig. 1), yet for H relies on the contribution of $\tilde{V}_{12;12} = -4$ to $\varepsilon_1 + \varepsilon_2 + \tilde{V}_{12;12} = \varepsilon_3$. A fermionic description of the full spectrum requires a deeper understanding of the extended symmetry.

Outlook—We obtained and analysed a long-range fermionic model with extended symmetry from the XXZ-type HS chain ("parent model") at $q = i$. It can be also viewed as a long-range alternating superspin chain. Periodicity, being incompatible with the symmetry, is modified to quasi-translation invariance. As for Heisenberg XXZ, we get a free-fermion Hamiltonian, H^{L}. Albeit quadratic in fermions from the start, its explicit diagonalization relies on quasi-translated fermions akin to [38]. Another conserved charge, H, has a four-fermion interaction realizing the statistical repulsion known from the isotropic HS chain. Both Hamiltonians have a Weyl-like spectrum on the lattice. Together, these features render the model rather simple, although a full understanding requires an explicit fermionic realization of the extended symmetry. This, and a systematic construction of the eigenvectors, which are known for the parent model, is left for future work.

In spin chains, even or odd length may lead to different properties due to the excitations' topological nature. The distinction is more pronounced for alternating $\mathfrak{gl}(1|1)$ spin chains, cf. [6]. Presently, it becomes extreme due to a pole in the parent model's long-range potential. For even N, the parent Hamiltonian *diverges* as $q \to i$, and regularization sets all energies to *zero*. Yet, the wave functions remain nontrivial, and numerics indicates Jordan blocks up to size $N/2 + 1$. While indecomposable representations are expected at central charge $\mathrm{E} = 0$, their dimensions signal that these are not just zigzag modules for systems with merely global symmetry [39]. We will soon report on even N [40]. A spin chain with a similar spectrum arose in AdS/CFT integrability [41].

Another important direction is the continuum limit, where we expect conformal invariance. Explicit identification of the CFT requires determining the extended symmetry. The isotropic HS chain suggests Kac-Moody symmetry in the CFT limit, perhaps level-1 $\mathfrak{gl}(1|1)$ [42]. It will be interesting to find the continuum counterpart of H^{L}, reminding the Virasoro generator L_0, and see what happens with the dispersion relation's staggering. The relativistic-like dispersions for odd N seems well adapted to the continuum limit, albeit at odds with the vanishing spectrum for even N. In the scaling limit, the fs will yield symplectic fermions, but the difference between even and odd length we see goes beyond that between the Ramond and Neveu-Schwarz sectors, requiring further insight.

For Heisenberg XXZ, other root-of-unity cases, notably $q^3 = 1$ [43], are special too. Their analogs for the parent model will also exhibit extended symmetry, quasi-translation invariance, nonunitarity and statistical level repulsion. More generally, it would be exciting to study root-of-unity points of the XXZ-type Inozemtsev chain [44], which interpolates between a quasiperiodic Heisenberg XXZ chain and our parent model.

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