Prepared for submission to JHEP

DESY-25-001
MITP-25-004
USTC-ICTS/PCFT-24-28

Rationalisation of multiple square roots in Feynman integrals

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Abstract: Feynman integrals are very often computed from their differential equations. It is not uncommon that the ε-factorised differential equation contains only dlog-forms with algebraic arguments, where the algebraic part is given by (multiple) square roots. It is well-known that if all square roots are simultaneously rationalisable, the Feynman integrals can be expressed in terms of multiple polylogarithms. This is a sufficient, but not a necessary criterium. In this paper we investigate weaker requirements. We discuss under which conditions we may use different rationalisations in different parts of the calculation. In particular we show that we may use different rationalisations if they correspond to different parameterisations of the same integration path. We present a non-trivial example – the one-loop pentagon function with three adjacent massive external legs involving seven square roots – where this technique can be used to express the result in terms of multiple polylogarithms.

Contents

1 Introduction

Feynman integrals play an essential role in precision calculations for particle physics phenomenology. Of particular importance is the question to which class of functions a particular Feynman integral evaluates. The simplest class of functions are the multiple polylogarithms. They are generalisations of the ordinary logarithm $\ln(x)$ and correspond to iterated integrals on a curve of genus zero. More complicated Feynman integrals are associated with curves of higher genus or geometries of higher dimension. The most prominent examples for the latter case are Calabi-Yau geometries. An important question is to decide, whether a given Feynman integral can be expressed in terms of functions from a given class of functions. Already for the simplest class of functions, i.e. the multiple polylogarithms, this is a non-trivial question.

A popular technique to compute Feynman integrals is the method of differential equations [1–4]. Within this approach, one first derives a system of differential equations for the yet unknown Feynman integrals and in a second step solve this system for the latter. The second step is straightforward, if the system of differential equations is in an ϵ-factorised form [5]. Thus, the task of computing Feynman integrals reduces to the task of finding a transformation for a system of differential equations to an ϵ-factorised form.

Now let us assume that we have an ϵ-factorised differential equation. If all entries of the connection matrix A are dlog-forms with arguments, which are rational functions of the kinematic variables x, it is straightforward to show that all Feynman integrals in this family can be expressed in terms of multiple polylogarithms to all orders in the dimensional regularisation parameter ϵ.

However, it is not uncommon that one has an ϵ-factorised differential equation, where all entries of the connection matrix A are dlog-forms with algebraic arguments and the algebraic part is given by (multiple) square roots. It is well-known that if all square roots are simultaneously rationalisable, the change of variables used in the rationalisation converts the dlog-forms with algebraic arguments to dlog-forms with rational arguments and – as a result – the Feynman integrals can again be expressed in terms of multiple polylogarithms. This argument shows that the requirement that all square roots are simultaneously rationalisable is a sufficient condition for expressing the Feynman integrals in terms of multiple polylogarithms. However, it is not a necessary condition. To see this, one may consider the example of the mixed QCD-electroweak corrections to the Drell–Yan process. It is known that the three square roots appearing in this example cannot be rationalised simultaneously [6]. However, the result can be expressed in terms of multiple polylogarithms [7]. The latter can be shown using either direct integration [8, 9] or symbol techniques [10, 11]. Hence, the requirement of being simultaneously rationalisable is not a necessary condition. The simplified differential equation approach [12–14] provides a further technique by introducing an auxiliary parameter.

On the other side it is also known, that there can be ϵ-factorised differential equation with dlog-forms with algebraic arguments where the resulting functions cannot be expressed in terms of multiple polylogarithms [15].

In view of these findings it is highly desirable to find weaker conditions under which Feynman integrals obeying an ϵ-factorised differential equation with dlog-forms with algebraic arguments can be expressed in term of multiple polylogarithms. This is the topic of this paper. We discuss under which conditions we may use different rationalisations in different parts of the calculation. To this aim we review path independence and parameterisation independence of iterated integrals. Path independence holds only if a certain condition due to Chen is met. In the context of Feynman integrals we may divide the full result into smaller subsets, consisting of specific linear combinations of iterated integrals, which are always path independent. We may use different rationalisations in different subsets.

Furthermore, any iterated integral is invariant under re-parameterisation of the same integration path. Hence, we may use within the same subset different rationalisations, if they correspond to different parameterisations of the same integration path. At first sight, this sounds very specific and it is not clear if there is actually a real application of this technique. However, we present a non-trivial example – the one-loop pentagon function with three adjacent massive external legs involving seven square roots – where this technique can be used to express the result in terms of multiple polylogarithms.

This paper is organised as follows: In the next section we introduce the general set-up and present the general theorems on path independence and parameterisation independence. In section 3 we present the one-loop pentagon integral with three adjacent massive external legs. In section 4 we show explicitly how the techniques of section 2 are applied to the one-loop pentagon integral. In section 5 we discuss the results for one-loop pentagon integral with three adjacent massive external legs. Finally, our conclusions are given in section 6. The article is supplemented by three appendices. In appendix A we give details on the momentum twistor representation, in appendix B we list the polynomials entering the definition of the letters and in appendix C we describe the content of the supplementary electronic file attached to the arxiv version of this article.

2 Multiple square roots

2.1 Set-up

We consider Feynman integrals, which depend on N kinematic variables $\vec{x} = (x_1, \dots, x_N)^T$. We view the kinematic variables as coordinates on the kinematic space X. Let $\vec{g} = (g_1, \dots, g_n)^T$ be a vector of n master integrals. We assume that the master integrals satisfy an ϵ-factorised differential equation

$$d\vec{g}\,(\vec{x}, \epsilon) = \epsilon \boldsymbol{A}\,(\vec{x})\,\vec{g}\,(\vec{x}, \epsilon)\,. \tag{2.1}$$

with an integrable connection $\boldsymbol{A}$:

$$\epsilon d\boldsymbol{A} - \epsilon^2 \boldsymbol{A} \wedge \boldsymbol{A} = 0. \tag{2.2}$$

The differential d is the differential in the kinematic variables $\vec{x}$:

$$d = \sum_{j=1}^{N} dx_j \frac{\partial}{\partial x_j} \tag{2.3}$$

As $\boldsymbol{A}$ is independent of ϵ, the ϵ^1-term and the ϵ^2-term in eq. (2.2) have to vanish separately and we get two individual equations

$$d\boldsymbol{A} \;=\; 0 \text{ and } \boldsymbol{A} \wedge \boldsymbol{A} \;=\; 0. \tag{2.4}$$

The first conditions states that the entries of the $(n \times n)$-matrix $\boldsymbol{A}$ are closed one-forms. We denote a basis of the differential one-forms appearing in $\boldsymbol{A}$ by $\omega_1, \dots, \omega_{N_L}$ and the $\mathbb{C}$-vector space spanned by those by $\Omega^1(X)$. We write

$$\boldsymbol{A} = \sum_{i=1}^{N_L} A_i \omega_i, \tag{2.5}$$

where the A_i's are $(n \times n)$-matrices, whose entries are constant numbers. We are in particular interested in the case, where the ω_i's are dlog-forms,

$$\omega_i \;=\; d \ln \left(W_i \left(\vec{x} \right) \right), \tag{2.6}$$

where in turn the $W_i(\vec{x})$'s are algebraic functions of the kinematic variables $\vec{x}$. The typical cases are that W_i is either a rational function of $\vec{x}$ or of the form

$$W_i \left(\vec{x} \right) \;=\; \frac{P_i \left(\vec{x} \right) - \sqrt{Q_i \left(\vec{x} \right)}}{P_i \left(\vec{x} \right) + \sqrt{Q_i \left(\vec{x} \right)}}, \tag{2.7}$$

where P_i and Q_i are polynomials in the variables $\vec{x}$.

We further assume that all master integrals have a Taylor expansion in the dimensional regularisation parameter ϵ, thus we may write

$$\vec{g} \;=\; \sum_{j=0}^{\infty} \epsilon^j \vec{g}^{(j)} \text{ and } g_k \;=\; \sum_{j=0}^{\infty} \epsilon^j g_k^{(j)}, \quad 1 \le k \le n. \tag{2.8}$$

Let $\vec{C} = (C_1, \dots, C_n)^T$ be the boundary constants at $x = 0$. They have a similar Taylor expansion in ϵ:

$$C_k \;=\; \sum_{j=0}^{\infty} \epsilon^j C_k^{(j)}, \quad 1 \le k \le n. \tag{2.9}$$

Given the differential equation (2.1) and the boundary constants, we may express the master integrals in terms of iterated integrals. Let

$$\gamma : [0,1] \to X \tag{2.10}$$

be a path in the kinematic space X with starting point $\gamma(0) = \vec{0}$ and endpoint $\gamma(1) = \vec{x}$. We write

$$f_j(\lambda)\, d\lambda = \gamma^* \omega_j \tag{2.11}$$

for the pull-back of ω_j to the interval $[0, 1]$. If $f_r(\lambda)$ is regular at $\lambda = 0$ we define the iterated integral by

$$I_{\omega_1,\dots,\omega_r}[\gamma] = \int\limits_0^\lambda d\lambda_1 f_1(\lambda_1) \int\limits_0^{\lambda_1} d\lambda_2 f_2(\lambda_2) \cdots \int\limits_0^{\lambda_{r-1}} d\lambda_r f_r(\lambda_r). \tag{2.12}$$

In case $f_r(\lambda)$ has a simple pole at $\lambda = 0$ we use the standard "trailing zero" or tangential base point prescription (see for example refs. [16–18]).

