Results on meson-meson scattering at large N_c

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We present results on the large N_c scaling of meson-meson scattering amplitudes. We work in a theory with $N_f = 4$ degenerate quark flavors and run lattice simulations with $N_c = 3 - 6$ and pion mass $M_\pi \approx 590$ MeV. We focus on three different scattering channels, two of which have the same quantum numbers as some tetraquark candidates recently found at LHCb. Finite-volume energies are extracted using a large set of operators, containing two-particle operators corresponding to two pions or two vector mesons, and local tetraquark operators. Using Lüscher's quantization condition, we constrain the infinite-volume scattering amplitudes and investigate subleading N_c corrections to the large N_c limit. For one of the channels, we find indications of a virtual bound state at $N_c = 3$, which may be related to one of the aforementioned exotic states.

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1. QCD in the large N_c limit

The 't Hooft or large N_c limit of QCD [1] is the limit in which the number of colors, N_c, is taken to infinity, while keeping the number of quark flavors, N_f, constant. QCD simplifies in this limit, but retains most of its non-perturbative features, such as asymptotic freedom, confinement and spontaneous chiral symmetry breaking. The limit also makes it possible to characterize how different observables scale with N_c and N_f through a perturbative analysis of correlation functions at large N_c. In many cases, large N_c predictions have proven accurate in the low-energy regime, and have been used in phenomenological approaches to QCD. In other cases, however, these predictions fail to reproduce experimental results, due to large subleading N_c corrections, which are hard to estimate analytically. The lattice regularization of QCD, on the other hand, enables the determination of these corrections from first principles.

Several works have studied the large N_c limit of QCD on the lattice—see ref. [2] and references therein. We have recently investigated the scaling of different meson observables using lattice QCD simulations at varying N_c in a theory with $N_f = 4$ degenerate quark flavors.[1] These observables include the pion mass, M_π, and decay constant, F_π, [3], non-leptoning kaon decays [4], leading to a better understanding of the $\Delta I = 1/2$ puzzle, and pion-pion scattering near threshold [5]—see also ref. [6] for another recent study of the large N_c scaling of the pion-pion scattering length.

In the context of meson-meson scattering, the large N_c limit has been used by different phenomenological investigations. A notorious example is ref. [7], where the large N_c limit was used in combination to the inverse amplitude method (IAM) [8, 9] to investigate the dependence of resonance poles with N_c. This study, however, neglects subleading N_c corrections.

Another timely question is the possible existence of tetraquark states in the large N_c limit. The standard lore, due to Witten [10] and Coleman [11] precludes their existence based on the factorizations property. According to this property, the correlation function of a tetraquark state would decompose into that of two non-interacting mesons at large N_c. Weinberg [12] recently presented counterarguments to this picture, indicating that tetraquarks could arise from poles in the subleading connected part of the correlation functions, and they would have a width $\Gamma_{\text{tetra}} \sim \mathcal{O}(N_c^{-1})$ analogously to other standard resonances. This latter claim was later refined in refs. [13, 14], finding that the width of tetraquark states would depend on their flavor content, with tetraquarks with four open flavors being narrower, $\Gamma_{\text{tetra}} \sim \mathcal{O}(N_c^{-2})$.

In this work, we study the N_c dependence of meson-meson scattering observables using lattice QCD simulations, extending the analysis in ref. [5] beyond threshold, with a two-fold objective. First, to characterize the N_c scaling of meson-meson scattering amplitudes, focusing on subleading corrections to the large N_c limit, and more specifically, to match lattice results to chiral perturbation theory (ChPT) and constrain the scaling of the relevant low-energy constants (LECs) from ChPT. Second, we aim to investigate the existence of tetraquark states in the limit and, if they are found, to characterize their nature as a function of N_c.

