Thermal Bootstrap for the Critical O(N) Model

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We propose a numerical method to estimate one-point functions and the free-energy density of conformal field theories at finite temperature by solving the Kubo-Martin-Schwinger condition for the two-point functions of identical scalars. We apply the method for the critical O(N) model for $N = 1, 2, 3$ in $3 \leq d \leq 4$. We find agreement with known results from Monte Carlo simulations and previous results for the 3D Ising model, and we provide new predictions for $N = 2, 3$.

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Introduction and summary—Finite-temperature phenomena in conformal field theories (CFTs) can be studied by placing the theory on the geometry $S^1_\beta \times \mathbb{R}^{d-1}$, where $\beta = 1/T$ is the inverse temperature. Thermal dynamics plays a crucial role, as quantum critical points in experimental systems occur at nonzero temperatures [1,2]. Additionally, it is essential to study CFTs at finite temperature to gain insights on anti–de Sitter black holes in the quantum regime [3].

The success of the conformal bootstrap in constraining zero-temperature CFT data (see, e.g., the reviews [4–6]), namely, conformal dimensions and structure constants, naturally raises the question of whether similar techniques can be applied to thermal CFTs [7,8]. Since the operator product expansion (OPE) of the original CFT remains valid locally [9], thermal correlation functions can be expressed in terms of zero-temperature CFT data and thermal one-point functions. The goal of the thermal bootstrap program is to compute these observables employing the zero-temperature data as an input, and the Kubo-Martin-Schwinger (KMS) condition [10,11], namely, the periodicity along the thermal circle, as a consistency constraint. Among all the operators, a special role is played by the stress-energy tensor, since its thermal one-point function is closely related to the free-energy density of the system [8,12].

In this Letter, we introduce a new efficient method to numerically estimate thermal one-point functions. We impose the KMS condition on a thermal two-point function of identical scalars near the KMS fixed point [13]. This generates an infinite set of equations with an infinite number of unknowns. The novelty of this work is to analytically approximate the contribution of heavy operators using an improved version of the Tauberian asymptotics proposed in [14], reducing the system to a finite set of unknowns.

The method can be tested in 4D free scalar theory, 2D Ising model and in the large N limit of the O(N) model, where numerical estimations can be compared with analytical results [15]. In the following, we apply it in the strongly coupled regimes of the critical O(N) models for $N = 1, 2, 3$. These correspond to the critical Ising model ($N = 1$), the XY model ($N = 2$), and the Heisenberg model ($N = 3$), which are relevant for understanding ferromagnetism and other physical phenomena [16–19]. Our key results are the free-energy density in $3 \leq d \leq 4$ (Fig. 1), the two-point function of the lightest scalar in the critical 3D Ising model (Fig. 2), and the one-point functions of several operators in the critical O(1), O(2), and O(3) models

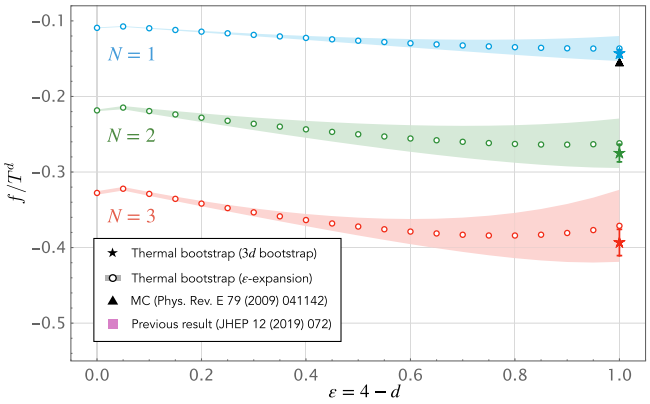

FIG. 1. Free-energy density of the critical O(N) models for $N = 1, 2, 3$ in $3 \leq d \leq 4$ (i.e., $0 \leq \varepsilon \leq 1$).