Usually we evaluate an iterated integral along a standard path. A typical example is given by the piecewise smooth path γ_{standard} consisting of a straight line from $(0, 0, \dots, 0, 0)$ to $(0, 0, \dots, 0, x_N)$, followed by a straight line from $(0, 0, \dots, 0, x_N)$ to $(0, 0, \dots, x_{N-1}, x_N)$, and the pattern continues in this way. The last segment is given by straight line from $(0, x_2, \dots, x_{N-1}, x_N)$ to $(x_1, x_2, \dots, x_{N-1}, x_N)$.

We may express the ϵ^r-term of the k-th master integral as a linear combination of iterated integrals of depth $\leq r$

$$g_k^{(r)}(\vec{x}) = \sum_{j=1}^{r} \sum_{i_1,\dots,i_j=1}^{N_L} c_{i_1\dots i_j}^{(k,r)} I_{\omega_{i_1},\dots,\omega_{i_j}}[\gamma]. \tag{2.13}$$

We stress that an iterated integral $I_{\omega_{i_1},\dots,\omega_{i_r}}[\gamma]$ is a functional of the path γ and not just a function of the endpoint $\vec{x}$. However, the linear combination of iterated integrals appearing on the right-hand side of eq. (2.13) is path-independent and defines therefore a function of the endpoint $\vec{x}$. It is therefore worth reviewing under which conditions a linear combination of iterated integrals is path-independent.

2.2 Path independence

Of particular interest are linear combinations of iterated integrals which for a fixed starting point $\vec{0}$ and fixed end point $\vec{x}$ are independent of the path connecting $\vec{0}$ with $\vec{x}$. A single iterated integral $I_{\omega_{i_1},\dots,\omega_{i_r}}[\gamma]$ is in general not path independent. A criterium to decide if a given linear combination of iterated integrals is path independent can be stated as follows [19]: There is a one-to-one correspondence between ordered sequences of differential one-forms ω_{i_1}, ω_{i_2}, $\dots$,ω_{i_r} and elements in the tensor algebra $(\Omega(X))^{\otimes r}$ (where $\Omega(X)$ denotes the vector space generated by the wedge products of $\omega_1, \dots, \omega_{N_L}$) of the form

$$\omega_{i_1} \otimes \omega_{i_2} \otimes \dots \otimes \omega_{i_r}. \tag{2.14}$$

It is customary to denote the latter as

$$[\omega_{i_1}|\omega_{i_2}|\dots|\omega_{i_r}] = \omega_{i_1} \otimes \omega_{i_2} \otimes \dots \otimes \omega_{i_r}. \tag{2.15}$$

In the tensor algebra we define (taking into account that all ω's are closed)

$$d\left[\omega_{i_1}|\omega_{i_2}|\dots|\omega_{i_r}\right] = \sum_{j=1}^{r-1} \left[\omega_{i_1}|\dots|\omega_{i_{j-1}}|\omega_{i_j} \wedge \omega_{i_{j+1}}|\omega_{i_{j+2}}|\dots|\omega_{i_r}\right]. \tag{2.16}$$

Let us now consider a linear combination of iterated integrals of depth $\leq r$ with constant coefficients

$$g = \sum_{j=1}^{r} \sum_{i_1,\dots,i_j=1}^{N_L} c_{i_1 \dots i_j} I_{\omega_{i_1},\dots,\omega_{i_j}}[\gamma] \tag{2.17}$$

and the corresponding element in the tensor algebra

$$B = \sum_{j=1}^{r} \sum_{i_1,\dots,i_j=1}^{N_L} c_{i_1 \dots i_j} \left[\omega_{i_1}|\dots|\omega_{i_j}\right]. \tag{2.18}$$

g is a homotopy functional (i.e. independent of small deformations of the integration path) if and only if

$$dB = 0. \tag{2.19}$$

The proof is due to Chen [19]. This is the sought-after criteria when a linear combination of iterated integrals is path independent. We call any B satisfying eq. (2.19) an integrable word.

While eq. (2.19) allows us to test easily if a given linear combination of iterated integrals is an integrable word, it does not lead automatically to a systematic procedure to construct such a linear combination.

However, we know from the start, that the connection $\boldsymbol{A}$ appearing in the differential equation is integrable (or flat). This guarantees that the linear combination of iterated integrals appearing in the solution for the k-th master integral g_k is path independent.

We may refine this statement: First of all, integrability holds for each order in the dimensional parameter ϵ independently, as we started from an ϵ-factorised differential equation where the one-forms ω_i are independent of ϵ. This implies that the linear combination of iterated integrals appearing in the solution for the ϵ^r-term of the k-th master integral $g_k^{(r)}(\vec{x})$ is path independent.

Secondly, there is a further refinement, which up to now and to the best of our knowledge has not been stated clearly in the literature: To prepare this statement let us first note that each iterated integral appearing in $g_k^{(r)}(\vec{x})$ comes with a coefficient proportional to some boundary constant $C_{k'}^{(j')}$. We may therefore consider the linear combination of iterated integrals appearing in $g_k^{(r)}(\vec{x})$ and being proportional to $C_{k'}^{(j')}$.

Proposition 1. *The linear combination of iterated integrals appearing in* $g_k^{(r)}(\vec{x})$ *and being proportional to* $C_{k'}^{(j')}$ *is by itself path independent.*

Proof. The proof is rather simple: The iterated integrals appearing in $g_k^{(r)}(\vec{x})$ are path independent for any value of the boundary constants, therefore the linear combination proportional to one particular boundary constant $C_{k'}^{(j')}$ must be path independent by itself. □

2.3 Parameterisation independence

In this section we consider different parameterisations of the same path. To this aim let

$$\begin{aligned} \gamma_1 \,:\, [0,1] &\rightarrow X \\ \lambda_1 &\rightarrow \gamma_1\left(\lambda_1\right) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \gamma_2 \,:\, [0,1] &\rightarrow X \\ \lambda_2 &\rightarrow \gamma_2\left(\lambda_2\right) \end{aligned} \tag{2.21}$$

be two different parameterisations of the same path. We may express λ_2 in terms of λ_1 and vice versa:

$$\lambda_2 \;=\; \gamma_2^{-1}\left(\gamma_1\left(\lambda_1\right)\right), \quad \lambda_1 \;=\; \gamma_1^{-1}\left(\gamma_2\left(\lambda_2\right)\right). \tag{2.22}$$

We require in addition that

$$\left.\frac{d\lambda_2}{d\lambda_1}\right|_{\lambda_1=0} = 1. \tag{2.23}$$

An iterated integral is independent under these re-parameterisations of the path:

$$I_{\omega_1,\dots,\omega_r}\left[\gamma_1\right] = I_{\omega_1,\dots,\omega_r}\left[\gamma_2\right]. \tag{2.24}$$

The additional condition in eq. (2.23) is required for iterated integrals with trailing zeros.

2.4 The treatment of multiple square roots

With these prerequisites at hand, we may now turn to our main problem: The treatment of multiple square roots in the arguments of the dlog-forms. We consider a situation where all differential one-forms ω_i are dlog-forms, but some arguments of the logarithms are algebraic functions of the kinematic variables involving square roots.

It is well-known that if all occurring square roots are simultaneously rationalisable, then the result can be expressed in terms of multiple polylogarithms. Being simultaneously rationalisable is a sufficient condition, but not a necessary condition. As we advance in Feynman integral calculations we encounter more and more examples with multiple square roots which cannot be rationalised simultaneously. This does not necessarily mean that the Feynman integrals cannot be epxressed in terms of multiple polylogarithms. Alternative methods like direct integration or symbol calculus might lead to a result in terms of multiple polylogarithms [7, 20]. As the method of differential equations is the most popular method for computing Feynman integrals, we will give below a weaker sufficient condition under which the master integrals can be expressed (to any order in the dimensional parameter ϵ) in terms of multiple polylogarithms.

The first observation is that often it is impossible that all square roots appear in a single iterated integral. A typical example is the situation, where a square root r_1 is associated with one particular sub-sector, a second square root r_2 with a second sub-sector with the same number of propagators and the two square roots appear only in these two sub-sectors. Then we will never have an iterated integral (to any order in the dimensional parameter ϵ) involving both square roots r_1 and r_2, as the differential equation does not couple sub-sectors with the same number of propagators.

We are therefore tempted to consider different rationalisations for the subsets of square roots which do occur in a single iterated integral. Within a given rationalisation we would like to use again a simple integration path (like γ_{standard}) for the integration. The complication we have to face is the following: A simple integration path in the new variables does not necessarily correspond to the integration path in the original variables. If we are going to use different integration paths for different iterated integrals, the question of path-independence becomes relevant.

With our previous results, we may now state a weaker sufficient condition under which we may express the master integrals in terms of multiple polylogarithms: We are free to use different integration paths (and hence different rationalisations) for any linear combination of iterated integrals, which are by themselves path-independent. For example, we may collect in the ϵ^r-term of the k-th master integral the iterated integrals proportional to the boundary constant $C_{k'}^{(j')}$. For each boundary constant we may use a different integration path and therefore a different rationalisation.

