Prepared for submission to JHEP YITP-25-32, DESY-25-041

Holographic Interfaces in Symmetric Product Orbifolds

Sebastian Harris,[a,b] **Yasuaki Hikida,**[c] **Volker Schomerus,**[a,b,d] **and Takashi Tsuda**[c]

[a] *Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, D-22607 Hamburg, Germany*
[b] *Zentrum für Mathematische Physik, Bundesstrasse 55, D-20146 Hamburg , Germany*
[c] *Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan*
[d] *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany*

E-mail: sebastian.harris@desy.de, yhikida@yukawa.kyoto-u.ac.jp, volker.schomerus@desy.de, takashi.tsuda@yukawa.kyoto-u.ac.jp

Abstract: The study of non-local operators in gauge theory and holography, such as line-operators or interfaces, has attracted significant attention. Two-dimensional symmetric product orbifolds are close cousins of higher-dimensional gauge theory. In this work, we construct a novel family of interfaces in symmetric product orbifolds. These may be regarded as two-dimensional analogues of Wilson-line operators or Karch-Randall interfaces at the same time. The construction of the interfaces entails the choice of boundary conditions of the seed theory. For a generic seed theory, we construct the boundary states associated to the interfaces via the folding trick, compute their overlaps and extract the spectrum of interface changing operators through modular transformation. Then, we specialise to the supersymmetric four-torus $\mathbb{T}^4$ and show that the corresponding interfaces of the symmetric product orbifold are dual to AdS_2 branes in the tensionless limit of type IIB superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$.

Contents

1 Introduction

Symmetric product orbifolds of two-dimensional conformal field theories (CFTs) have been studied extensively for more than thirty years starting with [1]. They provide families of two-dimensional CFTs whose central charge grows linearly with the number N of copies of the underlying seed theory $\mathcal{M}$. Correlation functions of local operators in these theories share many features with those of higher-dimensional gauge theories. In particular, they possess a diagrammatic large N expansion that is very similar to the 't Hooft expansion of non-Abelian gauge theories [2]. Just as it is the case for their higher-dimensional cousins, these features of symmetric product orbifolds are suggestive of their holographic correspondence with string theory in AdS_3 backgrounds, as was first suggested in [3–5]. The work of Eberhardt, Gaberdiel and Gopakumar [6] established that the symmetric product orbifold of a supersymmetric four-torus is dual to string theory on AdS_3 with one unit of pure NSNS flux. This has become one of the key examples for a holographic duality and it is one that could help to uncover the inner workings of the AdS/CFT correspondence.

The study of symmetric product orbifolds has largely focused on the spectrum and correlation functions of local fields. There are only relatively few papers that deal with boundary conditions, defects or interfaces, see [7–12]. This is in some contrast with higher-dimensional gauge theories, and in particular in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, where the study of e.g. Karch-Randell interfaces and Maldacena-Wilson lines has received very considerable attention, both for their roles in gauge theory and for the insights they provide on non-perturbative aspects of string theory in anti-de Sitter (AdS) space. In this work, we initiate a study of interfaces in symmetric product orbifolds that can be considered the lower-dimensional cousins of both Karch-Randell interfaces and of Wilson lines at the same time.

Before we go into the features of the interfaces between symmetric product orbifolds, it is instructive to briefly recall basic facts about the higher-dimensional analogues. The Karch-Randall interface connects two maximally SYM theories with gauge groups $\mathrm{SU}(N_+)$ and $\mathrm{SU}(N_-)$ on either side of the interface. It arises from a D3-D5 brane system [13], where D3 branes end on D5 branes.[1] Wilson lines in $\mathcal{N} = 4$ SYM theory possess different holographic realisations depending on a choice of representation of the gauge group. The most standard case is that of the fundamental representation in which case the brane realisation is in terms of a single fundamental string in AdS_5. For representations that arise through the tensor product of k fundamental representations with k of order N, which is more relevant in our context, the brane realisation uses D-branes instead, either D3 or D5 branes depending on whether the representation is symmetric or antisymmetric.[2] In both cases, the charge of k fundamental strings is carried by the electric flux of the D-brane, see e.g. [14] and references therein, in particular [15–18]. Such branes with non-vanishing electric flux approach the boundary of AdS at an angle that depends on the amount of flux.

Let us now turn to the two-dimensional case. Certain symmetric product orbifolds can be thought of as describing excitations of a system of Q_1 fundamental strings and Q_5 NS5 branes which wrap some four-dimensional compactification manifold, such as $\mathbb{T}^4$ or $K3$. We can create a defect in this theory by inserting a probe D1 brane into the dual AdS_3 background. We shall focus on the case in which this probe brane is localised along an AdS_2 inside AdS_3. More generally, we can consider $(n, 1)$-strings, i.e. bound states of a D1 brane and n fundamental strings, see e.g. [19]. As in the case of Wilson lines reviewed in the previous paragraph, one may think of these $(n, 1)$ strings as a D1 brane with a worldvolume electric field turned on with a strength that is determined by n. When we allow n fundamental strings to run along the worldvolume of the probe D1 brane, the number of fundamental strings that appear in the brane realisation of the symmetric

[1] More precisely, one considers configurations in which q of the N_+ D3 branes that realise the $\mathrm{SU}(N_+)$ SYM theory end on the D5 brane while $N_+ - q = N = N_-$ run through it to realise the $\mathrm{SU}(N_-)$ SYM theory on the other side.

[2] Other representations of the gauge group whose Young diagram have more than one row or column require multiple probe branes.

product orbifold can jump from one side of the D1 brane junction to the other, just as in the case of Karch-Randall interfaces in gauge theory.

In contrast to the higher-dimensional theories, string theory on AdS_3 with a pure NSNS-background is famously solvable using a description in terms (non-compact) Wess-Zumino-Novikov-Witten (WZNW) models. The related WZNW model on the sphere with hyperbolic target space H_3^+ was first solved by Teschner [20, 21]. With input from Teschner's CFT analysis, Maldacena and Ooguri were able to construct a theory of closed strings on AdS_3 background with pure NSNS flux [22, 23]. A central ingredient of their work was the inclusion of spectrally flowed representations of the underlying $\mathfrak{sl}(2)$ affine Kac-Moody algebra. These were needed in order to describe long strings . The worldsheet analysis of branes and open strings in these models was initiated in [19] with a classification of possible brane geometries. Maximally symmetric branes, i.e. branes that preserve the affine Kac-Moody algebra, can be localised along several different submanifolds. Here, we shall mainly deal with AdS_2 branes. In general, D-branes may be described by boundary states of the worldsheet CFT which encode all information about the open string spectrum and the couplings with closed strings. Boundary states for AdS_2 branes (along with spherical branes) have been constructed in [24, 25] building upon some preliminary studies in [26–30]. In agreement with the geometric intuition we recalled above, there is a continuous family of such boundary states that is parametrized by one real parameter that is related to the angle at which the AdS_2 brane approaches the boundary. Moreover, the branes support a family of 'half-winding' long open strings with arbitrarily large winding number. The half-winding is related to the fact that the AdS_2 brane approaches the asymptotic boundary of AdS_3 at two opposite sides which are mapped onto each other by a half-rotation of AdS_3. While the spectrum of open strings on the AdS_2 branes is rather rich, they only couple to a very small subset of the bulk states in the WZNW model, namely to states of winding number $w = 0$ only.

The goal of our work is to address the holographic description of AdS_2 branes and to construct the interface between symmetric orbifold theories that is created when an AdS_2 brane ends on the boundary of AdS_3, see also [9] for a very insightful previous discussion. More precisely, we construct a family of interfaces between any pair of symmetric product orbifolds of rank N_- and N_+ that are parametrised by some integer $p = 1, \ldots, \min(N_-, N_+)$. The interfaces $\mathcal{I}^{(p)}_{|a_\pm\rangle}$ that we define are transmissive in p components of the product theory and reflective in the remaining ones with a reflective behaviour that depends on the choice of a boundary conditions $a_\pm$ of the underlying seed conformal field theory $\mathcal{M}$. A precise definition is given in section 2.2, see eq. (2.38) and the explanation of notation below that formula. Our description uses the folding trick and hence it describes the interface in terms of a boundary state of the folded theory, as shown in figure 1. As usual, the boundary state encodes the entire set of one-point couplings between the interface and bulk fields of the CFT. From this description of the interfaces, we then determine the annulus amplitude and thereby the exact spectrum of interface changing operators between any pair of interfaces in section 2.4. This key result is stated in eqs. (2.54 - 2.59). As is familiar in the context symmetric product orbifolds since the seminal paper of Dijkgraaf-Moore-Verlinde-Verlinde

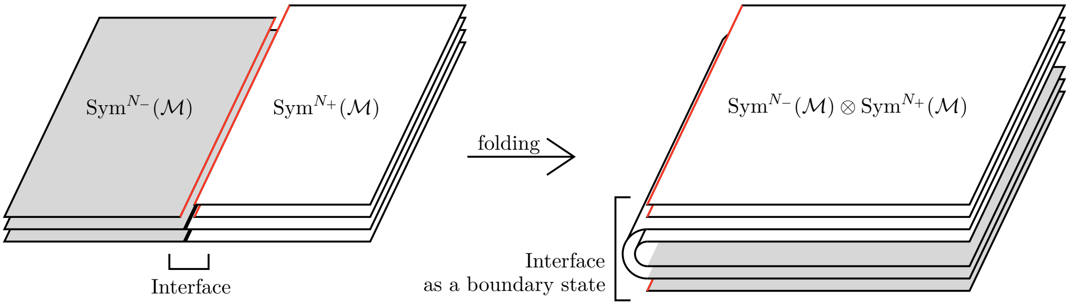

Figure 1. Interface between $\mathrm{Sym}^{N_-}\mathcal{M}$ and $\mathrm{Sym}^{N_-}\mathcal{M}$ as a boundary state of the folded theory. The red lines indicate reflective boundaries. This figure is for $N_- = 3$, $N_+ = 4$, $p = 2$.

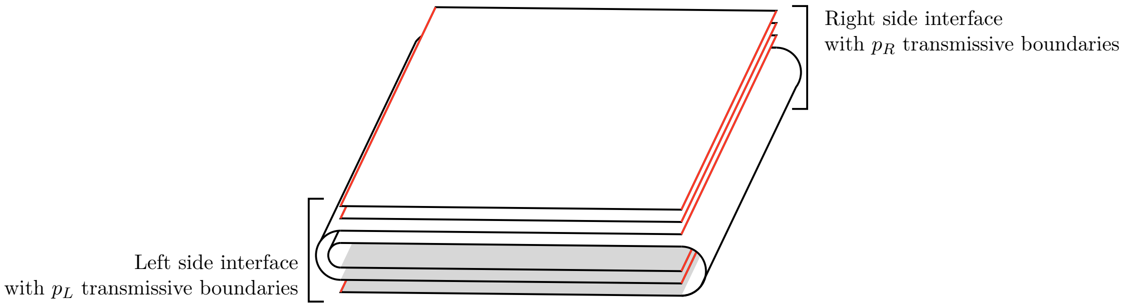

Figure 2. Contribution to the torus partition function in the presence of a pair of interfaces with parameters $N_- = 3$, $N_+ = 4$, $p_L = 2$, $p_R = 1$. The red lines indicate reflective boundaries. As discussed in Section 2.4, the full partition function is a symmetrised sum over terms of the type illustrated in this figure.

[31], it is easiest to state the result in terms of a grand canonical partition function. In the setup under consideration, this grand canonical partition function involves four chemical potentials $\mu_\pm$ and ρ_A with $A \in \{L, R\}$. The first two, i.e. μ_- and μ_+, correspond to the order $N_\pm$ of the symmetric product orbifold on the two sides of the interfaces. The other two chemical potentials, ρ_L and ρ_R, are associated with the numbers p_L and p_R to the left and the right of the interface changing operator, see figure 2.

A key objective of this paper is to show that the interfaces constructed in section 2.2 provide a holographic description of AdS_2 branes (and the open strings thereon) in type IIB superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$. In the presence of k units of NSNS-flux, the dual CFT is supposed to be given by a marginal deformation of symmetric product orbifold of $\mathbb{T}^4$. When $k = 1$, i.e. for minimal amount of NSNS-flux, the dual CFT has been argued to coincide with the orbifold fixed point [6], without any need for further deformation. As evidence, Gaberdiel et al. were able to show that the spectrum of the string theory agreed with that of the symmetric product orbifold CFT. Later, the matching was extended to string amplitudes and correlation functions. In particular it was understood that the string worldsheet in calculations of closed string amplitudes localises to the boundary of the AdS_3 target space. This has been exploited to calculate string amplitudes and compare them with the corresponding correlation functions in the dual CFT, see in particular [32, 33]. The holographic description of spherical branes in the tensionless limit of type IIB superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ was addressed in [8]. More concretely, it was found that (some) spherical branes are dual to special cases of the boundaries in symmetric product orbifolds that were studied extensively by Belin et al. in [7] by applying standard constructions of boundary CFT, see in particular [34]. With the exception of appendix C, which contains some new perspectives on correlation functions in the presence of spherical branes, our work will focus on AdS_2 branes. In the tensionless limit, these possess a few special features that deserve to be spelled out right away. While for generic values of the level k there exists a continuous family of AdS_2 branes, the tensionless limit of IIB superstring theory admit a unique AdS_2 brane in the AdS_3 factor of the target space.[3] Quite remarkably, the couplings of this unique AdS_2 brane to (physical) closed string states are trivial. To understand this, recall that AdS_2 branes only couple to bulk states with winding number $w = 0$. But the tensionless string has no non-trivial physical states with $w = 0$ and hence the AdS_2 brane appears transparent to closed strings. We shall discuss below how all these features emerge from the proposed holographic description. In order to do so we extend the match of bulk partition functions [6] to the case of AdS_2 branes. Our matching of spectra is based on formulas for the partition function of open strings on AdS_2 branes in (global and) thermal AdS_3 backgrounds that we derive in section 3, see in particular eq. (3.34). For a concise summary of the holographic relation we propose, the reader might also find the box on page 31 useful.

Let us now briefly outline the plan of this paper. In many respects, section 2 contains the main new results of this work. After a short review 2.1 of symmetric product orbifolds that we use to set up notations, section 2.2 constructs the interfaces $\mathcal{I}^{(p)}_{|a_\pm\rangle}$. As a small interlude, we compute the transmissivity and reflectivity of these interfaces in terms of the parameter p in section 2.3. Then, in section 2.4, we calculate the spectrum of interface changing operators. While our construction of the interface may seem somewhat artificial, it is guided by the intention to construct a holographic dual of AdS_2 branes in AdS_3. To show that this objective is indeed achieved, at least in the tensionless limit of type IIB superstrings on $AdS_3 \times S^3 \times \mathbb{T}^4$, we discuss the string theory of AdS_2 branes in section 3.1. The boundary states of the world sheet theory corresponding to AdS_2 branes have been constructed in [8]. We briefly review this construction and then, in section 3.2, use it to determine the spectrum of worldsheet boundary excitations associated to open string states on the AdS_2 branes in global AdS_3. The result serves as an input to section 3.3, where we perform an orbifold construction to pass to thermal AdS_3. After integration over the modular parameter of the toroidal worldsheet, we finally obtain the partition function of open superstrings in thermal AdS_3. The two strands of the discussion are merged in

[3] In [8] the authors constructed two boundary states. Here we only admit one special linear combination thereof in order to restore the rotations symmetry with respect to half-rotations of AdS_3. This additional feature makes the boundary state unique.

section 4 where we compare the string partition function with the "single particle" spectrum of our interface in the symmetric product orbifold of the supersymmetric four-torus to find complete agreement between these two quantities. The section also contains some qualitative discussion of correlation functions in the symmetric product orbifold and the corresponding scattering amplitudes of open and closed strings in the dual string theory. A more quantitative analysis will appear in forthcoming work. This and other future directions are discussed in the concluding section 5. The work contains three appendices in which we provide some explicit calculations to prove formulas that are stated in the main text.

2 Interfaces of symmetric product orbifolds

The arguably simplest type of interfaces are those that are purely reflecting or purely transmitting. Purely reflecting interfaces can be obtained by choosing boundary conditions for the two CFTs which are separated by the interface, resulting simply in a product of two essentially decoupled boundary theories. In the specific context of symmetric product orbifolds, a detailed analysis of the spectrum of boundary states has been provided by [7]. The most prominent example of a purely transmitting interface is the trivial interface between two identical CFTs. For symmetric product orbifolds, a broader class of purely transmitting defects was studied in [10].

In this section, we introduce a more general family of interfaces between two different symmetric product orbifolds $\mathrm{Sym}^{N_\pm}(\mathcal{M})$ of the same seed theory $\mathcal{M}$ that interpolates between the purely transmitting and purely reflecting extremes. For this purpose, section 2.1 establishes the general notation which we use to describe symmetric product orbifolds. Section 2.2 gives a precise definition of our interfaces. Section 2.3 serves the purpose of making our definitions more accessible by demonstrating how to compute the reflectivity and transmissivity coefficients of our interface. It does not contain any information that is strictly necessary for the rest of the discussion and may be skipped. Finally, section 2.4 determines partition functions of interface changing operators that act within the class of interfaces that we have proposed.

2.1 Notation and conventions

This section sets up the necessary notation to work with symmetric product orbifolds $\mathrm{Sym}^N(\mathcal{M})$ of a CFT $\mathcal{M}$. The Hilbert space of the seed theory $\mathcal{M}$ is denoted by $\mathcal{H}_\mathcal{M}$. It comes equipped with two commuting actions of the Virasoro algebra along with possibly further modes of some extended chiral algebra $\mathcal{W}$. Under the action of this chiral algebra, the state space decomposes into sectors which we label by letters $i, j, \dots$ of the Roman alphabet. In a unitary CFT, these are lowest weight representations. The partition function of the seed theory $\mathcal{M}$ is given by

$$Z(t,\bar{t}) = \mathrm{Tr}_{\mathcal{H}_\mathcal{M}}[x^{L_0-\frac{c_\mathcal{M}}{24}}\bar{x}^{\bar{L}_0-\frac{c_\mathcal{M}}{24}}], \quad x = e^{2\pi i t}, \quad \bar{x} = e^{-2\pi i\bar{t}}. \tag{2.1}$$

Here, $c_\mathcal{M}$ denotes the central charge of the Virasoro algebra, as usual. With these basic ingredients of the seed theory set up, we move on to the symmetric product orbifold.

States of the symmetric product orbifold $\mathrm{Sym}^N(\mathcal{M})$ are obtained by the standard rules of the orbifold construction. In a first step, we construct twisted sectors $\mathcal{H}^g$ for the N^{th} power $\mathcal{M}^{\otimes N}$ of the seed theory. These are labelled by group elements $g \in S_N$ of the symmetric group. For a given element g, we define the centraliser subgroup

$$\mathcal{C}_g = \mathcal{C}_g^N = \{\, \sigma \in S_N \,|\, \sigma g \sigma^{-1} = g \,\} \subseteq S_N . \tag{2.2}$$

The twisted sector $\mathcal{H}^g$ carries a representation of $\mathcal{C}_g$ and, as part of the orbifold procedure, we are instructed to project to its $\mathcal{C}_g$ invariant subspace. The projection can be performed by averaging over the orbits of $\mathcal{C}_g$. Hence, the torus partition function of the symmetric product orbifold is given by[4]

$$Z_N(t,\bar{t}) \coloneqq \frac{1}{N!} \sum_{g \in S_N} \sum_{h \in \mathcal{C}_g} \mathrm{Tr}_{\mathcal{H}^g} [h x^{L_0 - \frac{N c_\mathcal{M}}{24}} \bar{x}^{\bar{L}_0 - \frac{N c_\mathcal{M}}{24}}]. \tag{2.3}$$

One can straightforwardly express the partition function of the orbifold theory in terms of the seed theory partition function by directly evaluating the trace. The most elegant formula for the partition function [31] passes through the grand canonical ensemble

$$\mathcal{Z}(\kappa; t,\bar{t}) \coloneqq \sum_{N=0}^{\infty} \kappa^N Z_N(t,\bar{t}) = \exp\left[\sum_{k=1}^{\infty} \kappa^k T_k Z(t,\bar{t}) \right]. \tag{2.4}$$

The terms in the exponent on the right hand side are obtained from the partition function $Z(t,\bar{t})$ of the seed theory through application of the following Hecke operators

$$T_k Z(t,\bar{t}) \coloneqq \frac{1}{k} \sum_{w|k} \sum_{j=0}^{w-1} Z\left(\frac{kt}{w^2} + \frac{j}{w}, \frac{k\bar{t}}{w^2} + \frac{j}{w} \right). \tag{2.5}$$

By expanding the exponent on the right hand side in powers of κ, it is straightforward to obtain the partition functions Z_N of symmetric product orbifolds for any finite integer N.

It is important to discuss formula (2.3) in more detail. We have written the formula as a sum over all elements g of the symmetric group S_N. But many of the twisted sectors $\mathcal{H}^g$ we are summing over give rise to equivalent representations of the chiral algebra. By construction, two twisted sectors $\mathcal{H}^{g_1}$ and $\mathcal{H}^{g_2}$ give rise to equivalent representations if g_1 and g_2 belong to the same conjugacy class. We denote the conjugacy class of $g \in S_N$ by

$$[g] \coloneqq \{ hgh^{-1} \,:\, h \in S_N \} \subseteq S_N \tag{2.6}$$

and more generally

$$[g_1, g_2, \ldots, g_n] \coloneqq \{ (hg_1h^{-1}, hg_2h^{-1}, \ldots, hg_nh^{-1}) \,:\, h \in S_N \}. \tag{2.7}$$

Famously, the conjugacy classes $[g]$ of the symmetric group S_N are in one to one correspondence with integer partitions $(w_1, \ldots, w_\ell)$ of $N = \sum_{r=1}^{\ell} w_i$ or, equivalently, Young diagrams

[4] The normalisation factor is $1/N!$ because we divide both by the size of the centraliser $|\mathcal{C}_g|$ (due to the averaging over $\mathcal{C}_g$ orbits) as well as by the size of the conjugacy class of g.

Y with N boxes. The correspondence is easy to spell out more explicitly. Any element $g \in S_N$ possesses a unique (up to reordering of the factors) decomposition

$$g = \omega_1 \omega_2 \cdots \omega_\ell \tag{2.8}$$

into cyclic permutations ω_r of length w_r such that the cycles constitute a partition $\{\omega_r\}_{r=1}^{\ell}$ of $\{1, \ldots, N\}$ into ordered sets. The integer partition $\{w_r\}_{r=1}^{\ell}$ of N is graphically represented by the Young diagram Y_g associated to $g \in S_N$. Conversely, any two group elements that possess a factorisation of the form (2.8) with ℓ cycles of order w_r are conjugate to each other. Given a Young diagram Y with N boxes we can always pick some representative g_Y of the corresponding conjugacy class of S_N.