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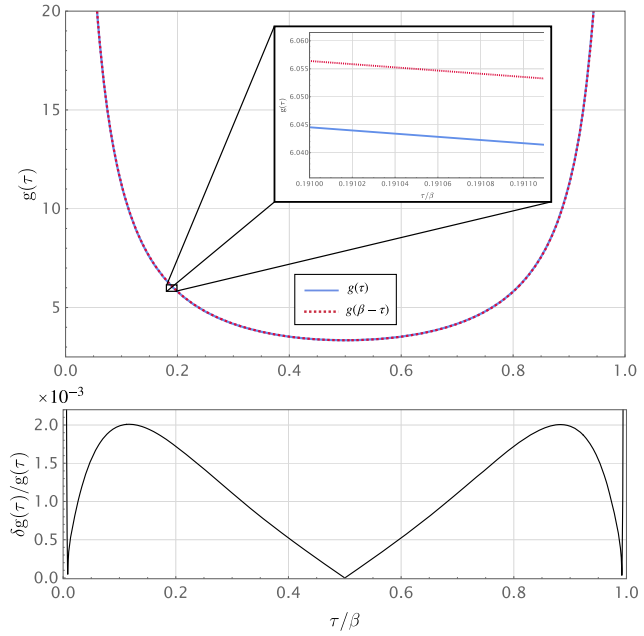

FIG. 2. The thermal two-point function $g(\tau)$ is shown alongside its KMS equivalent $g(\beta-\tau)$ in the Ising model. The second plot shows the difference between the two curves in the main plot. We observe an excellent agreement in the region around the KMS fixed point ($\tau/\beta = 1/2$).

(Figs. 3 and 4). In the case of the 3D Ising model, our results can be compared with previous numerical estimates [20] and Monte Carlo (MC) simulations [21–23], confirming the validity of our method [24]. The predictions for $N = 2, 3$ are new and could, in principle, be tested through further Monte Carlo simulations or experiments [25].

Thermal bootstrap—The starting point of our analysis is the KMS condition. For the two-point function of identical scalar operators $g(\tau) = \langle\phi(\tau)\phi(0)\rangle_\beta$, where the spatial distance between the two operators is set to zero, the

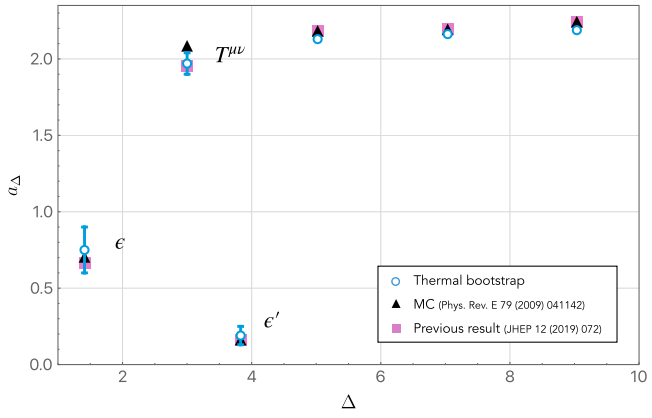

FIG. 3. OPE coefficients for the lightest operators of the critical 3D Ising model. The points with no error bar associated correspond to analytical Tauberian predictions, whose error is not the object of study of this Letter.

KMS condition results into a tower of constraints that take the form

$$0 = \frac{\partial^m}{\partial\tau^m}\left[g\left(\frac{\beta}{2}+\tau\right) - g\left(\frac{\beta}{2}-\tau\right)\right]_{\tau=0}, \tag{1}$$

where $m \in 2\mathbb{N}+1$. These constraints can be expressed as a set of sum rules [14]

$$\sum_\Delta a_\Delta \mathrm{F}(\Delta, \Delta_\phi, m) = 0, \tag{2}$$

where the sum is performed over all the operators in the OPE between the two operators ϕ. The kernel F, defined in Eq. (9) in [14], depends solely on zero-temperature CFT data, which we treat as input. Meanwhile, the coefficients a_Δ encode the thermal dynamical information

$$a_\Delta = \sum_{\mathcal{O}_\Delta} \frac{b_\mathcal{O} f_{\phi\phi\mathcal{O}}}{c_\mathcal{O}} \frac{J!}{2^J(\nu)_J} C_J^\nu(1), \tag{3}$$

where $\nu = (d-2)/2$, $C_J^{(\nu)}$ is a Gegenbauer polynomial and the sum is performed over operators sharing the same scaling dimension, but with different spins. Here, the coefficients $f_{\phi\phi\mathcal{O}}$ and $c_\mathcal{O}$ correspond, respectively, to the structure constants and to the two-point function normalization of the operator $\mathcal{O}$ at zero temperature. $b_\mathcal{O}$ is the thermal one-point function coefficient defined via [8,26]

$$\langle\mathcal{O}_\Delta^{\mu_1\ldots\mu_J}\rangle_\beta = \frac{b_\mathcal{O}}{\beta^\Delta}(e^{\mu_1}\ldots e^{\mu_J} - \text{traces}). \tag{4}$$

The ultimate goal of the thermal bootstrap program is to compute these observables completing the set of thermal CFT data.