Within each path-independent linear combination of iterated integrals we have to use the same integration path, but we may use different parameterisations of the same integration path. Different parameterisations of the same integration path may correspond to different rationalisations. A typical example is the following: Consider the integration path γ_{standard} and a transformation

$$x_1 = f\left(t_1, x_2, \dots, x_N\right), \tag{2.25}$$

such that $x_1 = 0$ corresponds to $t_1 = 0$ and

$$\left.\frac{dx_1}{dt_1}\right|_{t_1=0} = 1. \tag{2.26}$$

This will replace the variable x_1 by t_1. The transformation in eq. (2.25) may depend on the additional variables $x_2, \dots, x_N$. However, it does not change the integration path γ_{standard}: The path lying in the hyperplane $x_1 = 0$ is not affected at all and for the last segment from $(0, x_2, \dots, x_{N-1}, x_N)$ to $(x_1, x_2, \dots, x_{N-1}, x_N)$ it is just a re-parameterisation, as $x_2, \dots, x_N$ are constant along this segment.

In the following we present a highly non-trivial example where this happens in an actual calculation: The one-loop pentagon integral with three adjacent massive external legs is an example which involves

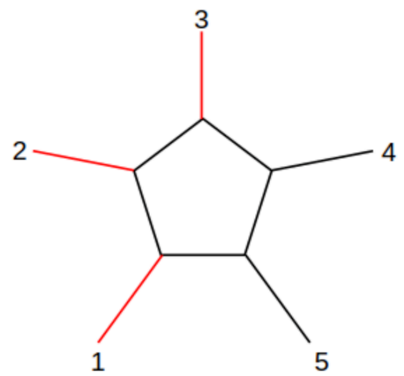

Figure 1. Pentagon topology with three adjacent massive external legs denoted in red. All other external legs and internal propagators are massless.

seven square roots.[1] We are not able to rationalise simultaneously all seven square roots. On combinatorial grounds only three square roots will ever appear in a single iterated integral. We show that any triple of occuring square roots can be rationalised by a suitable re-parameterisation of the standard path γ_{standard}. The re-parameterisations are of the form as in eq. (2.25). This establishes that the one-loop three-mass hard pentagon integral can be expressed to all orders in the dimensional regularisation parameter ϵ in terms of multiple polylogarithms.

3 The one-loop pentagon integral with three adjacent massive external legs

3.1 Definition of the Feynman integral

We consider the family of Feynman integrals shown in Fig. 1 and given by

$$G\left[\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\right] = e^{\epsilon \gamma_E} (\mu^2)^{\nu - \frac{D}{2}} \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}}, \tag{3.1}$$

where l is the loop momentum and D denotes its dimension. Unless stated otherwise we take $D = 4 - 2\epsilon$. The arbitrary scale μ is introduced to render the integral dimensionless. Three consecutive external legs are massive, $p_i^2 = m_i^2$ with $i = 1, 2, 3$, whereas $p_4^2 = p_5^2 = 0$. The inverse propagators D_i are

$$\begin{aligned} D_1 &= l^2, \qquad D_2 = (l - p_1)^2, \qquad D_3 = (l - p_1 - p_2)^2, \\ D_4 &= (l - p_1 - p_2 - p_3)^2, \qquad D_5 = (l - p_1 - p_2 - p_3 - p_4)^2, \end{aligned} \tag{3.2}$$

and there are eight (dimensionful) kinematic variables for this integral,

$$\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, m_2^2, m_3^2\}, \tag{3.3}$$

[1]The one-loop pentagon integral with three non-adjacent legs is simpler, as it involves five square roots, and has already been computed with the simplified differential equations approach [14].

where $s_{i,i+1} = (p_i + p_{i+1})^2$ denotes the Mandelstam variables. As usual, we may set μ^2 without loss of generality to a specific value. Doing so, the integral will depend on seven dimensionless kinematic variables. For example, setting $\mu^2 = -m_1^2$, the integral will depend on the seven dimensionless kinematic variables

$$\{\frac{s_{12}}{m_1^2}, \frac{s_{23}}{m_1^2}, \frac{s_{34}}{m_1^2}, \frac{s_{45}}{m_1^2}, \frac{s_{15}}{m_1^2}, \frac{m_2^2}{m_1^2}, \frac{m_3^2}{m_1^2}\}. \tag{3.4}$$

However, in this concrete example it is slightly more convenient to keep the dependence on the eight dimensionful kinematic variables, as we will use a twistor parameterisation for the latter.

3.2 Master integrals

All of the Feynman integrals in the pentagon integral family $G[\nu_1, \ldots, \nu_5]$ can be reduced to a set of master integrals with the help of integration-by-parts relations. With the help of e.g. `LiteRed` [21], these may be solved in terms of the 19 master integrals shown in Figure 2. As zero exponents are equivalent to removing the corresponding propagator or in other words contracting the corresponding edge, it is evident that this set of master integrals includes 8 bubble integrals, 5 triangle integrals, 5 box integrals, and 1 pentagon integral.

The master integrals of Figure 2 are not of uniform transcendental (UT) weight and do not lead to an ϵ-factorised differential equation. In the following we will present a basis of master integrals, which are of uniform transcendental weight. In the case of one-loop integrals this is straightforward. Integrals of uniform transcendental weight are expected to have constant leading singularities [22]. We choose the integer dimension of the loop momentum of n-point 1-loop integrals to be $2\lfloor \frac{n+1}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, and divide these integrals by their leading singularities [23]. Dimensional recurrence relations [24, 25] relate integrals with D and $D-2$ dimensions, thus if desired these can be used to re-express our basis in terms of integrals with only four integer dimensions.

We start with the bubble integrals. It is easy to show that the bubble integral $G[0, 0, 1, 0, 1]$ has leading singularity $\frac{1}{s_{34}}$. Thus the integral $s_{34}G[0, 0, 1, 0, 1]$ is uniformly transcendental in $D = 2 - 2\epsilon$, and with the help of dimensional recurrence relations we can trade this with the uniformly transcendental integral $s_{34}\, G[0, 0, 2, 0, 1]$ in $D = 4 - 2\epsilon$. In the same fashion, the eight bubble integrals in our UT basis will be,

$$\begin{aligned}
g_1 &= \epsilon\, \frac{s_{34}}{\mu^2}\, G[0, 0, 2, 0, 1], & g_2 &= \epsilon\, \frac{m_3^2}{\mu^2}\, G[0, 0, 2, 1, 0], \\
g_3 &= \epsilon\, \frac{s_{15}}{\mu^2}\, G[0, 2, 0, 0, 1], & g_4 &= \epsilon\, \frac{s_{23}}{\mu^2}\, G[0, 2, 0, 1, 0], \\
g_5 &= \epsilon\, \frac{m_2^2}{\mu^2}\, G[0, 2, 1, 0, 0], & g_6 &= \epsilon\, \frac{s_{45}}{\mu^2}\, G[2, 0, 0, 1, 0], \\
g_7 &= \epsilon\, \frac{s_{12}}{\mu^2}\, G[2, 0, 1, 0, 0], & g_8 &= \epsilon\, \frac{m_1^2}{\mu^2}\, G[2, 1, 0, 0, 0],
\end{aligned} \tag{3.5}$$

where the pre-factor ϵ ensures that the integrals are of uniform transcendental weight zero (if we count the

