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Chapter Article tagline: update of previous edition, reprint.

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Abstract

Monte Carlo event generators are in a modern terminology the digital twins of collider-based particle physics experiment. We give an introduction into the application of MC generators for particle physics, discuss their different components each simmulating a different aspect of physics. The main part is the hard scattering process, sampled over a commplicated phase space of kinematic variables, followed by simulation of strong and electromagnetic radiation in parton showers, and hadronization of the end products of the shower into mesons and baryons. These then undergo several levels of decays into those particles that are measured in the detectors – mostly electrons, photons, pions and kaons. For each step, the main techniques will be described and explained.

Keywords: Monte Carlo generators, Parton shower, Hadronization, phase space sampling, higher order corrections

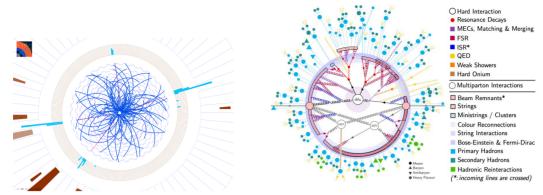


Fig. 1: Particle physics event display (from the CMS detector at the Large Hadron Collider (LHC) (left), and different components of its digital twin, the MC generator similation (right).

Objectives

- Monte Carlo event generators are introduced as the main working horses of collider physics experiments in particle
 physics.
- · Monte Carlo integration, sampling of multi-dimensional phase spaces and event generation are discussed.
- The generation of amplitudes as building blocks and their higher order extensions are introduced.
- The simulation of proton and lepton distribution functions is discussed.
- Parton showers and their matching to fixed order perturbation theory and hadronization are explained in detail.
- Some special topics, software development, performance and efficiency aspects are listed.

1 Introduction

Monte Carlo (MC) event generators take all of our knowledge about the fundamental interactions and matter in particle physics, quantum chromodynamics (QCD) for strong interactions and electroweak (EW) interactions (weak interactions and quantum electrodynamics (QED), and encode them in software tools to simulate collider or fixed-target experiments. There are big synergies with MC generators for air showers in cosmic ray physics (in terms of physics) and detector simulation (in terms of methods and also physics); both topics are beyond the scope of this section. Fig. 1 shows an event display of a particle physics (a Higgs candidate event from the CMS detector at the Large Hadron Collider LHC) on the left, while the right hand side shows the part simulated by a MC generator. All components on the left are within the small white circle in the center (and partially in the blue bent lines in the tracker volumes). The sketch on the right of Fig. 1 has been designed for hadron collisions, but can easily be transferred to electron-positron, muon or electron-hadron collisions, respectively.

We will first discuss the rationale of Monte Carlo generators for sampling high dimensional phase spaces and do weighted and unweighted event generation in Sec. 2. The core physics part is the so called hard scattering process, e.g. $gg \to H \to \gamma\gamma$ for the discovery channel of the Higgs boson at the LHC in 2012. In Sec. 3 we will discuss methods to generate code for matrix elements at fixed perturbative order (tree-level and beyond) in an effective manner. In addition, phase space integration over processes with high multiplicities of final states with complicated kinematical cuts will be explained. Furthermore, we will connect this to methods to analytically or semi-numerically resum perturbative corrections to all order. Then, in Sec. 4 we will explain the inclusion of parton distribution functions (PDFs) for hadron beams as well as perturbatively calculable lepton PDFs for the resummation of initial-state QED corrections in case of lepton beams. Also, we discuss the simulation of beam effects in MC generators, like beamstrahlung, beam energy spread, crossing angle and polarization of beams. The next section, Sec. 5 is devoted to parton showers, the simulation of the emission of up to a hundred strongly interacting partons and a single-digit number of photons. We will touch upon weak showers and discuss the combination of parton showers with fixed-order simulations in so-called matching procedures. After that, in Sec. 6 we explain how the final-state partons will be combined into bound states of mesons and baryons via hadronization, also called fragmentation.

In particle physics, there is not a single MC generator that natively accommodates all different aspects listed here, but there are a few multi-purpose generators, a term which is not exactly well-defined, sometimes restricted to those that simulate (par-

| Final state X | particles | dim(partonic) | dim(with PDFs) |
|---------------------------------|-----------|---------------|----------------|
| $\mu^+\mu^-$ | 2 | 2 | 4 |
| jjj | 3 | 5 | 7 |
| $\ell\ell bb$ | 4 | 8 | 10 |
| ℓℓbbj ℓνℓνbb | 5 | 11 | 13 |
| $\ell \nu \ell \nu b b$ | 6 | 14 | 16 |
| | | | |
| $\ell \nu \ell \nu b b j j j j$ | 10 | 26 | 28 |

Table 1: Dimensionality of the phase-space integration of several benchmark processes in particle physics. The left of the two dimensionality columns assumes a partonic process, while the rightmost column takes two variables from the energy fractions of the beam PDFs into account.

ton shower and) fragmentation, sometimes to multi-leg (parton-level) generators that can generate arbitrary processes of (almost) arbitrary multiplicity. The most well-known and most applied ones are HERWIG [1–3], MADGRAPGH5_AMC@NLO [4–6], PYTHIA [7–9], SHERPA [10–12], WHIZARD [13–15], with a medium number of additional generators like e.g. POWHEGBOX [16–18], COMPHEP/CALCHEP [19–21], GENEVA [22], PHOTOS [23–25], KKMC [26] and many more that have some specific focus on the type of interactions, the type of collider, or a particular type of perturbative corrections. More details will be found in the sections below. As described above, these generators cover all of particle physics, so such a pedagogical review clearly has to focus on certain key topics and cover some special aspects only in a very short way.

There are many excellent overview articles on MC event generators, both from lectures [27–29], and within the context of particle physics strategy updates [30]. Besides there extremely prominent role in the data analysis and interpretation of LHC data, in the past decade, MC generators have played a pivotal role as digital twins for the planning of the next generation of (e^+e^-) colliders [31–40] as well as muon colliders [41–43].

2 The rationale of Monte Carlo generators

2.1 The need of Monte Carlo integration methods

The main task of theoretical particle collider physics is to model $2 \rightarrow n$ scattering processes according to the differential cross section formula:

$$d\sigma_{\alpha \to \beta} = \frac{|\mathcal{M}_{\beta \alpha}|^2}{4\sqrt{(p_{a,1} \cdot p_{\alpha,2})^2 - m_{\alpha,1}^2 m_{\alpha,2}^2}} \left(\prod_{i=1}^n \frac{d^3 q_\beta}{(2\pi)^3 2q_{\beta,i}^0} \right) (2\pi)^4 \delta^4((p_{\alpha,1} + p_{\alpha,2} - q_{\beta,1} - \dots - q_{\beta,n}),$$
(1)

consisting of three components: the (squared) matrix element $\mathcal{M}_{\beta\alpha}$, the kinematic flux factor in the denominator and the integral over the kinematic final particle phase space variables. For an n-particle final state, this integral has dimension 3n-4 (and two more variables in case of convolutions over initial state parton distribution functions, PDFs), cf. Table 1. While there are explicit formulae for low-dimensional phase spaces [44], for four or more final state particles only numerical integration methods can be applied. Numerical methods like Newton or Gauss integrations do not work very well: (i) the dimensionality can become very high, with 6-10 particles for complicated processes, (ii) the squared matrix elements has singular or nearly singular structures due to propagator denominators close to their mass shells, and (iii) experimental selection cuts make the topology of the integration manifold highly complicated. Hence, Monte Carlo sampling is the only viable choice; besides being the only feasible path for performing these high-dimensional integrations, it also provides a method to generate events from a probablity density given directly by (1). Each kinematic configuration will be generated with a probability given by $d\sigma/dq$ which acts as a weight of the event. These weighted events can then be unweighted by applying a veto algorithm, keeping events with a probability given by the ratio of their weight divided by a maximal weight over all phase space. Finally, these unweighted events (which can be given uniform weight w=1 or $w=\sigma_{\text{tot}}$) then are generated with probabilities mirroring the ones in collider physics scattering experiments.