In order to solve the constraints (2), a naive approach consists in truncating the sum at a cutoff dimension $\Delta_{\max}$. However, this approach fails, as the contribution of the heavy operators cannot be discarded [27]. This issue can be circumvented by approximating the tail of heavy operators using the asymptotic behavior of the coefficients a_Δ [14]

$$a_\Delta^{\text{heavy}} = \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi+1)}\delta\Delta\left(1+\frac{c_1}{\Delta}+\ldots\right). \tag{5}$$

Here, $\delta\Delta$ represents the gap between the scaling dimension Δ and the scaling dimension of the operator below it in the OPE spectrum. The coefficient c_1 is theory dependent and corresponds to the first correction to the leading behavior. Let us comment that, in order to derive (5), it is necessary to add an analyticity assumption on a_Δ, since the Tauberian theorem fixes only the leading term [14]. Moreover, note that the power of Δ in the first correction is universal, but those of the subleading terms are theory dependent and currently unknown. Determining them is an important next

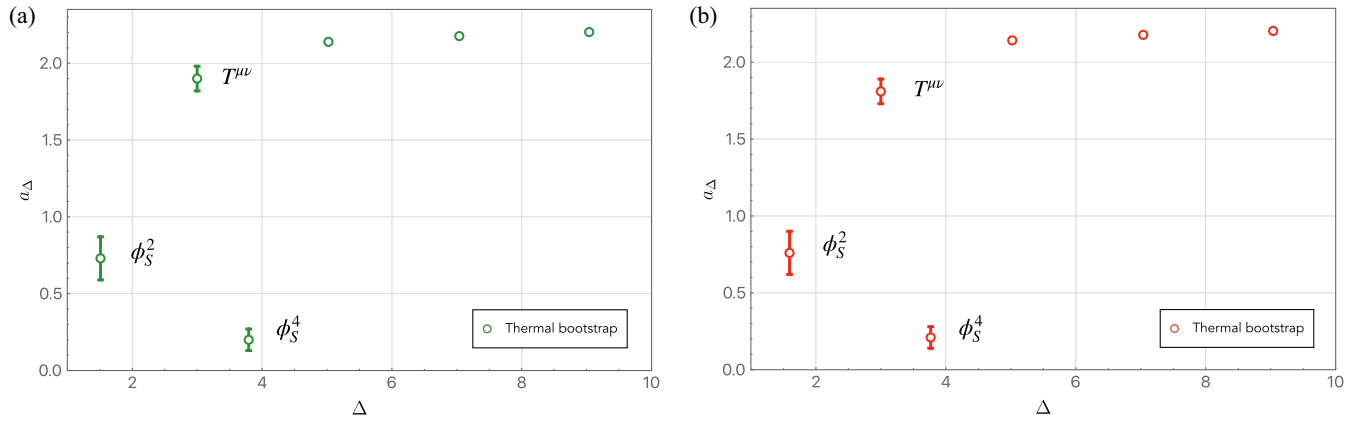

FIG. 4. The two plots present the results for the OPE coefficients associated to the lightest operators of the OPE spectrum for the O(2) (a) and O(3) (b) models. The points with no error bar associated correspond to analytical Tauberian predictions, whose error is not the object of study of this Letter.

goal. The constraints of Eq. (2) can be split into two terms

$$\mathtt{f}(m) = \sum_{\Delta \le \Delta_{\max}} a_\Delta \mathtt{F}(\Delta, \Delta_\phi, m) + \sum_{\Delta > \Delta_{\max}} a_\Delta^{\text{heavy}} \mathtt{F}(\Delta, \Delta_\phi, m). \tag{6}$$