We work in the same setup used in refs. [3–5], with $N_f = 4$ degenerate quark flavors. In this context, meson-meson scattering observables classify in seven different scattering channels,

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[1]All pseudoscalar mesons are degenerate in a theory with $N_f = 4$ degenerate quarks, and so we refer to them generically as "pions" (π).

corresponding to different irreducible representations (irreps) of the isospin group, $SU(4)_f$, two of which are degenerate [15]. We focus on three of them:

- The 84-dimensional irrep, known as the *SS* channel, which is analogous to the isospin-two channel of two-flavor QCD.
- The 20-dimensional irrep, known as the *AA* channel, which is antisymmetric in both quarks and antiquarks, and only exists for $N_f \geq 4$.
- One of the two degenerate 45-dimensional irreps, known as the *AS* channel, which is anti-symmetric in quarks and symmetric in antiquarks.

Both the *SS* and *AA* channels only contain even partial waves, while the *AS* channel contains odd partial waves.

For all these channels, the large N_c limit allows one to characterize the N_c and N_f dependence of the scattering amplitudes. For the *SS* and *AA* channels, one finds

$$\mathcal{M}_2^{SS,AA} = \mp \frac{1}{N_c}\left(a + b\frac{N_f}{N_c} \pm c\frac{1}{N_c}\right) + \mathcal{O}(N_c^{-3}). \tag{1}$$

Both channels have a common large N_c limit with opposite sign, while subleading corrections are separated in two terms: one proportional to N_f that appears with the same sign, and another one independent of N_f with opposite signs. The coefficients a, b and c are numerical constants depending on the kinematics. In general, they cannot be determined analytically, but can be constrained using lattice simulations. What is more, by combining lattice results for both channels, it is possible to disentangle the two subleading terms even if simulating at fixed $N_f = 4$, as was done in ref. [5]. For the *AS* channel, on the other hand,

$$\mathcal{M}_2^{AS} = d\frac{N_f}{N_c^2} + e\frac{1}{N_c^2} + \mathcal{O}(N_c^{-3}). \tag{2}$$

where d and e are again unknown constants. The scattering amplitude in this channel is suppressed by an additional power of N_c, with the cancellation of the leading $\mathcal{O}(N_c^{-1})$ term being related to the kinematic properties of this channel.

In ref. [5], both the *SS* and *AA* channels were investigated near threshold and results were matched to ChPT, successfully constraining subleading corrections to the large N_c scaling of the relevant LECs. The *AA* channel was found to have attractive interactions, making it a candidate to contain a tetraquark state. A qualitative study based on the IAM supported the presence of such a state at higher center-of-mass (CM) energies. However, a quantitative study of this possibility requires of a dedicated lattice study, on which we focus in this work.

The putative existence of a tetraquark resonance in some of the channels under study is further supported by recent experimental findings. In recent years, LHCb has reported on the discovery of several scalar tetraquark states: the $T_{cs0}^0(2900)$ in the invariant mass of $D^- K^+$ [16, 17], and the $T_{c\bar{s}0}^{++}(2900)$ and $T_{c\bar{s}0}^0(2900)$ in the invariant mass of $D_s^+ \pi^+$ and $D_s^+ \pi^-$, respectively [18, 19]. While all these states are expected to be part of some isospin triplets in nature, they all lie on the *AA* channel in a theory with $N_f = 4$ degenerate flavors. In addition, LHCb has a also found a vector tetraquark in the invariant mass of $D^- K^+$, known as $T_{cs1}^0(2900)$ [16, 17], which would lie on the *AS* channel in our setup. It is worth remarking that all these exotic states have been phenomenologically described as vector-meson molecules, due to their proximity to the $D^* K^*$ and $D_s^* \rho$ thresholds [20].