The Monte Carlo method is based on the central limit theorem of calculus that a (multi-dimensional) integral can be approximated by volume of the integration domain, V, times the mean value of the integrand, $\langle f \rangle$, where the error of the approximation is given by the product of the volume V and the variance, σ , divided by the square root of evaluation points N: $I = V \cdot \langle f \rangle \pm V \frac{\sigma}{\sqrt{N}}$. Here, $\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$, $\sigma^2 = \langle f^2 \rangle - \langle f \rangle^2$. There are two ways to improve the precision of a MC predictions, either to increase the number of random number points (MC "calls") or to reduce the variance to bring the MC error down. The most common method for variance reduction in MCs is importance sampling, i.e. to sample integrand

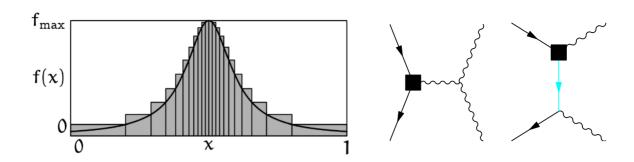


Fig. 2: Left: stratified sampling in one dimension; right: Multi-channel integration, two different kinematic channels for the process $e^+e^- \to W^+W^-$: s-channel diagram (left), t-channel diagram (right).

functions f much denser in peaked regions than in more "unimportant" domains of f. This is achieved by a change of integration variables by means of an invertible mapping (which is needed to calculate the Jacobian and to convert between original and new variables):

$$I = \int f \, dV = \int \frac{f}{g} dV = \langle \frac{f}{g} \rangle \pm \frac{1}{\sqrt{N}} \sqrt{\langle \left(\frac{f}{g}\right)^2 - \langle \frac{f}{g} \rangle^2} \qquad . \tag{2}$$

Now, instead of sampling a function f over the variables dV one is sampling the function f/g over variables gdV. This can be used to "map out" e.g. Breit-Wigner resonance peaks or radiation kinematics in QCD or QED that are enhanced for collinear or soft splittings. This precision is encoded in the accuracy of the final MC prediction for the total or fiducial cross section; this directly turns into the number of weighted event (phase space calls) that are needed to generate a single unweighted event (i.e. the unweighting efficiency becomes closer and closer to 100% the closer the phase space mapping turn the integrand into a constant function).

Besides importance sampling, there is also stratified sampling where the integration domain (e.g. in each dimension) is partitioned and this partitioning is optimized. This can autoated over many different dimensions and is codified in the VEGAS algorithm [45]. Fig. 2 shows an example of a one-dimensional stratification in the left panel. In general, immportance sampling outperforms stratified sampling by far, as stratified sampling samples functions where not where they are largest, but where they are changing most rapidly. More about these methods will be discussed in the following subsection.

For a general overview on Monte Carlo techniques in particle physics, cf. [46]. Finally, of course, MC techniques heavily rely on high-quality pseudo-random number generators (RNGs), that need to cover the high-dimensionl unit hypercube in a way as uniformly and densely as possible [47–54]. Especially, when phase space sampling is parallelized over many different cores, cf. Subsec. 2.3, it is crucial to make sure that random number sequences on different parallel instances are not correlated.

2.2 Multi-channel sampling of high-dimensional phase spaces

A generic phase space generator of an MC integrator consists of pseudo random number generator (RNG) and a setup of mappings from the variables of the unit hypercube to the kinematic moentum variables of the final state particles. The simplest examples are completely featureless and parameterize a cascade-like construction of the collision energy from massless final-state momenta; this is encoded in the RAMBO algorithm [55, 56].

Three avenues have been followed to more effectively sample high-dimensional phase spaces, constrained by complicated experimental cut selections: (i) importance sampling and subsection, (ii) stratified sampling have been mentioned already in the last subsection. The third avenue is multi-channel sampling, taking into account that the integrand function, the squared matrix element, has different features that cannot be mapped simultaneously with the same variable transformation to a smoother function, i.e. that the different resonance singularities are factorizable: $g(x) = g(x_1) \cdot g(x_2) \dots g(x_n)$. In such a case, the integral can be split up into a sum over different "phase space channels"

$$I = \int f(x)dV = \sum_{i} \alpha_{i} \int \frac{f(x)}{g(x)} g_{i}(x)dV \qquad .$$
(3)

The channel weights, α_i , constitute a partition of unity; in an adaptive procedure, they can be optimize according to the importance of different phase space channels within a single MC integral [57]. Fig. 2 shows two different phase space channels

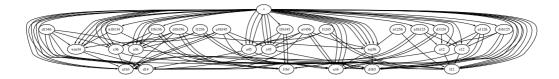


Fig. 3: Recursive algorithms for matrix elements reuse all possible subamplitudes and avoid all redundancies, from [80].

(s- and t-channel kinematics) for the process $e^+e^- \to W^+W^-$ (right panel). Most of the art of modern MC generators is in the clever choice of distribution into phase space channels and their particular mappings. Examples of modern phase space generators are MADEVENT [58], the generator of MADGRAPH5_AMC@NLO, where channels are chosen according to heuristically dominant Feynman diagrams. A combination of importance sampling with physics-driven phase space mappings, multi-channel adaptation and stratification is used in VAMP [59], the phase space sampler of WHIZARD. There are many more examples. Several of these algorithms can also handle overlapping singularities that cannot be factorized. More recently, algorithms of neural importance sampling based on invertible neural networks (INNs) have been implemented and are becoming interesting competitors to more classical algorithms [60–62].

2.3 Modern developments, validation and performance

Besides very computing-intensive higher order matrix elements, cf. Subsec. 3.1, phase space adaptation and sampling is the major bottleneck of MC generators for multi-leg processes. Parallelization over distributed or heterogeneous compute architectures was a must to make complicated processes accessible to the particle physics community. One aspect is the parallel evaluation of matrix elements during phase space sampling, the other aspect is the parallelization of the phase space sampling algorithm itself, while event generation can be usually trivially parallelized by dissecting the event samples into batches. For the parallel evaluation of matrix elements, several options exist, e.g. using OMP parallelization over different helicities or color flows in different threads. A major step is to parallelize the phase space adaptation parallel in different phase space channels, using either Message Processing Protocols like MPI [63] or graphics cards like GPUs [64].