With the help of the factorisation (2.8) of the group element g into a product of cyclic permutations, it is easy to give a more explicit description of the centraliser group $\mathcal{C}_g^N$. Concretely, $\mathcal{C}_g^N$ is given by

$$\mathcal{C}_g^N := \{\, \sigma = \pi \prod_{r=1}^{\ell} \omega_r^{\nu_r} | \nu_r \in \mathbb{Z} \text{ and } \pi \in S_\ell \text{ such that } w_{\pi r} = w_r \} \simeq \prod_{w=1}^{N} S_{m_w} \ltimes \mathbb{Z}_w^{m_w}, \tag{2.9}$$

where the sequence $(m_w)_{w \in \mathbb{N}}$ is the weight of the diagram Y_g i.e. m_w is the number of rows of length w. By the orbit-stabiliser theorem, the order $|S_N| = N!$ of the symmetric group factorises as

$$|S_N| = |\mathcal{C}_g^N| \cdot |Y_g|, \tag{2.10}$$

where $|Y_g|$ denotes the number of elements in the conjugacy class Y_g of g. After this preparation, it is natural to rewrite our formula (2.3) for the partition function of the symmetric product orbifold as

$$Z_N(t, \bar{t}) = \sum_{|Y|=N} \mathrm{Tr}_{\mathcal{H}^{g_Y}} [\Pi_0^{g_Y} x^{L_0 - \frac{N c_{\mathcal{M}}}{24}} \bar{x}^{\bar{L}_0 - \frac{N c_{\mathcal{M}}}{24}}] \quad \text{where} \quad \Pi_0^{g_Y} := \frac{1}{|\mathcal{C}_{g_Y}^N|} \sum_{\sigma \in \mathcal{C}_{g_Y}^N} \sigma. \tag{2.11}$$

In performing the sum over Young diagrams Y with N boxes, we can pick a single representative $g = g_Y \in S_N$ and then construct the twisted sector $\mathcal{H}^g$, the centraliser $\mathcal{C}_g$ and the projection Π_0^g for that representative. The result is obviously independent of the representative we choose. Thereby, we have rewritten the partition function of the symmetric product orbifold as a sum over partitions of N rather than group elements $g \in S_N$.

In order to illustrate the difference between the formulas (2.3) and (2.11), let us briefly discuss the example of $N = 3$. In this case, the symmetric group S_3 has six elements and the original sum in eq. (2.3) runs over all these six elements. But the six group elements fall into only three conjugacy classes which are associated with the three possible partitions of $N = 3$. The conjugacy class for the partition $Y = [1, 1, 1]$ consists of the unit element $g = e$ and its centraliser group $\mathcal{C}_e^3 \simeq S_3 \ltimes \mathbb{Z}_1^3$ is the entire symmetric group S_3. Next, let us consider the partition $Y = [1, 2]$. The associated conjugacy class contains three elements, namely $g_1 = (1)(23)$, $g_2 = (2)(13)$ and $g_3 = (3)(12)$. Their centraliser subgroups $\mathcal{C}_{g_i}^3 \cong \mathbb{Z}_2$ contain two elements each. Finally, for the trivial partition $Y = [3]$, the conjugacy class consists of the elements $\omega_1 = (123)$ and $\omega_2 = (132)$. Their centraliser group, on the other hand, is now given by $\mathcal{C}_{\omega_i}^3 \cong \mathbb{Z}_3$.

In the following, we shall continue to think of the twisted sectors $\mathcal{H}^g$ as being associated with elements $g \in S_N$ rather than with the Young diagrams Y_g. This is correct as long as we always remember not to sum but to average over the elements of a conjugacy class. Now given a group element g with the factorisation (2.8) of length ℓ, states of the associated twisted sector $\mathcal{H}^g$ take the form

$$|(\psi_1, \ldots, \psi_\ell)\rangle_g \quad \text{where} \quad \psi_r \in \mathcal{H}_\mathcal{M} \tag{2.12}$$

are ℓ states in the seed theory. By abuse of notation, we simply write

$$|\psi_r\rangle_g \quad \text{instead of} \quad |(\psi_r)_{r=1}^\ell\rangle_g. \tag{2.13}$$

Let us assume that these ℓ states ψ_r, $r = 1, \ldots, \ell$ of the seed theory have been chosen to be eigenstates of $(L_0, \bar{L}_0)$ with eigenvalues $(h_r, \bar{h}_r)$. Then the twisted sector state (2.12) is an eigenstate of the symmetric product orbifold Virasoro generators $(L_0, \bar{L}_0)$ with eigenvalues

$$h^g = \sum_{r=1}^{\ell} \left(\frac{h_r}{w_r} + \frac{c_\mathcal{M}}{24} \left(w_r - \frac{1}{w_r} \right) \right) \qquad \text{and} \qquad \bar{h}^g = \sum_{r=1}^{\ell} \left(\frac{\bar{h}_r}{w_r} + \frac{c_\mathcal{M}}{24} \left(w_r - \frac{1}{w_r} \right) \right). \tag{2.14}$$

In particular, its spin is given by

$$J^g = \bar{h}^g - h^g = \sum_{r=1}^{\ell} \left(\frac{\bar{h}_r - h_r}{w_r} \right). \tag{2.15}$$

Only states (2.12) for which the spin is an integer correspond to local operators of the symmetric product orbifold. In order to find the subset of physical states within the twisted sector $\mathcal{H}^g$, one needs to impose the projection to $\mathcal{C}_g$ invariants, see the construction in Π_0 in eq. (2.11). The action of $\sigma \in \mathcal{C}_g$ on twisted sector states (2.12) is given by

$$\sigma|\psi_r\rangle_g = \prod_{r=1}^{\ell} e^{2\pi i \frac{\nu_r}{w_r}(\bar{h}_r - h_r)} |\psi_{\pi r}\rangle_g = |e^{2\pi i \nu_{\pi r}(\bar{L}_0 - L_0)} \psi_{\pi r}\rangle_g. \tag{2.16}$$

Here, we have represented $\sigma \in \mathcal{C}_g$ as a product $\sigma = \pi \omega_1^{\nu_1} \cdots \omega_\ell^{\nu_\ell}$ as explained in eq. (2.9). Furthermore, the Virasoro elements on the right hand side are understood to be those of the symmetric product orbifold i.e. operators with a spectrum as given in (2.14). The exponents ν_r of the single cycles ω_r only enter the phase factor, while π permutes the r labels. In order to help the reader to get used to the notation for symmetric product orbifolds introduced in this section, we use it in appendix A to compute the torus partition function.

2.2 Definition of the interfaces

This section constructs a class of interfaces between $\mathrm{Sym}^{N_-}(\mathcal{M})$ and $\mathrm{Sym}^{N_+}(\mathcal{M})$. We think of these interfaces as separating the upper and lower half of the complex plane with $\mathrm{Sym}^{N_-}(\mathcal{M})$ living in the lower half plane and $\mathrm{Sym}^{N_+}(\mathcal{M})$ in the upper. The definition of our interfaces involves selecting a pair of boundary states $|a_\pm\rangle$ of the seed theory $\mathcal{M}$. These boundary states of the seed theory give rise to a purely reflecting interface, in which all the $N_-[N_+]$ copies of the seed theory in the lower[upper] half plane are simply reflected[5] back. This is obviously a bit too trivial to be interesting. Instead, we propose to consider interfaces in which $p \leq \min(N_-, N_+)$ components can pass through the interface while the remaining ones are reflected.

Qualitative overview of the construction. Before we discuss the precise construction of the interfaces, let us first give a qualitative description of all steps that constitute it. To spell out a concrete formula for the interfaces $\mathcal{I} = \mathcal{I}^{(p)}_{|a_\pm\rangle}$, we utilize the folding trick and describe the interfaces as boundary states of

$$\mathrm{Sym}^{N_-}(\mathcal{M}) \otimes \mathrm{Sym}^{N_+}(\mathcal{M}) = \mathcal{M}^{N_- + N_+} / S_{N_-} \times S_{N_+}. \tag{2.17}$$

In our construction, we first choose a gauge in which p components in each of the theories that we want to be transmitting are singled out. This procedure manifestly breaks the $S_{N_-} \times S_{N_+}$ symmetry of the folded theory down to $S_{N_- - p} \times S_p \times S_p \times S_{N_+ - p}$.

After this first gauge choice, we impose the reflecting boundary condition $|a_-\rangle$ along $N_- - p$ components from the first factor and likewise $|a_+\rangle$ along $N_+ - p$ components from the second factor. The remaining copies are glued together with transmitting boundary conditions. Such a boundary condition is stabilised by the subgroup

$$S_{N_- - p} \times S_{p,\mathrm{diag}} \times S_{N_+ - p} \subseteq S_{N_- - p} \times S_p \times S_p \times S_{N_+ - p} \subseteq S_{N_-} \times S_{N_+} \tag{2.18}$$

and thus the selection of a particular way to transmit the selected p copies of the seed theory in the lower half plane to p copies in the upper half plane constitutes a second gauge choice that breaks $S_p \times S_p$ to the diagonal subgroup $S_{p,\mathrm{diag}} := \{(g,g) | g \in S_p\} \subseteq S_p \times S_p$. To restore the full $S_{N_-} \times S_{N_+}$ symmetry, we finally need to average over the gauge orbits.

In the construction of boundary states of the folded theories, we can select characters for each of the three factors of the stabiliser subgroup within the orbifold group $S_{N_-} \times S_{N_+}$. We denote these characters by (χ_-, χ_p, χ_+). The boundary states we are about to construct are denoted by

$$|p, a_\pm; \chi_\pm, \chi_p\rangle = |p, a_\pm; \chi_\pm, \chi_p\rangle^{N_\pm}_{\mathcal{M}}. \tag{2.19}$$

We shall often drop the sub- and superscripts that refer to the bulk data. In addition, throughout most of our discussion we will set the characters $\chi_\pm, \chi_p$ to be trivial, drop them from the arguments and write

$$|p, a_\pm; N_\pm\rangle = |p, a_\pm; \chi_\pm = \mathbf{1}, \chi_p = \mathbf{1}\rangle^{N_\pm}_{\mathcal{M}}. \tag{2.20}$$

The interfaces obtained from the trivial representation of the stabiliser subgroup turn out to be the ones that are relevant for holography.[6] This is not merely an a posteriori observation, but rather to be expected from first principles: From the string perspective, the representations associated to the characters $\chi_\pm$ and χ_p are the representations of the

[5] See section 2.3 below for a more precise definition of reflectivity and transmissivity in this context.

[6] There is some tension between this observation and the perspective advocated for in [7]. We comment more on this issue in Appendix C where, in the context of correlation functions, it becomes especially relevant.

symmetric group that govern the statistics of multi-string states. If we would like to describe multi-string ensembles with Bose statistics, we should hence make use only of the trivial characters in the dual symmetric orbifold, see also [12]. Moreover, the restriction to fully symmetric representations of the 'gauge group' S_N is also consistent with the situation for Maldacena-Wilson lines in four-dimensional $N = 4$ SYM theory. In the higher dimensional context, Wilson lines in other than symmetric traceless tensor representations require to consider multiple $D3$ branes in the bulk [35] (or $D5$ branes).

Reflective part of the interfaces. After this first qualitative description of our interfaces, let us now proceed with their systematic construction. The boundary states $|a_\pm\rangle$ of the seed theory constitute the most non-trivial data that enters the construction. Let us recall that the sectors of the seed theory, i.e. the irreducible representations of its chiral algebra $\mathcal{W}$, are labelled by some index j. Since the gluing condition for chiral fields relates the label of the right- and left-moving representations, the index j also labels the Ishibashi states $|j\rangle\rangle$ that can contribute to the boundary state.[7] Thus, a boundary state $|a\rangle$ of the seed theory is described by a set of coefficients a_j in the expansion

$$|a\rangle = \sum_j a_j |j\rangle\rangle. \tag{2.21}$$

We can uplift this boundary state of the seed theory to a boundary state of the symmetric product orbifold $\mathrm{Sym}^M(\mathcal{M})$ for any $M \in \mathbb{N}$. Specifically we are interested in the cases $M = N_- - p$ or $M = N_+ - p$. Once again, the boundary state of the symmetric product orbifold is a sum over Ishibashi states. For a fixed gauge $\rho \in S_M$, the relevant Ishibashi state is given by

$$|j_r\rangle\rangle_\rho^M := |\{|j_r\rangle\rangle\}\rangle_\rho^M. \tag{2.22}$$

Here, ρ is assumed to possess a factorisation of the form (2.8) and the twisted sector states that appear on the right hand side are obtained from ℓ Ishibashi states $|j_r\rangle\rangle$, $r = 1, \ldots \ell$ of the seed theory in the spirit of eq. (2.12). The only difference is that Ishibashi states live in some appropriate completion of $\mathcal{H}^\rho$ rather than $\mathcal{H}^\rho$ itself.[8] The action of elements $\sigma \in \mathcal{C}_\rho$ in the centraliser subgroup on the Ishibashi states takes the form

$$\sigma|j_r\rangle\rangle_\rho = |j_{\pi r}\rangle\rangle_\rho\,, \tag{2.23}$$

where $\pi = \pi_\sigma$ is a permutation that exchanges two cycles of the same length. Note that the action of the factors $\omega_r^{\nu_r}$ in the factorisation formula for σ is trivial since contributions from holomorphic and anti-holomorphic components cancel each other. The overlap of any two of these Ishibashi states is given by

$${}_{\rho'}\langle\langle j'_r|\sigma \hat{x}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j_r\rangle\rangle_\rho = \delta^{(\ell)}_{j'_r, j_{\pi r}}\,\delta_{\rho',\rho}\prod_{r=1}^{\ell}\chi_{j_r}\big(\tfrac{\hat{t}}{w_r}\big). \tag{2.24}$$

[7] To be more precise, the sectors of the seed theory are labelled by pairs $(j,\bar{\jmath})$ of representations for the (anti-)holomorphic chiral fields. The specification of a boundary state involves picking a gluing automorphism Ω for the chiral algebra. This automorphism induces a map on representation labels $j \mapsto j_\Omega$. Given the choice of Ω, Ishibashi states only exist for those sectors of the theory for which $\bar{\jmath} = j_\Omega$. In this sense, we only need to specify a single representation label in order to specify the Ishibashi state, see [36] for details.

[8] For instance, they may be viewed as discontinuous linear functionals on $\mathcal{H}$.

Here, $\sigma \in \mathcal{C}_\rho^M$ is in the centraliser subgroup of ρ and $\pi = \pi_\sigma$, as before. Furthermore[9],

$$\chi_j(t) \coloneqq \mathrm{Tr}_{\mathcal{H}_{j,\bar{j}}} x^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}. \tag{2.25}$$

Let us stress that the Ishibashi states $|j_r\rangle\!\rangle_\rho$ we have introduced are not yet projected to the subspace of $\mathcal{C}_\rho$ invariant states. Given our Ishibashi states $|j_r\rangle\!\rangle_\rho$, we construct the linear combination

$$|a\rangle_\rho = \sum_{\{j_r\}} \frac{a_{j_1}\cdots a_{j_\ell}}{\sqrt{|\mathcal{C}_\rho^M|}} |j_r\rangle\!\rangle_\rho^M \tag{2.26}$$

with coefficients formed from products of the coefficients that appeared in the boundary state $|a\rangle$ of the seed theory. The overlap of any two of these states can be easily computed from the overlap (2.24). Note that the states $|a\rangle_\rho$ are actually within the subspace of $\mathcal{C}_\rho$ invariant elements, i.e. the projector Π_0^ρ acts trivially on these states. The normalisation we chose is the one that is appropriate for Cardy consistent boundary states, see e.g. [7] or Section 4.3 of [8]. The states (2.26) constitute the first ingredient we shall use in constructing our interfaces.

Transmissive part of the interfaces. As a second ingredient of our construction, we need to discuss conformal interfaces for $\mathrm{Sym}^p(\mathcal{M})$. After the folding trick, such an interface is described as a boundary state in which the holomorphic fields of the theory on the upper half plane are glued to the anti-holomorphic fields of the second theory that was folded up from the lower half plane and vice versa. Ishibashi states for the "permutation boundary states" [34] in $\mathrm{Sym}^p(\mathcal{M}) \times \mathrm{Sym}^p(\mathcal{M})$ will be denoted by

$$|j_r, i_s\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} \tag{2.27}$$

with $\tau_\pm \in S_p$ that are conjugate to each other, i.e. $[\tau_-] = [\tau_+]$. More concretely, if we formally[10] write

$$|j_r\rangle\!\rangle_\tau = |j_r\rangle_\tau \overline{|j_r\rangle}_\tau, \tag{2.28}$$

then

$$|j_r, i_s\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} = |j_r\rangle_{\tau_-} \overline{|i_s\rangle}_{\tau_-} |i_s\rangle_{\tau_+} \overline{|j_r\rangle}_{\tau_+}. \tag{2.29}$$

The action of the centraliser of τ_+ and τ_- on these Ishibashi states is given by

$$\sigma_+\sigma_-\, |j_r, i_s\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} = |\alpha^{\nu^-_{\pi^- r}} j_{\pi^- r}\rangle_{\tau_-} \overline{|\alpha^{\nu^-_{\pi^- s}} i_{\pi^- s}\rangle}_{\tau_-} |\alpha^{\nu^+_{\pi^+ s}} i_{\pi^+ s}\rangle_{\tau_+} \overline{|\alpha^{\nu^+_{\pi^+ r}} j_{\pi^+ r}\rangle}_{\tau_+}, \tag{2.30}$$

where

$$\alpha = e^{-2\pi i L_0}. \tag{2.31}$$

[9] In accordance with footnote 7, we could also have written j_Ω instead of $\bar{j}$ in eq. (2.25).

[10] Of course $|j_r\rangle\!\rangle_\tau$ is not a pure tensor product of left and right movers, but a sum of such products. Our notation captures the structure of the individual pure tensor summands.

The overlaps of these Ishibashi states take the form

$$ {}_{\tau'_\pm}\langle\!\langle j'_r, i'_s | \sigma_- \sigma_+ \hat{x}^{L_0+\bar{L}_0-\frac{c}{12}} | j_r, i_s \rangle\!\rangle_{\tau_\pm} = \delta^{(2\ell)}_{i'_k, i_{\pi_\pm k}} \delta^{(2\ell)}_{j'_k, j_{\pi_\pm k}} \delta^{(2)}_{\tau'_\pm, \tau_\pm} \chi_{i,\sigma_+,\sigma_-}(t) \chi_{j,\sigma_-,\sigma_+}(t). \tag{2.32} $$

In particular, they are only non-vanishing if $j_{\pi^- r} = j_{\pi^+ r}$. To spell out $\chi_{j,\sigma_-,\sigma_+}(t)$, recall that $\pi^\pm$ is a product of permutations that shuffle twisted sectors associated to cyclic permutations of length w. Hence, if we write $\pi^\pm$ as a product $\pi^\pm = \pi_1^\pm \pi_2^\pm \dots$ of cyclic permutations, then each permutation $\pi_a^\pm$ has two integers associated to it, namely the winding $w_{\pi_a^\pm}$ of the sectors it permutes and the length $\ell_{\pi_a^\pm} = |\pi_a^\pm|$.

The same is true for $\pi := \pi_+ \pi_-^{-1} = \pi_1 \dots \pi_m$, i.e. for each π_a we have pair (w_a, ℓ_a) of integers. The condition $j_{\pi^- r} = j_{\pi^+ r}$ is equivalent to the statement that $j_r = j_{\pi r}$ i.e. $j_r = j_{\pi_a r}$ for all a. In particular, if we denote by $\mathbf{a}$ the set of labels appearing in the cycle π_a, then we can introduce $j_{\mathbf{a}}$ as being equal to j_r for some non further specified $r \in \mathbf{a}$ and this is a well defined prescription. In terms of

$$ \chi_j(t, \bar{t}) := \mathrm{Tr}_{\mathcal{H}_{j,\bar{j}}}\, x^{L_0 - \frac{c}{24}} \bar{x}^{\bar{L}_0 - \frac{c}{24}}, \tag{2.33} $$

$\chi_{j,\sigma_-,\sigma_+}(t)$ can then be expressed as

$$ \chi_{j,\sigma_-,\sigma_+}(t) = \prod_{a=1}^{m} \chi_{j_{\mathbf{a}}} \left(\frac{\ell_a t + \sum_{k \in \mathbf{a}} \nu_k^-}{w_a}, \frac{\ell_a t + \sum_{k \in \mathbf{a}} \nu_k^+}{w_a} \right). \tag{2.34} $$

Once again, we introduce a linear combination of these twisted Ishibashi states by summing over the labels j_r, i_s. But in this case, the sum over the labels is not sufficient in order to ensure $\mathcal{C}_{\tau_\pm}$ invariance. Hence, we also have to perform the relevant projection

$$ |\mathbb{I}\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} = \Pi_0^{\tau_+} \Pi_0^{\tau_-} \sum_{j_r, j_s} |j_r, i_s\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} \qquad \text{where} \qquad \Pi_0^{\tau_\pm} = \frac{1}{|\mathcal{C}^p_{\tau_\pm}|} \sum_{\sigma_\pm \in \mathcal{C}^p_{\tau_\pm}} \sigma_\pm. \tag{2.35} $$

The overlap between any two such states can be computed with the help of formula (2.32). This completes the discussion of the second ingredient for our construction of interfaces.

Averaging over gauge orbits. As the final step in the construction of the boundary states (2.19), we now perform the averaging over gauge orbits. This gives the gauge invariant state

$$ |p, a_\pm\rangle^{N_\pm}_{\mathscr{O}_\pm} = \frac{1}{\sqrt{|\mathscr{O}_-||\mathscr{O}_+|}} \sum_{(\rho_\pm, \tau_\pm) \in \mathscr{O}_\pm} |a_-\rangle_{\rho_-} \cdot |\mathbb{I}\rangle\!\rangle^{p,p}_{\tau_-,\tau_+} \cdot |a_+\rangle_{\rho_+}, \tag{2.36} $$

where

$$ \mathscr{O}_\pm := [\tau, \rho_\pm]_p^{N_\pm} = \{ (h \tau h^{-1}, h \rho_\pm h^{-1}) : h \in S_{N_\pm} \} \tag{2.37} $$

are $S_{N_\pm}$ gauge orbits associated to a choice of $\rho_\pm \in S_{N_\pm - p}$ and $\tau \in S_p$. We added the subscript p to remind ourselves, that τ and $\rho_\pm$ are elements of $S_{N_\pm - p}$ and S_p and that the associated subgroup is an extra piece of data that we explicitly keep track of. That is, we distinguish for instance between $id_{S_p} \in S_p$ and $h \cdot id_{S_p} \cdot h^{-1} = id_{hS_p} \in hS_p$.

Let us briefly comment on the choice of normalisation that we made in the definition of $|p, a_\pm\rangle^{N_\pm}_{\mathscr{O}_\pm}$. Ultimately, we do not have an a priori principle that tells us what the "correct" normalisation should be. We can only justify our choice by showing that its overlaps lead to the physical partition functions that we are interested in, which indeed will turn out to be the case. However, we can at least argue, already before any computation, that our guess is very natural one: Since we normalised appropriately the mutually orthogonal states $|a_\pm\rangle_{\rho_\pm}$ and $|\mathbb{I}\rangle_{\tau_-,\tau_+}$ to represent reflective/transmissive boundaries, we should simply divide out the square root of the size of the orbits that we sum over such that the sum does not alter the already correct norm.

Now that we have constructed the boundary states $|p, a_\pm\rangle^{N_\pm}_{\mathscr{O}_\pm}$ associated to some choice of $S_{N_\pm}$ orbits, we finally perform a weighted sum over all choices of $\mathscr{O}_\pm$ to obtain the boundary state

$$|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle = \sum_{\tau \in S_p} \sum_{\rho_\pm \in S_{N_\pm - p}} \frac{\chi_-(\rho_-)\chi_p(\tau)\chi_+(\rho_+)}{|[\tau]||[\rho_+]||[\rho_-]|} |p, a_\pm\rangle^{N_\pm}_{\mathscr{O}_\pm}. \tag{2.38}$$

As advertised with eq. (2.19) at the end of our first qualitative description of the construction, formulating eq. (2.38) is the purpose of this section.