2. Lattice results for the finite-volume energy spectra

Lattice simulations make it possible to determine two-particle finite-volume energies, which can be later used to constrain infinite-volume scattering observables [21, 22]. In this work, we use four different ensembles with $N_c = 3 - 6$ at fixed $M_\pi \approx 590$ MeV and lattice spacing $a \approx 0.075$ fm, generated using the Iwasaki gauge action and $N_f = 4$ flavors of dynamical clover-improved Wilson fermions with periodic boundary conditions. Correlation functions are measured using a unitary setup with identical action on the valence sector. The lattices have spacial and temporal extent $(L/a)^3 \times (T/a) = 24^3 \times 48$ ($N_c = 3$) and $20^3 \times 36$ ($N_c = 4 - 6$). Simulations are performed using the HiRep code [23, 24].

To determine finite-volume energies, we use a large set of operators to compute a matrix of correlation functions. More concretely, we three types of operators: operators with the form of two-meson states with different relative momenta, either two pions ($\pi\pi$) or two vector mesons ($\rho\rho$), and local tetraquark operators. In all cases, we consider several values of the total momentum, $\boldsymbol{P}$, and study all cubic group irreps that contain pion-pion states with s-wave or p-wave, for the SS and AA, and AS channels, respectively. Operators with the form of two vector mesons are included to accurately determine the finite-volume energy spectrum below the four-pion inelastic threshold, since vector mesons are stable in our ensembles, $M_\rho \sim (1.7 - 2)M_\pi$. The use of vector-meson and tetraquark operators, moreover, could be very important to identify the presence of a tetraquark resonance [25]. To compute correlation functions involving a two-particle operator at source, we use time- and spin-diluted $\mathbb{Z}_2 \times \mathbb{Z}_2$ stochastic sources. For those cases with a tetraquark operator at source, on the other hand, we make use of point sources located on a sparse lattice [26],

$$T_{\rm sp}(x) = \sum_{\boldsymbol{x} \in \Lambda_{\rm S}} {\rm e}^{-i\boldsymbol{P}\boldsymbol{x}} \left[\bar{q}_{f_1}(x)\Gamma_1 q_{f_2}(x)\bar{q}_{f_3}(x)\Gamma_2 q_{f_4}(x) \right] , \tag{3}$$

where $x = (\boldsymbol{x}, t)$, Γ_i are products of Dirac gamma matrices, flavor indices are used to project to the channels of interest and

$$\Lambda_{\rm S}(t) = \{ a(t/a, s_1 + dn_1, s_2 + dn_2, s_3 + dn_3) \mid n_i \in \mathbb{Z}, 0 \leq n_i < (L/a)/d, 0 \leq s_i < d \} , \tag{4}$$

is the sparse lattice, with $d = 4$ and s_i some offsets that are randomly chosen on all configurations and ensure we recover the correct correlation functions.

After computing the matrix of correlation functions, we use a generalization of the shift-reweighting technique [27] to mitigate thermal effects related to the finite time extent of our lattices. We then solve a generalized eigenvalue problem (GEVP) [28, 29] and extract the finite-volume energies from one-state correlated fits to the eigenvalues of the GEVP. These fits are repeated over several fit ranges and the final results are determined from an average using weights based on the Akaike Information Criterium [30]. The eigenvectors of the GEVP are related to the overlap of the operators into the finite-volume states, and are used to identify those states that are predominantly two pions, on which we focus our amplitude study.