MC generators need to be validated and during their development being continuously tested for consistency, validity and efficiency. Modern generators profit tremendously from cutting-edge programming paradigms like encapsulation, object-oriented design, test-driven programming, and validation paradigms like continuous integration and continuous deployment. In the past, e.g. for LEP and Tevatron, different generators have been validated by hand against other, while nowadays automated software frameworks exist for continuous validation of generator versions [65].

3 The hard scattering process

In this section, we discuss the generation of matrix elements at tree level and higher orders, Subsec. 3.1, which exhibit infrared or mass singularities for (quasi) massless particles in virtual or real corrections. In MC generators they have to be treated with so called subtraction methods, which will be detailed in Subsec. 3.2. In Subsec. 3.3 we briefly comment on combining MC simulations with analytic resummation to all orders.

3.1 Fixed-order perturbation theory matrix elements

Fixed-order matrix elements are the main building blocks of the hard scattering process: they are functions of the external momenta of the incoming and outgoing particles as well as their quantum numbers like helicities and color degrees of freedom.

These amplitudes are either based on a diagrammtic expansion with a subsequent common subexpressions elimination (CSE), or by using recursive algorithms are avoiding redundancies from the very beginning [66–68]. The recursion, shown in Fig. 3 can be based on functional programing paradigms [69]. QCD quantum numbers need to be accounted for, for which several algorithms exist, e.g. the color-flow formalism [70–74], as well as relative sign of subamplitudes according to Fermi statistics. This can be fully automated recursively, but is quite intricate for fermion-number non-conserving interactions (e.g. supersymmetric models and models with Majorana fermions) [75].

Automated generators of fixed-order matrix elements: many of these matrix element generators are historically attached to a MC generator framework (listed in parentheses below), but most if not all of them are in principle standalone and could be used interchangeably within different MC frameworks: Alpha [76] (Alpgen), Amegic/Comix [77, 78] (Sherpa)

MADGRAPH [79] (MADGRAPH5_AMC@NLO O'MEGA [80] (WHIZARD) COMPHEP/CALCHEP [19–21] (same MC name), and many more.

Basically all automated matrix element generators provide code in a compilable computer language like C++ or Fortran, or more recently with Cuda support for GPUs [81–84], which is then compiled and linked into dynamic libraries. For complicated processes, this process code and the binaries can become quite large. Alternatives are to transfer matrix elements into bytecode instructions (wavefunctions to be multiplied with vertex factors and propagators) which are interpreted by a single binary, where such an interpreter acts as a "virtual machine" [85].

For the Standard Model (SM), there is a well-defined subset of building blocks in terms of Lorentz representations of external particles and propagators, Lorentz structures for vertices, and all that. Extending the SM towards higher-dimensional operators in a framework like SM effective field theory (SMEFT) or studying an arbitrary framework beyond the SM (BSM) necessitates an almost infinite plethora of building blocks. In the past, BSM models needed to be hard-coded and the validation of these implementations was tedious (a famous example is the minimal supersymmetric SM, MSSM [86]). The situation was vastly improved with Lagrangian-level tools, where BSM models can be typed in using a textbook-like format and get automatically exported to MC generators. Examples are Lanhep [87], SARAH [88] or Feynrules [89]. Now, still for each of these tools the interface to each specific MC generator needs to be validated and maintained, cf. e.g. [90]. Hence, an intermediate layer has been invented, the Universal Feynman Output format [91, 92], such that now each Lagrangian tool and each MC generator only has to validate and maintain a single (UFO) interface. Connected to this interfaces there are convenience tools for automated parameter scans, automated calculations of decay widths and branching ratios.

3.2 Higher orders and subtraction methods

Higher order corrections in a perturbative series in quantum field theory contain either real or virtual corrections. At next-to-leading order (NLO), real corrections consist of the square of the real-emission diagrams and the interference of the tree-level with the one-loop amplitude. At the next order, NNLO, there are double-virtual contributions, the interference of the tree-level with the two-loop amplitude, double-real corrections (the square of double-emission diagrams) and real-virtual corrections (intereferences of loop diagrams with diagrams with a single real emission). Real-emission diagrams are simply tree-level diagrams of higher mulitplicity, while loop diagrams can be constructed by their analytic properties from tree-level diagrams by so-called unitarity cut methods [93, 94]. The loop integrals over internal momenta can then be algebraically reduced to basic "master integrals" [95–97]. Due to the relative size of the QCD coupling and the QED/weak coupling $0.1\alpha_s \sim \alpha$, for hadron colliders like the LHC generically NLO EW corrections are considered to be of the same importance as NNLO QCD corrections, explaining the focus on the QCD corrections. For lepton colliders, this is almost reversed as most signal processes are free from QCD corrections and NLO or NNLO QED/EW corrections dominate the theory uncertainties.

An important concept of quantum field theories with massless particles is that of infrared safety: only observables that are defined in a way such that they are not affected by soft or collinear emissions of massless particles, yield finite results in perturbation theory. This is encoded in the Kinoshita-Lee-Nauenberg (KLN) theorem [98, 99]. This is closely related to the concept of QCD jets, which are bundles of strongly interacting particles defined by a geometric procedure in phase space in order to ensure infrared safety, a so called jet algorithm, e.g. the Cambridge-Aachen, k_T or anti- k_T algorithm [100–106]. For more details cf. the section on "Perturbative QCD" [107]. The emission of soft or/and collinear QCD particles in both virtual and real corrections generate mass singularities that cancel each other in infrared quantities. In analytic calculations, these singularities are regularized in an algebraic manner, e.g. dimensional regularization. This is not possible for MC generators. Until ca. 2000, complicated partitions of phase space ("phase space slicing") had been applied, where for each process at higher order the independence of the final results on the partitioning of phase space (slicing parameters) had to be proven. By the end of the 1990s, a more sophisticated framework had been developed in terms of subtraction algorithms:

$$d\sigma_{2\to n}^{\rm NLO} = d\Phi_n \left[\mathcal{B}_n + \mathcal{V}_n + \mathcal{B}_n \otimes S \right] + d\Phi_{n+1} \left[\mathcal{R}_{n+1} - \mathcal{B}_n \otimes dS \right] \tag{4}$$

from the real emissions \mathcal{R}_{n+1} to a $2 \to n$ Born process \mathcal{B}_n , their soft/collinear singular parts dS are subtracted in a way that they can be analytically (or semi-numerically) be integrated over the phase space variables of the QCD radiation. This enables an analytic integration of the singular parts which can be added to the virtual components \mathcal{V}_n of a higher-order calculation, such that each component is separately infrared finite. Subtraction is based on the property that in the soft and/or collinear limit, quantum field theoretic n+1 particle amplitudes factorize into n particle amplitudes times a $1 \to 2$ splitting function. In addition, also the n+1 particle phase space factorizes into the n particle phase space times a one-particle radiation phase space. The two main subtraction algorithms are Catani-Seymour (CS) [108, 109] and Frixione-Kunszt-Signer (FKS) subtraction [110, 111], while there are also more numerical approaches [112].