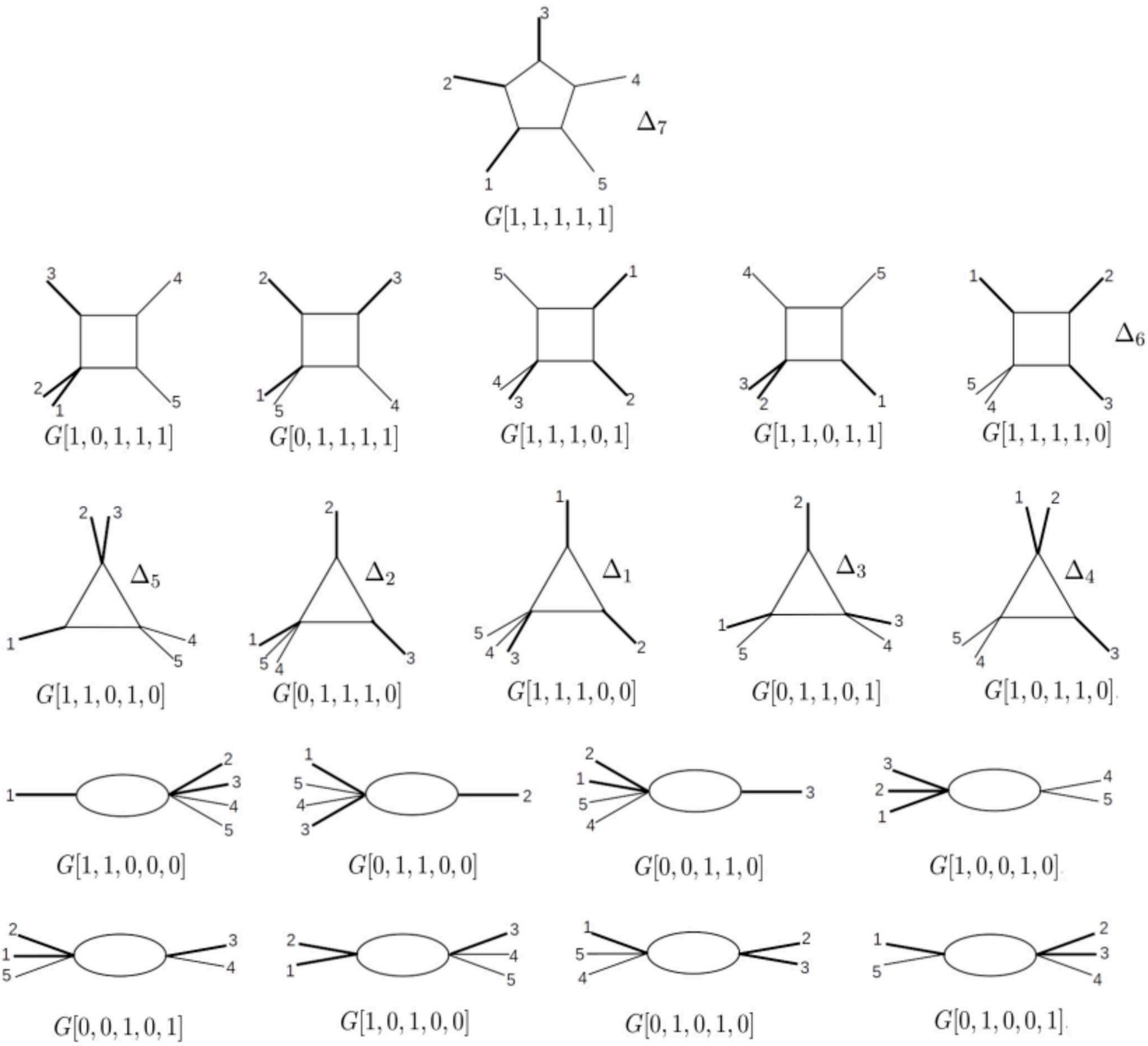

Figure 2. All master integrals of the one-loop three-mass hard pentagon topology, with thick lines denoting the massive external legs. The leading singularities of a subset thereof will depend on the square roots (3.9), as indicated in the figure.

dimensionless parameter with weight -1).

The external legs of each triangle integral are all massive, and the square roots shown in the first five lines of eq. (3.9) below appear in the denominators of their leading singularities. The triangle integrals in our basis are

$$g_9 = \epsilon^2 \frac{\Delta_3}{2\mu^2} G[0,1,1,0,1], \qquad g_{10} = \epsilon^2 \frac{\Delta_2}{2\mu^2} G[0,1,1,1,0],$$
$$g_{11} = \epsilon^2 \frac{\Delta_4}{2\mu^2} G[1,0,1,1,0], \qquad g_{12} = \epsilon^2 \frac{\Delta_5}{2\mu^2} G[1,1,0,1,0],$$
$$g_{13} = \epsilon^2 \frac{\Delta_1}{2\mu^2} G[1,1,1,0,0]. \tag{3.6}$$

Similarly, the box integrals in our basis are

$$\begin{aligned}
g_{14} &= \epsilon^2 \frac{s_{23}s_{34} - m_3^2 s_{15}}{2\mu^4} G[0,1,1,1,1], \\
g_{15} &= \epsilon^2 \frac{s_{34}s_{45}}{2\mu^4} G[1,0,1,1,1], \\
g_{16} &= \epsilon^2 \frac{s_{45}s_{15}}{2\mu^4} G[1,1,0,1,1], \\
g_{17} &= \epsilon^2 \frac{s_{15}s_{12} - m_1^2 s_{34}}{2\mu^4} G[1,1,1,0,1], \\
g_{18} &= \epsilon^2 \frac{\Delta_6}{2\mu^4} G[1,1,1,1,0],
\end{aligned} \tag{3.7}$$

where g_{15} and g_{16} have two massive external legs, g_{14} and g_{17} have three massive external legs, and g_{18} has all external legs massive, with Δ_6 also given in eq. (3.9).

Finally, the pentagon integral has leading singularity $\frac{1}{\Delta_7}$ in $D = 6 - 2\epsilon$, and the corresponding UT integral is

$$g_{19} = \epsilon^3 \frac{\Delta_7}{\mu^4} G^{D=6-2\epsilon}[1,1,1,1,1], \tag{3.8}$$

noting in particular the integer dimension choice indicated at the beginning of this subsection. With the help of dimensional recurrence relations [24, 25] we can always write this integral as a linear combination of integrals in $D = 4 - 2\epsilon$, hence this merely corresponds to a convenient notation.

Finally, the square roots appearing in the above formulas are given by

$$\begin{aligned}
\Delta_1 &= \sqrt{m_1^4 - 2m_1^2 m_2^2 + m_2^4 - 2m_1^2 s_{12} - 2m_2^2 s_{12} + s_{12}^2}, \\
\Delta_2 &= \sqrt{m_2^4 - 2m_2^2 m_3^2 + m_3^4 - 2m_2^2 s_{23} - 2m_3^2 s_{23} + s_{23}^2}, \\
\Delta_3 &= \sqrt{m_2^4 - 2m_2^2 s_{15} + s_{15}^2 - 2m_2^2 s_{34} - 2s_{15}s_{34} + s_{34}^2}, \\
\Delta_4 &= \sqrt{m_3^4 - 2m_3^2 s_{12} + s_{12}^2 - 2m_3^2 s_{45} - 2s_{12}s_{45} + s_{45}^2},
\end{aligned}$$

$$
\begin{aligned}
\Delta_5 &= \sqrt{m_1^4 - 2m_1^2 s_{23} + s_{23}^2 - 2m_1^2 s_{45} - 2s_{23}s_{45} + s_{45}^2}, \\
\Delta_6 &= \sqrt{m_1^4 m_3^4 - 2m_1^2 m_3^2 s_{12}s_{23} + s_{12}^2 s_{23}^2 - 2m_1^2 m_2^2 m_3^2 s_{45} - 2m_2^2 s_{12}s_{23}s_{45} + m_2^4 s_{45}^2}, \\
\Delta_7 &= \sqrt{\Lambda(p_1,p_2,p_3,p_4)} = \sqrt{\det(-p_i \cdot p_j)}.
\end{aligned} \tag{3.9}
$$

which unfortunately cannot be rationalized simultaneously [26, 27].

3.3 The alphabet

The differential equation for the basis $\vec{g} = \{g_1, ..., g_{19}\}$ has the ϵ-factorised form

$$
d\,\vec{g} = \epsilon\,\boldsymbol{A}(\vec{v})\,\vec{g}\,, \quad \boldsymbol{A}(\vec{v}) = \sum_{i=1}^{57} A_i\,d\log\left(\frac{W_i}{(\mu^2)^{\alpha_i}}\right)\,, \tag{3.10}
$$

where the matrix $\boldsymbol{A}$ is independent of ϵ, and its dependence on $\vec{v}$ comes only through the *letters* W_i, whereas their coefficients A_i are constant matrices. The exponent α_i is chosen such that $\frac{W_i}{(\mu^2)^{\alpha_i}}$ is dimensionless. To make contact with the previous section we set

$$
\omega_i = d\log\left(\frac{W_i}{(\mu^2)^{\alpha_i}}\right). \tag{3.11}
$$

For this integral family we have 57 letters in the differential equation. We will provide their explicit form in a moment, but together with $\boldsymbol{A}$ these may also be found in the ancillary file attached to the arxiv version of this article, see appendix C for details. We set

$$
\mathcal{A} = \{\omega_1, \ldots, \omega_{57}\}\,. \tag{3.12}
$$

Expressed in terms of the kinematic variables v_i, the total differential in the above equation takes the form

$$
d = \sum_{i=1}^{8} dv_i\,\frac{\partial}{\partial v_i}. \tag{3.13}
$$

The 57 letters $\{W_i\}$ appearing in the matrix $\boldsymbol{A}$ of the differential equation (3.10) may be obtained from the general results for one-loop alphabets worked out in [23, 28, 29]. In particular, by taking the limit of the generic to the three-mass hard pentagon integral with the help of the ancillary file of [29], and eliminating any multiplicative dependence, we arrive at the following alphabet:

$$
\begin{aligned}
W_1 &= -m_1^2, \quad W_2 = -m_2^2, \quad W_3 = -m_3^2, \quad W_4 = -s_{12}, \\
W_5 &= m_1^4 - 2m_2^2 m_1^2 - 2s_{12}m_1^2 + m_2^4 + s_{12}^2 - 2m_2^2 s_{12}, \quad W_6 = m_1^2 - s_{15}, \\
W_7 &= -s_{15}, \quad W_8 = s_{15} - s_{23}, \quad W_9 = -s_{23}, \quad W_{11} = m_3^2 - s_{34},
\end{aligned}
$$

$$W_{10} = m_2^4 - 2m_3^2m_2^2 - 2s_{23}m_2^2 + m_3^4 + s_{23}^2 - 2m_3^2s_{23}, \qquad W_{12} = s_{12} - s_{34},$$