It is instructive to study the impact on the results for the finite-volume energies of varying the operator set used to compute the matrix of correlation functions. In fig. 1 we show the finite-volume energy spectrum for the AA channel with $N_c = 3$, for three choices of the operator set. Horizontal solid and dashed lines are the free energies of two pions and two vector mesons, respectively,
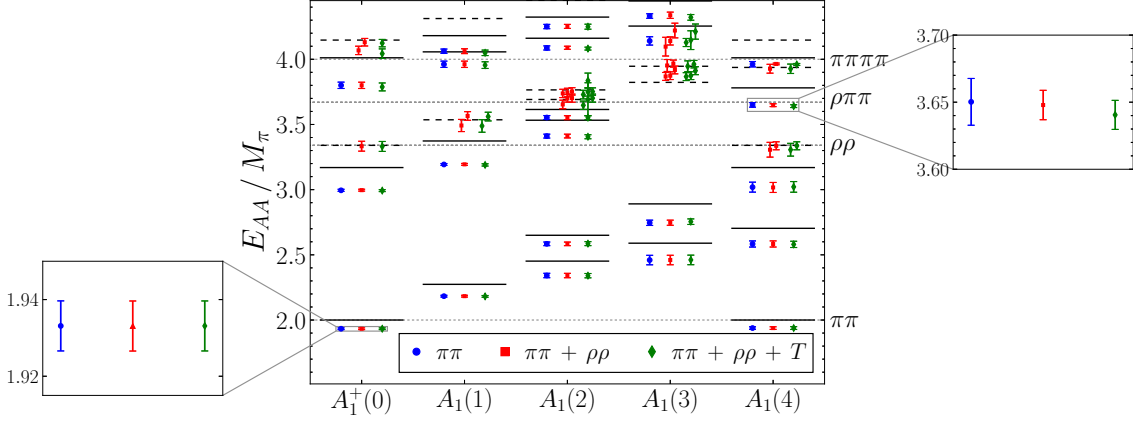
Figure 1: Preliminary results for the CM finite-volume energy spectra of the AA channel with $N_c = 3$. Each column corresponds to a different irrep of the cubic group and momentum frame, with the number in parenthesis indicating $|\boldsymbol{P}|^2$ in units of $(2\pi/L)^2$, and each marker type refers to a different set of operators used to solve the GEVP. Horizontal segments are the free energies of two pions (solid) and two vector mesons (dashed), while dotted horizontal lines are the most relevant inelastic thresholds.

and dotted horizontal lines indicate relevant inelastic thresholds. We observe that the inclusion of vector-meson and tetraquark operators has little effect on the determination of the finite-volume energies of two-pion states, although it leads to the appearance of states that can be assigned to two vector mesons. The effect is completely negligible in the case of the lowest-lying states, although the inclusion of additional operators helps at reducing the errors for excited states. This can be observed in the zoomed-in panels in fig. 1.

3. Results for the finite-volume scattering amplitude

Two-particle finite-volume energies can be used to constrain infinite-volume scattering amplitudes using the so-called two-particle quantization condition (QC). This was first developed for two identical scalar particles [21, 22], and has since be extended to include any two-particle system [31–39]. In general, the two-particle QC takes the form of a matrix equation,

$$\det\left[\mathcal{K}_2^{-1}(E) + F(L, \boldsymbol{P}; E)\right] = 0 \,, \tag{5}$$

where $\mathcal{K}_2$ is the two-particle K-matrix, related to the infinite volume scattering amplitude, and F is a geometric factor that only depends on the lattice geometry and the total momentum, and contains power-law finite-volume effects. The solutions to this equation are the finite-volume energies.

When considering the simple case of single-channel scattering in the lowest partial wave ℓ, eq. (5) can be reduced to an algebraic equation that relates the finite-volume energies to the scattering phase shift, δ_ℓ. In the case of s-wave, it takes the form,

$$k \cot \delta_0 = \frac{2}{\gamma L \pi^{1/2}} \mathcal{Z}_{00}^{\boldsymbol{P}}\left(\frac{kL}{2\pi}\right) \,, \tag{6}$$