We do not have access to the spectrum in the heavy sector, and for this reason, a further approximation is required. In this Letter, we restrict our consideration to the leading trajectories of operators $[\phi\phi]_{n,\ell}$ in the second term of Eq. (6), which, by channel duality, correspond to the identity [34]. These operators take the classical form $\phi\partial^\ell\Box^n\phi$, and their conformal dimensions can be approximated by the mean-field theory result $\Delta_{n,\ell} = 2\Delta_\phi + 2n + \ell$. Thus, there are two sources of error: one arises from the omission of subleading trajectories, the other from the anomalous dimensions of these operators. The former is negligible with respect to the latter: we estimate both in [15]. In this approximation, only a finite number of unknown coefficients are left: the coefficients a_Δ associated with the light operators $\Delta \le \Delta_{\max}$, and the corrections to the leading behavior (5), namely $c_1, \ldots$. The constraints (2) can be formulated as the minimization of the cost function

$$\eta(\{\omega_i\}) = \sum_{m \le m_{\max}} \omega_m \mathtt{f}(m)^2, \tag{7}$$

where $m_{\max}$ determines the maximum number of derivatives considered and $\omega_i \in (0, 1)$ is a set of random number weights, which allows us to test the numerical stability of the algorithm as previously done, e.g., in [31]. The minimization process results in estimations for the unknown parameters, which are affected by numerical errors stemming from two contributions: (i) a statistical error, estimated by the square root of the variance over multiple runs of the minimization of (7); (ii) a systematic error, due to the approximation of the contribution of the heavy operators using (5), estimated in [15].

The uncertainties in the zero-temperature input data propagate into the thermal one-point functions. When using the 3D bootstrap results as input, these errors remain negligible. However, when the zero-temperature conformal data comes from the ε expansion, the associated error increases with ε. This effect is estimated in [15] and illustrated in Fig. 1 for $0 \le \varepsilon \le 1$. The errors presented in this Letter should be regarded as estimations rather than rigorous error bounds, adopting the terminology of [38].

The free-energy density of the system is determined by the one-point function coefficient of the stress-energy tensor through $f = b_T/d$, with d the number of spacetime dimensions [8]. The structure constant $f_{\phi\phi T}$, appearing in (3), is fixed by the Ward identity [39], and therefore,

$$f = -a_d \frac{\Gamma(d/2)}{2\pi^{d/2}(d-1)\Delta_\phi} \frac{c_T}{c_{T,\text{free}}}, \tag{8}$$

where $c_{T,\text{free}} = d\Gamma(d/2)^2/[4\pi^d(d-1)]$. The method presented here can be tested on simple examples and is found to produce accurate results for the free scalar field in 4D, the 2D Ising model, and the O(N) model at large N [15]. More details and illustrative examples can be found in [40].

Ising, XY, and Heisenberg models—The method presented above can be used to study the O(N) model in $3 \le d \le 4$. We consider, in (2), the lightest scalar ϕ_i ($i = 1, \ldots, N$) as external operator. We use two distinct sets of zero-temperature input: the results obtained from the ε expansion [41] and gathered in [42], and the results from the (zero-temperature) 3D bootstrap, given in [43–48] for $N = 1, 2, 3$. To approximate the tail of heavy operators, we consider only the operators $[\phi\phi]_{n,\ell}$ in the second term of Eq. (6), corresponding to the identity by channel duality. We consider the contribution of the identity operator and of the three lightest operators in the spectrum, and one

TABLE I. OPE coefficients a_Δ of light operators in the 3D Ising model, compared to Monte Carlo results (MC) and previous results (PR). The value for the Tauberian correction is $c_1 \sim -0.065$, for which the error is negligible.

$\mathcal{O}$	$\Delta_\mathcal{O}$ [44,45]	This Letter	MC [21–23]	PR [20]
ϵ	1.412 625(10)	0.75(15)	0.711(3)	0.672(74)
$T_{\mu\nu}$	3	1.97(7)	2.092(13)	1.96(2)
ϵ'	3.829 51(61)	0.19(6)	0.17(2)	0.17(2)

correction to the Tauberian approximation [15]. This results in four unknowns: the three nontrivial one-point functions and the correction to the Tauberian approximation c_1. All our calculations are performed by setting $m_{\max} = 7$ in (7), which corresponds to having four constraints of the type (2). Increasing $m_{\max}$ would result in an increased error from the Tauberian approximation, which would, in turn, require the inclusion of additional corrections in (5).