Summary of the construction. Let us finally summarise all ingredients of eq. (2.38) in a concise manner. $\chi_\pm, \chi_p$ are characters of the $S_{N_\pm - p}$ and S_p, respectively. The normalising prefactors involving the numbers $|[\rho_\pm]|$ and $|[\tau]|$ of elements in the conjugacy classes of $\rho_\pm \in S_{N_\pm - p}$ and $\tau \in S_p$, respectively, appear here because we are summing over group elements rather than conjugacy classes. The states we sum over are defined in eq. (2.36) through a weighted average over elements $g_\pm$ of $S_{N_\pm}$. Their definition involves a product of three states. Two of them are the states $|a_\pm\rangle\rangle_{\rho'_\pm}$, which are purely reflective with reflection coefficients that are specified by the coefficients of boundary states $|a_\pm\rangle$ in the seed theory, see eq. (2.26). The third factor in the states we average over is purely transmitting, see eq. (2.35).

2.3 Reflectivity and transmissivity of the interfaces

Now that we have defined the interfaces in eq. (2.38) let us pause for a moment and compute a first physical quantity that may help to better understand them. Concretely, we determine the (stress-energy) reflectivity and transmissivity as proposed in [37]. It measures e.g. how much energy gets transmitted through the interface or, after passing to the folded setup, how much energy gets passes from e.g. from +-components of the folded CFT to the −-component and vice versa. The interpretation of the reflectivity is similar. More concretely, after folding the theory to the upper half plane we need to compute correlations of the product $T^\eta(z)\bar{T}^\eta(\bar{z})$ with $\eta \in \{+,-\}$ in the presence of the interface boundary state (2.38). Replacing the Virasoro fields by the corresponding states, the quantity in question is

$$R_{\eta\eta'} = \frac{\langle 0|L_2^\eta \bar{L}_2^{\eta'}|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle}{\langle 0|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle}. \tag{2.39}$$

The overlaps that appear in this expressions are not that difficult to compute. The simplest is clearly the overlap in the denominator which is obviously given by

$$\langle 0|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle = \chi_{r_-}(id)\chi_p(id)\chi_{r_+}(id)\sqrt{\binom{N_-}{p}\binom{N_+}{p}}\frac{(a_-)_0^{N_--p}(a_+)_0^{N_+-p}}{\sqrt{(N_--p)!(N_+-p)!}}. \tag{2.40}$$

The overlap in the numerator is not that much harder to compute. Note that the Virasoro fields $T^\pm$ and $\bar{T}^\pm$ of the two symmetric product orbifold CFTs are in the untwisted sector and hence the overlap in the numerator sees essentially the same coefficients as that in the denominator. In formulas this means only the term with $\tau = \rho_\pm = id$ in the expansion of the state (2.38) can contribute to give

$$\langle 0|L_2^\eta \bar{L}_2^{\eta'}|p, a_\pm; \chi_-, \chi_p, \chi_+\rangle = \chi_{r_-}(id)\chi_p(id)\chi_{r_+}(id)\langle 0|L_2^i \bar{L}_2^j|p, a_\pm\rangle^{N_\pm}_{[id,id]_p^{N_\pm}}. \tag{2.41}$$

The state on the right hand side is a special case of eq. (2.36) which consists of a single term only and has trivial coefficient,

$$|p, a_\pm\rangle^{N_\pm}_{[id,id]_p^{N_\pm}} = |a_-\rangle \cdot |\mathbb{I}\rangle^{p,p} \cdot |a_+\rangle. \tag{2.42}$$

Let us now first address the off-diagonal elements of the reflection matrix R, i.e. the matrix elements $R_{+-} = R_{-+}$. Each of the $N_- - p$ components within the state $|a_\pm\rangle$ contributes a factor of $(a_\pm)_0$. Moreover, every transmitting component gives rise to a term $c_\mathcal{M}2(2^2 - 1)/12 = c_\mathcal{M}/2$ from the commutations relations of the Virasoro generators in the seed theory. Since we need to sum over all the transmitting components of which there are p, we find the simple relation

$$\langle 0|L_2^\pm \bar{L}_2^\mp|p, a_\pm; \chi\rangle = p\frac{c_\mathcal{M}}{2}\langle 0|p, a_\pm; \chi\rangle. \tag{2.43}$$

between the overlaps in the numerator and denominator. Here, we used the shorthand $\chi = (\chi_-, \chi_p, \chi_+)$ to denote the triple of characters. For the diagonal elements of R, the transmissive boundaries do not contribute at all. The contribution of all the reflective factors, on the other hand, sum to

$$\langle 0|L_2^\pm \bar{L}_2^\pm|p, a_\pm; \chi\rangle = (N_\pm - p)\frac{c_\mathcal{M}}{2}\langle 0|p, a_\pm; \chi\rangle. \tag{2.44}$$

Plugging the previous two relations back into the definition (2.39) of the reflection matrix R we conclude that

$$R = \frac{c_\mathcal{M}}{2}\begin{pmatrix} N_- - p & p \\ p & N_+ - p \end{pmatrix}. \tag{2.45}$$

From the matrix elements, we read off the reflectivity of our interface, which is given by

$$\mathcal{R} := \frac{2}{c_\mathcal{M}(N_- + N_+)}(R_{--} + R_{++}) = 1 - \frac{2p}{N_- + N_+}, \tag{2.46}$$

and the transmissivity

$$\mathcal{T} = \frac{2p}{N_- + N_+}. \tag{2.47}$$

Note that the "unitarity condition" $\mathcal{R} + \mathcal{T} = 1$ is indeed satisfied. We also observe that $p = 0$ corresponds to a purely reflective interface and $N_- = N_+ = p$ is purely transmissive, as expected. Let us end this short interlude with three minor comments on the result.

First, note that the formula (2.45) for R may be brought into its standard form

$$R = \frac{c_{\mathcal{M}}}{2} \frac{N_- N_+}{N_- + N_+} \left[\begin{pmatrix} \frac{N_-}{N_+} & 1 \\ 1 & \frac{N_+}{N_-} \end{pmatrix} + \omega_b \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \tag{2.48}$$

from which we read off

$$\omega_b = 1 - \frac{p}{N_-} - \frac{p}{N_+}. \tag{2.49}$$

Second, note that the reflectivity and transmissivity coefficients are insensitive to most of the data that we used to construct the interfaces in the previous subsection. In fact, only the number p of transmissive components and the numbers $N_\pm - p$ of reflective components enter. The choice of the boundary conditions $a_\pm$ or the characters $\chi_\pm$ and χ_p on the other hand has no effect. The reason for this is rather clear from the derivation: The stress-energy reflectivity and transmissivity only receive contributions from the untwisted sector of the symmetric product orbifold. In this sense, $\mathcal{R}$ and $\mathcal{T}$ are very coarse grained characterisations of the interfaces $\mathcal{I}^{(p)}_{|a_\pm\rangle}$.

Let us finally make a somewhat related observation. We decided to determine the matrix R above from the concrete formula (2.38). However, the authors of [37] noted already that for a certain class of interfaces, the matrix R is rather easy to compute by more general arguments. They considered a situation in which the chiral algebras $\mathcal{A}_-$ and $\mathcal{A}_+$ of the two CFTs the interface interpolates between possess a common subalgebra $\mathcal{C}$. If the interface $\mathcal{I}$ in question is completely transmissive for the common subalgebra $\mathcal{C}$ and reflective for all other degrees of freedom, then the chiral algebra of the folded theory is broken down to a product

$$\mathcal{A}_- \otimes \mathcal{A}_+ \to \mathcal{A}_-/\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{A}_+/\mathcal{C}. \tag{2.50}$$

by the presence of the interface $\mathcal{I}$. Denoting the corresponding boundary state by $|\mathcal{I}\rangle$, we can easily compute

$$\begin{aligned} \langle 0|L_2^\pm \bar{L}_2^\mp|\mathcal{I}\rangle &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})_\pm} + L_2^{\mathcal{C}_\pm})(\bar{L}_2^{(\mathcal{A}/\mathcal{C})_\mp} + \bar{L}_2^{\mathcal{C}_\mp})|\mathcal{I}\rangle \\ &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})_\pm} + L_2^{\mathcal{C}_\pm})(L_{-2}^{(\mathcal{A}/\mathcal{C})_\mp} + L_{-2}^{\mathcal{C}_\pm})|\mathcal{I}\rangle = \frac{c}{2} \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} \langle 0|L_2^\pm \bar{L}_2^\pm|\mathcal{I}\rangle &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})_\pm} + L_2^{\mathcal{C}_\pm})(\bar{L}_2^{(\mathcal{A}/\mathcal{C})_\pm} + \bar{L}_2^{\mathcal{C}_\pm})|\mathcal{I}\rangle \\ &= \langle 0|(L_2^{(\mathcal{A}/\mathcal{C})_\pm} + L_2^{\mathcal{C}_\pm})(L_{-2}^{(\mathcal{A}/\mathcal{C})_\pm} + L_{-2}^{\mathcal{C}_\mp})|\mathcal{I}\rangle = \frac{c_\pm - c}{2}, \end{aligned} \tag{2.52}$$

where $c_\pm$ are the central charges of $\mathcal{A}_\pm$ and c the central charge of the chiral subalgebra $\mathcal{C}$. In the two short calculations we have split the Virasoro elements $L^\pm$ and $\bar{L}^\pm$ of the chiral algebras $\mathcal{A}_\pm$ according the factorisation (2.50). Then, we used the Ishibashi conditions of the interface boundary state $|\mathcal{I}\rangle$ to replace anti-holomorphic components by holomorphic ones in passing from the first to the second line. In the last step we used commutation relations of the Virasoro elements to evaluate the overlap in terms of the central charges. In our case, the chiral algebras $\mathcal{A}_\pm$ are those of the symmetric product orbifolds $\mathrm{Sym}^{N_\pm}(\mathcal{M})$ while the common subalgebra $\mathcal{C}$ that is preserved by the interface is the chiral algebra of $\mathrm{Sym}^p(\mathcal{M})$. Plugging in the associated values of the central charges $c_\pm = c_\mathcal{M} N_\pm$ and $c = c_\mathcal{M} p$ reproduces eq. (2.45).

2.4 Partition functions of interface changing operators

A very interesting object related to the interfaces that we defined in section 2.2 is the partition function that enumerates interface changing operators. To be concrete, let us choose one interface $|p^R, a^R_\pm; \chi^R\rangle$ to the right of the origin and another $|p^L, a^L_\pm; \chi^L\rangle$ to the left. The associated partition function for interface changing operators is obtained from the associated overlap as follows

$$\mathcal{Z}^{N_\pm}_{(p^L,a^L_\pm;\chi^L),(p^R,a^R_\pm,\chi^R)}(t) = \mathrm{Tr}[x^{L_0-\frac{c}{24}}] = \langle p^L, a^L_\pm; \chi^L|\hat{x}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|p^R, a^R_\pm; \chi^R\rangle. \tag{2.53}$$

Here, $c = (N_+ + N_-)c_\mathcal{M}$ is the total central charge of the folded theory and the parameter $\hat{x}$ on the right is related to $x = \exp(2\pi i t)$ by modular transformation $\hat{x} = \exp(2\pi i\hat{t})$ with $\hat{t} = -1/t$. We continue to use the shorthand χ for the entire triple (χ_-, χ_p, χ_+) of characters. These overlaps could certainly be computed in full generality, but since we have no need for such a general result, we restrict to the case in which all the characters $\chi^{L/R}$ are trivial. This turns out to be the only one that is relevant for the holographic relation with AdS_2 branes.

As usual, the partition functions of the type defined in eq. (2.53) are somewhat inconvenient to describe individually, but as whole organise into a rather simple grand canonical partition function. Concretely, we propose that the grand canonical partition function

$$\mathcal{Z}_{a^{L/R}_\pm}[\mu_\pm, \rho_{L/R}; t] := \sum_{N_\pm=0}^{\infty} \sum_{p^{L/R}=0}^{\min(N_\pm)} \mu_+^{N_+}\mu_-^{N_-}\rho_L^{N_++N_--2p^L}\rho_R^{N_++N_--2p^R}\mathcal{Z}^{N_\pm}_{(p^L,a^L_\pm;\mathbf{1}),(p^R,a^R_\pm,\mathbf{1})}(t) \tag{2.54}$$

is given by the exponential

$$\mathcal{Z}_{a^{L/R}_\pm}[\mu_\pm, \rho_{L/R}; t] = \exp(\mathcal{Z}_C[\mu_\pm, \rho_{L/R}; t] + \mathcal{Z}_O[\mu_\pm, \rho_{L/R}; t]) \tag{2.55}$$

of the sum of a "single closed string" part

$$\mathcal{Z}_C[\mu_\pm, \rho_{L/R}; t] = \sum_{k=1}^{\infty} \mu_-^k \mu_+^k T_k Z_c(t) \tag{2.56}$$

and a "single open string" part

$$\mathcal{Z}_O[\mu_\pm, \rho_{L/R}; t] = \sum_{A,B\in\{L,R\}} \sum_{k=1}^{\infty} \sum_{w|k} \rho_A^w \rho_B^w \mu_-^{k-w\delta^R_A\delta^L_B} \mu_+^{k-w\delta^L_A\delta^R_B} \frac{1}{w} \hat{Z}_o^{AB}\left(\frac{(2k-w+w\delta^B_A)\hat{t}}{w^2}\right). \tag{2.57}$$

$\mathcal{Z}_C$ and $\mathcal{Z}_O$ involve two types of partition functions for the bulk theory. On the one hand, there is the "closed string" partition function Z_c defined as

$$Z_c(t) = Z_{\mathcal{M}}(t,\bar{t})_{t=\bar{t}} \tag{2.58}$$

which is the modular invariant bulk partition function of the seed theory $\mathcal{M}$ restricted to the diagonal $t = \bar{t}$. On the other hand, there are the open string partition functions Z_o^{AB} defined as

$$Z_o^{LL} = Z_{a_-^L,(a_+^L)^*}, \quad Z_o^{LR} = Z_{a_-^L,a_-^R}, \quad Z_o^{RL} = Z_{a_+^L,a_+^R} \quad \text{and} \quad Z_o^{RR} = Z_{(a_+^R)^*,a_-^R} \tag{2.59}$$

where $Z_{a,b}(t) = Z_{b*,a*}(t)$ is the partition function of the annulus with boundary conditions a and b imposed along the two components of the boundary. It is famously related to the overlap of the boundary states in the seed theory through

$$Z_{a,b}(t) = \langle a|\,\hat{x}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c_{\mathcal{M}}}{12})}\,|b\rangle = \hat{Z}_{a,b}(\hat{t}). \tag{2.60}$$

with $\hat{t} = -1/t$ as usual. Strictly speaking, formula (2.55) is a conjecture. We decided against a rigorous derivation of the full formula in this work. However, we do prove it carefully for the special case $\rho_L = 0$ in appendix B. Considering this special case is necessary in order to verify that we made the correct normalisation choices in section 2.2. It is also sufficient in the sense that the case of $\rho_L = 0$ is fully sensitive to all normalisations that we chose in the construction of the interface (2.38). Assuming that, with the correct choice of normalisation, the grand canonical partition function exponentiates, eq. (2.55) then follows by computing the "single string" sums $\mathcal{Z}_C$ and $\mathcal{Z}_O$ in the exponent. The calculation of these "single string" sums is the most illuminating part of the computation. This is why we dedicate the remainder of the current section to this task

Let us start by discussing the limit $\rho_L = \rho_R = 0$ in which only transmissive boundaries are present. When both ρ parameters vanish, so does $\mathcal{Z}_O$. This means that only the first term in the exponent contributes in the limit. Hence, our result (2.55) simplifies to

$$\sum_{N=0}^{\infty} \kappa^N \langle N| x^{\frac{1}{2}(L_0^{(1)}+\bar{L}_0^{(2)}-\frac{c}{12})}\bar{x}^{\frac{1}{2}(\bar{L}_0^{(1)}+L_0^{(2)}-\frac{c}{12})}|N\rangle = \exp\left[\sum_{k=1}^{\infty}\mu_-^k\mu_+^k T_k Z(t,\bar{t})\right], \tag{2.61}$$

where $Z = Z_c$ is the torus partition function of the seed theory, as before, T_k is the usual Hecke operator (see eq. (2.5)), and $|N\rangle$ is the fully transmissive boundary state

$$|N\rangle = |p = N, a_\pm; N_+ = N = N_-\rangle. \tag{2.62}$$

Obviously, this state does not depend on the choice of $|a_\pm\rangle$ since we do not allow for any reflection. The state $|N\rangle$ is associated with the trivial defect line of the symmetric product orbifold. Hence, we expect the partition function to coincide with the torus partition function (2.4), as shown in figure 3. This is indeed what we observe.

To verify eq. (2.61), we start by plugging in the definition (2.38) of $|N\rangle$ on the left hand side of the equation and simplifying, which gives

$$\langle N|x^{\frac{1}{2}(L_0^{(1)}+\bar{L}_0^{(2)})}\bar{x}^{\frac{1}{2}(\bar{L}_0^{(1)}+L_0^{(2)})}|N\rangle = \frac{1}{N!}\sum_{\tau\in S_N}\sum_{\sigma\in\mathcal{C}_\tau^N}\sum_{j_r,i_s} {}_{\tau,\tau}^{N,N}\langle\!\langle j_r,i_s|x^{L_0}\bar{x}^{\bar{L}_0}\sigma|j_r,i_s\rangle\!\rangle_{\tau,\tau}^{N,N}. \tag{2.63}$$

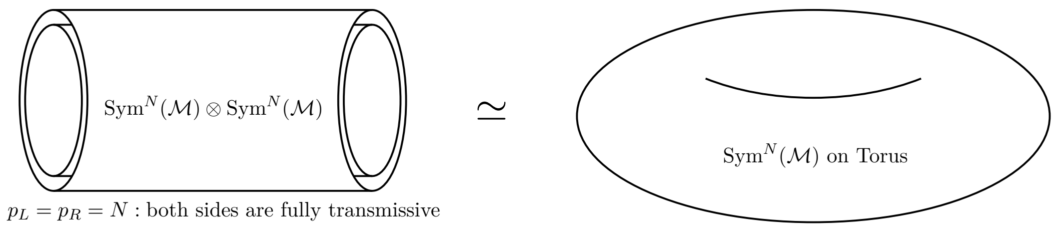

Figure 3. Coincidence of partition function of fully transmissive boundaries and torus partition function.

This formula for the overlap manifestly is a sum of contributions each associated to a permutation $\tau \in S_N$ and an element σ of the centraliser of τ. As explained in section 2.1, σ includes a factor π_σ that permutes j and i labels associated to cycles of the same length within the permutation τ. Only those terms in the sum for which $j_r = j_{\pi_\sigma r}$ and $j_s = j_{\pi_\sigma s}$ are non-vanishing. The full answer factorises into individual contributions associated to the cycles of τ and π_σ. This suggests[11] that the full grand canonical partition function is an exponential of sums of single cycle contributions $Z_{(w,\ell)}$ associated to a π_σ that is a single cycle of length ℓ and a τ that is a product of ℓ cycles of length w such that $N = w\ell$. To compute the single cycle contributions $Z_{(w,\ell)}$, let us define

$$\tau_{(w,\ell)} = \prod_{k=0}^{\ell-1}(kw+1, kw+2, \ldots, kw+w). \tag{2.64}$$

From eq. (2.63), we can then directly conclude that

$$Z_{(w,\ell)} = \frac{|[\tau_{(w,\ell)}]|}{N!} \sum_{\substack{\sigma\in\mathcal{C}_{\tau_{(w,\ell)}} \\ |\pi_\sigma|=\ell}} \sum_{j_r,i_s} {}^{N,N}_{\tau_{(w,\ell)},\tau_{(w,\ell)}}\langle\!\langle j_r, i_s| x^{L_0-\frac{c}{24}} \bar{x}^{\bar{L}_0-\frac{c}{24}} \sigma | j_r, i_s\rangle\!\rangle^{N,N}_{\tau_{(w,\ell)},\tau_{(w,\ell)}}. \tag{2.65}$$

The action of σ on the states was spelled out in eq. (2.30) which leads to

$$Z_{(w,\ell)} = \frac{1}{w^\ell \ell!}(\ell-1)! \sum_{i_1=0}^{w-1} \cdots \sum_{i_\ell=0}^{w-1} Z_{\mathcal{M}}\left(\frac{\ell t + \sum_{k=1}^{\ell} i_k}{w}, \frac{\ell \bar{t} + \sum_{k=1}^{\ell} i_k}{w}\right). \tag{2.66}$$

Using that the partition function $Z_{\mathcal{M}}$ of the bulk theory is invariant under modular T transformations, i.e. under shifts of the argument by arbitrary integers, the previous formula simplifies to

$$Z_{(w,\ell)} = \frac{1}{w\ell} \sum_{i=0}^{w-1} Z_{\mathcal{M}}\left(\tfrac{\ell t + i}{w}, \tfrac{\ell \bar{t} + i}{w}\right) = \frac{1}{N} \sum_{i=0}^{w-1} Z_{\mathcal{M}}\left(\tfrac{Nt}{w^2} + \tfrac{i}{w}, \tfrac{Nt}{w^2} + \tfrac{i}{w}\right). \tag{2.67}$$

This is what we wanted to show.

[11] As mentioned before, the computation in appendix B proves the claim rigorously.

Let us now look at the more generic case, where only ρ_L is sent to 0 in our general formula (2.55). In this case, we obtain

$$\mathcal{Z}_{a^{L/R}}[\mu_\pm, 0, \rho_R; \hat{t}] = \exp\left(\sum_{k=1}^{\infty} \mu_-^k \mu_+^k T_k Z_c(\hat{t}) + \sum_{k=1}^{\infty} \sum_{w|k} \mu_-^k \mu_+^k \rho_R^{2w} \frac{1}{w} \hat{Z}_o^{RR}\left(2\tfrac{k\hat{t}}{w^2}\right)\right). \tag{2.68}$$

We now sketch how this result is derived. Again, just by plugging in the definition (2.38) and performing a few elementary simplification steps, one can deduce that

$$\langle N|\hat{x}^{L_0}|p,a\rangle = \frac{1}{p!(N-p)!} \sum_{\tau\in S_p} \sum_{\rho\in S_{N-p}} \sum_{\sigma\in\mathcal{C}_{\tau\rho}^N} \sum_{j_r,i_s} {}^{N,N}_{\rho\tau,\tau\rho}\langle\!\langle j_r, i_s|\hat{x}^{L_0}\sigma|a\rangle_\rho |\mathbb{I}\rangle_{\tau,\tau}|a\rangle_\rho, \tag{2.69}$$

where σ acts only on the states of the symmetric orbifold living in the lower half plane. At this point, we can also unfold using the notation introduced in eq. (2.29) and obtain

$$\langle N|\hat{x}^{L_0}|p,a\rangle = \frac{1}{p!(N-p)!} \sum_{\tau\in S_p} \sum_{\rho\in S_{N-p}} \sum_{\sigma\in\mathcal{C}_{\tau\rho}^N} \sum_{j_r,i_s} {}_\rho\langle a|\langle i_s|^p_\tau \overline{\langle j_r|}^p_\tau \hat{x}^{L_0}\sigma|i_s\rangle^p_\tau \overline{|j_r\rangle}^p_\tau |a\rangle_\rho. \tag{2.70}$$

The overlap can be interpreted as a sum of products of "open string" contributions (which involve reflecting boundaries) and "closed string" contributions (which do not involve reflecting boundaries) that arise from the interplay of different τ, ρ and σ. In the special case $p = 0$, there are no closed string contributions.