where γ is the boost factor to the CM frame, k is the magnitude of the relative momentum in the CM frame and $\mathcal{Z}$ is the generalized zeta function [21, 22, 32]. Similar algebraic relations also hold for p-wave interactions, although the exact form depends on the cubic group irrep [25].
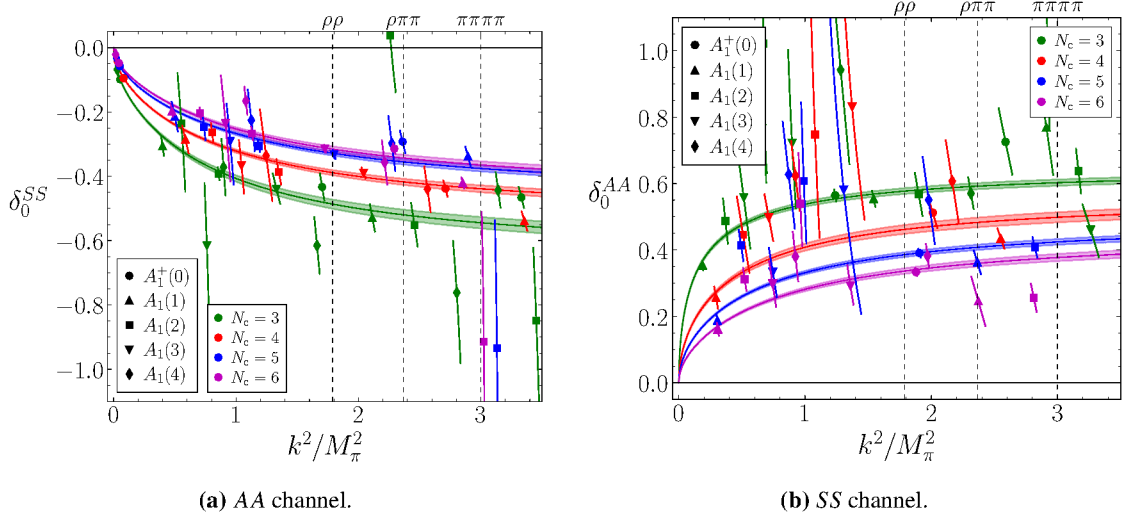
(a) *AA* channel.   (b) *SS* channel.

Figure 2: Preliminary results for the *s*-wave pion-pion scattering phase shift, for different N_c. Dotted lines represent inelastic thresholds, as indicated over the figures, computed using the lightest value of M_ρ among all ensembles. Solid lines and bands are the best-fit results to eq. (7) with $z = M_\pi^2$.

In figs. 2a and 2b, we present preliminary results for the pion-pion scattering phase shift in the *SS* and *AA* channels, respectively, obtained using eq. (6). We also indicate some of the most relevant inelastic thresholds, computed using the lightest mass of the vector and axial mesons among all ensembles. For both channels, we observe that the phase shift grows rapidly in magnitude above threshold, but depends weakly on the energy above $k \gtrsim M_\pi$. We also observe that the interactions become weaker as N_c increases, as expected from large N_c arguments.

To analyze the results in more detail, we constrain the scattering amplitude by fitting the finite-volume energies to predictions obtained with the QC for a given phase-shift parametrization. We perform fits to a modified effective range expansion [40, 41], that includes the effect from an Adler zero,

$$\frac{k}{M_\pi} \cot \delta_0 = \frac{M_\pi E}{E^2 - 2z} \left(B_0 + B_1 \frac{k^2}{M_\pi^2} + ... \right), \tag{7}$$

where we keep the location of the pole fixed at $z = M_\pi^2$ and set higher-order terms to zero. Fits are performed at fixed N_c and the results are presented as solid lines and one-sigma error bands in figs. 2a and 2b.

In both channels, this parametrization leads to an accurate description of the lattice data. Our results for the *AA* channel do not indicate the presence of any tetraquark resonance above threshold, but results for $N_c = 3$ are consistent with a virtual bound state at $E_b/M_\pi = 1.741(13)$. The nature of this state and its possible relation to experimental findings is under investigation.