$$W_{13} = s_{34}, \qquad W_{14} = s_{12}s_{15} - m_1^2s_{34}, \qquad W_{15} = m_3^2s_{15} - s_{23}s_{34},$$

$$W_{16} = m_2^4 - 2s_{15}m_2^2 - 2s_{34}m_2^2 + s_{15}^2 + s_{34}^2 - 2s_{15}s_{34},$$

$$W_{17} = -s_{34}m_1^4 - s_{34}^2m_1^2 + s_{12}s_{15}m_1^2 + m_2^2s_{34}m_1^2 + s_{12}s_{34}m_1^2 + s_{15}s_{34}m_1^2 - s_{12}s_{15}^2 - m_1^2m_2^2s_{12} - s_{12}^2s_{15} + m_2^2s_{12}s_{15} - m_2^2s_{15}s_{34} + s_{12}s_{15}s_{34},$$

$$W_{18} = -s_{15}m_3^4 - s_{15}^2m_3^2 + m_2^2s_{15}m_3^2 - m_2^2s_{23}m_3^2 + s_{15}s_{23}m_3^2 + s_{15}s_{34}m_3^2 + s_{23}s_{34}m_3^2 - s_{23}s_{34}^2 - s_{23}^2s_{34} - m_2^2s_{15}s_{34} + m_2^2s_{23}s_{34} + s_{15}s_{23}s_{34},$$

$$W_{20} = s_{15}^2 - m_1^2s_{15} - s_{23}s_{15} + s_{45}s_{15} + m_1^2s_{23}, \qquad W_{19} = -s_{45},$$

$$W_{21} = s_{34}^2 - m_3^2s_{34} - s_{12}s_{34} + s_{45}s_{34} + m_3^2s_{12},$$

$$W_{22} = m_3^4 - 2s_{12}m_3^2 - 2s_{45}m_3^2 + s_{12}^2 + s_{45}^2 - 2s_{12}s_{45},$$

$$W_{23} = m_1^4 - 2s_{23}m_1^2 - 2s_{45}m_1^2 + s_{23}^2 + s_{45}^2 - 2s_{23}s_{45},$$

$$W_{24} = \frac{m_1^2 + m_2^2 - s_{12} - \Delta_1}{m_1^2 + m_2^2 - s_{12} + \Delta_1}, \qquad W_{25} = \frac{m_1^2 - m_2^2 + s_{12} - \Delta_1}{m_1^2 - m_2^2 + s_{12} + \Delta_1},$$

$$W_{26} = \frac{m_2^2 + m_3^2 - s_{23} - \Delta_2}{m_2^2 + m_3^2 - s_{23} + \Delta_2}, \qquad W_{27} = \frac{m_2^2 - m_3^2 + s_{23} - \Delta_2}{m_2^2 - m_3^2 + s_{23} + \Delta_2},$$

$$W_{28} = \frac{R_{28} - s_{34}m_1^2\Delta_1 + s_{12}s_{15}\Delta_1}{R_{28} + s_{34}m_1^2\Delta_1 - s_{12}s_{15}\Delta_1}, \qquad W_{29} = \frac{R_{29} + s_{15}m_3^2\Delta_2 - s_{23}s_{34}\Delta_2}{R_{29} - s_{15}m_3^2\Delta_2 + s_{23}s_{34}\Delta_2},$$

$$W_{30} = \frac{m_2^2 + s_{15} - s_{34} - \Delta_3}{m_2^2 + s_{15} - s_{34} + \Delta_3}, \qquad W_{31} = \frac{m_2^2 - s_{15} + s_{34} - \Delta_3}{m_2^2 - s_{15} + s_{34} + \Delta_3},$$

$$W_{32} = \frac{R_{32} + s_{12}\Delta_3s_{15} - m_1^2s_{34}\Delta_3}{R_{32} - s_{12}\Delta_3s_{15} + m_1^2s_{34}\Delta_3}, \qquad W_{33} = \frac{R_{33} + m_3^2\Delta_3s_{15} - s_{23}s_{34}\Delta_3}{R_{33} - m_3^2\Delta_3s_{15} + s_{23}s_{34}\Delta_3},$$

$$W_{34} = \frac{m_3^2s_{12}s_{15}^2 - m_1^2m_3^2s_{34}s_{15} - s_{12}s_{23}s_{34}s_{15} + m_2^2s_{34}s_{45}s_{15} + m_1^2s_{23}s_{34}^2}{R_{34}},$$

$$W_{35} = \frac{R_{35}}{m_1^4m_3^4 - 2m_1^2s_{12}s_{23}m_3^2 - 2m_1^2m_2^2s_{45}m_3^2 + s_{12}^2s_{23}^2 + m_2^4s_{45}^2 - 2m_2^2s_{12}s_{23}s_{45}},$$

$$W_{36} = \frac{m_3^2 + s_{12} - s_{45} - \Delta_4}{m_3^2 + s_{12} - s_{45} + \Delta_4}, \qquad W_{37} = \frac{m_3^2 - s_{12} + s_{45} - \Delta_4}{m_3^2 - s_{12} + s_{45} + \Delta_4},$$

$$W_{38} = \frac{m_1^2 + s_{23} - s_{45} - \Delta_5}{m_1^2 + s_{23} - s_{45} + \Delta_5}, \qquad W_{39} = \frac{m_1^2 - s_{23} + s_{45} - \Delta_5}{m_1^2 - s_{23} + s_{45} + \Delta_5},$$

$$W_{40} = \frac{-2m_3^2s_{12} + s_{34}s_{12} + m_3^2s_{34} - s_{34}s_{45} - s_{34}\Delta_4}{-2m_3^2s_{12} + s_{34}s_{12} + m_3^2s_{34} - s_{34}s_{45} + s_{34}\Delta_4},$$

$$W_{41} = \frac{m_1^2s_{15} + s_{23}s_{15} - s_{45}s_{15} - \Delta_5s_{15} - 2m_1^2s_{23}}{m_1^2s_{15} + s_{23}s_{15} - s_{45}s_{15} + \Delta_5s_{15} - 2m_1^2s_{23}},$$

$$
\begin{aligned}
W_{42} &= \frac{m_1^2 m_3^2 + s_{12}s_{23} - m_2^2 s_{45} - \Delta_6}{m_1^2 m_3^2 + s_{12}s_{23} - m_2^2 s_{45} + \Delta_6}, \qquad W_{43} = \frac{m_1^2 m_3^2 - s_{12}s_{23} + m_2^2 s_{45} - \Delta_6}{m_1^2 m_3^2 - s_{12}s_{23} + m_2^2 s_{45} + \Delta_6}, \\
W_{44} &= \frac{R_{44} - \Delta_1\Delta_6}{R_{44} + \Delta_1\Delta_6}, \qquad W_{45} = \frac{R_{45} - \Delta_2\Delta_6}{R_{45} + \Delta_2\Delta_6}, \qquad W_{46} = \frac{R_{46} - \Delta_4\Delta_6}{R_{46} + \Delta_4\Delta_6}, \\
W_{47} &= \frac{R_{47} - \Delta_5\Delta_6}{R_{47} + \Delta_5\Delta_6}, \qquad W_{48} = \frac{R_{48} + \Delta_7}{-R_{48} + \Delta_7}, \qquad W_{49} = \frac{R_{49} - \Delta_1\Delta_7}{R_{49} + \Delta_1\Delta_7}, \\
W_{50} &= \frac{R_{50} - \Delta_7 s_{15}}{R_{50} + \Delta_7 s_{15}}, \qquad W_{53} = \frac{R_{53} - \Delta_2\Delta_7}{R_{53} + \Delta_2\Delta_7}, \qquad W_{54} = \frac{R_{54} - \Delta_3\Delta_7}{R_{54} + \Delta_3\Delta_7}, \\
W_{51} &= \frac{R_{51} + \Delta_7 m_1^2 - s_{15}\Delta_7}{R_{51} - \Delta_7 m_1^2 + s_{15}\Delta_7}, \qquad W_{52} = \frac{R_{52} + \Delta_7 s_{15} - s_{23}\Delta_7}{R_{52} - \Delta_7 s_{15} + s_{23}\Delta_7}, \\
W_{55} &= \frac{R_{55} - \Delta_4\Delta_7}{R_{55} + \Delta_4\Delta_7}, \qquad W_{56} = \frac{R_{56} - \Delta_5\Delta_7}{R_{56} + \Delta_5\Delta_7}, \qquad W_{57} = \frac{R_{57} - \Delta_6\Delta_7}{R_{57} + \Delta_6\Delta_7},
\end{aligned}
\tag{3.14}
$$

where the R_i denote large polynomials in the kinematic variables, whose precise form has been relegated to appendix B.