Furthermore, for $p = 0$, the only "single string" contributions come from the conjugacy class $[\rho] = [(1\dots N)]$ of the maximal cyclic permutation. The choice of σ does not matter: every element of the commutator acts trivially. Accordingly, these contributions to the partition are simply

$$\frac{|[\rho]||\mathcal{C}_\rho|}{N!}\, {}_\rho\langle a|\hat{x}^{2L_0-\frac{c}{12}}|a\rangle_\rho = {}_\rho\langle a|\hat{x}^{2L_0-\frac{c}{12}}|a\rangle_\rho = \frac{1}{N}\hat{Z}_o\left(\tfrac{2t}{N}\right), \tag{2.71}$$

where Z_o is defined in eq. (2.59). The factor $1/N$ on the right hand side of the equation originates from our choice to normalise $|a\rangle_\rho$ by $1/\sqrt{\mathcal{C}_\rho}$, see eq. (2.26).

More generally, we can find single particle contributions at $p = w(\ell-1)$ for $N = w\ell$. They arise for

$$|\pi_\sigma| = \ell, \quad [\tau] = [\tau_{(w,\ell-1)}], \quad [\rho] = [((\ell-1)w+1\dots\ell w)]. \tag{2.72}$$

Their contribution to the partition function is

$$\frac{|[\tau]||[\rho]||\{\sigma\in\mathcal{C}_{\tau\rho}^N : |\pi_\sigma| = \ell\}|}{p!(N-p)!|\mathcal{C}_\rho^{N-p}|}\hat{Z}_o\left(\tfrac{2\ell t}{w}\right) = \frac{|\{\sigma\in\mathcal{C}_{\tau\rho}^N : |\pi_\sigma| = \ell\}|}{|\mathcal{C}_\tau^p||\mathcal{C}_\rho^{N-p}|^2}\hat{Z}_o\left(\tfrac{2\ell t}{w}\right) = \frac{1}{w}\hat{Z}_o\left(\tfrac{2Nt}{w^2}\right). \tag{2.73}$$

Assuming that the grand canonical partition function is obtained by the exponentiation of these single particle contributions, we obtain eq. (2.68). Appendix B provides a detailed computation that establishes this result more rigorously.

Before we close this section, let us add a few comments concerning the formula (2.57). First of all, we have introduced $\mu_\pm$ and $\rho_{L/R}$ in order to unify the discussion of the various special cases obtained by sending some of the chemical potentials to zero in this section. However, we will ultimately be especially interested in the case

$$\mathcal{Z}_O[\mu,\hat{t}] := \mathcal{Z}_O[\mu,\mu,1,1,\hat{t}] = \sum_{A,B\in\{L,R\}} \sum_{k=1}^{\infty} \sum_{w|k} \frac{1}{w} \mu^{2k+w\delta_A^B - w} \hat{Z}_o^{AB}\left(\frac{(2k-w+w\delta_A^B)\hat{t}}{w^2}\right), \tag{2.74}$$

which is more conveniently described by summing over

$$\ell = 2\frac{k}{w} + \delta_A^B - 1 \tag{2.75}$$

instead of k. Indeed,

$$\boxed{\mathcal{Z}_O[\mu,\hat{t}] = \sum_{A,B\in\{L,R\}} \sum_{\ell,w=1}^{\infty} \tfrac{1}{2w}(1 - e^{i\pi(\ell+\delta_A^B)})\mu^{w\ell} \hat{Z}_o^{AB}\left(\tfrac{\ell\hat{t}}{w}\right).} \tag{2.76}$$

In this equation, as in all other partition functions above, we have used the "closed string" modular parameter $\hat{t}$ as appropriate for the overlap of boundary states. Given the interpretation as a counting function for boundary/interface changing operators, it may be more natural to perform a modular S transformation and rewrite the grand canonical partition function in terms of the dual modular parameter $t = -1/\hat{t}$. Using $Z_o(t) = \hat{Z}_o(\hat{t})$, we obtain

$$\mathcal{Z}_O[t] = \sum_{A,B\in\{L,R\}} \sum_{\ell,w=1}^{\infty} \tfrac{1}{2\ell}(1 - e^{i\pi(w+\delta_A^B)}) Z_o^{AB}\left(\tfrac{\ell t}{w}\right). \tag{2.77}$$

In rewriting the expression we also exchanged w and ℓ.

In the remainder of this work, our focus is on the holographic relation with AdS_2 branes for tensionless superstring theory in $AdS_3 \times S^3 \times \mathbb{T}^4$. In this theory, the spacetime symmetry is enhanced from the usual Virasoro algebra to the small $\mathcal{N} = 4$ superconformal algebra and hence states of the CFT can be distinguished by their R-symmetry charge that is given by the eigenvalue of the Cartan generators K_0 and $\bar{K}_0$ of the R-symmetry group. It is then natural to include these charges in the counting functions for states. At the level of the seed theory, this means that the bulk and boundary partition functions (2.58) and (2.59) become functions of another variable ζ in addition to t, i.e.

$$Z_c(t) \;\to\; Z_c(t,\zeta), \quad Z_o^{AB}(t) \;\to\; Z_o^{AB}(t,\zeta). \tag{2.78}$$

The corresponding change to formula (2.57) is given by the substitution

$$\hat{Z}_o^{AB}\left(\frac{\ell\hat{t}}{w}\right) \longrightarrow \hat{Z}_o^{AB}\left(\frac{\ell\hat{t}}{w}, \ell\hat{\zeta}\right). \tag{2.79}$$

The function $\hat{Z}$ is obtained from the partition function Z through the prescription

$$\hat{Z}(\hat{t},\hat{\zeta}) = Z(-1/\hat{t}, \hat{\zeta}/\hat{t}) = Z(t,\zeta). \tag{2.80}$$

We complement these qualitative remarks with more explicit formulas in section 4.1 after establishing some more background on the supersymmetric four-torus in section 3.2.

3 AdS_2 branes in tensionless AdS_3 backgrounds

In this section, we turn to the dual side and discuss the string theory of AdS_2 branes in AdS_3, or more concretely, in type IIB superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ with one unit of NS-NS flux. In the hybrid formulation or Berkovits-Vafa-Witten [38], the latter involves a $\mathfrak{psu}(1,1|2)_k$ WZNW model at level $k = 1$ along with a topologically twisted four-torus and some ghosts, see below.

In the first subsection, section 3.1, we review the relevant boundary states for AdS_2 branes in the $\mathfrak{psu}(1,1|2)_1$ WZNW model following [8]. After adding the remaining factors of the worldsheet model, i.e. the topologically twisted $\mathbb{T}^4$ and the ghost factors, we, in section 3.2, compute the overlaps of the full boundary states and interpret the resulting quantity as a counting function for boundary operators. The worldsheet partition function we end up with counts physical open string vertex operators in the tensionless $AdS_3 \times S^3 \times \mathbb{T}^4$ background. In order to compute the partition function of the associated spacetime string theory, we must integrate over the modulus τ of the worldsheet torus. We do this first for global AdS_3 and then, anticipating our comparison with the dual CFT we studied in section 2, for thermal AdS_3 in section 3.3. Note that the boundary of global AdS_3 is a sphere whereas the boundary of thermal AdS_3 is a torus. The latter is the geometry that is relevant for comparison with the CFT partition functions we computed above.

3.1 Boundary states for AdS_2 branes in tensionless AdS_3

According to the work of Berkovits, Vafa and Witten [38], the description of superstrings in $AdS_3 \times S^3$ in the hybrid formulation involves a $\mathfrak{psu}(1,1|2)_1$ WZNW model. The latter has been studied in the past, see in particular [39, 40] and references therein to earlier work in string theory and quantum Hall plateaux transitions, as well as [41] for some later extensions. Here, we restrict to the case of $k = 1$. Our exposition is purposefully kept minimalistic and we refer the reader to sections 3 and 4 of [8] for more detail.

The even part of the $\mathfrak{psu}(1,1|2)_1$ current algebra is generated by the affine currents $J^a, a = 1,2,3$, of an $\mathfrak{sl}(2,\mathbb{R})_1$ WZNW model along with the currents $K^a, a = 1,2,3$, of an $\mathfrak{su}(2)_1$ WZNW model. These six bosonic currents are associated with the six directions in the bosonic base of the supergroup PSU$(1,1|2)$. In addition, the supergroup also has eight fermionic directions which give rise to the fermionic currents $S^{\alpha\beta\gamma}$ in the WZNW model. Here α,β,γ are spinor indices that take the values $\alpha = \pm 1$, respectively. These currents satisfy the relations of an affine $\mathfrak{psu}(1,1|2)_1$ Kac-Moody algebra at level $k = 1$. Since we do not need these relations, we do not spell them out here, see e.g. section 3 of [6] for a complete list.

The state space of the bulk theory is a direct sum of representations of this worldsheet current algebra. Following standard conventions, we shall denote by $\mathcal{F}_\lambda$ the representations of the $\mathfrak{psu}(1,1|2)_1$ algebra whose spectrum of conformal weights is bounded from below. Here, the index $\lambda \in [0,1[$ determines the quantization of the zero mode J^3_0 of the current J^3 of the non-compact current algebra $\mathfrak{sl}(2,\mathbb{R})_1$. For generic level k, the representations $\mathcal{F}_\lambda$ would carry other labels that keep track of angular momenta but these are all removed by the null-vectors at $k = 1$. In addition to the representations $\mathcal{F}_\lambda$, the worldsheet model also includes representations $\mathcal{F}_\lambda^w$ that are obtained by application of the spectral flow automorphism σ^w, that is $\mathcal{F}_\lambda^w = \sigma^w(\mathcal{F}_\lambda)$. Recall that the space of ground states from which $\mathcal{F}_\lambda$ is obtained by the action of the negative $\mathfrak{psu}(1,1|2)_1$ modes is the short multiplet

$$\begin{array}{ccc} & (\mathscr{C}_\lambda^{\frac{1}{2}}, \mathbf{2}) & \\ (\mathscr{C}^1_{\lambda+\frac{1}{2}}, \mathbf{1}) & & (\mathscr{C}^0_{\lambda+\frac{1}{2}}, \mathbf{1}). \end{array} \tag{3.1}$$

Here, the symbol $(\mathscr{C}_\lambda^j, \mathbf{m})$ denotes an irreducible representation of the subalgebra $\mathfrak{sl}(2,\mathbb{R})_1 \oplus \mathfrak{su}(2)_1$ which is the maximal bosonic subalgebra of $\mathfrak{psu}(1,1|2)_1$. These representations are constructed from the $\mathbf{m}$-dimensional highest weight representation $\mathbf{m}$ of the compact algebra $\mathfrak{su}(2)_1$, along with the continuous series representations $\mathscr{C}_\lambda^j$ of the non-compact $\mathfrak{sl}(2,\mathbb{R})_1$. The labels j and λ of $\mathscr{C}_\lambda^j$ capture the eigenvalue $-j(j-1)$ of the quadratic Casimir element and the fractional part of the J_0^3 eigenvalues, respectively. As we stressed before, in the case $k = 1$, the label j is entirely fixed and hence we dropped it from the symbol $\mathcal{F}_\lambda$. In terms of representations of the $\mathfrak{psu}(1,1|2)_1$ current algebra, the Hilbert space of the $\mathfrak{psu}(1,1|2)_1$ WZNW is the direct sum

$$\mathcal{H}^{\text{WZNW}} = \bigoplus_{w\in\mathbb{Z}} \int_0^1 d\lambda\ \sigma^w(\mathcal{F}_\lambda) \otimes \overline{\sigma^w(\mathcal{F}_\lambda)}. \tag{3.2}$$

Let us now turn to the boundary theory. The gluing conditions for the $\mathfrak{sl}(2,\mathbb{R})$ currents $J^a(z)$, the $\mathfrak{su}(2)$ currents $K^a(z)$ and the Fermionic currents $S^{\alpha\beta\gamma}(z)$ associated to the AdS_2 branes take the form

$$J^a(z) = \bar{J}^a(\bar{z}), \quad K^a(z) = \bar{K}^a(\bar{z}), \quad S^{\alpha\beta\gamma}(z) = \varepsilon \bar{S}^{\alpha\beta\gamma}(\bar{z}) \tag{3.3}$$

at $z = \bar{z}$. The parameter ε that enters the gluing conditions we impose on the fermionic currents is a sign i.e. $\varepsilon \in \{+,-\}$. The corresponding Ishibashi states $|w,\lambda,\varepsilon\rangle\!\rangle$ in the sectors of the bulk decomposition (3.2) are characterised by

$$(J_n^a + \bar{J}_{-n}^a)|w,\lambda,\varepsilon\rangle\!\rangle = 0, \quad (K_n^a + \bar{K}_{-n}^a)|w,\lambda,\varepsilon\rangle\!\rangle = 0, \quad (S_n^{\alpha\beta\gamma} + \varepsilon\, \bar{S}_{-n}^{\alpha\beta\gamma})|w,\lambda,\varepsilon\rangle\!\rangle = 0. \tag{3.4}$$

These conditions imply $w = 0$ and $\lambda \in \{0, 1/2\}$. In [8], Gaberdiel et al. suggested to consider the following two linear combinations of Ishibashi states

$$\|\varepsilon\rangle\!\rangle_A = \frac{1}{\sqrt{2}} \sum_{\lambda=0,1/2} e^{2\pi i(\lambda-1/2)\delta_A^L} |0,\lambda,\varepsilon\rangle\!\rangle, \tag{3.5}$$

where the label[12] A is chosen from $A \in \{L, R\}$. The two boundary states impose Dirichlet conditions for the angular coordinate of AdS_3 forcing open strings to end "left" or "right" of the centre of the AdS_2 brane respectively. In this sense, they only describe half branes and the worldsheet boundary state corresponding to the full AdS_2 brane is

$$\|\varepsilon\rangle\!\rangle = \|\varepsilon\rangle\!\rangle_L + \|\varepsilon\rangle\!\rangle_R. \tag{3.6}$$

[12] In [8], this label is called Θ and chosen from $\{0, 1\}$.

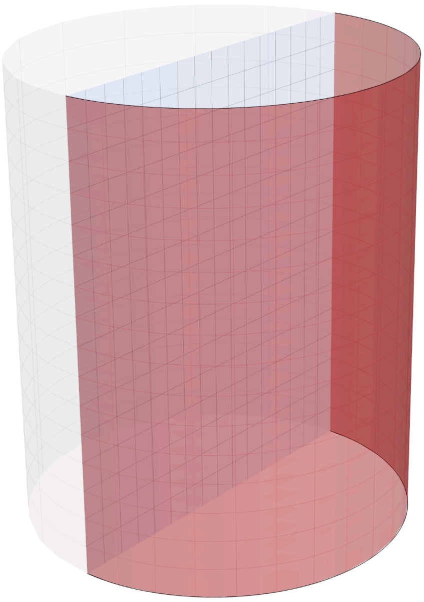

Figure 4. Open string winding half way around AdS_3 at $k = 1$. Since the worldsheet (red) is pinned to ∂AdS_3, the end points of the string remain on either of the two asymptotics boundaries of the AdS_2 brane (blue).

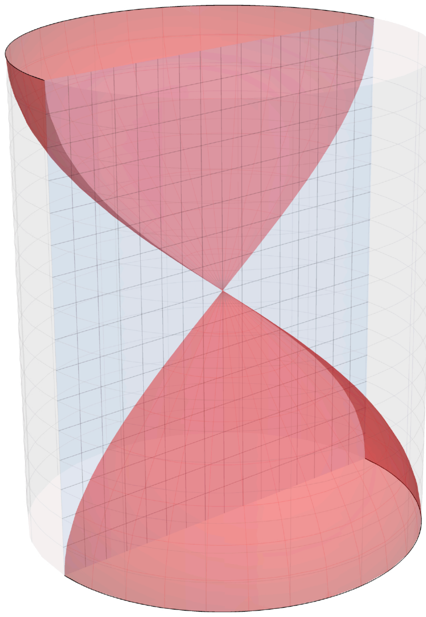

Figure 5. Open string winding half way around AdS_3 for large k. The string can fall into the interior of AdS_3 and re-emerge with its end-point on opposite asymptotic boundaries of the AdS_2 brane.

For generic $k > 1$, the splitting of $\|\varepsilon\rangle\rangle$ into $\|\varepsilon\rangle\rangle_L$ and $\|\varepsilon\rangle\rangle_R$ is unnatural since the ends of open strings can cross the centre of AdS_3. It is a special feature of the tensionless limit, that it is useful to keep track of $\|\varepsilon\rangle\rangle_L$ and $\|\varepsilon\rangle\rangle_R$ separately. Indeed, since long open strings in the tensionless limit cannot probe the centre of AdS_3, an open string end on the left/right asymptotic end of the AdS_2 brane will always stay on the left/right. We illustrate the situation in figures 4 and 5.

As we stressed above, the full worldsheet description of superstrings in the hybrid formulation also involves a topologically twisted four-torus and ghosts. Hence, the boundary states of the $\mathfrak{psu}(1,1|2)_1$ WZNW model we have described here must still be multiplied with further contributions from these additional sectors to obtain the following boundary states for AdS_2 branes in $AdS_3 \times S^3 \times \mathbb{T}^4$,

$$\|u,\varepsilon\rangle\rangle = \|\varepsilon\rangle\rangle\, \|u,\mathrm{R},\varepsilon\rangle\rangle_{\mathbb{T}^4}\, \|\mathrm{ghost},\varepsilon\rangle\rangle. \tag{3.7}$$

Corresponding to the splitting (3.6), we also define

$$\|u,\varepsilon\rangle\rangle_A = \|\varepsilon\rangle\rangle_A\, \|u,\mathrm{R},\varepsilon\rangle\rangle_{\mathbb{T}^4}\, \|\mathrm{ghost},\varepsilon\rangle\rangle. \tag{3.8}$$

The additional factors are discussed in detail in appendix B.1 of [8]. Here, it suffices to say that the letter u refers to the freedom we have in selecting a boundary condition on the four-torus $\mathbb{T}^4$, such as the brane's dimension, orientation and position. It is not necessary for us to specify this choice any further.

Regarding the R label, recall that in the hybrid formalism, the fermions of the original sigma model on $\mathbb{T}^4$ with small $\mathcal{N} = (4,4)$ supersymmetry mix with the currents of the $\mathfrak{sl}(2,\mathbb{R})_1$ and $\mathfrak{su}(2)_1$ WZNW models to generate the supercurrents of the $\mathfrak{psu}(1,1|2)_1$ model. The subsequent decoupling of torus degrees of freedom from the $\mathfrak{psu}(1,1|2)_1$ WZNW model leaves us with a topologically twisted $\mathbb{T}^4$. While the bosonic degrees of freedom of the topologically twisted sigma model coincide with that of the untwisted theory, part of the fermionic degrees of freedom are removed and we are essentially only left with the RR sector. Consequentially, the boundary condition that we pick for the topologically twisted sector is a boundary condition for RR fermions. The purpose of the label R is to remind us of this fact.

3.2 Partition function for AdS_2 branes in global AdS_3

Our goal in this subsection is to compute the overlap of any two of the boundary states (3.7) for AdS_2 branes and to apply a modular transformation in order to interpret this overlap in terms of the boundary spectrum. More precisely, the quantity in question is

$$\hat{Z}^{\text{WS}}_{u|v}(\hat{t},\hat{\zeta};\hat{\tau}) := \langle\!\langle u,\mp\| \hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\hat{x}^{\frac{1}{2}(J^3_0-\bar{J}^3_0)}\hat{y}^{\frac{1}{2}(K^3_0-\bar{K}^3_0)}\|v,\pm\rangle\!\rangle = \sum_{A,B\in\{L,R\}} \hat{Z}^{\text{WS}}_{A,u|B,v}(\hat{t},\hat{\zeta};\hat{\tau}), \quad (3.9)$$

where on the right hand side, we split the brane into a left and right half according to (3.6), that is

$$\hat{Z}^{\text{WS}}_{A,u|B,v}(\hat{t},\hat{\zeta};\hat{\tau}) := {}_A\langle\!\langle u,\mp\| \hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\hat{x}^{\frac{1}{2}(J^3_0-\bar{J}^3_0)}\hat{y}^{\frac{1}{2}(K^3_0-\bar{K}^3_0)}\|v,\pm\rangle\!\rangle_B \quad (3.10)$$

counts the open strings that stretch between the A-half of the u-brane and the B-half of the v-brane. The overlap factorises into three contributions from the $\mathfrak{psu}(1,1|2)_1$ WZNW model, the four-torus $\mathbb{T}^4$ and the ghosts respectively. In order to spell these out explicitly, we need a bit of notation. It is convenient to use the conventions of [42] for Jacobi theta-functions. That is, we define

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\zeta|\tau) := \sum_{n\in\mathbb{Z}} \exp\left(i\pi(n+\alpha)^2\tau + 2\pi i(n+\alpha)(\zeta+\beta)\right). \quad (3.11)$$

Together with the standard Dedekind η function, one can use these theta-functions to compute the supercharacter [8]

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t,z;\tau) := \text{tr}_{\mathcal{F}_\lambda}\left[(-1)^F q^{L_0-\frac{c}{24}} x^{J^3_0} y^{K^3_0}\right] = \sum_{r\in\mathbb{Z}+\lambda} \frac{x^r}{\eta(\tau)^4}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\tfrac{t+\zeta}{2}\middle|\tau\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\tfrac{t-\zeta}{2}\middle|\tau\right), \quad (3.12)$$

which by performing the sum over r can also be written as

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t,z;\tau) = x^\lambda \sum_{m\in\mathbb{Z}} \frac{\delta(t-m)}{\eta(\tau)^4}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\tfrac{t+\zeta}{2}\middle|\tau\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\tfrac{t-\zeta}{2}\middle|\tau\right). \quad (3.13)$$

By the identity

$$\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\left(\tfrac{m\pm\zeta}{2}\middle|\tau\right) = e^{i\pi\left[\frac{m}{2}\right]}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1+e^{i\pi m}}{4}\end{bmatrix}\left(\pm\tfrac{\zeta}{2}\middle|\tau\right), \quad (3.14)$$

this simplifies further to

$$\widetilde{\text{ch}}[\mathcal{F}_\lambda](t,z;\tau) = \sum_{m\in\mathbb{Z}} e^{2\pi i\lambda m}\frac{\delta(t-m)}{\eta(\tau)^4}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1+e^{i\pi m}}{4}\end{bmatrix}\left(\tfrac{\zeta}{2}\middle|\tau\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1+e^{i\pi m}}{4}\end{bmatrix}\left(-\tfrac{\zeta}{2}\middle|\tau\right). \tag{3.15}$$

The supercharacter can be used to determine the overlap of the boundary states for AdS_2 branes in the $\mathfrak{psu}(1,1|2)_1$ WZNW model,

$$\langle\!\langle 0,\lambda',\mp|\,\hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\hat{x}^{\frac{1}{2}(J_0^3-\bar{J}_0^3)}\hat{y}^{\frac{1}{2}(K_0^3-\bar{K}_0^3)}\,|0,\lambda,\pm\rangle\!\rangle = \delta_{\lambda,\lambda'}\,\widetilde{\text{ch}}[\mathcal{F}_\lambda](\hat{t},\hat{\zeta};\hat{\tau}). \tag{3.16}$$