In addition, the fit results allow us to qualitatively study the sensitivity to subleading N_c corrections. In fig. 3 we present these fit results multiplied by a factor that eliminates the expected leading N_c scaling. In both channels we observe that, after cancelling the leading N_c factors, the phase shift is of similar magnitude for all N_c. However, we also note that we are sensitive to subleading corrections. In the *SS* channel, results for $N_c = 6$ seem to be unnaturally separated from the rest, which we are currently investigating.
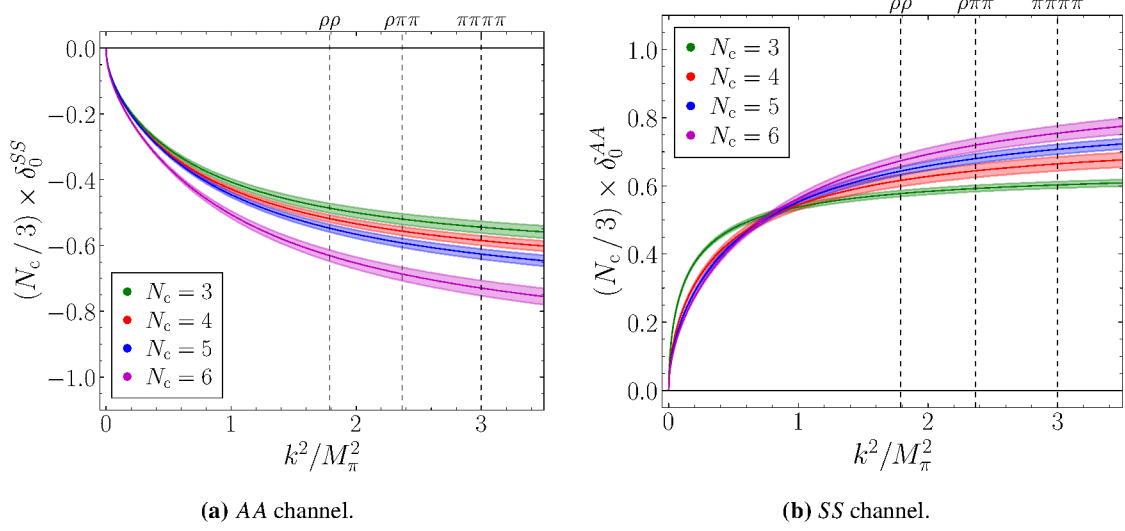
(a) *AA* channel.   **(b)** *SS* channel.

Figure 3: Preliminary results for the s-wave pion-pion scattering phase shift, multiplied by $N_c/3$ to eliminate the leading N_c dependence.

We also analyze the sensitivity to subleading corrections by studying the N_c scaling of different scattering observables. In fig. 4 we present results for the scattering length, a_0, and effective range, r_0, which can be related to the parameters in eq. (7). Results for the scattering length are normalized by leading-order (LO) prediction from ChPT. For both quantities, we observe that our results for $N_c = 4 - 6$ are well described by a linear extrapolation towards the large N_c limit, with large $\mathcal{O}(N_c^{-2})$ corrections seemingly present for $N_c = 3$. However, our results for a_0 are not consistent between the two channels at large N_c as expected from eq. (1). This, however, could be related to mismatches in
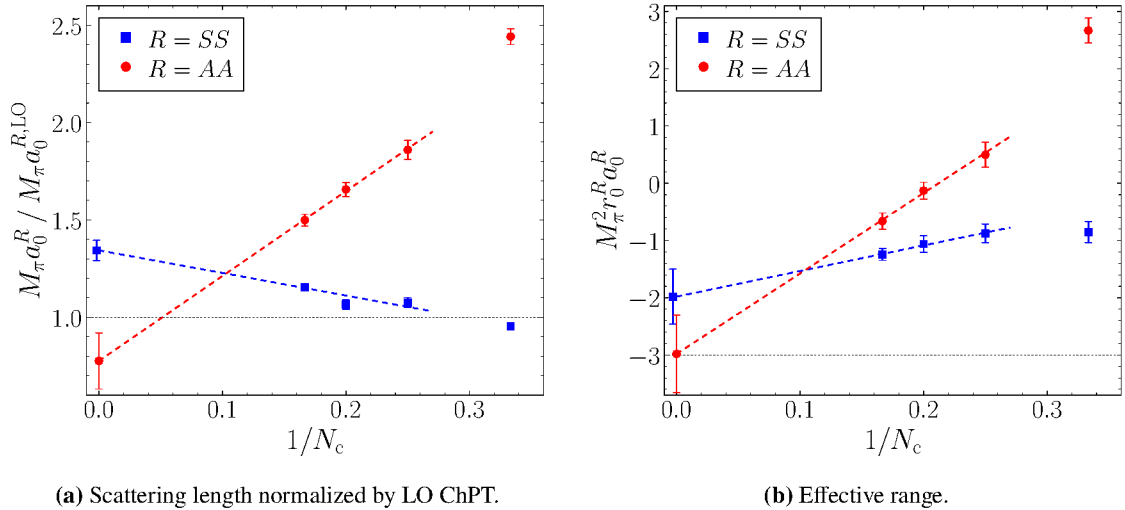