A typical MC generator that provides automated NLO calculations/simulations then consists of a framework of automatically generating all the subtraction terms with the corresponding phase-space setup, providing tree-level matrix elements for the Born process, the real emissions, the color- and spin-correlated matrix elements for the subtraction terms in the soft and collinear regions, respectively. Either the MC generator has its own one-loop matrix element generator or receives the virtual amplitudes from a one-loop provider (OLP). These tools consist of libraries that contain the tensor reduction into master integrals and libraries for the scalar master integrals [113–117]. The interface between MC generators and OLP tools has been standardized in the Binoth Les Houches Accord (BLHA) [118, 119]. Examples of these interfaces can be found in [14, 120].

As discussed above, unitarity cuts, tensor reduction to master integrals and libraries of all needed master integrals have triggered the famous "NLO revolution", the automation of NLO matrix element generation (and their usage via subtraction formalism in MC generators) starting from ca. 2010. Automated one-loop generators are GoSam [121, 122], Madloop [123], Openloops [124, 125] and Recola [126]. Using these OLP tools, NLO QCD+EW, i.e. full NLO SM processes have been automated in Madgraph5_AMC@NLO [5, 6], Sherpa [12, 120] and Whizard [127, 128].

This NLO paradigm can be in principle extended towards (automated) NNLO tools, however, two-loop integrals, especially with many legs and/or many different internal mass scales are highly complicated and not automated yet, and though there is tremendous progress for subtraction schemes at NNLO for QCD, there is no completely automated NNLO subtraction scheme for QCD and EW interactions. The work on general subtraction schemes for NNLO QCD is a very active field of research [129–153]. More details can be found in [107]. In addition, mixed QCD-EW corrections have become possible [154–157]. As of now, there is no automated matrix element generator for NNLO QCD, but there are libraries of the most important processes like Higgs production, Drell-Yan, dibosons and top pairs, cf. e.g. [158–164] or event shapes in e^+e^- [165]. The MATRIX framework [166] combines a larger list of NNLO QCD processes. NNLO EW corrections only exist for very few processes, e.g. dominant corrections for $e^+e^- \to WW$ [167] and the full corrections $e^+e^- \to ZH,WW$ [168] and attempts for the automation are still in its infancy, note however [169]. For simulation tools for low-energy experiments like MUonE, the situation becomes simpler as all leptons can be treated massive and only soft singularities remain: this makes the subtraction much simpler. This has been used in the McMule tool which aims at NNLO accuracy [170]. An extensive overview for tools for low-energy experiments can be found here [171].

There are several places in an NLO or NNLO calculation which could produce negative weights: (i) NLO or higher order fits of PDFs need not necessarily be positive definite any more, (ii) outside the strict soft or collinear limits, cancellation between the real matrix elements and their soft-collinear approximation can turn negative, and (iii) there regions of phase space where the cancellation of subtractions and their integrated terms is numerically imperfect. Negative weights normally average each other out with positive weights in binned distributions, but they hamper the efficiency of NLO event generation, as for several negative weights events a corresponding number of positive weight events need to generated. This greatly enlarges the number of needed events to reach a current precision. Therefore, algorithms to reduce the number of negative weights, e.g. by more cleverly grouping them or resampling techniques [172–178].

A special case is the top quark when produced close to threshold, as it exhibits effects of a quasi-bound state similar to hadronization effects (cf. Sec. 6), which, however, need to be handled in the framework of the hard fixed-order process. In MC generators, this has been first addressed for the top threshold in e^+e^- collisions [179, 180], while recently it has been realized that this effect is experimentally visible also at LHC [181].

3.3 Combining fixed order with resummation tools

For many aspects, fixed-order calculations within a perturbative series are not sufficient to achieve a precision goal for theoretical/MC predictions. Due to large kinematic scale separations (e.g. typical jet energy scales of hundreds of GeV at the LHC and the hadronization scale of $\Lambda_{\rm QCD} \sim 1$ GeV, large logarithms occur, such that for a small coupling $\alpha_s \ll 1$ the product $\alpha_s \log Q_1/Q_0 \sim 1$ or even larger than one. A typical example are QCD Sudakov logarithms which are e.g. addressed in Sec. 5. Especially for more inclusive quantities like total or single-differential cross sections, it is possible to resum such logarithms to all orders, where such resummation is performed using methods from effective field theories (EFTs) which have the same low-energy behavior than the full theory (here e.g. QCD) but are much simpler in the ultraviolet. Such theories are e.g. soft-collinear EFT (SCET). While such resummation calculations very often have to be done in a process-specific manner, there are several tools that have been developed that take care of certain universal features of such resummations. Examples of interfacing resummation tools to fixed order simulations, are [182, 183], very often in the context of jet or event shape variables. For more information see the section on jets in e^+e^- [184].

Another example of such large logarithms are electroweak (EW) Sudakov logarithms [185, 186]: these originate from incomplete cancellations between virtual and real corrections as initial or final states are not EW singlet states or because the event selection do not include explicit EW real radiation. For TeV-scale processes like the production of EW bosons

at the LHC or basically any process at a future 3-10 TeV muon collider, these logarithms are quite large and can reduce the Born cross section by 20 to 100 per cent. Given the external lines of a process, tools as add-ons to MC generators can automatically generate such EW Sudakov logarithms and dress MC processes with them [187–190]. Applications of EW Sudakov resummations for LHC can be found in [191, 192] and in [193] for the muon collider.

4 Parton distribution functions and beam simulations

Most beam particles within collisions in particle physics are not pointlike particles: this is definitely true for proton collisions, where the beams consists of bound states of quarks and gluons, but also of leptons like electron and muons. The latter undergo QED or weak interactions and appear at high energies as well as a collinear bunch of electromagnetically or weakly interacting particles. In this section, we will discuss the inclusion of parton distribution functions in MC generators, first for proton beams in Subsec. 4.1, then for lepton beams in Subsec. 4.3, and finally comment on aspects like beam spectra of lepton or photon beams, crossing angles and polarization in Subsec. 4.4.

4.1 Parton distribtion functions of the proton

Parton distribution functions (PDFs) of quarks and gluons within protons, modelled in a Lorentz frame fully collinear with the incoming beams, are being fitted from data, from deep-inelastic scattering experiment like the electron-proton collider HERA or from proton-proton collisions like LHC. The errors from these fits, especially for the gluon PDF which is the most important for many LHC processes, are for many measurements one of the largest sources of uncertainties. There are several different collaborations each doing independent fits, like ABM [194], CTEQ [195–197], HeraPDF [198], MSTW/MMHT [199, 200], NNPDF [201, 202], cf. as well [203, 204]. For the MC generator, the details of these different fits do not really matter as long as these PDF fits are available in a fast and standardized manner: this is done in a library like LHAPDF [205] (standardized through an effort during Les Houches workshops and the NLO-MC BLHA interface or LHA/LHE event formats). The generator calls the PDF as function of the parton flavor, the energy fraction of the incoming beam particle ("Bjorken x"), the factorization scale, the perturbative QCD order at which the PDF had been performed and certain validaty boundaries of the fits. Generally the parton flavor has 11 components, the gluon and five quark and anti-quark components each, with the top (anti-)quark content assumed to be zero. Many PDFs now include the photon content inside the proton [206], which becomes important for high-charge ion collisions generating a huge photon flux. More recently, also lepton content in proton PDFs have been considered, e.g. from photon-to-lepton pair splitting at NNLO QED level [207].