Combining the explicit expression (3.15) for the super character with the definition (3.5) of the boundary states representing the AdS_2 brane, we obtain

$$\begin{aligned}\hat{Z}^{\mathfrak{psu}}_{A|B}(\hat{t},\hat{\zeta};\hat{\tau}) &= \frac{1}{2}\sum_{\lambda=0,1/2} e^{2\pi i(1/2-\lambda)(\delta^L_A-\delta^L_B)}\,\widetilde{\text{ch}}[\mathcal{F}_\lambda](\hat{t},\hat{\zeta};\hat{\tau})\\ &= \sum_{m=1}^{\infty}\frac{e^{i\pi m}-e^{i\pi\delta^B_A}}{2\eta(\hat{\tau})^4}\delta(\hat{t}-m)\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{\delta_{A,B}}{2}\end{bmatrix}\left(-\tfrac{\hat{\zeta}}{2}\middle|\hat{\tau}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{\delta_{A,B}}{2}\end{bmatrix}\left(+\tfrac{\hat{\zeta}}{2}\middle|\hat{\tau}\right).\end{aligned} \tag{3.17}$$

The two remaining contributions from the overlaps of the boundary states in the four-torus and the ghost factor are the same as in the case of spherical branes that were fully analysed in [8]. We use the same conventions and only slightly different notation. For instance, we write the four-torus factor in the overlap using the notation for Jacobi theta-functions introduced in eq. (3.11) as

$$\hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\hat{\zeta}|\hat{t}) := \frac{\hat{\Theta}^{\mathbb{T}^4}_{u|v}(\hat{t})}{\eta(\hat{t})^6}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(+\tfrac{\hat{\zeta}}{2}\middle|\hat{t}\right)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(-\tfrac{\hat{\zeta}}{2}\middle|\hat{t}\right). \tag{3.18}$$

Here, the quantity $\hat{\Theta}^{\mathbb{T}^4}_{u|v}$ in the numerator of the first factor stems from the bosonic directions on the four-torus and it depends on the choice of boundary states on $\mathbb{T}^4$, see appendix B.1 of [8] for details. Instead of the bracket notation it is also common to write, as Gaberdiel et al. do in [8],

$$\hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \hat{Z}^{\mathbb{T}^4}_{u|v,\tilde{\text{R}}},\quad \hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix} = \hat{Z}^{\mathbb{T}^4}_{u|v,\text{R}},\quad \hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}0\\ 0\end{bmatrix} = \hat{Z}^{\mathbb{T}^4}_{u|v,\text{NS}},\quad \hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix} = \hat{Z}^{\mathbb{T}^4}_{u|v,\widetilde{\text{NS}}}, \tag{3.19}$$

where R and NS refer to the Ramond and Neveu-Schwarz sector of the fermions, respectively, and the tilde on top indicates whether we count fermions with additional signs or not, i.e. whether we insert a factor $(-1)^F$ in the trace. Since

$$\|u,\text{R},\mp\rangle\!\rangle = (-1)^F\,\|u,\text{R},\pm\rangle\!\rangle, \tag{3.20}$$

the insertion of $(-1)^F$ is effectively achieved by choosing opposite ε for the two states whose overlap we compute. With the notations in place, we can now state that the relevant overlap of the boundary states on the four-torus is

$$\langle\!\langle u,\text{R},\mp\|\hat{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}\|v,\text{R},\pm\rangle\!\rangle = \frac{\hat{\Theta}^{\mathbb{T}^4}_{u|v}(\hat{\tau})}{\eta(\hat{\tau})^6}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\hat{\tau})\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\hat{\tau}). \tag{3.21}$$

The last contribution we need is that from the overlap of boundary states in the ghost sector which we also take from [8]

$$\hat{Z}_{\text{ghost}}(\hat{\tau}) = \frac{\eta(\hat{\tau})^4}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\hat{\tau})\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0|\hat{\tau})}\,. \tag{3.22}$$

Multiplying eqs. (3.22), (3.21) and (3.17), we conclude that the overlap (3.10) is given by

$$\hat{Z}^{\text{WS}}_{A,u|B,v}(\hat{t},\hat{\zeta};\hat{\tau}) = \sum_{m=1}^{\infty}\tfrac{1}{2}(e^{i\pi m} - e^{i\pi\delta_A^B})\delta(\hat{t}-m)\hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\frac{1}{2}\\ \frac{\delta_{A,B}}{2}\end{bmatrix}(\hat{\zeta}|\hat{\tau}). \tag{3.23}$$

We placed a superscript WS on this quantity to remind us that this is a partition of the worldsheet theory that describes strings in $AdS_3\times S^3\times\mathbb{T}^4$. In particular, the modular parameter $\hat{\tau} = -1/\tau$ is that of the annulus worldsheet.

The overlap we have just computed is still written in terms of closed string parameters $\hat{t},\hat{\zeta}$ and $\hat{\tau}$. In order to reinterpret the overlap in terms of open string vertex operators, we need to perform a modular S transformation. This transformation acts as

$$\hat{\tau} = -\frac{1}{\tau}, \qquad \hat{t} = \frac{t}{\tau}, \qquad \hat{\zeta} = \frac{\zeta}{\tau} \tag{3.24}$$

on the chemical potentials. We can rewrite our result (3.23) as a function of the dual variables with the help of the following transformation formula for the overlap of boundaries on the four-torus

$$\hat{Z}^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\hat{\zeta}|\hat{\tau}) = e^{\frac{\pi i\zeta^2}{2\tau}}Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}(\zeta|\tau) \tag{3.25}$$

with $\alpha,\beta\in\{0,\frac{1}{2}\}$. Here, following [8], we absorb the sign from the modular transformation of the theta functions into the definition

$$Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\zeta|t) := e^{4\pi i\alpha\beta}\frac{\Theta^{\mathbb{T}^4}_{u|v}(t)}{\eta(t)^6}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(+\tfrac{\zeta}{2}\middle|t\right)\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(-\tfrac{\zeta}{2}\middle|t\right). \tag{3.26}$$

This allows us to rewrite eq. (3.23) as

$$\hat{Z}^{\text{WS}}_{A,u|B,v}(\hat{t},\hat{\zeta};\hat{\tau}) = \sum_{m=1}^{\infty}\tfrac{1}{2}(e^{i\pi m} - e^{i\pi\delta_A^B})\delta(\tfrac{t}{\tau}-m)e^{\frac{\pi i\zeta^2}{2\tau}}Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\frac{\delta_{A,B}}{2}\\ \frac{1}{2}\end{bmatrix}(\zeta|\tau). \tag{3.27}$$

As usual, we interpret the right hand side as the partition function that counts operators that can be inserted along the boundary of the worldsheet or, equivalently, the state space of the theory in an infinite strip, i.e.

$$\begin{aligned} Z^{\text{WS}}_{A,u|B,v}(t,\zeta;\tau) :=& \text{Tr}_{\mathcal{H}^{\text{WZNW}}_{A,u|B,v}}\left((-1)^F q^{L_0-\frac{c}{24}}\,e^{2\pi i tJ_0^3}\,e^{2\pi i\zeta K_0^3}\right)\\ =&\frac{\tau^2}{t}\sum_{w=1}^{\infty}\tfrac{1}{2}(e^{i\pi w} - e^{i\pi\delta_A^B})\delta(\tfrac{t}{w}-\tau)e^{\frac{\pi i\zeta^2}{2\tau}}Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix}\frac{\delta_{A,B}}{2}\\ \frac{1}{2}\end{bmatrix}(\zeta|\tau). \end{aligned} \tag{3.28}$$

Relative to the previous result in eq. (3.27), there is a prefactor $\frac{\tau^2}{t}$ that results from rewriting the argument of the δ function.

Integrating over the worldsheet modulus τ using the integral measure $d\hat{\tau} = d\tau/\tau^2$ yields

$$\int_0^{i\infty} \frac{d\tau}{\tau^2} Z^{\text{WS}}_{A,u|B,v}(t,\zeta;\tau) = \frac{1}{t}\sum_{w=1}^{\infty} \tfrac{1}{2}(e^{i\pi w} - e^{i\pi\delta^B_A}) e^{\frac{\pi i w \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta|\tfrac{t}{w}). \tag{3.29}$$

Formulas (3.28) and its integrated cousin (3.29) are the main result of this subsection. We discuss these further in section 4 when we compare with the expressions for the partition function of the spacetime CFT.

3.3 Partition function for AdS_2 brane in thermal AdS_3

We finally compute the boundary spectrum of AdS_2 branes in thermal AdS_3 along the lines of [43]. As is well known, thermal AdS is obtained from the global one through an orbifold construction with the orbifold group $\mathbb{Z}$. The $\mathbb{Z}$ action enforces a periodic identification along the Euclidean time direction. Our starting point for the construction is the boundary partition function we computed in eq. (3.28). The associated orbifold partition function is obtained as a sum over all twisted sector contributions,

$$\ell\underset{0}{\square} = \mathcal{N} Z^{\text{WS}}_{A,u|B,v}(\ell t, \ell\zeta;\tau), \tag{3.30}$$

corresponding to strings that wrap the thermal cycle of thermal AdS_3 with winding number ℓ. Here, we inserted a normalisation constant $\mathcal{N}$ that we shall determine later. Summing over ℓ, we obtain the worldsheet partition function

$$Z^{\text{WS}\ AdS_3^T}_{A,u|B,v}(t,\zeta;\tau) = \mathcal{N}\frac{\tau^2}{\ell t}\sum_{w,\ell=1}^{\infty} \tfrac{1}{2}(e^{i\pi w} - e^{i\pi\delta^B_A})\delta(\tfrac{\ell}{w}t - \tau) e^{\frac{\pi i \ell^2\zeta^2}{2\tau}} Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{bmatrix}(\ell\zeta|\tau). \tag{3.31}$$

Upon integration over the modulus τ of the worldsheet, we obtain

$$Z^{\text{ST}\ AdS_3^T}_{A,u|B,v}(t,\zeta) = \frac{\mathcal{N}}{t}\sum_{w,\ell=1}^{\infty} \tfrac{1}{2\ell}(e^{i\pi w} - e^{i\pi\delta^B_A}) e^{\frac{\pi i w \ell \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{bmatrix}(\ell\zeta|\tfrac{\ell}{w}t). \tag{3.32}$$

In terms of $N = w\ell$, we can also rewrite this as

$$Z^{\text{ST}\ AdS_3^T}_{A,u|B,v}(t,\zeta) = \frac{\mathcal{N}}{t}\sum_{N=1}^{\infty}\sum_{w|N} \frac{w}{N}\tfrac{1}{2}(e^{i\pi w} - e^{i\pi\delta^B_A}) e^{\frac{\pi i N \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{bmatrix}(\tfrac{N}{w}\zeta|\tfrac{N}{w^2}t). \tag{3.33}$$

Finally, we conclude that the full open string partition function, counting strings that can start and end on both halves of the AdS_2 brane, is

$$Z^{\text{ST}\ AdS_3^T}_{u|v}(t,\zeta) = \frac{\mathcal{N}}{t}\sum_{A,B\in\{L,R\}}\sum_{w,\ell=1}^{\infty} \tfrac{1}{2\ell}(e^{i\pi w} - e^{i\pi\delta^B_A}) e^{\frac{\pi i w \ell \zeta^2}{2t}} Z^{\mathbb{T}^4}_{u|v}\begin{bmatrix} \frac{\delta_{A,B}}{2} \\ \frac{1}{2} \end{bmatrix}(\ell\zeta|\tfrac{\ell}{w}t). \tag{3.34}$$

Here, t is the modular parameter of the spacetime torus and ζ is a chemical potential associated with the R-charge. The only non-trivial ingredient in the summands are the functions $Z^{\mathbb{T}^4}_{u|v}$ that we defined in eq. (3.26).

Before carrying on and matching the result of this string computation to the symmetric orbifold grand canonical partition function in section 4, let us pause and give a geometric interpretation of the formulas we obtained. Looking back at our formula (3.31), we infer from the argument of the δ function that the worldsheet modulus τ localises to $\frac{\ell}{w}t$ with two integers ℓ and w. We can relate this observation to the geometric interpretation that was given to a similar closed string computation in [43]. Eberhardt explained that the one-to-one correspondence between solutions to the equation

$$t = \frac{a\tau + b}{b\tau + d}, \quad \tau = \frac{dt - b}{-ct + a} \tag{3.35}$$

with integer coefficients $a, b, c, d \in \mathbb{Z}$ and holomorphic coverings of the t torus by the τ torus can be understood by realising that such a covering must lift to an affine map $\tilde{\Gamma} : \mathbb{C}^2 \to \mathbb{C}^2$, $z \mapsto \alpha z + \beta$ such that $\tilde{\Gamma}(\Lambda_\tau) \subseteq \Lambda_t$ holds. The latter condition in particular implies $\alpha, \beta \in \Lambda$ and $\alpha\tau + \beta \in \Lambda_t$ and hence enforces that τ is a fraction of elements of Λ_t. Consequentially, τ can be written as

$$\tau = \frac{dt - b}{-ct + a}. \tag{3.36}$$

For the situation at hand, we describe coverings of a cylinder by a cylinder obtained by cutting the tori along the imaginary axis (because the boundary of the AdS_2 brane is a thermal cycle in the boundary torus of thermal AdS_3). Thus, we want that $\tilde{\Gamma}(i\mathbb{R}) \subseteq i\mathbb{R}$, which implies $\alpha \in \mathbb{R}$ and $\beta \in i\mathbb{R}$. However, in combination with the fact that t is purely imaginary, $\alpha \in \mathbb{R}$ implies that c is zero. But if c is zero and t, τ are purely imaginary then b must also be zero. Hence, we are restricted to

$$\tau = \frac{d}{a}t. \tag{3.37}$$

Furthermore, the prefactor of $\frac{1}{\ell}$ in the partition function is to be expected. w counts the winding of the strings around the spatial cycle, while ℓ counts the winding around the thermal cycle. The translation symmetry along the spatial cycle is broken by the presence of the AdS_2 brane. But we still have full translation symmetry along the thermal cycle. Hence, there is a $\mathbb{Z}_\ell$ symmetry that the partition function needs to reflect by a factor of $\frac{1}{\ell}$.

4 Holographic matching of interfaces with AdS_2 branes

The goal of this section is to collect evidence supporting the claim that the interfaces we have constructed in section 2 indeed provide a holographic description of AdS_2 branes in a theory of tensionless type IIB superstrings in $AdS_3 \times S^3 \times \mathbb{T}^4$. In order to do so, we first, in section 4.1, compare our results on open string spectra, most notably formula (3.34) for the partition function of single open string states for AdS_2 branes in thermal AdS_3, with the counting function (2.57) for interface changing operators. Moreover, since closed strings do not couple to the AdS_2 brane in the tensionless limit, the holographic matching between symmetric orbifold correlators and closed string amplitudes trivially reduces to the well established duality in the absence of branes – some aspects of which we review in section 4.2. Finally, section 4.3 establishes the basis for the holographic duality between amplitudes involving open strings and corresponding correlation functions of the symmetric orbifold that involve certain interface changing operators. Concretely, we show that the large N expansion of correlation functions in the presence of our interfaces takes the form of a string theoretic genus expansion. We also establish the analogous claim for correlation functions in the presence of the boundary conditions studied in [8] to describe the dual of spherical branes in AdS_3. However, since this work is mainly about AdS_2 branes, we have moved the analysis of the spherical brane case into appendix C.

4.1 Partition functions and string amplitudes

As promised at the end of section 2.4, we now use the notation for partition functions of the supersymmetric four-torus established in section 3.2, to spell out eq. (2.76) more concretely for the special case in which the seed theory $\mathcal{M}$ is given by the supersymmetric four-torus $\mathbb{T}^4$. This entails making a particular choice for the boundary conditions $a_\pm^{L/R}$. Our proposal is

$$|a_-^L\rangle = \|u, \mathrm{R}, \varepsilon\rangle\!\rangle_{\mathbb{T}^4} \quad \text{and} \quad |a_+^L\rangle = \|v^*, \mathrm{R}, \varepsilon\rangle\!\rangle_{\mathbb{T}^4} \tag{4.1}$$

as well as

$$|a_-^R\rangle = -\|v, \mathrm{R}, -\varepsilon\rangle\!\rangle_{\mathbb{T}^4} \quad \text{and} \quad |a_+^R\rangle = -\|u^*, \mathrm{R}, -\varepsilon\rangle\!\rangle_{\mathbb{T}^4}, \tag{4.2}$$

which leads to

$$\hat{Z}_o^{LL}(\hat{t},\hat{\zeta}) = \hat{Z}_o^{RR}(\hat{t},\hat{\zeta}) = \hat{Z}_{u|v}^{\mathbb{T}^4}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\hat{\zeta}|\hat{t}) \quad \text{and} \quad \hat{Z}_o^{LR}(\hat{t},\hat{\zeta}) = \hat{Z}_o^{RL}(\hat{t},\hat{\zeta}) = -\hat{Z}_{u|v}^{\mathbb{T}^4}\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(\hat{\zeta}|\hat{t}). \tag{4.3}$$

We can thus conclude that

$$\mathcal{Z}_O[\hat{t},\hat{\zeta}] = \sum_{A,B\in\{L,R\}}\sum_{\ell,w=1}^{\infty}\tfrac{1}{2w}(e^{i\pi\ell} - e^{i\pi\delta_A^B})\hat{Z}_{u|v}^{\mathbb{T}^4}\begin{bmatrix}\frac{1}{2}\\ \frac{\delta_{A,B}}{2}\end{bmatrix}(\ell\hat{\zeta}|\tfrac{\ell}{w}\hat{t}) \tag{4.4}$$

and applying eq. (3.25), as well as swapping w and ℓ, gives

$$\mathcal{Z}_O[t,\zeta] = \sum_{A,B\in\{L,R\}}\sum_{\ell,w=1}^{\infty}\tfrac{1}{2\ell}(e^{i\pi\ell} - e^{i\pi\delta_A^B})e^{\frac{\pi i w\ell\zeta^2}{2t}}Z_{u|v}^{\mathbb{T}^4}\begin{bmatrix}\frac{\delta_{A,B}}{2}\\ \frac{1}{2}\end{bmatrix}(\ell\zeta|\tfrac{\ell}{w}t). \tag{4.5}$$

This partition function is manifestly identical to eq. (3.34) if we choose the normalisation $\mathcal{N} = t$. Furthermore, eq. (2.56) directly tells us that $\mathcal{Z}_C$ is identical to the $t = \bar{t}$ restriction of the space time CFT torus partition function. Making the appropriate choice of spin structure, the latter has been matched with the thermal AdS_3 closed string partition function in [43]. The rest of this section is devoted to a qualitative description of the holographic duality underlying this equality.

The equality of partition functions is equivalent to the statement that the spectrum (i.e. the conformal weights and the eigenvalues of K_0 as well as the dimensions of the eigenspaces) of open strings stretching between two AdS_2 branes coincides with the spectrum of interface changing operators of our symmetric orbifold interfaces. Combining this observation for the open strings with the fact that there is no coupling of closed strings to the brane leads to a natural proposal for the holographic description of the tensionless string dynamics in the presence of an AdS_2 brane. Let us first sketch this proposal in a rough qualitative manner and then explore its precise quantitative implications.

1. We recall from [44] that in the tensionless limit, long *closed superstrings* of type IIB theory on $AdS_3 \times S^3 \times \mathbb{T}^4$ possess a holographic description in terms of twisted sector bulk operators in the symmetric product orbifolds $\mathrm{Sym}^N(\mathbb{T}^4)$.

2. Given that tensionless closed superstrings do not couple to an AdS_2 brane, we propose that the holographic dual of the *AdS_2 brane* is given by the unique defect in symmetric product orbifolds that does not couple to any bulk fields: The trivial defect.

3. Based on the equality between the partition functions (3.34) and (4.5), we propose that the *open strings* that end on the AdS_2 brane are dual to interface changing operators between interfaces of $\mathrm{Sym}^{N_\pm}(\mathbb{T}^4)$ which we constructed in section 2.2.

In order to simplify the navigation through the rest of this section, we divided it into short paragraphs that provide the interested reader with further details on various aspects of our proposal. The paragraphs can be read somewhat independently.

Thermal versus global AdS_3. In the three claims above, we do not specify whether we consider string theory on thermal or global AdS_3. Let us start our discussion of the claims by filling in some more detail on this. Recall that we constructed the open string spectrum on thermal AdS_3 from the partition function of global AdS_3, which we computed in eq. (3.29), by including additional twisted sectors that wind the thermal cycle $\ell = N/w$ times. Vice versa, we can pass from the thermal AdS_3 open string partition function (3.34) to its analogue for global AdS_3 simply by dropping all the $\ell > 1$ terms[13]. Because of this, the above holographic matching of the thermal AdS_3 partition function directly implies that we can also obtain the global AdS_3 partition function from our interface construction. What this requires is to adjust the boundary states (2.38) that describe our interfaces or, more precisely, their building blocks defined in eq. (2.36). These states involve the boundary states (2.26) which contain a sum in the ρ-twisted sectors. In order to recover the partition function of global AdS_2, we need to restrict the sum over $\rho_\pm$ to the trivial term $\rho_\pm = id$. This manifestly removes the $\ell > 1$ terms in the partition function of defect changing operators.

Absence of disk contributions. Note that, in sharp contrast to the bulk discussion [45] as well as that of spherical branes [8], our holographic matching of the open string

[13] As also explained at the end of section 3.3, the situation for the open strings is in this sense simpler than the closed string analogue discussed in [43]. Indeed, for the closed strings it is not true that the global AdS_3 partition function is obtained from the thermal AdS_3 one by simply dropping the $\ell > 1$ terms.

partition function did not rely on the extrapolation of large k results for (disconnected) genus 0 contributions to $k = 1$. Instead, the one-loop (annulus) string computation directly reproduces the exact symmetric orbifold partition function of interface changing operators (up to an overall normalisation factor)[14]. In fact, this remarkable absence of the somewhat problematic disk contributions is to be expected on simple geometric grounds. As opposed to the spherical branes, the AdS_2 branes are not contractible inside thermal AdS_3. Thus, the existence of contractible world-sheets whose boundary winds one of the boundaries of the AdS_2 brane is topologically obstructed.

The set of string scattering states. Having commented on the absence of disk contributions and clarified the roles of thermal and global AdS_3, let us now expand more generally on part three of our proposal. In order to formulate an actual prescription for holographic computations of open string scattering amplitudes in the tensionless limit of superstring theory on $AdS_3 \times S^3 \times \mathbb{T}^4$, the next paragraphs describe how to label and construct the interface changing operators dual to string scattering states. Recall from eqs. (2.55 - 2.59) that these operators fall into two classes, corresponding to the two classes of string states. On the one hand, the string theory contains long closed strings. Their quantum numbers include the winding numbers (w, ℓ). As in the previous sections of this paper, w counts how many times the string winds the spatial cycle of the boundary torus. Similarly, ℓ is the winding number associated to the thermal cycle. In addition, the string theory also contains long open strings. These possess two new labels $A, B \in \{L, R\}$ as well as the familiar pair (w, ℓ) of winding numbers. The labels $A, B \in \{L, R\}$ refer to the two components of the boundary of the AdS_2 brane to which two endpoints of the open strings are attached. While the interface changing operators that are dual to long closed strings are simply obtained by restricting familiar bulk operators to the one-dimensional locus of the interface, those that are associated with long open strings are new. In the following few paragraphs we will therefore mostly focus on the latter.