The seven roots $\Delta_1 - \Delta_7$ are associated to specific sub-sectors, in particular the roots $\Delta_1 - \Delta_5$ are associated to the triangle diagrams. From the block triangular structure of the matrix $\boldsymbol{A}$ it follows, that the roots $\Delta_1 - \Delta_5$ can never occur simultaneously in any given iterated integral. Furthermore, as the triangle $G[0,1,1,0,1]$ is not a sub-sector of the box integral $G[1,1,1,1,0]$ it follows that the roots Δ_3 and Δ_6 can never occur simultaneously in any given iterated integral. Therefore, the only sub-sets of roots which will appear in any given iterated integral are

$$
\begin{aligned}
&(\Delta_1,\Delta_6,\Delta_7), \quad (\Delta_2,\Delta_6,\Delta_7), \quad (\Delta_4,\Delta_6,\Delta_7), \quad (\Delta_5,\Delta_6,\Delta_7), \\
&(\Delta_1,\Delta_7), \quad (\Delta_2,\Delta_7), \quad (\Delta_3,\Delta_7), \quad (\Delta_4,\Delta_7), \quad (\Delta_5,\Delta_7), \quad (\Delta_6,\Delta_7), \\
&(\Delta_1,\Delta_6), \quad (\Delta_2,\Delta_6), \quad (\Delta_4,\Delta_6), \quad (\Delta_5,\Delta_6), \qquad (\Delta_1), \quad \dots \quad (\Delta_7).
\end{aligned}
\tag{3.15}
$$

4 Conversion to multiple polylogarithms

With the ϵ-factorised differential equation (3.10) at hand we may express the master integrals $\vec{g}$ in terms of iterated integrals

$$
I_{\omega_{i_1},\dots,\omega_{i_r}}[\gamma], \tag{4.1}
$$

where $\omega_{i_j} \in \mathcal{A}$ and γ is a path from a chosen boundary point to the desired point in kinematic space. In this section we show that all iterated integrals at any weight can be expressed algorithmically in terms of multiple polylogarithms. We do this in two steps. In the first step we perform a change of variables from our original kinematic variables $\vec{v}$ to a new set of variables obtained from a momentum twistor parameterization. This will rationalise three of the seven square roots. We then fix an integration path. The remaining non-rationalised square roots are associated with triangle integrals and in any given iterated integral there will occur at most a single non-rationalised square root. In the second step we use different re-parameterisations of the same integration path to rationalise the remaining square roots.

4.1 Step 1: Momentum twistor parameterization

In the first step we express with the help of momentum twistors (see appendix A for details) the kinematic variables $\vec{v} = \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, m_2^2, m_3^2\}$ in terms of new variables $\boldsymbol{x} = \{x_1, x_2, ..., x_8\}$. The transformation is given by

$$
\begin{aligned}
m_1^2 &= \frac{\mu^2}{x_4}, \\
m_2^2 &= -\frac{x_1(1+x_6)}{(-1+x_1)x_6 - x_1(-1+x_2)x_7}\mu^2, \\
m_3^2 &= -\frac{N_3}{(x_5-x_7)\big[x_1(-1+x_2)x_7-(-1+x_1)x_6\big]\big[x_4+(-1+x_2)x_8\big]\big[-1+x_7-x_6(1+x_8)\big]}\mu^2, \\
s_{12} &= \frac{\big[-1+x_1(1+x_4)\big](-1+x_7)}{x_4\big[(-1+x_1)x_6-x_1(-1+x_2)x_7\big]}\mu^2, \\
s_{23} &= -\frac{(-1+x_5-x_6)(1+x_6-x_7)(1+x_8)^2}{(x_5-x_7)\big[x_4+(-1+x_2)x_8\big]\big[-1+x_7-x_6(1+x_8)\big]}\mu^2, \\
s_{34} &= -\frac{N_{34}}{x_4(x_5-x_7)\big[x_1(-1+x_2)x_7-(-1+x_1)x_6\big]\big[-1+x_7-x_6(1+x_8)\big]}\mu^2, \\
s_{45} &= \frac{(-1+x_7)x_8\big[-(-1+x_2)(x_5-x_7)x_8-x_4\big(1-x_7+x_8-x_5x_8+x_6(1+x_8)\big)\big]}{x_4(x_5-x_7)\big[x_4+(-1+x_2)x_8\big]\big[-1+x_7-x_6(1+x_8)\big]}\mu^2, \\
s_{15} &= \frac{(x_3-x_5)(1+x_6-x_7)(1+x_8)}{x_4(x_5-x_7)\big[-1+x_7-x_6(1+x_8)\big]}\mu^2,
\end{aligned}
\tag{4.2}
$$

where N_3 and N_{34} are shorthands for

$$
\begin{aligned}
N_3 = (-1+x_7)\Big[&(-1+x_5-x_6)(1+x_6-x_7)(1+x_8) \\
&+ x_1(1+x_6)(1+x_8+x_6(1+x_8)+x_7(-1+(-2+x_2-x_4)x_8)) \\
&- x_1x_5\big[1+x_6-x_7+x_2x_8+x_2x_6x_8-x_7x_8+x_4\big(1+x_6-x_7(1+x_8)\big)\big]\Big], \\
N_{34} = (-1+x_7)\Big[&-\big[-1+x_1(1+x_4)\big]\big[(1+x_6)x_7x_8+x_5\big(1+x_6-x_7(1+x_8)\big)\big] \\
&+ x_3(-1-x_6+x_7-x_8+x_5x_8-x_6x_8) \\
&+ x_1x_3\big[1+x_8-x_2x_5x_8+x_6(1+x_8)+x_7\big(-1+(-1+x_2)x_8\big)\big]\Big].
\end{aligned}
\tag{4.3}
$$

It is easily checked that this defines for generic values of $\vec{v}$ (or $\boldsymbol{x}$) an invertible transformation. The above transformation has the following properties:

1. The square roots Δ_5, Δ_6 and Δ_7 are rationalized.
2. The remaining square roots Δ_1 - Δ_4 are of degree 2 in x_1.
3. Restricted to the hypersurface $x_1 = 0$, the remaining square roots Δ_1 - Δ_4 are rationalized on this hypersurface.

We now fix the integration path to be the piecewise smooth path $\gamma = \gamma_{\text{standard}}$ consisting of a straight line from $(0, 0, \dots, 0, 0)$ to $(0, 0, \dots, 0, x_8)$, followed by a straight line from $(0, 0, \dots, 0, x_8)$ to $(0, 0, \dots, x_7, x_8)$, and the pattern continues in this way. The last segment is given by straight line from $(0, x_2, \dots, x_7, x_8)$ to $(x_1, x_2, \dots, x_7, x_8)$. Due to property 3 from the list above we will encounter the remaining square roots only on the last segment from $(0, x_2, \dots, x_7, x_8)$ to $(x_1, x_2, \dots, x_7, x_8)$. The integration over all other segments will yield multiple polylogarithms in a straightforward way.

4.2 Step 2: Re-parameterisation of the integration path

It remains to consider the last integration segment from $(0, x_2, \dots, x_7, x_8)$ to $(x_1, x_2, \dots, x_7, x_8)$. Our task is to convert the iterated integrals along this segment to multiple polylogarithms. The remaining non-rationalized square roots are $\Delta_1, \Delta_2, \Delta_3$ and Δ_4. These square roots are associated to triangle sub-sectors and from discussion at the end of section 3.3 it follows that there will be never two or more of them in any given iterated integral. Hence, any given iterated integral will have at most one square root. The variables $x_2, \dots, x_8$ are constant along the last integration segment. This implies in particular, that square roots, which only depend on the variables $x_2, \dots, x_8$, but not on x_1 are unproblematic. In the variables $\boldsymbol{x}$ the square roots $\Delta_1 - \Delta_4$ factor into square roots of perfect squares (which are un-problematic) and a square roots of a degree 2 polynomial in x_1. The last square root can always be rationalized by a re-definition of the x_1-variable. As we do not change $x_2, \dots, x_8$, which are kept as constant parameters, this corresponds to a re-parameterisation of the integration path.

Let us assume that we would like to rationalize the square root

$$\sqrt{(x_1 - a)(x_1 - b)}, \tag{4.4}$$

where a and b are functions of the variables $x_2, \dots, x_8$ and may contain square roots which only depend on $x_2, \dots, x_8$. The substitution

$$x_1 = \frac{bt\left[(a-b)t + 4ab\right]}{(a-b)t^2 + 4abt + 4ab^2} \tag{4.5}$$

rationalizes this square root with respect to the new variable t. The transformation will produce $\sqrt{ab}$, but this is un-problematic, as this quantity depends only on $x_2, \dots, x_8$. The transformation in eq. (4.5) has the property that $t = 0$ corresponds to $x_1 = 0$ and

$$\left.\frac{\partial x_1}{\partial t}\right|_{t=0} = 1. \tag{4.6}$$

square roots	variables
no square root	$\{x_1, x_2, \ldots, x_8\}$
Δ_1	$\{t_1, x_2, \ldots, x_8\}$
Δ_2	$\{t_2, x_2, \ldots, x_8\}$
Δ_3	$\{t_3, x_2, \ldots, x_8\}$
Δ_4	$\{t_4, x_2, \ldots, x_8\}$

Table 1. Parameterizations of the (same) integration path. If no square roots occurs in an iterated integral, the first parameterization is used. If the square root Δ_i occurs, the parameterization with the variable t_i is used.