These PDF fits are provided as grids of certain values with specific interpolation and extrapolation routines between and beyond. Though this looks trivial from the point of the MC generator, it is crucial that these calls to PDF values are fast and efficient: the PDFs are not only used for convolutions with matrix elements, but also as ratios of PDFs in the construction of subtraction terms and in Sudakov factors of initial-state parton showers.

4.2 Underlying event and multi-parton interactions

Very briefly, multi-parton interactions (MPI, do not confuse with the Message Processing Interface) in Subsec. 2.3) is specific to hadron collisions: there can be more than one parton undergoing a hard interaction. For the MC generator this is a double invocation of its routines; the main new feature is to access the probabilities for the double-parton splitting out of a hadron, for which special PDFs exist that need to be interfaced to the generator. Secondly, there is the "underlying event" which collects all additional effects that are connected to the initial state: the beam remnant which is either a color-(anti)sextet in the case of a (anti-)quark or a color octet in the case of a gluon as a parton entering the hard scattering process. The beam remnant also undergoes QCD radiation simulated by the parton shower and needs to be transferred into a system that can be processed by the hadronization, including the initial conditions for the kinematics of the beam remnant. In addition, also the generation of "intrinsic" p_T is counted with the underlying event: while PDFs are fitted in the strictly collinear limits, the parton splitting can also be generate transverse momentum (as it does e.g. in transverse-momentum dependent PDFs, TMDs). This "intrinsic" transverse momentum contributes to the p_T distribution of jets and electroweak particles at hadron colliders in the lowest bins of a few to 10-15 GeV. For more details see the discussions in Sec. 10 of [7].

4.3 Electron/lepton parton distribution functions

At high enough lepton beam energies, Q_{ℓ} , even for the smaller QED coupling constant, $\alpha \sim \alpha_s/10$, the quantity $\alpha \log Q_{\ell}^2/m_{\ell}^2$ becomes of an order where resummation is necessary. In contrast to QCD partons, the initial conditions need not be fitted from experimental data, but can be perturbatively calculated from first principles. The result is an object

that gives the probability of finding a "partonic" lepton, anti-lepton or photon within the beam photon with a certain energy fraction $0 \le x \le 1$, $f_{\ell}(x, Q_{\ell})$. These objects historically have been called lepton structure functions, but nowadays are mostly called lepton PDFs: structure functions are considered to be based on kinematic approximations, while lepton PDFs are field-theoretic objects that obey renormalization group or DGLAP equations.

In the derivation of lepton PDFs, soft photons can be resummed into a compact formula to all orders [208, 209], most easily using the formalism of Mellin transforms, to which hard-collinear corrections have been added to first, second [210, 211] and third order in α [212–214]. All of these formulae resum the leading-logarithmic (LL) corrections. In the past years, anticipating the needs of a future e^+e^- Higgs factory, a lot of effort has been put into resumming the NLL terms [215–218], and calculating the fixed-order NNLO QED contributions [219] to these PDFs. For a review on these topics, cf. [220]. The biggest phenomenogical differences to QCD PDFs are (1) that the rise in the infrared is less steep for lepton PDFs as QED is not asymptotically free so the rise simply comes from the masslessness of the degrees of freedom, and (2) that the lepton PDFs exhibit an integrable singularity in the limit of Bjorken-x $x \to 1$ (while QCD PDFs vanish in this limit). For the MC generator this singularity results in a huge challenge regarding numerical stability, which is manageable in convolutions with tree-level matrix elements, but becomes quite intricate when combined with subtraction terms for higher order calculations. To provide a stable automated framework for NLO EW calculations in MC generators is an active field of research.

Note that there are several dedicated programs for NLO EW corrections for specific processes, that include the effects of these lepton PDFs at LL or NLL accuracy [221, 222]. In addition, it is, of course, also possible to do NLO EW calculations in MC generators for lepton colliders using massive initial state leptons, e.g. [14, 223].

At very high energies, e.g. future 10 TeV parton level colliders like the muon collider MuC or FCC-hh, one can consider the full SM as partons, generalizing the concept of QED lepton PDFs to EW lepton PDFs. As EW interactions are chiral, partons need to be considered polarized, and all of them are coupled to the DGLAP equations of the full SM. Counting all degrees of freedom yields 59 in the SM (including interferences induced by EW symmetry breaking) [224, 225]. These EW PDFs have been put into the same framework like proton PDFs by numerically solving and interpolating those coupled DGLAP equations; they are the initial state counterparts of the full soft-collinear realm of the SM to the EW parton showers in Subsec. 5.3. In this framework they are available in the MC generators (cf. e.g. to the framework in [226]).

4.4 Beam simulations of lepton and photon colliders

There are several aspects of the physics at lepton colliders that are different from hadron colliders. Electron-positron colliders at high energies and/or high luminosities exhibit beamstrahlung, classical radiation caused by the electromagnetic fields from the other bunch shortly before the collisions. This has to be taken into account in order to carefully plan radiation occupancies in detectors under development and to estimate systematic uncertainties for reconstruction and analysis. In addition, these effects lead to luminosity smearing which is a convolution of three different effects: the natural beam energy spread resulting from the machine design (of the order of 0.02 - 0.1 %), beamstrahlung determined from classical electrodynamics which depends on the beam optics in front of the collision point, especially the final focus magnets, and QED initial state radiation. There exist dedicated simulation tools developed in accelerator physics like Guinea Pig [227, 228], Cain [229], Fluka [230] or XSuite [231]. At synchrotrons like LEP, CEPC and FCC-ee this leads mostly to a Gaussian beam spread, while for linear colliders like LCF, ILC, CLIC or C3 it leads to sizeable effects that need to be taken into account. While synchrotrons mostly exhibit transverse polarization of the electrons due to the Sokolov-Ternov effect from synchrotron radiation, linear colliders allow for longitudinally polarized beams [232]. All major multi-leg MC generators allow for the simulation of longitudinally polarized beams (also with polarization fractions in event generation different from 100%), while the Whizard framework allows for arbitrary polarization (i.e. mixtures of longitudinal and transverse polarizations or arbitrary spin quantization axes, in general a completely general initial state spin density matrix).