Interface changing operators dual to open strings. When we discussed bulk fields of symmetric product orbifolds in section 2, we pointed out that it is useful to work with a larger set of twist fields that are labelled by group elements of the orbifold group rather than the physical fields which are labelled by conjugacy classes. The physical operators are obtained from the former by averaging over the orbits of the $S_{N_\pm}$ action. We follow a similar strategy for the interface changing operators and describe the physical operators

$$\sigma^{AB}_{(w,\ell)} = \mathcal{N} \sum\nolimits_{\gamma} \sigma^{A,B}_{\gamma} \tag{4.6}$$

as a sum over objects that are labelled by a sequence $\gamma = (g_i)_{i=1}^{w}$ of length w. The entries g_i of the sequence γ are cycles of length ℓ with the additional condition that they alternate between the two orbifold groups S_{N_-} and S_{N_+}, i.e. if $g_i \in S_{N_\pm}$ then $g_{i+1} \in S_{N_\mp}$. The labels A, B determine the permutation group from which we pick the first and last entry g_1 and g_w of our sequence. If $A = L$, we agree to choose $g_1 \in S_{N_-}$ while $g_1 \in S_{N_+}$ in case $A = R$. The rules apply in reverse for the last entry g_w, i.e. $g_w \in S_{N_+}$ if $B = L$ and $g_w \in S_{N_-}$ for

[14] We thank Andrea Dei for encouraging us to emphasise this point more explicitly.

$B = R$. In the special case of $\ell = 1$ our sequence γ is simply a sequence of integers that we chose alternating between the two sets $\underline{N}_\pm = \{1, \dots, N_\pm\}$ with appropriate conditions imposed on the first and last entry. The labelling of open string states with $\ell = 1$ through sequences of integers was described in [9] already. Martinec used red and blue colours to distinguish between elements of $\underline{N}_\pm$.

Having described the label γ, it is rather straightforward to state how the actual operator $\sigma_\gamma^{A,B}$ acts. Recall that the interfaces described in section 2 are gauge invariant sums (2.36) of states that couple to one specific pair $g_\pm = \rho_\pm \tau_\pm \in S_{N_\pm}$ of twisted sectors in the bulk of the product theory. To each cycle in g_-, these gauge fixed states either associate a reflective boundary condition or a cycle in g_+ to which it is transmitted. In order to specify the operator σ_γ^{AB}, we pick a twisted sector with twists elements $g_\pm$ of the folded bulk theory to which the interface to the left (the L-interface) and to the right (the R-interface) couples. The cycles of $g_\pm$ are related to the sequence $\gamma = (g_i, i = 1, \dots, w)$ in the following way:

$$g_\mp = \prod_{i \text{ odd}} g_i \quad , \quad g_\pm = \prod_{i \text{ even}} g_i \quad \text{for } A = \begin{matrix} L \\ R \end{matrix} \, . \tag{4.7}$$

Regarding reflective boundary conditions, we demand that the first cycle g_1 is reflected at the A-interface and the last cycle g_w is reflected at the B-interface. All other cycles are to be transmitted. Regarding transmissive boundary conditions, our prescription is as follows. If $A = L$ then the sectors that are associated with the twist elements g_i and g_{i+1} for odd integer i are glued together at the R interface. If the index i is even, on the other hand, the two twisted sectors are glued at the L interface. For $A = R$ the prescription is reversed.

Interface changing operators dual to closed strings. Now that we have completed our description of the interface changing operators we propose to be dual to open strings, let us briefly comment on the operators $\sigma_{w,\ell}$ dual to closed strings. These are more familiar. Again, they possess a representation akin to eq. (4.6) but the nature of the label γ is a bit different. For $\ell = 1$ we are dealing with the operators that are dual to strings that can propagate in global AdS_3. In this case, γ is the product of two cycles of length $w/2$, one in S_{N_-} the other in S_{N_+}. More generally, for $\ell > 1$, the sequence γ of cycles g_i of length ℓ that we described in our discussion of the operators $\sigma_{(w,\ell)}^{AB}$ are now periodically identified, i.e. we extend the sequence such that $g_{w+1} = g_1$. Since we had chosen neighbouring entries from the upper and lower half plane in an alternating fashion, we see that w must be even in order to admit the periodic identification.

The operators σ_γ that contribute to $\sigma_{(w,\ell)}$ now admit a description that is very reminiscent of our description of $\sigma_\gamma^{A,B}$ in the previous comment except that they do no longer involve reflections but rather are purely transmissive. Put differently, twist elements $g_i \in S_{N_+}$ and $g_{i+1} \in S_{N_-}$ are glued together at the L interface and similarly twist elements $g_i \in S_{N_-}$ and $g_{i+1} \in S_{N_+}$ are glued together at the R interface. This time, the prescription also applies to the endpoints of the sequence γ using the periodic identification $g_{w+1} = g_1$.

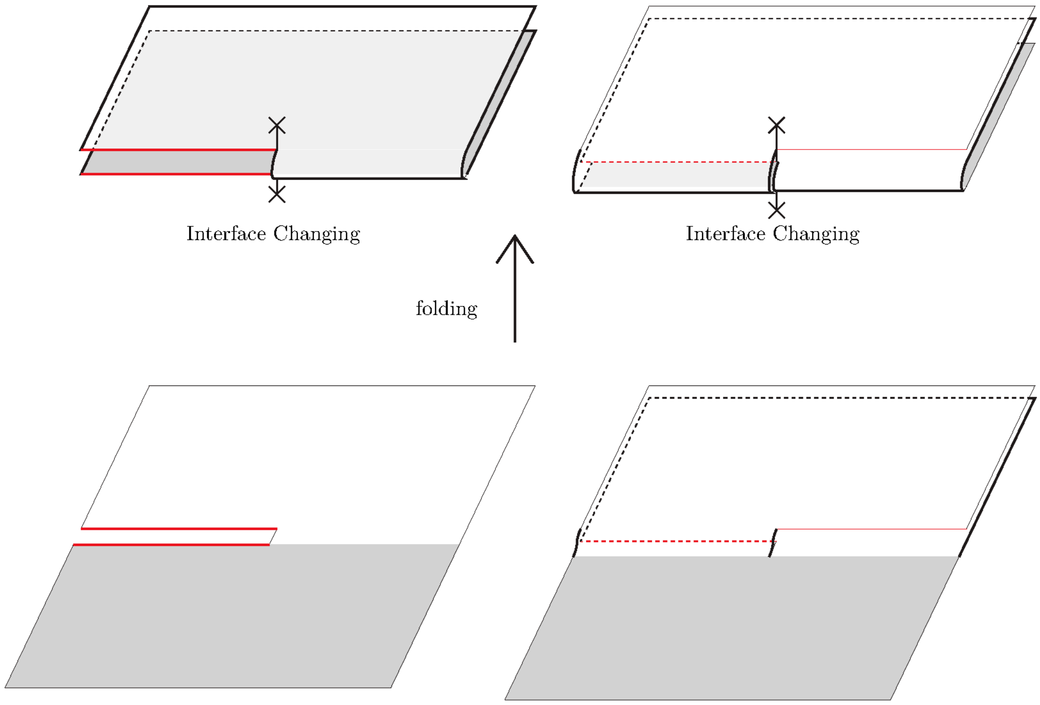

Figure 6. Visualisation of interface changing operators. Red lines indicate reflective boundaries. Gray sheets are representing the lower half-plane on which $\mathrm{Sym}^{N_-}(\mathcal{M})$ lives. White sheets are upper half-planes that carry $\mathrm{Sym}^{N_+}(\mathcal{M})$. On the left hand side, we portray the operators $\sigma^{LL}_{(w=2,\ell=1)}$. The right hand side represents $\sigma^{RL}_{(3,1)}$.

Taking stock. Let us pause a moment to take stock of what we have formulated in the previous paragraphs. We have constructed two types of interface changing operators $\sigma^{AB}_{(w,\ell)}$ and $\sigma_{(w,\ell)}$ that carry the same labels as open strings ending on AdS_2 branes and closed strings winding the boundary of AdS_3 respectively.

Moreover, all interface changing operators between any two of the interfaces described in section 2 can be obtained by taking products of the elementary operators we described. Hence, the grand canonical partition function counting these general interface changing operators is an exponential of a "single particle" partition function that counts the $\sigma^{AB}_{(w,\ell)}$ and $\sigma_{(w,\ell)}$. We have seen that this "single particle" partition function coincides with the partition function of open and closed strings propagating on thermal AdS_3. This provides strong evidence for the conjecture that the two families of operators we have introduced in the previous paragraphs are dual to open and closed strings on thermal AdS_3.

General string scattering amplitudes. Of course, the proposed duality should eventually be supported through a comparison of arbitrary scattering amplitudes of open and closed strings with correlation functions of the interface changing operators $\sigma^{AB}_{(w,\ell)}$ and $\sigma_{(w,\ell)}$ we introduced above. At this point, we can at least state somewhat more explicitly what this relation is expected to look like.

On the string side, we encounter scattering amplitudes that can be written as integrals over the moduli space $\mathcal{M}_{b,g,n_c,n_o}$ of Riemann surfaces Σ_{b,g,n_c,n_o} of genus g with b boundary components as well as n_c punctures in the interior and n_o punctures on the boundary. The associated integrand is a correlation function of n_c closed string vertex operators and n_o open string vertex operators of the worldsheet theory. Thus, such a string scattering amplitude takes the form

$$\mathcal{A} = \int_{\mathcal{M}_{b,g,n_c,n_o}} \langle \prod_{i=1}^{n_c} V_{w_i,h_i,\bar{h}_i}(z_i,\bar{z}_i,t_i) \prod_{j=1}^{n_o} V^{A,B}_{w_j,\Delta_j}(y_j,t_j)\rangle_{\Sigma_{b,g,n_c,n_o}} . \tag{4.8}$$

Here, w_i label the winding of the closed strings around the cycles of the Riemann surface, $(h_i,\bar{h}_i)$ are its holomorphic and anti-holomorphic space time conformal weights, $(z_i,\bar{z}_i)$ labels a point in the interior of the Riemann surface and t_i labels a point on the boundary of the AdS_2 brane in the target space. The role of t_i is completely analogous to that of $(x_i,\bar{x}_i)$ in the x-basis vertex operators of [32], i.e. they are introduced by an appropriate deformation of the ordinary Vertex operators. In the closed string scenario discussed in [32], the deformation is generated by the zero modes J_0^+ and $\bar{J}_0^+$ of one holomorphic and one anti-holomorphic current of the $SL(2,\mathbb{R})$ WZNW model. The AdS_2 boundary condition glues these currents together along the boundary so that only one zero modes survives. This zero mode generates the deformation by the parameter t. Moreover, the open string vertex operators are inserted at the points y_j on the boundary of the worldsheet and they carry only a single conformal weight Δ_j.

On the symmetric orbifold side, these string scattering amplitudes should be reproduced by correlation functions

$$\langle \prod_{i=1}^{n_c} \sigma_{w_i,h_i,\bar{h}_i}(t_i) \prod_{j=1}^{n_o} \sigma^{A,B}_{w_j,\Delta_j}(t_j)\rangle\Big|_{g,b} \tag{4.9}$$

of interface changing operators. Here, we have not only included the labels which specify the ground state of the twisted sectors in which the operators live, but also added labels $h_i,\bar{h}_i$ and Δ_j that keep track of the excitations. The label g,b instructs us to restrict to the contribution to the correlator computed through covering maps whose covering space is an element of $\mathcal{M}_{b,g,n_c,n_o}$.

We will formulate the conjecture in more detail and collect evidence for it in forthcoming work. Here, we shall content ourselves with a few basic statements. Concretely, in the remaining two paragraphs of this section, we discuss two special classes of string amplitudes for which the correspondence is trivially satisfied. In section 4.3 we then show, for a special subset of boundary operators, that the N scaling of the large N-expansion of the boundary symmetric orbifold indeed matches the scaling of the dual string scattering amplitudes in the string coupling.

Closed string scattering amplitudes One simple type of amplitudes that we can compare right away are the scattering amplitudes of closed string states in the absence of open strings.

On the symmetric orbifold side, the relevant correlation functions only involve interface changing operators of type $\sigma_{(w,\ell)}$. These are simply correlators of bulk twist fields whose insertion points are restricted to the one-dimensional (trivial) interface of the orbifold theory. Such correlation functions are known to be dual to closed string amplitudes in the absence of an AdS_2 brane.

To match this with the string theory side, it is crucial to recall that the only closed strings that can couple to AdS_2 branes are those with vanishing winding number. But tensionless strings in $AdS_3 \times S^3 \times \mathbb{T}^4$ contain no physical states with zero winding number. Hence, the coupling to AdS_2 branes is trivial and without the insertion of extra open strings, the closed string scattering amplitudes are not affected by the presence of the brane, just as predicted by the previous statements about the dual correlation functions.

Scattering of one closed and one open string. We can actually go one step further and study scattering amplitudes of a single open and a single closed string. These should be related to the two-point functions

$$\langle \sigma_{w_c,h_c,\bar{h}_c}(t_c)\sigma^{A,B}_{w_o,\Delta_o}(t_o)\rangle \tag{4.10}$$

in the symmetric product CFT. We claim that all these two-point functions of interface changing operators vanish. This is not difficult to see. Indeed, around the insertion point t_c of the field σ_{w_c}, all boundary conditions must be transmissive. On the other hand, inserting operators of the form $\sigma^{A,B}_{w_o}$ requires that some components of the symmetric product are reflected, at least on one side of the interface changing operator. Hence, correlation functions with a single insertion of $\sigma^{A,B}_{w_0}$ vanish.

We expect that this behaviour is matched in string theory. Indeed, for tensionless strings in $AdS_3 \times S^3 \times \mathbb{T}^4$, closed string amplitudes factorise on the set of closed string states simply because there are no non-trivial closed string states that couple to the AdS_2 brane. This makes it appear plausible that there are no processes in which a single closed string can split open to convert into a single open string. A more precise derivation using localisation of the worldsheet along with properties of branching functions will be given in a forthcoming paper.

4.2 Bulk correlation functions and amplitudes - a review

As mentioned towards the end of the last section, now that we have successfully matched the string theory and space time CFT partition functions, the natural next step is to show that the duality also extends to correlators. Here, we shall content ourselves with the very first step of such an extension. Namely, we show that the large N expansion of correlation functions involving bulk and interface changing operators in the presence of our interfaces takes the form of a string theoretic genus expansion. This is reminiscent of 't Hooft's analysis of the large N limit for gauge theories. A detailed comparison of the particular amplitudes with those of string theory in AdS_3 will be performed in a forthcoming publication. For bulk correlations of symmetric product orbifolds the large N behaviour was studied by Pakman, Rastelli and Razamat [2]. To set up notations and illustrate the main ideas, we shall review their arguments first before including interface changing operators in the next subsection.

The covering space method of[15] Lunin and Mathur [47] computes correlation functions

$$\langle \sigma_{g_1}(z_1, \bar{z}_1) \ldots \sigma_{g_{n_c}}(z_n, \bar{z}_n) \rangle \tag{4.11}$$

of gauge fixed twist fields in (bulk) symmetric product orbifolds on some two-dimensional base space M from the data of branched coverings

$$\gamma : \Sigma \to M \tag{4.12}$$

of the base space M by a covering space Σ. Here, we shall assume that the elements $g_\nu \in S_N$ are cyclic permutations of length w_ν. The general case can be treated similarly and in fact follows directly from that of cyclic g_ν by taking suitable OPEs. We place a subscript c on the number n_c of bulk field insertions to remind us that these should be in one-to-one correspondence with insertions of vertex operators of closed strings in the dual theory.

Let us explain how to determine the covering space topology from the choice of cyclic permutations g_i. To this end, it is useful to introduce the concept of "active colours" [2]. Given any $g \in S_N$, the subset A_g of active colours consists of all elements in $\{1, \ldots, N\}$ on which g acts non-trivially i.e. $A_g := \{i | gi \neq i\}$. Furthermore, we define the active colours A_S of any subset $S \subseteq S_N$ of the permutation group to be the union of the active colours of all its elements. This definition is now applied to the set $S = \{g_1, \ldots, g_{n_c}\}$ of cyclic permutations that appear in the correlation function (4.11). We call the associated set A_S the *set of active colours of the correlation functions.*

The permutations g_ν that appear in the correlation function we want to evaluate generate a subgroup $H_S \subseteq S_N$ of the permutation group. By construction, H_S acts faithfully on the corresponding set A_S of active colours. Under this action, the set A_S decomposes into orbits which we denote by $\mathscr{O}_1, \ldots, \mathscr{O}_m$. Once we have determined the orbits, it is rather easy to infer key features of the covering surface Σ. More specifically, it turns out that

1. The orbits $\mathscr{O}$ are in one-to-one correspondence with the connected components $\Sigma_{\mathscr{O}}$ of the covering surface Σ. In particular, the number of connected components of Σ is equal to the number $m = m(S)$ of orbits.

2. The genus $g_{\mathscr{O}}$ of the connected component $\Sigma_{\mathscr{O}}$ that is associated to the orbit $\mathscr{O}$ can be computed in terms of the number of elements $|\mathscr{O}|$ of the set $\mathscr{O}$ as

$$g_{\mathscr{O}} = 1 - |\mathscr{O}| + \tfrac{1}{2} \sum_{A_{g_\nu} \subseteq \mathscr{O}} (w_\nu - 1). \tag{4.13}$$

   The sum runs over all those indices $\nu \in \{1, \ldots, n_c\}$ for which the set of active colours A_{g_ν} of the group element g_ν is a subset of the orbit $\mathscr{O}$.

Hence, the covering surface Σ is connected if and only if there is exactly one orbit. Put differently, in order for Σ to be simply connected the subgroup $H_S \subseteq S_N$ that is generated

[15] See also [46] for an earlier application of covering space methods to orbifold CFTs.

by the elements of $S = \{g_1, \ldots, g_{n_c}\}$ must act transitively on the active colours A_S of the correlator. Denoting the number of active colours of the correlator by $|A_S| = n$, the formula for the genus then becomes

$$g = 1 - n + \tfrac{1}{2}\sum_{\nu=1}^{n_c}(w_\nu - 1). \tag{4.14}$$

The physical correlation functions to consider in a symmetric product orbifold are of course not the gauge fixed correlators that we have discussed so far, but rather the correlation functions of gauge invariant twist fields

$$\sigma_w = \frac{1}{\sqrt{|[(1\ldots w)]|}}\sum_{g\in[(1\ldots w)]}\sigma_g = \sqrt{\tfrac{(N-w)!w}{N!}}\sum_{g\in[(1\ldots w)]}\sigma_g. \tag{4.15}$$

Clearly, the expansion of the gauge invariant twist fields into gauge fixed twist fields allows us to immediately express correlators of the former in terms of correlators of the latter,

$$\langle\prod_{\nu=1}^{n_c}\sigma_{w_\nu}(z_\nu,\bar{z}_\nu)\rangle = \left(\prod_{\nu=1}^{n_c}\sqrt{\tfrac{(N-w_\nu)!w_\nu}{N!}}\right)\sum_{g_\nu\in[(1\ldots w_\nu)]}\langle\sigma_{g_1}(z_1,\bar{z}_1)\ldots\sigma_{g_{n_c}}(z_{n_c},\bar{z}_{n_c})\rangle. \tag{4.16}$$

The connected part of such a correlator is defined as the restriction of the above sum to those terms that are associated to connected covering spaces Σ. According to the comments that we made earlier in this section, we can state this definition as

$$\langle\prod_{\nu=1}^{n_c}\sigma_{w_\nu}(z_\nu,\bar{z}_\nu)\rangle_c := \left(\prod_{\nu=1}^{n_c}\sqrt{\tfrac{(N-w_\nu)!w_\nu}{N!}}\right)\sum_{\substack{g_\nu\in[(1\ldots w_\nu)]\\ m(\{g_1,\ldots,g_{n_c}\})=1}}\langle\sigma_{g_1}(z_1,\bar{z}_1)\ldots\sigma_{g_{n_c}}(z_{n_c},\bar{z}_{n_c})\rangle. \tag{4.17}$$

The restricted sum now runs over all sets $S = \{g_1, \ldots, g_{n_c}\}$ of cyclic permutations g_ν for which the active colours A_S of the correlations function form a single orbit under the action of H_S. Remarkably, the large N expansion of the connected part of the correlator takes the form of a string genus expansion. In order to understand the precise scaling with N, we point out that the restricted sum on the right hand side of eq. (4.17) contains combinatorial enhancements from the number of ways in which one can pick n active colours from the N colours that are available. Thus, the connected correlator scales as

$$\prod_{\nu=1}^{n_c}\sqrt{\tfrac{(N-w_\nu)!w_\nu}{N!}}\,\frac{N!}{n!(N-n)!} \;\sim\; N^{n-\frac{1}{2}\sum_\nu^{n_c}w_\nu} \sim g_s{}^{-2n+\sum_\nu^{n_c}w_\nu} = g_s^{-2+2g+n_c}\;. \tag{4.18}$$

In the first step, we used Stirling's formula to determine the leading asymptotics of the prefactor as N becomes large. Then, we used the standard relation $g_s^2 \sim 1/N$ between the string coupling g_s and the total number N of colours. Finally, we inserted eq. (4.14) in order to rewrite the exponent in terms of the genus g of the covering surface Σ. The right hand side of eq. (4.18) can be recognised as the usual dependence of closed string scattering amplitudes on the string coupling.

4.3 Boundary genus expansion for AdS_2 branes

In this subsection, we extend the previous analysis to correlation functions involving both bulk and interface changing fields and show that the large N expansion of correlation functions in the presence of our interfaces organises itself in the fashion of a string theory genus expansion. Just as in section 4.2, (the connected part of) a gauge invariant correlation function is a sum of gauge dependent correlators each of which can be associated to a covering space with a certain genus g and with n_c punctures associated to the (bulk) operators inserted inside the correlator. In contrast to the covering spaces described in section 4.2, the covering spaces that we encounter in this subsection additionally have a possibly non vanishing number b of boundaries, as well as a number n_o of punctures on the boundary.