We gather our results for the 3D Ising model ($N = 1$) in Table I and compare them to the Monte Carlo values and the previous results, which relied on a different thermal bootstrap approach. All our results are consistent with previous bootstrap findings [8]. Both our results and those of [8] align with Monte Carlo predictions, with the sole exception of the thermal OPE coefficient of the stress-energy tensor. [49] As already observed in [8], the value of the stress-energy tensor contribution is close to the large N approximation, where $b_T \sim -0.459N$ and $a_T \sim 1.923$ [51,52]. The results obtained with the ε expansion and the 3D conformal bootstrap as an input are shown in Fig. 1 for the free energy density. Notice that the error estimated on the coefficient a_d propagates nontrivially on the free energy; in particular, it is multiplied by N. We also estimated the thermal two-point function $g(\tau)$ by inputting the numerical results in the OPE: Fig. 2 shows a comparison between the two KMS-dual channels. The results for the OPE coefficients are presented in Fig. 3.

Also, for the XY model ($N = 2$), many zero-temperature results have been obtained through the ε expansion and the conformal bootstrap. We find the following predictions for the OPE coefficients in 3D:

$$a_{\phi_s^2} = 0.73(14), \qquad (\Delta_{\phi_s^2} = 1.51136(22)), \tag{9}$$

$$a_T = 1.90(8), \qquad (\Delta_T = 3), \tag{10}$$

$$a_{\phi_s^4} = 0.20(7), \qquad (\Delta_{\phi_s^4} = 3.794(8)). \tag{11}$$

The value for the Tauberian correction is $c_1 \sim -0.0539$, for which the error is negligible. The free-energy density can be calculated using Eq. (8), and the results are shown in Fig. 1.

We performed the same calculations for the Heisenberg model ($N = 3$), using the input from the ε expansion and the conformal bootstrap. We obtain the following results for the OPE coefficients in 3D:

$$a_{\phi_s^2} = 0.76(14), \qquad (\Delta_{\phi_s^2} = 1.59489(59)), \tag{12}$$

$$a_T = 1.81(8), \qquad (\Delta_T = 3), \tag{13}$$

$$a_{\phi_s^4} = 0.21(7), \qquad (\Delta_{\phi_s^4} = 3.7668(100)). \tag{14}$$

The value for the Tauberian correction is $c_1 \sim -0.0471$, for which the error is negligible. As for the other cases, we show the free-energy density in Fig. 1. The results for the OPE coefficients of the XY and the Heisenberg models are presented in Fig. 4. Note, again, that the values of a_T for these models closely follow the large N prediction. The asymptotic behavior is not strictly monotonic, nonetheless, the qualitative dependence of a_T on N aligns with the findings of [50].

Discussion—In this Letter, we propose a numerical method for computing thermal OPE coefficients, which we apply to the critical O(N) models for $N = 1, 2, 3$. In particular, we extract the free-energy density of the system in $3 \leq d \leq 4$ as well as the OPE coefficients of the lightest operators. In the case of the 3D Ising model ($N = 1$), our results can be compared with previous studies, while for $N = 2, 3$, we produce new predictions.

There are several directions to explore following this work. The methods presented here can be applied to different models. Motivated by recent progress in the context of holographic black holes [54–58], it would be interesting to study the thermal $\mathcal{N} = 4$ super Yang-Mills and ABJM theories, for which a plethora of zero-temperature CFT data is available in the literature [59–63]. Moreover, it was shown in [12] that the bootstrap problem in the presence of a temporal line defect is very similar to the one discussed in this Letter. The exploration of this direction is crucial because of low-energy applications [64–66] and holographic interpretations [3]. In the case of the Maldacena-Wilson line [67], a great amount of CFT data has been extracted recently [68,69]. Furthermore, we are currently working on improving the precision on the numerical results in tandem with developing an analytical approach which will provide the next corrections in the Tauberian approximation [70].

The strategy of this Letter could be adapted to all these configurations, which also provide a good stage for improving the precision on the numerical results.

Finally, recently, many different directions to study finite temperature effects in CFTs were proposed [71–79]. It would be interesting to compare and possibly incorporate these techniques with the method proposed in this Letter.

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