For beam simulations and beam spectra there are three levels of sophistications: (i) the simulation of a two-sided or one-sided Gaussian beam spread, (ii) a parameterized spectra and (iii) an MC generator based on a two-dimensional histogrammed fit. Option (i) is available in almost all MC generators for e^+e^- colliders and is very likely sufficent for muon colliders and synchrotrons at lower energies (while FCC-ee at 365 GeV shows deviations from the Gaussian profile). The parameterized spectra (option ii) assumes that the beam spectra of the e^- and e^+ can be factorized and each of them can be described by a smeared delta peak and a polynomial tail, $D_i(x) \approx \alpha \delta_{\epsilon}(x) + \gamma_i x^{\alpha_i} (1-x)^{\beta_i}, i=1,2$. This formalism is sufficient for low-energy ILC-type and C3-type machines (with energies ≤ 500 GeV), and it is implemented in [233] and the first version in [234]. Option (iii) is the most versatile approach: it does not assume that the effects from the two beams can be factorized. For this, it uses a two-dimensional grid fitted to the low-statistics outputs of the machine simulation tools listed above. To compensate artifacts of this low statistics a smoothing with a Gaussian filter is applied. In addition, the interior of the square $(x1, x2) \sim ([0, 1] \times [0, 1])$ and the boundary have to be fitted independently in order to avoid artificial beam energy spreads. This formalism is encoded in the CIRCE2 algorithm [234] included in the WHIZARD [13]. Such a

description is needed for drive-beam accelerators like CLIC, plasma-wakefield accelerators and also photon colliders based on Compton backscattering of electrons from laser photons. Of course, it covers also ILC machine setups, and it has been applied to the beam simulation of CEPC and FCC-ee (where all four energy stages and all four interaction points).

5 Parton Showers

In this section we discuss the generation of QCD, electromagnetic and weak radiation via parton showers and the matching to fixed-order matrix elements. Additional information can be found on the section on jets at electron-positron colliders [184].

5.1 Parton showers: rationale and implementation

Typical events at high-energy hadron collider contain up to a hundred different final state particles (e.g. constituents of jets), and even high energy lepton collider events contain 10-20 particles. There are two main obstacles to simulate such multiplicities with the methods of the last two sections: (i) the dimensionality of phase space extends the capacity of MC methods, and (ii) matrix element generation even for tree-level processes hits its limitations for 15-20 external particles. On the other hand, it is also not really necessary to simulate most of these components with complete matrix elements, as these emissions are dominated by soft and/or collinear emissions. We can give here only a short introduction into parton showers, for a general overview cf. the lecture notes [235]. Historically, the concept of parton showers have been developed together with the first physics results on jets at the PETRA storage ring at DESY. As for subtraction terms in fixed order calculations (cf. Subsec. 3.2), parton showers are based on the structure of quantum field theories in the infrared where n+1 particle amplitudes factorize into the convolution of n particle amplitude times splitting function. In the soft-/collinear limit, not only amplitudes factorize, but also the phase space: $d\Phi_{n+1} \approx d\Phi_n \cdot d(\Phi_{\rm rad})$. In this limit, emissions become independent of each other and can be exponentiated, where the exponential gives the total probability for any number of emissions to happen. The inverse of this exponential, the Sudakov form factor, then yields the no-emission probability: $\Delta(t,t_0) = \exp\left[-\int_{t_0}^t \frac{dt'}{t'}\int_{z_-}^{z_+} dz \frac{\alpha_s(z,t)}{2\pi} P(z,t)\right]$. It can be used to implement a veto algorithm: a random number is drawn and compared to the Sudakov form factor to decide whether to do an emission or not. t is the "shower time" variable: different parton shower algorithms use different kinematic variables, the emission angle θ , the parton virtuality Q, transverse momentum p_{\perp} . Final state showers perform a forward evolution in shower time from large to low scales, while initial-state showers do a backward evolution. For the initial-state shower, the Sudakov form factor contains ratios of the PDFs, as one has to divide out the PDF of the Born parton without emission and multiply with the PDF of the parton before the emission. The variable z is a kinemiatic variable of the splitting, usually the energy fraction of the original parton, while P(z,t) is the DGLAP splitting function, which obeys a QCD evolution equation. An important concept of parton showers is color coherence by (large angle) soft emissions: this is connected to angular ordering, where fist emissions have to be carried out at largest angles. This guarantees the dominant color coherence effects. There exist many different parton shower algorithms, based on different concepts of the shower evolution or splitting variable, the included order, the recoil scheme etc. Some examples are [236–239]. Multi-purpose event generators, that have their own parton shower implementations, are Herwig [1, 3], Pythia [9], Sherpa [11, 12] and Whizard. There are other, stand-alone shower implementations like ARIADNE [240], DEDUCTOR [241] etc.

The shower "evolution" is a unitary evolution, i.e. the parton shower emissions (when being fully inclusive, summing over all emission multiplicities and integrating over all of phase space) do not change the total cross section of the Born process (total cross sections, however, have to change when matching parton shower emissions to emissions from hard matrix elements, cf. Subsec. 5.2. Parton showers do resum Sudakov logarithms [242] (which had been discovered for analogous QED emissions), which make them the "exclusive part" of resummation algorithms, cf. Subsec. 3.3. Due to their probabilistic (Markovian) nature, the assessment of theoretic uncertainties is much harder than for an analytic resummation. The past decade has seen a lot of progress quantifying the uncertainties of parton showers beyond just running different shower algorithms and comparing the differences. Angular-ordered showers [243–245] play an important role, and with a careful bookkeeping of the kinematics of the emissions [246], such angular-ordered showers maintain NLL accuracy accuracy for global observables, i.e. not too differential observables. These efforts of the past years, many efforts have been devoted to go beyond the quasi-classical approximation of leading splitting kernels (i.e. independing emissions at the LL order) to achieve next-(next-)to-leading logarithmic accuracy [246–255]. This is achieved by including more exact kinematics, by incorporating different recoil schemes and to include higher orders in spin and color correlations. It is technically quite challenging to prove NLL or NNLL accuracy of parton showers, using known results from analytic resummation. Fig. 4 shows in the left plot a comparison of two different parton showers (Alaric and Dire from the Sherpa framework) with LEP data.

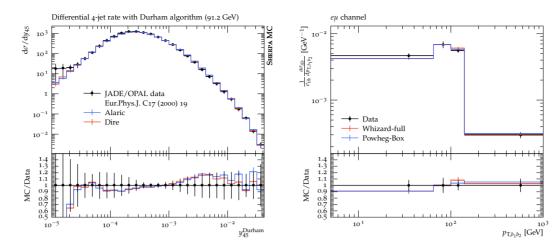


Fig. 4: Left: comparison of different parton shower to LEP data for the differential distribution in the $4 \to 5$ jet transition variable y_{45} , from [248]. Right: comparison of the two NLO generators POWHEG-Box and WHIZARD on the matched $pp \to b\bar{b}e^-\mu^+\bar{\nu}_e\nu_\mu$ process at 13 TeV to ATLAS data.

5.2 Matching fixed-order processes to parton showers

With dedicated approaches, it is possible to preserve the accuracy of the hard scattering process even at higher perturbative orders. At NLO, there exist techniques to match NLO calculations with parton showers and obtain NLO+PS predictions that preserve NLO accuracy for integrated distributions and Born-like quantities. In essence, these algorithms provide a way to exactly describe one hard emission and fill the remaining phase space with a shower, avoiding possible double counting. There are different strategies achieve this goal: veto emissions from the parton shower into phase space regions better described by emissions from the hard matrix elements (and potentially reweight by Sudakov non-emission probabilities), or explicitly subtract from the fixed-order calculations the parts described by the parton shower. An incarnation of the first approach is the POWHEG method [16, 17]), of the latter approach is the MC@NLO method [256–258]. The right panel in Fig. 4 shows an comparison of different incarnations of the POWHEG matching algorithm to ATLAS data for the process $pp \to b\bar{b}e^-\mu^+\bar{\nu}_e\nu_\mu$ at 13 TeV LHC Run2. Other algorithms can be found in [259, 260].