In this paper, we do not provide a full analysis of all interface changing correlation functions, but rather restrict to a special class of boundary operators. While somewhat more tedious to spell out, the generalisation to arbitrary interface changing operators is conceptually straight forward. Concretely, we restrict to fields $\sigma^{\pm}$ that interpolate between $\mathcal{I}^{(p)}_{|a_\pm\rangle}$ and $\mathcal{I}^{(p\pm1)}_{|a_\pm\rangle}$ and possess an expansion of the form

$$\sigma^{\pm} = N_-^{-\frac{1}{4}} N_+^{-\frac{1}{4}} \sum_{i^-=1}^{N_-} \sum_{i^+=1}^{N_+} \sigma^{\pm}_{i^{\mp} i^{\pm}}. \tag{4.19}$$

Here, the operator $\sigma^-_{i^- i^+}$ cuts the i^-th copy of the seed theory in the lower hemisphere from the i^+th copy in the upper hemisphere and turns on an $|a_\pm\rangle$ boundary for both. Conversely, the operator $\sigma^+_{i^+ i^-}$ removes two copies of the boundary $|a_\pm\rangle$ and glues the i^-th copy of the seed theory in the lower hemisphere to the i^+th copy in the upper hemisphere. Consequently

$$\sigma^+ = \sigma^{RR}_{(2,1)} \quad \text{and} \quad \sigma^- = \sigma^{LL}_{(2,1)} \tag{4.20}$$

in the notation used in section 4.1. Note that the order of the indices i^+ and i^- is different in σ^+ and σ^-. This is a choice of notation that is convenient in describing the computation of correlation functions

$$\langle \prod_{\nu=1}^{n_o} \sigma^{s_\nu}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \rangle \,, \tag{4.21}$$

where $s_\nu \in \{-,+\}$. We assume the ordering $\nu < \mu \Rightarrow x_\nu < x_\mu$. Since the background in which this correlation function is computed just consists of the trivial interface, any boundaries that should be glued together by a σ^+ first had to be cut open by the insertion of a σ^-. More generally, a correlator with n_o (ordered) interface changing operators $\sigma^{\pm}$ can only be non-vanishing if

$$\sum_{i=1}^{n_o} s_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} s_i \leq 0 \tag{4.22}$$

for every $k < n_o$. As for the correlation functions of bulk twist fields, we can express the correlator (4.21) as a sum over gauge fixed terms. Concretely, inserting the definition (4.19) of $\sigma^{\pm}$, we obtain

$$\langle \prod_{\nu=1}^{n_o} \sigma^{s_\nu}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \rangle = N^{-\frac{n_o}{2}} \langle \prod_{\nu=1}^{n_o} \sum_{i_\nu^{\pm}}^{N} \sigma^{s_\nu}_{i_\nu^{-s_\nu}, i_\nu^{s_\nu}}(t_\nu) \prod_{\mu=1}^{n_c} \sigma_{w_\mu}(x_\mu, \bar{x}_\mu) \rangle. \tag{4.23}$$

For the individual terms on the right hand side not to vanish, all indices $i_\nu^{\pm}$ must be contracted according to the following rule:

Wick contractions for $\sigma^{\pm}$ colour indices: The left index i_ν^+ of every boundary creation operator $\sigma^-_{i_\nu^+, i_\nu^-}$ needs to be contracted with the right index i_μ^+ of a boundary annihilation operator $\sigma^+_{i_\mu^-, i_\mu^+}$ such that $x_\nu < x_\mu$. In complete analogy, the left index i_μ^- of a boundary annihilation operator inserted at x_μ needs to be contracted with the right index i_ν^- of a boundary creation operator inserted at $x_\nu < x_\mu$.

Following this contraction procedure, starting from an arbitrary boundary creation operator $\sigma^-_{c_1^+ c_1^-}$, we zigzag back and forth along the boundary line, collecting a string of boundary creation and annihilation operators $\sigma^-_{c_1^+ c_1^-} \sigma^+_{c_1^- c_2^+} \sigma^-_{c_2^+ c_3^-} \dots \sigma^+_{c^-_{w_{n_c}} c_1^+}$ that ultimately has to close into some cycle bringing us back to c_1^+. Keeping only the c_i^-, we obtain a cyclic permutation $g_{n_c+1} = (c_1^- c_2^- \dots c^-_{w_{n_c+1}}) \in S_{N_-}$ whose length we denote by w_{n_c+1}. This notation reflects the fact that upon fusing all of the operators that formed the cycle, we obtain twist fields in the g_{n_c+1} twisted sector. The latter are analogous to the bulk operators that are indexed by n_c cycles g_ν of length w_ν.

Effectively, the Wick contractions among interface changing operators we just performed thus gave rise to an additional cycle. If the latter did not include all the interface changing operators, we pick one of the remaining ones and form a second cycle g_{n_c+2} of length w_{n_c+2}. We continue this process until we reach the last cycle g_{n_c+b} of length w_{n_c+b}, i.e. until no uncontracted interface changing operators remain.

Once we have completed this contraction process, we are now left with $n_c + b$ cycles g_ν of length w_ν. These data are identical to those that would appear if we had to calculate a $(n_c + b)$-point correlation function of bulk operators. Consequently, we can now follow essentially the same steps we outlined in the previous subsection. As we did there, we first form the set $S = \{g_1, g_2, \dots g_{n_c+b}\}$ and the group H_S that is generated by the cycles g_μ. We then let H_S act on the active colours. We shall say that a term in the sum on the right hand side of eq. (4.23) contributes to the connected part of the correlation functions if the group H_N acts transitively on the active colours. Furthermore, the terms that are contributing to the connected part of our correlation functions are associated with a connected covering space whose genus g is given by[16]

$$g = 1 - n + \frac{1}{2} \sum_{\mu=1}^{n_c+b} (w_\mu - 1) \tag{4.24}$$

where n denotes the number of active colours as before. By construction, the cycles that are associated with the interface changing operators are in one-to-one correspondence with the

[16] This follows directly from the fact that the interface changing operators that are associated to a cycle g_{n_c+i} fuse into g_{n_c+i} twist fields upon performing the OPE, which we have already alluded to above.

connected components of the boundary of the covering surface. In particular, the number of connected components of the boundary is b.

We are finally able to compute the large N scaling behaviour of the connected part of the correlation function. There are three factors that contribute. As in the case of bulk correlation functions, we need to select the n active colours from the N colours that are available. In addition, we need to collect the normalisations of bulk and interface changing fields. The large N behaviour of the first two factors we evaluated in eq. (4.18) already. Since the interface changing fields we consider each contribute a factor of $N^{-1/2}$, the large N asymptotics of the prefactor reads

$$N^{n-\frac{1}{2}\sum_{\nu}^{n_c} w_\nu - \frac{n_o}{2}} = N^{n-\frac{1}{2}\sum_{\nu}^{n_c} w_\nu - \frac{1}{2}\sum_{\nu=1}^{b} w_{n_c+\nu} - \frac{n_o}{4}} . \tag{4.25}$$

In going to the right hand side of this equations, we have used the fact that

$$\sum_{\nu=1}^{b} w_{n_c+\nu} = \frac{n_o}{2}, \tag{4.26}$$

which follows from the construction of the w_μ with $\mu > n_c$ since they count the number of interface changing operators that are combined into a cycle and non-vanishing correlators require that all the fields are part of some cycle. Now we can proceed by inserting equation (4.24) to express the sums over the cycle lengths w_μ through the genus g of the branching surface. The result is

$$N^{n-\frac{1}{2}\sum_{\nu}^{n_c} w_\nu - \frac{1}{2}\sum_{\nu=1}^{b} w_{n_c+\nu} - \frac{n_o}{4}} = N^{1-g-\frac{b}{2}-\frac{n_c}{2}-\frac{n_o}{4}} \sim g_s^{-2+2g+b+n_c+\frac{n_o}{2}} . \tag{4.27}$$

Once again, we have used that $g_s^2 \sim 1/N$ to express the final result for the large N asymptotics of the correlation function in terms of the string coupling g_s rather than $N \sim 1/g_s^2$. The final result indeed has the g_s dependence of a string amplitude for a surface of genus g with b boundary components, n_c bulk punctures and n_o boundary punctures. This is what we wanted to show.

5 Conclusion and outlook

In this work, we have constructed a new family of interfaces $\mathcal{I}^{(p)}_{|a_\pm\rangle}$ between two symmetric product orbifolds $\mathrm{Sym}^N(\mathcal{M})$ with $N = N\pm$. These interfaces are associated with a pair of boundary states $|a_\pm\rangle$ of the seed theory $\mathcal{M}$. The integer $p \le \min(N_-, N_+)$ controls the transmissivity of the interface. More precisely, the transmissivity (2.47) is proportional to p and takes its largest value for $p = \min(N_-, N_+)$. For $p = 0$, on the other hand, the interface is purely reflecting. A precise formula for the interface was given in section 2, see eq. (2.38) and the paragraph below that equation for notations. The overlaps between the associated boundary states and hence the spectrum of interface changing operators was computed in section 2.4. As in the case of the bulk spectrum of symmetric product orbifolds which is elegantly encoded in the grand canonical partition function found in [31], we stated the results in terms of a generalised grand canonical partition function which also involves chemical potentials for the indices p^L and p^R of transmissivity on both sides of the interface changing operator, see eqs. (2.54-2.59). The modular transform of the central formula (2.57), or rather of its restriction (2.76) to special values of the chemical potentials, was given in eq. (2.77). The latter formula contains contributions from interface changing operators of arbitrary twist. Single particle states with odd-integer twist require the orders N_+ and N_- of the symmetric product orbifolds on the two sides of the interface to be different.

The fact that our interfaces support operators in twisted sectors with arbitrary twist is a key feature that hints toward a possible holographic description in terms D-branes in AdS_3. Indeed, AdS_2 branes have been argued to support long open strings of arbitrary winding numbers [28]. In the case of the supersymmetric four-torus $\mathcal{M} = \mathbb{T}^4$, we have in fact provided significant evidence for such a holographic duality with AdS_2 branes in type IIB theory of tensionless strings in $AdS_3 \times S^3 \times \mathbb{T}^4$. In section 3, we computed the partition function (3.34) of AdS_2 branes in thermal AdS_3. The resulting function was shown in section 4.1 to match precisely with the corresponding terms in the exponent of the grand canonical partition function (2.76) of our interfaces in symmetric product orbifolds. In addition, the triviality of scattering amplitudes of a single tensionless closed string in the presence of an AdS_2 brane matches our description of the brane as a trivial defect in the dual symmetric orbifold.

In section 4.3 and appendix C, we also suggested how to match more general string amplitudes involving both closed and open strings, leaving a fully conclusive analysis for a forthcoming publication. As we have explained above, scattering amplitudes with at most one open string insertion are correctly reproduced by the holographic dual. In order to fully establish the correspondence between the interfaces $\mathcal{I}^{(p)}_{|a_\pm\rangle}$ and the AdS_2 branes for tensionless strings, it would therefore suffice to compare three-point functions of three interface changing operators with the scattering amplitudes of three open string states in AdS_3. Even though the three-point couplings are non-trivial, their computation and comparison is expected to closely follow the related analysis that was carried out for bulk operators in symmetric product orbifolds and closed strings, see [32]. We will work out the details in a forthcoming paper.

While the full proof of the holographic duality for tensionless closed and open strings that we described in the previous paragraph relies on the explicit calculation of quantities on both sides of the correspondence, it would also be very interesting to uncover the mechanism of the holographic relationship in the spirit of [48]. In that work, two of the authors considered an extension of string theory in AdS_3 to arbitrary values k of NSNS background flux. According to a proposal of Eberhardt in [49] such a string theory is dual to some non-rational symmetric product orbifold with a certain Liouville like interaction turned on, see also [50, 51]. In [48] this interacting CFT_2 was used as a starting point and its correlators were rewritten (for any number of bulk insertions) in terms of scattering amplitudes of closed strings in some AdS_3 string theory. The main idea was to uplift the Liouville direction of the interacting non-rational symmetric product orbifold to the radial direction of AdS_3 by reversing an intriguing relation between the H_3^+ WZNW model and Liouville field theory [52–55], see also [56–59] for related developments. Thereby the holographic correspondence was established for arbitrary correlation functions without ever computing a single correlation function or scattering amplitude. It should be possible to extend this type of derivation to include branes and open strings along the lines of [60].

A particularly interesting aspect that deserves attention in going away from the case with $k = 1$ arises because AdS_2 branes are generically not unique but rather carry an additional parameter r. In the spacetime description, the parameter r is related to the angle at which the brane ends on the boundary of AdS_3. The case of $r = 0$ corresponds to an AdS_2 brane that runs straight through the centre of AdS_3 so that it approaches the boundary at 90 degrees. For other non-vanishing values of r, the angle is non-trivial. The analogy with Karch-Randall interfaces in higher dimensions suggests that the parameter r should be related to the difference $N_+ - N_-$. In [9] Martinec also proposed that $r \sim (N_- - N_+)/(N_+ + N_-)$.

A very interesting challenge for the type of holographic relation we described here is to make contact with the geometric supergravity regime by turning on RR background flux. In the symmetric product orbifold this corresponds to a marginal deformation by some particular operator, see e.g. [61] for an early review. This deformation is somewhat similar to switching on the interaction in four-dimensional $\mathcal{N} = 4$ SYM theory. In the latter case, the leading perturbative corrections to the spectrum of the free field theory (and eventually the entire deformation into the geometric regime of infinite 't Hooft coupling λ) can be computed using integrability, see e.g. [62] for first order calculations (and [63] for further results from integrability). For the two-dimensional cousins such powerful tools to reach the geometric regime are not available (yet), even though there exists a few attempts to start an integrability based approach to the problem, see e.g. [64, 65]. One of the issues that complicates the analysis of the perturbative spectrum near the symmetric product orbifold is the mixing of left- and right-moving modes in the bulk. It might therefore be advantageous to study the deformation for open strings on AdS_2 probe branes instead. Note that their open string spectrum is as rich as that of closed strings with long strings of arbitrary winding number w and one might hope that these spectra can be deformed away from the tensionless limit all the way to the geometric regime using the ideas developed in [66, 67]. It would also be interesting to extend other integrability based studied of string theory in $AdS_3 \times S^3 \times \mathbb{T}^4$, see e.g. [68, 69] and references therein, to line defects as was done for Wilson lines in $\mathcal{N} = 4$ SYM theory [70, 71]. When combined with the toolbox of integrability, the interfaces we introduced here could turn out to be useful probes of emerging geometries.

Acknowledgments

We thank Alejandra Castro, Stefan Fredenhagen, Kazuo Hosomichi, Bob Knighton, Emil Martinec, Ingo Runkel, Bogdan Stefanski, Yu-ki Suzuki, Gerard Watts, and especially Andrea Dei and Torben Skrzypek for useful comments and discussions. This project received funding from the German Research Foundation DFG under Germany's Excellence Strategy - EXC 2121 Quantum Universe - 390833306 and the Collaborative Research Center - SFB 1624 "Higher structures, moduli spaces and integrability" - 506632645. SH is further supported by the Studienstiftung des Deutschen Volkes. YH is supported by JSPS KAKENHI

Grant Numbers JP21H05187 and JP23K25867. TT is supported by JSPS KAKENHI Grant Number JP24KJ1374. Sketches in this work have been created using the Adobe Illustrator plug-in LaTeX2AI (https://github.com/isteinbrecher/latex2ai).

A Torus partition function of the symmetric product orbifold

In this appendix, we derive eq. (2.4). To do so, we need to acknowledge that fixed (g,h) contributions to the partition function (2.3) can be identified with ways of wrapping collections of disconnected tori around the spacetime torus. In this picture, we should view $g \in [w_1,\ldots,w_\ell]$ as providing the data that, if we were to cut all tori open along a spatial slice, then the corresponding covering of cylinders would be one where the covering space consists of ℓ disconnected cylinders that are wrapped around the spacetime cylinder w_i times respectively. h on the other hand tells us how these cylinders are glued together at the spatial cycles where we performed the cut. The exponential in the grand canonical partition function comes from the fact that the full partition function is obtained from the "single particle" part i.e. the part corresponding to coverings with a connected covering space by exponentiation. To be precise, the contribution

$$\frac{1}{N}\sum_{j=0}^{w-1} Z\left(\tfrac{Nt}{w^2}+\tfrac{j}{w}\right) \tag{A.1}$$

comes from covering the space time torus with a torus that winds w times around the spatial cycle and $\ell = N/w$ times the thermal cycle (where w necessarily has to divide N). That is, it comes from terms in the trace where $g \in [g_{(w,\ell)}] \subseteq S_N$ with

$$g_{(w,\ell)} = \prod_{k=0}^{\ell-1}(wk+1 \ wk+2 \ \ldots \ wk+w) \tag{A.2}$$

and where furthermore $h \in S_\ell \ltimes (\mathbb{Z}_w)^\ell$ has a S_ℓ part π_h that is just a single cycle of length ℓ. There are $(\ell-1)!$ such cycles π_h and hence we get that the (w,ℓ) winding covers contribute

$$Z_{(w,\ell)} = \frac{1}{N!}\sum_{g\in[g_{(w,\ell)}]}\sum_{\substack{h\in\mathcal{C}_g\\ |\pi_h|=\ell}} \mathrm{Tr}_{\mathcal{H}^g}\left[h q^{L_0-\frac{Nc_{\mathcal{M}}}{24}}\right] = \frac{|[g_{(w,\ell)}]|}{N!}(\ell-1)!\sum_{j_1,\ldots,j_\ell=0}^{w-1} Z\left(\frac{\ell t+\sum_{k=1}^{\ell} j_k}{w}\right). \tag{A.3}$$

But $|[g_{(w,\ell)}]| = \frac{N!}{\ell! w^\ell}$ and, since the spectrum only contains integer spin states, the seed partition function is $\mathbb{Z}$ periodic. Hence,

$$Z_{(w,\ell)} = \frac{1}{\ell w}\sum_{j=0}^{w-1} Z\left(\frac{\ell t + j}{w}\right) \tag{A.4}$$

and summing over ℓ and w as well as exponentiating, we get the grand canonical partition function

$$\sum_{N=0}^{\infty}\kappa^N Z_N = \exp\left(\sum_{\ell,w=1}^{\infty} Z_{(w,\ell)}\kappa^{w\ell}\right), \tag{A.5}$$

which upon replacing ℓ by N/w is the same as eq. (2.4).

B Interface changing operator partition function: detailed computation

In this appendix, we provide a more detailed derivation of eq. (2.68). The starting point is eq. (2.69), which written as a sum over conjugacy classes instead of individual group elements, takes the form

$$\langle N|\hat{x}^{L_0}|p,a\rangle = \sum_{[\tau]\in[S_p]}\sum_{[\rho]\in[S_{N-p}]}\sum_{\sigma\in\mathcal{C}^N_{\tau\rho}}\sum_{j_b,i_d}\frac{1}{|\mathcal{C}^{N-p}_\rho||\mathcal{C}^p_\tau|}\,{}^{N,N}_{\rho\tau,\tau\rho}\langle\!\langle j_b,i_d|\hat{x}^{L_0}\sigma|a\rangle_\rho|\mathbb{I}\rangle_{\tau,\tau}|a\rangle_\rho. \tag{B.1}$$

We can describe equivalence classes $[\tau]$ and $[\rho]$ with sequences[17] $(t_w)_{w\in\mathbb{N}}$ and $(r_w)_{w\in\mathbb{N}}$ such that t_w gives the number of cycles of length w in $[\tau]$. In this context, let

$$|t| := \sum_{w=1}^{\infty} t_w w. \tag{B.2}$$

Using this notation,

$$\mathcal{Z}_a[\mu_\pm,0,\rho_R;\hat{t}] = \sum_{t,r}\mu^{|t+r|}\rho_R^{2|r|}\sum_{\sigma\in\mathcal{C}_{r+t}}\frac{1}{|\mathcal{C}_t||\mathcal{C}_r|}\sum_{j_b,i_d}{}^{N,N}_{\rho\tau,\tau\rho}\langle\!\langle j_b,i_d|\hat{x}^{L_0}\sigma|a\rangle_\rho|\mathbb{I}\rangle_{\tau,\tau}|a\rangle_\rho, \tag{B.3}$$

where $\mu := \mu_-\mu_+$ and

$$\mathcal{C}_u := \prod_{w=1}^{\infty}\left(S_{u_w}\ltimes\mathbb{Z}_w^{u_w}\right), \tag{B.4}$$

while ρ and τ are arbitrary representatives of the conjugacy class determined by r and t. Now the contribution of fixed t and r factorises into a product of components that correspond to different cycle lengths according to

$$\mathcal{Z}_a = \sum_{t,r}\prod_{w=1}^{\infty}\left(\mu^{w(t_w+r_w)}\rho_R^{2wr_w}\sum_{\sigma_w\in S_{r_w+t_w}\ltimes\mathbb{Z}_w^{r_w+t_w}}\frac{\prod_{s\in S}Z_c\left(\frac{\ell^c_s\hat{t}+\sum_{i=1}^{\ell^c_s}z_i}{w}\right)\prod_{j=1}^{r_w}\hat{Z}_o\left(\frac{2\ell^o_j\hat{t}}{w}\right)}{t_w!w^{t_w}r_w!^2w^{2r_w}}\right). \tag{B.5}$$

Here, for $\sigma_w = (\pi,(z_1,\ldots,z_{r_w+t_w}))$, define ℓ^o_j with $j=1,\ldots,r_w$ as the smallest integer $k>0$ such that $\pi^k j \le r_w$. Furthermore, let $\pi = \pi_1\ldots\pi_L$ be the decomposition of π into cycles. We denote the elements of the cycle π_s by $(\pi_s^1\ldots\pi_s^{\ell^c_s})$ and let $S\subseteq\{1,\ldots,L\}$ be the set of those indices s for which $\{\pi_s^1,\ldots,\pi_s^{\ell^c_i}\}$ has an empty overlap with $\{1,\ldots,r_w\}$. Note that

$$\sum_{s\in S}\ell^c_s + \sum_{j=1}^{r_w}\ell^o_j = r_w + t_w. \tag{B.6}$$

[17]Since the modular parameter relevant for this section will always be $\hat{t}$ and not the S transformed modular parameter t, we hope that the reader will not be offended by our choice to instead use t in this appendix to refer to the sequence associated to τ.