Expressing t as a function of x_1 we have to choose the sign of a square root. We always choose the sign such that $x_1 = 0$ corresponds to $t = 0$. For the case at hand we have four remaining square roots of the type as in eq. (4.4) and at most only one square root occurs in any given iterated integral. We introduce four new variables t_1, t_2, t_3 or t_4 in the way discussed above such that t_i rationalizes the square root Δ_i. Hence, we obtain five different parametrizations of the same integration path, which we will use as shown in table 4.2. The explicit expressions for the transformations from x_1 to the t_i's are rather long and may be found in the ancillary file attached to the arxiv version of this article. By using the appropriate parameterization we ensure that we only encounter dlog-forms with rational arguments, hence all iterated integrals can be expressed in terms of multiple polylogarithms.

5 Results

With the methods discussed above we have expressed up to weight four all master integrals for the one-loop pentagon integral with three adjacent massive external legs in terms of multiple polylogarithms. These include the lower-point integrals whose analytic ε-expansions have been discussed extensively in the literature. We note that the lower-point integrals include in particular the 1-loop box with all external legs massive. For simplicity we focus on the Euclidean region. The boundary values have been obtained with the help of the `AMFlow` package [30–32] and the PSLQ algorithm [33–36]. The explicit expressions for the master integrals in terms of multiple polylogarithms are rather long and given up to weight four in the ancillary file attached to the arxiv version of this article.

As a cross-check we evaluate the master integrals at a point $\vec{v}_0$ given by

$$\begin{aligned} m_1^2 &= -\frac{50}{51} & m_2^2 &= -\frac{175}{3798} & m_3^2 &= -\frac{1102585325}{883407397698} & s_{12} &= -\frac{67855}{129132}, \\ s_{23} &= -\frac{55296000}{232598051} & s_{34} &= -\frac{795086350}{2944306449} & s_{45} &= -\frac{1898617750}{11862500601} & s_{15} &= -\frac{4800}{12019}. \end{aligned} \tag{5.1}$$

In terms of the momentum twistor variables defined in appendix A, this point corresponds to

$$x_1 = -\frac{7}{20} \qquad x_2 = -\frac{3}{5} \qquad x_3 = -\frac{19}{50} \qquad x_4 = -\frac{51}{50},$$

$$x_5 = \frac{11}{5} \qquad x_6 = -\frac{6}{5} \qquad x_7 = \frac{9}{50} \qquad x_8 = 95, \tag{5.2}$$

and $\mu^2 = 1$. The numerical results for g_{18} and g_{19} up to weight four read

$$\begin{aligned} g_{18} &= 7.8968007697753055...\,\epsilon^2 + 48.74234942348662...\,\epsilon^3 + 166.59728539785323...\,\epsilon^4 \\ g_{19} &= -1.2884632786357...\,\epsilon^3 - 6.1292127077565...\,\epsilon^4. \end{aligned} \tag{5.3}$$

We verified that our result agrees with `AMFlow`.

6 Conclusions

In this work we considered families of Feynman integrals, which have an ε-factorised differential equation which contains only dlog-forms with algebraic arguments and where the algebraic part is given by (multiple) square roots. These families occur frequently in precision calculations. It is well-known that if all square roots are simultaneously rationalisable, the Feynman integrals can be expressed in terms of multiple polylogarithms. This is a sufficient, but not a necessary criterium. In this paper we presented weaker requirements. We review path independence and parameterisation independence of iterated integrals. In the context of Feynman integrals we may divide the full result into smaller path-independent subsets, and we may use different rationalisations in different subsets. Subsets, which are naturally path-independent, are characterised by proposition 1. Secondly, any iterated integral is invariant under re-parameterisation of the same integration path. Hence, we may use within the same subset different rationalisations, if they correspond to different parameterisations of the same integration path. We presented a non-trivial example, the one-loop pentagon function with three adjacent massive external legs involving seven square roots, where this technique can be used to express the result in terms of multiple polylogarithms. The technique will be useful for other Feynman integrals as well. For example, we expect that it will be useful in an attempt to prove that any 1-loop generic Feynman integral evaluates to multiple polylogarithms to all orders in the dimensional regulator, starting from the known canonical differential equations for these integrals [23, 28, 29]. Furthermore, this approach may potentially be extended to higher-loop cases [37–41].

Acknowledgements

The work of GP was supported by the UK Science and Technology Facilities Council grant ST/Z001021/1. KW is very grateful for the hospitality provided by the Institut für Physik, Mainz. KW is supported by the Helmholtz-OCPC International Postdoctoral Exchange Fellowship Program. YZ thanks the support from the NSF of China through Grant No. 12075234, 12247103, and 12047502. This research was supported in part by the Munich Institute for Astro-, Particle and BioPhysics (MIAPbP) which is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany´s Excellence Strategy – EXC-2094 – 390783311. We also thank the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support during the completion of this work.

A Momentum twistor parametrization

In this appendix we provide details on the momentum twistor parametrization [42–47]. The five external momenta of the pentagon integral shown in Fig. 1 can be expressed by eight massless momenta

$$p_1 = q_1 + q_2, \quad p_2 = q_3 + q_4, \quad p_3 = q_5 + q_6, \quad p_4 = q_7, \quad p_5 = q_8. \tag{A.1}$$

where the p_i's denote the original external momenta of the pentagon integral, while the q_i's denote massless momenta: $q_i^2 = 0$ $(i = 1, ..., 8)$. For the momenta q_i we introduce the dual coordinates as $y_{i+1} - y_i = q_i$ with $y_{i+8} = y_i$,[2] The eight kinematic variables in $\vec{v}$ of the pentagon integral (3.1) can then be expressed as follows:

$$\begin{aligned} &s_{12} = y_{1,5}^2, \quad s_{23} = y_{3,7}^2, \quad s_{34} = y_{5,8}^2, \quad s_{45} = y_{1,7}^2, \\ &s_{15} = y_{3,8}^2, \quad m_1^2 = y_{1,3}^2, \quad m_2^2 = y_{3,5}^2, \quad m_3^2 = y_{5,7}^2, \end{aligned} \tag{A.2}$$

where $y_{i,j} = y_j - y_i$. Although we work with dimensional regularisation, the external momenta lie always in a four-dimensional sub-space. We may therefore use eight momentum twistor variables $Z_i = (\lambda_i, \mu_i)$ to encode the corresponding q_i $(i = 1, ..., 8)$. The $y_{i,j}^2$ have the following expressions in terms of the momentum twistor variables:

$$y_{i,j}^2 = \frac{\langle i-1 i j-1 j \rangle}{\langle i-1\, i\, I_\infty \rangle \langle j-1\, j\, I_\infty \rangle} \mu^2. \tag{A.3}$$

Here, the four bracket $\langle ijkl \rangle$ is defined as the determinant of four momentum twistors. Explicitly it is given by

$$\langle ijkl \rangle = \epsilon^{ijkl} Z_i Z_j Z_k Z_l. \tag{A.4}$$

The I_∞ refers to two auxiliary momentum twistors Z_9 and Z_{10}: $I_\infty = (Z_9, Z_{10})$. It is convenient to introduce the scale μ in eq. (A.3), this ensures that the twistors are dimensionless. The momentum twistor parameterization is highly redundant and we are allowed to make specific choices to reduce this redundancy. Specifically, we choose the following momentum twistor parametrization in our calculation

$$(Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}) = \begin{pmatrix} 1 & 0 & 0 & 0 & x_6+1 & 1 & x_5 & x_3 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{x_1} & x_2 & z_{2,6} & 1 & x_4+1 & 0 & 1 \\ 0 & 0 & 1 & x_7 & 1 & z_{3,6} & z_{3,7} & x_7 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \tag{A.5}$$

[2] The standard notation of dual coordinates is x_i, but we use y_i to avoid the confusion with the free variables in Z.

where

$$
\begin{aligned}
z_{2,6} &= \frac{x_2x_5x_8 - x_2x_8 + x_4x_5 - x_4 + x_5 - x_6x_8 - x_6 - 1}{(x_8+1)(x_5 - x_6 - 1)}, \\
z_{3,6} &= -\frac{x_6x_7x_8 + x_6x_7 - x_7^2 - x_7x_8 + x_7 + x_8}{(x_8+1)(-x_6 + x_7 - 1)}, \\
z_{3,7} &= -\frac{-x_5x_7x_8 + x_5x_8 + x_6x_7x_8 + x_6x_7 - x_7^2 + x_7}{(x_8+1)(-x_6 + x_7 - 1)},
\end{aligned} \tag{A.6}
$$

in which $x_1, ..., x_8$ are free variables. Note that eq. (4.2) is the explicit expression for the parametrization in eq. (A.5). It is easily checked that this defines for generic values of $\vec{v}$ an invertible transformation between the eight kinematic variables of $\vec{v}$ and $x_1, \ldots, x_8$. While some matrix entries in Z are not so simple rational expressions of these variables, this particular choice has the benefit that it rationalizes three square roots, Δ_5, Δ_6 and Δ_7, whereas a generic momentum twistor parametrization only rationalizes Δ_7.