Since the mid-2010s, many efforts have been undertaken to combine NNLO calculations with parton showers through algorithms [261] that have been carried also into MC frameworks and tools like MiNNLO_{PS} [262, 263] or GENEVA [22, 264]. With the advent of the first N3LO calculations (e.g. for Higgs production in gluon function, cf. the section in [107]), parton showers also need to be matched to such cases where up to three emissions can come from the fixed order calculation [265].

For many studies at LHC, it has turned out beneficial to use inclusive multi-jet samples which contain a minimal number of selected jets (zero, one, two), but are inclusive in events with all higher jet multiciplities. In order to generate such samples, MC generators need to merge samples with exclusive jet multiplicities at different orders in the underlying fixed-order calculations (mostly LO and NLO). There are several different merging algorithms for inclusive multi-jet samples, again either discarding events where the shower would populate already existing multiplicities, or re-using as many events as possible and reweighting to get the normalization as close to the experimental data (examples at LO are [266–269], while NLO applications are [270–272]).

Another important information that needs to be carried from the hard process to the parton shower are decays of intermediate resonances like the W, the Z, the Higgs boson and the top quark. Even when the decays are included completely in full matrix elements (e.g. $2 \to 6$ for top pairs or $2 \to 4$ for hadronic WW), the MC generator needs to provide a resonance history as close as possible to the correct probabilities of the intermediate resonances: e.g. in $e^+e^- \to jjjj$ at 240 GeV, roughly 79% comes from W pairs, 18% from Z pairs and 2% is non-resonant QCD background. As the parton shower does rearrange kinematics in order to implement certain recoil schemes in its splittings or tries to bring partons close in phase space before hadronization, it would shift invariant masses of combinations of partons away from their original resonance masses. So the MC generator needs to take care of this information. An example for this is [273].

5.3 QED showers and their matching, weak showers

There are many collider observables, where a very precise simulation of photon radiation and other QED effects are needed to match the experimental precision: two paradigm examples are (i) the W mass measurements at hadron colliders like

Tevatron and LHC, where the transverse momentum distribution of the decay lepton is very sensitive to final state QED showering, and (ii) the luminosity measurement at e^+e^- colliders relying on the most precise predictions of Bhabha scattering, $e^+e^- \to e^+e^-$, as well as two-photon production, $e^+e^- \to \gamma\gamma$. Many of these specialized QED shower programs have been developed for the combined LEP1+2 program and the flavor factories, BaBar and Belle [274, 275].

Also, many algorithms and implementations for QED showers originated during the LEP time [276]. Like for QCD, it is important to properly combine these showers with fixed-order calculations without double counting [277]. One of the frameworks is Yennie-Frautschi-Suura (YFS) resummation based on purely soft emissions [278]. This formalism allows both to resum soft emissions to all orders in an inclusive way as well as exclusive emissions of (soft) photons as a shower. This is the basis for the implementations of many different LEP MC generators [26, 279–283] for two-particle final states, where the interference with hard emissions in a coherent picture between initial- and final-state emissions is possible [284]. More recently, the non-coherent formalism has been automated for arbitrary processes [285]. A review of the corresponding tools can be found here [286]. One of the more ubiquitous tools for QED showering which is very often attached to the simulations of multi-purpose generators, is Photos [23–25]. Like for QCD, also for QED photon emissions from the shower need to be properly matched to the fixed order hard process. Similar algorithms as for QCD can be applied for QED, but there are also dedicated algorithms that take the characterics of the emissions from the analytic lepton PDFs (cf. Subsec. 4.3) into account [287]. Examples for matching of NLO EW corrections to QED showers for e^+e^- colliders can be found here [288, 289].

With the plans for a very high-energy hadron collider of the order of 70-100 TeV (FCC-hh) or lepton colliders at the range of 10 TeV center-of-mass energy, also the inclusion of weak radiation has been studied [290, 291]. Weak splitting kernels can be included in the Sudakov factors, and the showers can be interleaved with the QCD and QED radiation. The DGLAP evolution then in principle connects all components of the Standard Model (SM). An important feature is the fact that weak splittings are chiral, so within weak showers partons automatically become polarized.

6 Hadronization

QCD becomes strongly interacting at low energy scales, and quarks (with the exception of the top) and gluons form bound states, mesons, baryons or more exotic objects (they "hadronize" or "fragment"). While it is possible to calculate very inclusive quantities analytically (cf. Sec. 7 in [184]), exclusive events need to be simulated by MC generators. As this is an intrinsincially nonperturbative phenomenon, no miscroscopic description from first principles exists and the fragmentation needs to be modelled phenomenologically, with the parameters of these models then being tuned to data. In Subsec. 6.1, we discuss the two main existing fragmentation models, before we briefly comment in Sec. Subsec. 6.2 on the simulation of hadronic decays and QED radiation of hadrons.

6.1 Fragmentations of partons into hadrons in MC generators

After a first era of very crude models of hadronization, using e.g. flux tubes as modelling of jet masses initiated by Feynman in the early 1970s, the first serious hadronization model was the so called independent fragmentation [292, 293], encoded in the first hadronization program Isajet [294] in 1979. This algorithm creates quark pairs from the vacuum to dress bare quarks and uses a Gaussian distribution for the generation of transverse momentum, p_T , but suffers from several problem, especially Lorentz covariance and infrared safety issues; also notoriously, the last open quark needed special treatment. Independent fragmentation is still used for determining total hadronic cross sections.

With the early 1980s, the two models which dominate fragmentation simulations until today, were developed and put into codes: fragmentation based on the Lund string model [295] and cluster fragmentation All MC generators today use variants of one of these two formalisms. The Lund string model became the basis of the Jetset [296, 297] event generator which became fused into the Pythia [7, 298] parton shower simulation. All fragmentation models are based on the two fundamental concepts of (1) local parton-hadron locality [299], assuming that fragmmentation is process with low momentum transfer, such that energy momentum and flavor quantum numbers of the produced hadrons follow closely those of the underlyind partons; and (2) of a universal low-energy strong coupling α_s used in all jets and branchings [243, 300].

The Lund string fragmentation model is based on the first attempt to understand strong interactions in terms of quantized strings, which can be connected in the framework of QCD as the fact that QCD field lines at low energies of a few $\Lambda_{\rm QCD}$ and below become compressed tubelike regions that correspond to the dynamics of string-like objects. In that picture, the linear confinement potential can be derived. Partons (quarks) that are close in phase space are ordered and get connected by rubber-band like string objects. The masses of the generated mesons are roughly proportional to the area that these strings trace in their time evolution. Non-perturbative effects appear as Lund string breakings which are simulated as tunneling processes, where in the breakup a new quark-antiquark pair pops out of the vacuum. This fragementation starts

in the middle of the strings and spreads outward. The flavor decomposition in fragmentation is roughly modelled as $u\bar{u}:d\bar{d}:s\bar{s}:c\bar{c}\approx 1:1:0.3:10^{-11}$ while bottom (and top) quarks are not generated in fragmentation. Though the Lund string model aims at a universal description of data (e.g. $e^+e^-\to hadrons$, it has a plethora of free parameters (roughly $\mathcal{O}(1)$ per known hadron, and it has difficulties in precisely describing baryon production (which tries to interpret antiquark as diquark to group them with a quarks, modelled as string junctions).