To simplify further, we can now perform the summation over $\mathbb{Z}_w^{r_w+t_w}$. This gives

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\mu^{w(t_w+r_w)} \rho_R^{2wr_w} \sum_{\sigma_w \in S_{r_w+t_w}} \frac{w^{t_w-|S|} \prod_{s\in S} Z_c\left(\frac{\ell_s^c \hat{t} + \sum_{i=0}^{w-1} i}{w}\right) w^{r_w} \prod_{j=1}^{r_w} \left(\hat{Z}_o\left(\frac{2\ell_j^o \hat{t}}{w}\right)\right)}{t_w! w^{t_w} r_w!^2 w^{2r_w}} \right). \tag{B.7}$$

By the use of eq. (B.6), this can be simplified further to

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\sum_{\sigma_w \in S_{r_w+t_w}} \frac{\prod_{s\in S} \frac{\mu^{w\ell_s^c}}{w} Z_c\left(\frac{\ell_s^c \hat{t} + \sum_{i=0}^{w-1} i}{w}\right) \prod_{j=1}^{r_w} \frac{\mu^{w\ell_j^o} \rho_R^{2w}}{w} \hat{Z}_o\left(\frac{2\ell_j^o \hat{t}}{w}\right)}{t_w! r_w!^2} \right). \tag{B.8}$$

Now we perform the sum over $S_{r_w+t_w}$. As a first step towards this goal, we should replace the labels ℓ_j^o and ℓ_s^c of individual cycle lengths by labels L^o and L^c that just count how many cycles of certain length are present. That is

$$L_\ell^o := |\{j \in \{1, \dots, r_w\} : \ell_j^o = \ell\}|, \quad L_\ell^c := |\{s \in S : \ell_s^c = \ell\}|. \tag{B.9}$$

Just as we notationally suppressed the dependence of ℓ^o, ℓ^c on σ, we will suppress the dependence of L^o, L^c on σ. In terms of the new labels, the sum becomes

$$\mathcal{Z}_a = \sum_{t,r} \prod_{w=1}^{\infty} \left(\sum_{\sigma_w \in S_{r_w+t_w}} \frac{\prod_{\ell=1}^{\infty} \left(\frac{\mu^{w\ell}}{w} Z_c\left(\frac{\ell \hat{t} + \sum_{i=0}^{w-1} i}{w}\right)\right)^{L_\ell^c} \prod_{\ell=1}^{\infty} \left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o\left(\frac{2\ell \hat{t}}{w}\right)\right)^{L_\ell^o}}{t_w! r_w!^2} \right). \tag{B.10}$$

Now, since the terms which we sum only depend on the L^o and L^c labels, we can introduce an equivalence relation $\sim$ on $S_{r_w+t_w}$ that identifies permutations with equal L^o, L^c labels, trading the sum over σ_w with a sum over equivalence classes $[L^o L^c]$. To correctly compute the sum, we have to determine the size of $[L^o L^c]$. Let us first take a step back and try to count permutations with fixed ℓ^o. They need to satisfy $\sigma_w^k(1) > r_w$ for all $0 < k < \ell_1^o$ and $\sigma_w^{\ell_1^o}(1) \leq r_w$. This leads to $\frac{t_w!}{(t_w - \ell_1^o + 1)!} r_w$ choices for $\{\sigma_w^k(1)\}_{k=1}^{\ell_1^o}$. Continuing a counting like this for all $\{\sigma_w^k(i)\}_{k=1}^{\ell_i^o}$ with $1 \leq i \leq r$ gives a factor of

$$r_w! \frac{t_w!}{\left(t_w - \sum_{i=1}^{r_w} (\ell_i^o - 1)\right)!} = r_w! \frac{t_w!}{\left(t_w - \sum_{\ell=0}^{\infty} L_\ell^o (\ell - 1)\right)!} \tag{B.11}$$

contributing to the size of $[L^o L^c]$. Additionally, we need to consider how many ways there are to realise the label L^o with labels ℓ^o. This leads to an extra factor of $\frac{r_w!}{\prod_\ell L_\ell^o!}$. Hence, the choice of L^o contributes a factor of

$$\frac{r_w!^2}{\prod_\ell L^o_\ell!}\frac{t_w!}{\left(t_w-\sum_{\ell=0}^{\infty}L^o_\ell(\ell-1)\right)!} \tag{B.12}$$

to the size of $|[L^oL^c]|$. Finally, we need to multiply by the size of the conjugacy class labelled by L^c to obtain

$$|[L^oL^c]| = \frac{r_w!^2}{\prod_{\ell=1}^{\infty} L^o_\ell!}\frac{t_w!}{\left(t_w-\sum_{\ell=1}^{\infty}L^o_\ell(\ell-1)\right)!}\frac{\left(\sum_{\ell=1}^{\infty}\ell L^c_\ell\right)!}{\prod_{\ell=1}^{\infty}L^c_\ell!\ell^{L^c_\ell}}. \tag{B.13}$$

Now

$$\sum_{\ell=1}^{\infty}\ell(L^c_\ell+L^o_\ell)=r_w+t_w \quad \text{and} \quad \sum_{\ell=1}^{\infty}L^o_\ell=r_w \tag{B.14}$$

implies

$$t_w-\sum_{\ell=1}^{\infty}L^o_\ell(\ell-1)=\sum_{\ell=1}^{\infty}\ell L^c_\ell \tag{B.15}$$

and therefore

$$|[L^oL^c]| = \frac{r_w!^2t_w!}{\prod_{\ell=1}^{\infty}L^o_\ell!L^c_\ell!\ell^{L^c_\ell}}. \tag{B.16}$$

This implies

$$\mathcal{Z}_a=\sum_{t,r}\prod_{w=1}^{\infty}\left(\sum_{[L^oL^c]}\prod_{\ell=1}^{\infty}\frac{\left(\frac{\mu^{w\ell}}{w\ell}Z_c\left(\frac{\ell\hat{t}+\sum_{i=0}^{w-1}i}{w}\right)\right)^{L^c_\ell}}{L^c_\ell!}\prod_{\ell=1}^{\infty}\frac{\left(\frac{\mu^{w\ell}\rho_R^{2w}}{w}\hat{Z}_o\left(\frac{2\ell\hat{t}}{w}\right)\right)^{L^o_\ell}}{L^o_\ell!}\right), \tag{B.17}$$

where the $[L^oL^c]$ sum runs over the set of equivalence classes $S_{r_w+t_w}/\sim$. Finally, we trade the unconstrained sum over the sequences t and r, which in turn constrains L^o and L^c to correspond to equivalence classes in $S_{r_w+t_w}$ for an unconstrained sum over L^o and L^c without t,r sum. This is achieved by reinterpreting the constraints (B.14) as fixing t_w and r_w in terms of L^o and L^c, leading to

$$\mathcal{Z}_a=\prod_{w,\ell=1}^{\infty}\left(\sum_{L^c_\ell=0}^{\infty}\left(\frac{\left(\frac{\mu^{w\ell}}{w\ell}Z_c\left(\frac{\ell\hat{t}+\sum_{i=0}^{w-1}i}{w}\right)\right)^{L^c_\ell}}{L^c_\ell!}\right)\sum_{L^o_\ell=0}^{\infty}\left(\frac{\left(\frac{\mu^{w\ell}\rho_R^{2w}}{w}\hat{Z}_o\left(\frac{2\ell\hat{t}}{w}\right)\right)^{L^o_\ell}}{L^o_\ell!}\right)\right). \tag{B.18}$$

The sums now simply give exponential functions

$$\mathcal{Z}_a[\mu_\pm, 0, \rho_R; \hat{t}] = \exp\left(\sum_{w,\ell=1}^{\infty} \left(\frac{\mu^{w\ell}}{w\ell} Z_c \left(\frac{\ell\hat{t} + \sum_{i=0}^{w-1} i}{w} \right) \right) + \sum_{w,\ell=1}^{\infty} \left(\frac{\mu^{w\ell} \rho_R^{2w}}{w} \hat{Z}_o \left(\frac{2\ell\hat{t}}{w} \right) \right) \right). \tag{B.19}$$

Replacing the summation variable ℓ by $k = w\ell$ finally gives us the result (2.68) that we wanted to prove.

C Boundary symmetric orbifold genus expansion for spherical brane

The main subtlety in establishing the genus expansion for connected correlation functions in the presence of the boundary

$$\|a\rangle\rangle = \frac{1}{\sqrt{N!}} \sum_{g \in S_n} |a\rangle_g \tag{C.1}$$

is to identify the correct definition of the word "connected" in this context. One naive attempt motivated by the definition of the connected part in the case without boundary would be to define the connected correlator

$$\langle\langle a \| \sigma_{w_1}(z_1, \bar{z}_1) \dots \sigma_{w_{n_c}}(z_{n_c}, \bar{z}_{n_c}) |0\rangle_c \tag{C.2}$$

directly as the restriction of the sum

$$\left(\prod_{i=1}^{n_c} \sqrt{\frac{(N-w_i)! w_i}{N!}} \right) \sum_{g_i \in [(1 \dots w_i)]} \langle\langle a \| \sigma_{g_1}(z_1, \bar{z}_1) \dots \sigma_{g_{n_c}}(z_{n_c}, \bar{z}_{n_c}) |0\rangle \tag{C.3}$$

to those terms for which the subgroup of S_N generated by $\{g_i\}_{i=1}^{n_c}$ acts transitively on the active colours. For the case without a boundary, this prescription is intuitively sound: All contributions to the correlator from the passive (i.e. not active) colours are trivial and hence, though the passive colours arguably could be seen as contributing "disconnected" parts to the correlator, we may discard them on the basis that they are undetectable in the gauge dependent correlators.

The situation in the presence of the boundary (C.1) is different from this. Even in the absence of any bulk operator insertions, i.e. even with a vanishing number of active colours, the correlation function has a non trivial dependence on N. Concretely,

$$\langle\langle a|0\rangle = \frac{1}{\sqrt{N!}} \sim N^{-N/2}. \tag{C.4}$$

Hence, we see that every passive colour contributes a factor of $N^{-1/2}$ to the overall N scaling. To obtain the true connected correlator, we should divide out the contributions of all passive colours. This leads to the definition[18]

$$\langle\!\langle a\| \prod_{i=1}^{n_c} \sigma_{w_i}(z_i,\bar{z}_i)|0\rangle_c := \sum_{\substack{g_i\in[(1\ldots w_i)] \\ \{g_1,\ldots g_{n_c}\} \text{ generates a transitive subgroup of } S_n}} \frac{\langle\!\langle a\| \prod_{i=1}^{n_c} \sqrt{\frac{(N-w_i)!w_i}{N!}}\sigma_{g_i}(z_i,\bar{z}_i)|0\rangle}{\langle\!\langle a|0\rangle^{\frac{N-n}{N}}}, \tag{C.5}$$

where n is the number of active and hence $N-n$ the number of passive colours.

Let us now compute the N scaling of the individual gauge dependent correlators. To do so, we first simply observe that

$$\frac{\langle\!\langle a\| \prod_{i=1}^{n_c} \sqrt{\frac{(N-w_i)!w_i}{N!}}\sigma_{g_i}(z_i,\bar{z}_i)|0\rangle}{\langle\!\langle a|0\rangle^{\frac{N-n}{N}}} \sim N^{-\frac{n}{2}} {}_{g_{n_c+1}}\langle a| \prod_{i=1}^{n_c} N^{-\frac{w_i}{2}}\sigma_{g_i}(z_i,\bar{z}_i)|0\rangle, \tag{C.6}$$

where

$$g_{n_c+1} = (g_1\ldots g_{n_c})^{-1}. \tag{C.7}$$

Now there are many terms in the sum (C.5) that give the same contribution to the overall correlator. This has to be accounted for in order to get the correct scaling with N with which an individual term ${}_{g_{n_c+1}}\langle a| \prod_{i=1}^{n_c} \sigma_{g_i}(z_i,\bar{z}_i)|0\rangle$ contributes to the overall correlator. In order to obtain the N scaling, we do not need to work out the precise combinatorics. All we need is an overall factor of N^n accounting for the choice of the n active colours. The detailed combinatorics just provides order 1 corrections, since it originates from a counting problem of S_n. Hence, ${}_{g_{n_c+1}}\langle a| \prod_{i=1}^{n_c} \sigma_{g_i}(z_i,\bar{z}_i)|0\rangle$ contributes with a weight of

$$N^{\frac{n}{2}-\sum_{j=1}^{n_c}\frac{w_j}{2}} \tag{C.8}$$

to the overall correlator. Thus, what we would like to show is

$$\frac{n}{2} - \sum_{j=1}^{n_c}\frac{w_j}{2} = 1-g-\frac{b}{2}-\frac{n_c}{2}. \tag{C.9}$$

In order to do so, we have to express g and b in terms of the group theoretic data. Now the number b of boundaries is simply given by the number of cycles in g_{n_c+1}. That is, the decomposition of g_{n_c+1} into cycles (including cycles of length 1), takes the form

$$g_{n_c+1} = (g_{n_c+1})_1\ldots(g_{n_c+1})_b. \tag{C.10}$$

Let us denote the length $|(g_{n_c+1})_i|$ of the i^{th} cycle by w_{n_c+i}. To determine the genus, we can imagine to shrink the boundary circles down to punctures, corresponding to the insertion

[18] Note that the large N behaviour of one-point functions in the presence of boundaries like the one discussed here was also considered by Bellin, Biswas, Sully in [7]. They however did not only divide out all the passive disconnected pieces to the correlator, but the full overlap of the boundary state with the defect. This leads to an enhancement of the one-point functions with N that, as they also observe in their paper, is incompatible with a geometric bulk dual interpretation.

of order w_{n_c+i} twist fields. The genus of the covering surface can then be determined by the formula for the genus in the absence of any boundaries as

$$g = 1 - n + \frac{1}{2}\sum_{j=1}^{n_c+b}(w_j - 1). \tag{C.11}$$

With this knowledge, we can now prove eq. (C.9). Indeed,

$$\begin{aligned} 1 - g - \frac{b}{2} - \frac{n_c}{2} &= 1 - \left(1 - n + \frac{1}{2}\sum_{j=1}^{n_c} w_j - \frac{n_c}{2} + \frac{1}{2}\sum_{j=n_c+1}^{n_c+n_b} w_j - \frac{b}{2}\right) - \frac{b}{2} - \frac{n_c}{2} \\ &= n - \frac{1}{2}\sum_{j=n_c+1}^{n_c+n_b} w_j - \frac{1}{2}\sum_{j=1}^{n_c} w_j. \end{aligned} \tag{C.12}$$

But

$$\frac{1}{2}\sum_{j=n_c+1}^{n_c+n_b} w_j = \frac{n}{2}, \tag{C.13}$$

which finishes the proof.

References

[1] A. Klemm and M.G. Schmidt, *Orbifolds by cyclic permutations of tensor product conformal field theories*, *Phys. Lett. B* **245** (1990) 53.

[2] A. Pakman, L. Rastelli and S.S. Razamat, *Diagrams for symmetric product orbifolds*, *JHEP* **10** (2009) 034 [0905.3448].

[3] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS_3*, *JHEP* **12** (1998) 026 [hep-th/9812046].

[4] F.M. Haehl and M. Rangamani, *Permutation orbifolds and holography*, *JHEP* **03** (2015) 163 [1412.2759].

[5] A. Belin, C.A. Keller and A. Maloney, *Permutation orbifolds in the large N Limit*, 1509.01256.

[6] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *The worldsheet dual of the symmetric product CFT*, *JHEP* **04** (2019) 103 [1812.01007].

[7] A. Belin, S. Biswas and J. Sully, *The spectrum of boundary states in symmetric orbifolds*, *JHEP* **01** (2022) 123 [2110.05491].

[8] M.R. Gaberdiel, B. Knighton and J. Vošmera, *D-branes in $AdS_3 \times S^3 \times \mathbb{T}^4$ at $k = 1$ and their holographic duals*, *JHEP* **12** (2021) 149 [2110.05509].

[9] E.J. Martinec, *A defect in AdS_3/CFT_2 duality*, *JHEP* **06** (2022) 024 [2201.04218].

[10] M. Gutperle, Y.-Y. Li, D. Rathore and K. Roumpedakis, *Non-invertible symmetries in S_N orbifold CFTs and holography*, *JHEP* **09** (2024) 110 [2405.15693].

[11] M. Gutperle, Y.-Y. Li, D. Rathore and K. Roumpedakis, *A note on entanglement entropy and topological defects in symmetric orbifold CFTs*, *JHEP* **09** (2024) 010 [2406.10967].

[12] B. Knighton, V. Sriprachyakul and J. Vošmera, *Topological defects and tensionless holography*, 2406.03467.

[13] A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *JHEP* **06** (2001) 063 [hep-th/0105132].

[14] Y. Imamura and M. Inoue, *Brane expansions for anti-symmetric line operator index*, *JHEP* **08** (2024) 020 [2404.08302].

[15] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J. C* **22** (2001) 379 [hep-th/9803001].

[16] J.M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[17] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109].

[18] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [hep-th/0603208].

[19] C. Bachas, J. de Boer, R. Dijkgraaf and H. Ooguri, *Permeable conformal walls and holography*, *JHEP* **06** (2002) 027 [hep-th/0111210].

[20] J. Teschner, *On structure constants and fusion rules in the $SL(2,\mathbb{C})/SU(2)$ WZNW model*, *Nucl. Phys. B* **546** (1999) 390 [hep-th/9712256].

[21] J. Teschner, *Operator product expansion and factorization in the H_3^+ WZNW model*, *Nucl. Phys. B* **571** (2000) 555 [hep-th/9906215].

[22] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200].

[23] J.M. Maldacena, H. Ooguri and J. Son, *Strings in AdS_3 and the $SL(2,\mathbb{R})$ WZW model. Part 2. Euclidean black hole*, *J. Math. Phys.* **42** (2001) 2961 [hep-th/0005183].

[24] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the Euclidean AdS_3*, *JHEP* **02** (2002) 016 [hep-th/0112198].

[25] P. Lee, H. Ooguri and J.-w. Park, *Boundary states for AdS_2 branes in AdS_3*, *Nucl. Phys. B* **632** (2002) 283 [hep-th/0112188].

[26] A. Giveon, D. Kutasov and A. Schwimmer, *Comments on D-branes in AdS_3*, *Nucl. Phys. B* **615** (2001) 133 [hep-th/0106005].

[27] P.M. Petropoulos and S. Ribault, *Some remarks on anti-de Sitter D-branes*, *JHEP* **07** (2001) 036 [hep-th/0105252].

[28] P. Lee, H. Ooguri, J.-W. Park and J. Tannenhauser, *Open strings on AdS_2 branes*, *Nucl. Phys. B* **610** (2001) 3 [hep-th/0106129].

[29] Y. Hikida and Y. Sugawara, *Boundary states of D branes in AdS_3 based on discrete series*, *Prog. Theor. Phys.* **107** (2002) 1245 [hep-th/0107189].

[30] A. Rajaraman and M. Rozali, *Boundary states for D-branes in AdS_3*, *Phys. Rev. D* **66** (2002) 026006 [hep-th/0108001].

[31] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [hep-th/9608096].

[32] L. Eberhardt, M.R. Gaberdiel and R. Gopakumar, *Deriving the AdS_3/CFT_2 correspondence*, *JHEP* **02** (2020) 136 [1911.00378].

[33] A. Dei, M.R. Gaberdiel, R. Gopakumar and B. Knighton, *Free field world-sheet correlators for AdS_3*, *JHEP* **02** (2021) 081 [2009.11306].

[34] A. Recknagel, *Permutation branes*, *JHEP* **04** (2003) 041 [hep-th/0208119].

[35] J. Gomis and F. Passerini, *Holographic Wilson Loops*, *JHEP* **08** (2006) 074 [hep-th/0604007].

[36] A. Recknagel and V. Schomerus, *Boundary conformal field theory and the worldsheet approach to D-branes*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2013).

[37] T. Quella, I. Runkel and G.M.T. Watts, *Reflection and transmission for conformal defects*, *JHEP* **04** (2007) 095 [hep-th/0611296].

[38] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [hep-th/9902098].

[39] G. Gotz, T. Quella and V. Schomerus, *The WZNW model on $PSU(1,1|2)$*, *JHEP* **03** (2007) 003 [hep-th/0610070].

[40] T. Quella and V. Schomerus, *Free fermion resolution of supergroup WZNW models*, *JHEP* **09** (2007) 085 [0706.0744].

[41] M.R. Gaberdiel and S. Gerigk, *The massless string spectrum on AdS_3 x S^3 from the supergroup*, *JHEP* **10** (2011) 045 [1107.2660].

[42] R. Blumenhagen, D. Lüst and S. Theisen, *Basic concepts of string theory*, Theoretical and Mathematical Physics, Springer, Heidelberg, Germany (2013), 10.1007/978-3-642-29497-6.

[43] L. Eberhardt, *Partition functions of the tensionless string*, *JHEP* **03** (2021) 176 [2008.07533].

[44] M.R. Gaberdiel and R. Gopakumar, *Tensionless string spectra on AdS_3*, *JHEP* **05** (2018) 085 [1803.04423].

[45] L. Eberhardt and M.R. Gaberdiel, *String theory on AdS_3 and the symmetric orbifold of Liouville theory*, *Nucl. Phys.* **B948** (2019) 114774 [1903.00421].

[46] S. Hamidi and C. Vafa, *Interactions on orbifolds*, *Nuclear Physics B* **279** (1987) 465.

[47] O. Lunin and S.D. Mathur, *Correlation functions for M^N/S_N orbifolds*, *Commun. Math. Phys.* **219** (2001) 399 [hep-th/0006196].

[48] Y. Hikida and V. Schomerus, *Engineering perturbative string duals for symmetric product orbifold CFTs*, *JHEP* **06** (2024) 071 [2312.05317].

[49] L. Eberhardt, *A perturbative CFT dual for pure NS–NS AdS_3 strings*, *J. Phys. A* **55** (2022) 064001 [2110.07535].

[50] B. Balthazar, A. Giveon, D. Kutasov and E.J. Martinec, *Asymptotically free AdS_3/CFT_2*, *JHEP* **01** (2022) 008 [2109.00065].

[51] S. Chakraborty, A. Giveon and D. Kutasov, *Effective AdS_3/CFT_2*, 2501.09119.

[52] S. Ribault and J. Teschner, *H_3^+-WZNW correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048].

[53] S. Ribault, *Knizhnik-Zamolodchikov equations and spectral flow in AdS_3 string theory*, *JHEP* **09** (2005) 045 [hep-th/0507114].

[54] Y. Hikida and V. Schomerus, *H_3^+ WZNW model from Liouville field theory*, *JHEP* **10** (2007) 064 [0706.1030].

[55] Y. Hikida and V. Schomerus, *The FZZ-duality conjecture: A proof*, *JHEP* **03** (2009) 095 [0805.3931].

[56] Y. Hikida and T. Liu, *Correlation functions of symmetric orbifold from AdS_3 string theory*, *JHEP* **09** (2020) 157 [2005.12511].

[57] B. Knighton, S. Seet and V. Sriprachyakul, *Spectral flow and localisation in AdS_3 string theory*, *JHEP* **05** (2024) 113 [2312.08429].

[58] S. Demulder, S. Driezen, B. Knighton, G. Oling, A.L. Retore, F.K. Seibold et al., *Exact approaches on the string worldsheet*, *J. Phys. A* **57** (2024) 423001 [2312.12930].

[59] B. Knighton and V. Sriprachyakul, *Unravelling AdS_3/CFT_2 near the boundary*, 2404.07296.

[60] T. Creutzig, Y. Hikida and P.B. Rønne, *The FZZ duality with boundary*, *JHEP* **09** (2011) 004 [1012.4731].

[61] J.R. David, G. Mandal and S.R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549 [hep-th/0203048].

[62] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $\mathcal{N} = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[63] N. Beisert et al., *Review of AdS/CFT integrability: An overview*, *Lett. Math. Phys.* **99** (2012) 3 [1012.3982].

[64] M.R. Gaberdiel, R. Gopakumar and B. Nairz, *Beyond the tensionless limit: Integrability in the symmetric orbifold*, *JHEP* **06** (2024) 030 [2312.13288].

[65] M.R. Gaberdiel, D. Kempel and B. Nairz, *$AdS_3 \times S^3$ magnons in the symmetric orbifold*, 2412.02741.

[66] V. Mitev, T. Quella and V. Schomerus, *Principal chiral model on superspheres*, *JHEP* **11** (2008) 086 [0809.1046].

[67] C. Candu, V. Mitev, T. Quella, H. Saleur and V. Schomerus, *The sigma model on complex projective superspaces*, *JHEP* **02** (2010) 015 [0908.0878].

[68] A. Cavaglià, S. Ekhammar, N. Gromov and P. Ryan, *Exploring the quantum spectral curve for AdS_3/CFT_2*, *JHEP* **12** (2023) 089 [2211.07810].

[69] S. Frolov, D. Polvara and A. Sfondrini, *On mixed-flux worldsheet scattering in AdS_3/CFT_2*, *JHEP* **11** (2023) 055 [2306.17553].

[70] N. Gromov and F. Levkovich-Maslyuk, *Quantum spectral curve for a cusped Wilson line in $\mathcal{N} = 4$ SYM*, *JHEP* **04** (2016) 134 [1510.02098].

[71] D. Grabner, N. Gromov and J. Julius, *Excited states of one-dimensional defect CFTs from the quantum spectral curve*, *JHEP* **07** (2020) 042 [2001.11039].