(a) Scattering length normalized by LO ChPT.   **(b)** Effective range.

Figure 4: Preliminary results for the large N_c dependence of the scattering length and effective range in the SS and AA channels, together with the LO predictions from ChPT. Results for $N_c = 4 - 6$ are used to linearly extrapolate to $N_c \to \infty$.

the pion masses between ensembles. To more accurately constrain the large N_c limit, one would need to incorporate the pion-mass dependence, which we plan to do via a fit to one-loop ChPT.

Finally, we present results for the *AS* channel in fig. 5, obtained using the *p*-channel counterparts of eq. (6). In this case, we observe the phase shift is very close to zero, indicating very weak interactions in this channel. This is in agreement with the large N_c limit, that predicts the scattering amplitude to be suppressed as $\mathcal{O}(N_c^{-2})$—see eq. (2)—, and also with ChPT, in which the amplitude is zero at LO, $\mathcal{M}_2^{AS,\text{LO}} = 0$.
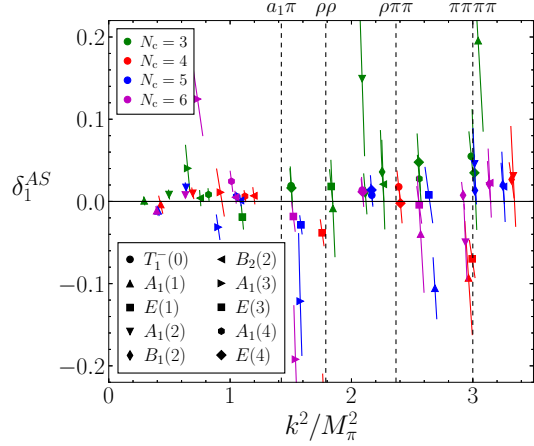
Figure 5: Preliminary results scattering phase shift in the *AS* channel, for $N_c = 3 - 6$.

4. Summary and outlook

We have presented preliminary results for an ongoing study of the N_c dependence of meson-meson scattering amplitudes using lattice QCD, working in a theory with $N_f = 4$ degenerate quark flavors. Using a large set of operators, containing two-particle and local tetraquark operators, we have determined the finite-volume energy spectra for the *SS*, *AA* and *AS* channels with $N_c = 3 - 6$. These results have been used to constrain the infinite-volume pion-pion scattering phase shift up to the four-pion inelastic threshold.

Preliminary results show sensitivity to subleading N_c corrections and we aim at matching these results to ChPT to constrain the N_c scaling of the LECs appearing in the effective theory. Our results also suggest the presence of a virtual bound state in the *AA* channel for $N_c = 3$, which is currently under investigation. In particular, its existence needs to be established also with other parametrizations of the scattering amplitude. Should strong evidence of its existence be found, a compelling future venue could be the study of the dependence of the state on the pion mass, which might shed some light into the nature of the tetraquark states found at LHCb.

Acknowledgments

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