The second major hadronization formalism is cluster fragmentation [244, 301, 302] which became fused with the HER-WIG multi-purpose MC generator [303], and in a different incarnation provides the hadronization module of the SHERPA framework [304]. Cluster fragmentation makes use of the fact that parton showers (at least their leading-logarithmic or leading-color) part order partons in color space, such that also color partners get generated close to each other in phase space, which motivates taking the $N_c \to \infty$ limit where planar Feynman diagrams dominate [305]. The clusters in this model are a recursively defined model of high-mass resonances whose spectrum is defined by the parton shower, so perturbation theory, where the primary cluster is independent of the production mechanism. An important parameter is the scale where the parton shower stops and hadronization sets is. The original algorithm is pure kinematics, i.e. phase space, no spin info, while now there are attempts to also include spin effects.

In the recent years, there are several attempts to use machine learning techniques in order to learn some of the core components of fragmentation modelling algorithms from (LHC) data. This would replace the fragmentation functions which are parameterized by a certain functional form and fitted to data by numerical distributions that come out of pattern recognition algorithms adapted to LHC data. This is a computationally interesting path, but it is open whether that can ever lead to a better conceptual understanding of non-perturbative low-energy regime of strong interactions.

6.2 Hadronic decays and final-state QED radiation

The main hadronization modelling creates a setup of hadronic resonances (mesons, baryons and also several classes of more exotic bound states like tetraquarks, pentaquarks, hadronic molecules etc.). Except for the proton, none of these states is stable on a macroscopic scale, so MC generators need to model the transitions of these states and also their decays. Over a century, these decays have been measured in dozens of larger and smaller collider and fixed target experiments and have been catalogued by the Particle Data Group (PDG), [306] (and older editions). All of these decays of ca. 200 resonances in the PDG with thousands of decay channels need to encoded in the decay modules of multi-purpose event generators to guarantee a precise simulation of the hadronic part of particle physics. Let us illustrate the complexity and variety with an example of a hadronic decay chain:

$$B^{*0} \xrightarrow{(1)} \gamma B^0 \xrightarrow{(2)} \overline{B}^0 \xrightarrow{(3)} e^{-\overline{\nu}_e} D^{*+} \xrightarrow{(4)} \pi^+ D^0 \xrightarrow{(5)} K^- \rho^+ \xrightarrow{(6)} \pi^+ \pi^0 \xrightarrow{(7)} e^+ e^- \gamma \tag{5}$$

The first step, (1), is a radiative electromagnetic decay, step (2) is a weak mixing of B mesons, (3) is a weak decay, step (4) a strong decay, (5) another weak decay with a broadly smeared ρ resonance peak, step (6) is a decay of the ρ^+ that has been produced polarized and where angular correlations are crucial, while the final step is a 3-particle "Dalitz" decay with a sharply peaked m_{ee} . All of the charged particles in these cascade $(D^{*\pm}, K^{\pm}, \rho^{\pm}, \pi^{\pm})$ undergo QED radiation, cf. Sec. Subsec. 5.3. There is a plethora of different, mostly, kinematic features like form factors, Dalitz correlations (phase space correlations), resonance peaks (sometimes with deviations from Breit-Wigner due to interference with QCD continuum), but also effects like spin correlations, weak and strong mixing, etc.

Among the leptons, tau decays are quite important because they allow information on the spin quantum numbers of their mother particles, especially from their hadronic decays, so these decays need to be simulated with great care, including spin correlations, QED radiation and exact kinematics [307].

Implementing all of these features is a major task for any generator, and given the fact, that LHC in the runs up to now has discovered new hadronic resonances and many more rare decay channels an equally important task to maintain and update the hadronic databases of each generator.

7 Conclusions

In this Chapter, we have provided a pedagogical introduction into the complexity of Monte Carlo event generators for collider applications in particle physics. They are the main workhorses for the data analysis and interpretation of all existing collider experiments and are equally indispensable for planning future collider experiments and determining their physics potential. Generally, they are two classes of such tools: (i) multi-purpose MC event generators which aim at covering all aspects; hence, they are very extensive and complicated structures as they encode all of our knowledge about particle physics; and (ii) dedicated MC tools that focus on a single or a few specific aspects. The main parts of a multi-purpose generator

are the hard scattering process at lowest and higher fixed orders in perturbation theory, the interface to simulating beam structure and PDF for hadron and lepton colliders, the underlying events, the parton showers, the hadronization and the decay of all unstable particles and resonances. MC generators constitute a sizeale fraction of the computing ressources of collider experiments (for ATLAS and CMS each ca. 20% for the current LHC runs): constant optimization and porting to heterogeneous computing architectures is necessary to enhance their speed and reduce their power consumption. Collider physics experiments run over several decades, and analyses need to be revisited decays after the end of the runtime of collider experiments; this leads to a heavy burden in maintenance and reproducability which cannot lay on a few or even a single scientist: MC generators are the prime examples for collaborational efforts in theoretical particle physics at the interplay between physics and computational science.

Acknowledgements. Over the years I am very much indebted for many fruitful discussions on Monte Carlo event generators and the physics connected to them to Guido Altarelli (†), Alexander Belyaev, Thomas Binoth (†), Eduard Boos, Simon Braß, Pia Bredt, Jonathan Butterworth, Matteo Cacciari, Stefan Dittmaier, Silvia Ferrario Ravasio, Rikkert Frederix, Stefano Frixione, Stefan Gieseke, Tao Han, Stefan Höche, Staszek Jadach (†), Tomaš Ježo, Stefan Kallweit, Alexander Karlberg, Wolfgang Kilian, Frank Krauss, Jonas Lindert, Leif Lönnblad, Fabio Maltoni, Olivier Mattelaer, Kirill Melnikov, Pier Monni, Stephen Mrenna, Zoltan Nagy, Paolo Nason, Thorsten Ohl, Davide Pagani, Mathieu Pellen, Fulvio Piccinini, Simon Plätzer, Tilman Plehn, Stefan Prestel, Alan Price, Alexander Pukhov, Peter Richardson, Gavin Salam, Steffen Schumann, Frank Siegert, Torbjörn Sjöstrand, Peter Skands, George Sterman, James Stirling (†), Melissa van Beekveld, André van Hameren, Bennie Ward, Zbigniew Was, Bryan Webber, Marius Wiesemann, Malgorzata Worek, and Marco Zaro, and many, many more.

Funding is acknowledged from the DFG under the German Excellence Strategy-EXC 2121 "Quantum Universe"- 390833306, and the National Science Centre (Poland) un- der OPUS research project no. 2021/43/B/ST2/01778. Furthermore, we acknowledge support from the COMETA COST Action CA22130, by EAJADE - Europe-America-Japan Accelerator Development and Exchange Programme (101086276), and from the Joint Research Programme by the International Center for Elementary Particle Physics (ICEPP), the University of Tokyo. We thank the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support during the completion of this work.

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