B The R_i terms of the letters

All the R_i that are shown in the letters in eq. (3.14) have the explicit expressions as follows:

$$
\begin{aligned}
R_{28} &= -s_{34}m_1^4 - 2m_2^2s_{12}m_1^2 + s_{12}s_{15}m_1^2 + m_2^2s_{34}m_1^2 + s_{12}s_{34}m_1^2 - s_{12}^2s_{15} + m_2^2s_{12}s_{15}, \\
R_{29} &= -s_{15}m_3^4 + m_2^2s_{15}m_3^2 - 2m_2^2s_{23}m_3^2 + s_{15}s_{23}m_3^2 + s_{23}s_{34}m_3^2 - s_{23}^2s_{34} + m_2^2s_{23}s_{34}, \\
R_{32} &= -s_{12}s_{15}^2 + m_2^2s_{12}s_{15} + m_1^2s_{34}s_{15} - 2m_2^2s_{34}s_{15} + s_{12}s_{34}s_{15} - m_1^2s_{34}^2 + m_1^2m_2^2s_{34}, \\
R_{33} &= -m_3^2s_{15}^2 + m_2^2m_3^2s_{15} - 2m_2^2s_{34}s_{15} + m_3^2s_{34}s_{15} + s_{23}s_{34}s_{15} - s_{23}s_{34}^2 + m_2^2s_{23}s_{34}, \\
R_{34} &= m_3^4m_1^4 + s_{34}^2m_1^4 - 2m_3^2s_{34}m_1^4 - 2s_{23}s_{34}^2m_1^2 - 2m_3^4s_{15}m_1^2 + 2m_3^2s_{12}s_{15}m_1^2 - 2m_3^2s_{12}s_{23}m_1^2 \\
&\quad + 2m_3^2s_{15}s_{34}m_1^2 - 2s_{12}s_{15}s_{34}m_1^2 + 2m_3^2s_{23}s_{34}m_1^2 + 2s_{12}s_{23}s_{34}m_1^2 - 2s_{34}^2s_{45}m_1^2 \\
&\quad - 2m_2^2m_3^2s_{45}m_1^2 + 2m_3^2s_{15}s_{45}m_1^2 + 2m_2^2s_{34}s_{45}m_1^2 + 2m_3^2s_{34}s_{45}m_1^2 + 2s_{15}s_{34}s_{45}m_1^2 \\
&\quad - 4s_{23}s_{34}s_{45}m_1^2 + m_3^4s_{15}^2 + s_{12}^2s_{15}^2 - 2m_3^2s_{12}s_{15}^2 + s_{12}^2s_{23}^2 + s_{23}^2s_{34}^2 + m_2^4s_{45}^2 + s_{15}^2s_{45}^2 \\
&\quad + s_{34}^2s_{45}^2 - 2m_2^2s_{15}s_{45}^2 - 2m_2^2s_{34}s_{45}^2 - 2s_{15}s_{34}s_{45}^2 - 2s_{12}^2s_{15}s_{23} + 2m_3^2s_{12}s_{15}s_{23} - 2s_{12}s_{23}^2s_{34} \\
&\quad - 2m_3^2s_{15}s_{23}s_{34} + 2s_{12}s_{15}s_{23}s_{34} - 2m_3^2s_{15}^2s_{45} - 2s_{12}s_{15}^2s_{45} - 2s_{23}s_{34}^2s_{45} + 2m_2^2m_3^2s_{15}s_{45} \\
&\quad + 2m_2^2s_{12}s_{15}s_{45} - 4m_3^2s_{12}s_{15}s_{45} - 2m_2^2s_{12}s_{23}s_{45} + 2s_{12}s_{15}s_{23}s_{45} - 4m_2^2s_{15}s_{34}s_{45} \\
&\quad + 2m_3^2s_{15}s_{34}s_{45} + 2s_{12}s_{15}s_{34}s_{45} + 2m_2^2s_{23}s_{34}s_{45} + 2s_{12}s_{23}s_{34}s_{45} + 2s_{15}s_{23}s_{34}s_{45}, \\
R_{35} &= -m_3^4m_1^2 + m_2^2m_3^2m_1^2 - m_2^2s_{12}m_1^2 + m_3^2s_{12}m_1^2 + m_3^2s_{23}m_1^2 + s_{12}s_{23}m_1^2 + m_2^2s_{45}m_1^2 + m_3^2s_{45}m_1^2 \\
&\quad - s_{23}s_{45}m_1^2 - s_{12}s_{23}^2 - m_2^2s_{45}^2 - m_1^4m_3^2 - s_{12}^2s_{23} - m_2^2m_3^2s_{23} + m_2^2s_{12}s_{23} + m_3^2s_{12}s_{23} \\
&\quad - m_2^4s_{45} + m_2^2m_3^2s_{45} + m_2^2s_{12}s_{45} - m_3^2s_{12}s_{45} + m_2^2s_{23}s_{45} + s_{12}s_{23}s_{45},
\end{aligned}
$$

$$
\begin{aligned}
R_{44} = {} & m_2^2m_3^2m_1^2 - 2m_2^2s_{12}m_1^2 + m_3^2s_{12}m_1^2 + s_{12}s_{23}m_1^2 + m_2^2s_{45}m_1^2 - m_1^4m_3^2 - s_{12}^2s_{23} + m_2^2s_{12}s_{23} \\
& - m_2^4s_{45} + m_2^2s_{12}s_{45}, \\
R_{45} = {} & -s_{45}m_2^4 + m_1^2m_3^2m_2^2 - 2m_3^2s_{23}m_2^2 + s_{12}s_{23}m_2^2 + m_3^2s_{45}m_2^2 + s_{23}s_{45}m_2^2 - m_1^2m_3^4 - s_{12}s_{23}^2 \\
& + m_1^2m_3^2s_{23} + m_3^2s_{12}s_{23}, \\
R_{46} = {} & -m_1^2m_3^4 + m_1^2s_{12}m_3^2 + s_{12}s_{23}m_3^2 + m_1^2s_{45}m_3^2 + m_2^2s_{45}m_3^2 - 2s_{12}s_{45}m_3^2 - m_2^2s_{45}^2 - s_{12}^2s_{23} \\
& + m_2^2s_{12}s_{45} + s_{12}s_{23}s_{45}, \\
R_{47} = {} & m_3^2s_{23}m_1^2 + s_{12}s_{23}m_1^2 + m_2^2s_{45}m_1^2 + m_3^2s_{45}m_1^2 - 2s_{23}s_{45}m_1^2 - s_{12}s_{23}^2 - m_2^2s_{45}^2 - m_1^4m_3^2 \\
& + m_2^2s_{23}s_{45} + s_{12}s_{23}s_{45}, \\
R_{48} = {} & -m_1^2m_3^2 + s_{15}m_3^2 + s_{12}s_{15} - s_{12}s_{23} + m_1^2s_{34} - 2s_{15}s_{34} + s_{23}s_{34} + m_2^2s_{45} - s_{15}s_{45} - s_{34}s_{45}, \\
R_{49} = {} & -s_{34}m_1^4 + m_2^2m_3^2m_1^2 - 2m_2^2s_{12}m_1^2 + m_3^2s_{12}m_1^2 + m_3^2s_{15}m_1^2 + s_{12}s_{15}m_1^2 + s_{12}s_{23}m_1^2 + m_2^2s_{34}m_1^2 \\
& - 2m_3^2s_{34}m_1^2 + s_{12}s_{34}m_1^2 + s_{23}s_{34}m_1^2 + m_2^2s_{45}m_1^2 - s_{15}s_{45}m_1^2 + s_{34}s_{45}m_1^2 - m_1^4m_3^2 - s_{12}^2s_{15} \\
& - m_2^2m_3^2s_{15} + m_2^2s_{12}s_{15} + m_3^2s_{12}s_{15} - s_{12}^2s_{23} + m_2^2s_{12}s_{23} - 2s_{12}s_{15}s_{23} - m_2^2s_{23}s_{34} + s_{12}s_{23}s_{34} \\
& - m_2^4s_{45} + m_2^2s_{12}s_{45} + m_2^2s_{15}s_{45} + s_{12}s_{15}s_{45} + m_2^2s_{34}s_{45} - s_{12}s_{34}s_{45}, \\
R_{50} = {} & -m_3^2s_{15}^2 - s_{12}s_{15}^2 + s_{45}s_{15}^2 + m_1^2m_3^2s_{15} + s_{12}s_{23}s_{15} + m_1^2s_{34}s_{15} + s_{23}s_{34}s_{15} - m_2^2s_{45}s_{15} \\
& - s_{34}s_{45}s_{15} - 2m_1^2s_{23}s_{34}, \\
R_{51} = {} & m_3^2m_1^4 - s_{34}m_1^4 - 2m_3^2s_{15}m_1^2 + s_{12}s_{15}m_1^2 - s_{12}s_{23}m_1^2 + s_{15}s_{34}m_1^2 + s_{23}s_{34}m_1^2 - m_2^2s_{45}m_1^2 + s_{15}s_{45}m_1^2 \\
& + s_{34}s_{45}m_1^2 + m_3^2s_{15}^2 - s_{12}s_{15}^2 + s_{12}s_{15}s_{23} - s_{15}s_{23}s_{34} - s_{15}^2s_{45} + m_2^2s_{15}s_{45} - 2s_{12}s_{15}s_{45} + s_{15}s_{34}s_{45}, \\
R_{52} = {} & -m_3^2s_{15}^2 + s_{12}s_{15}^2 - s_{45}s_{15}^2 + m_1^2m_3^2s_{15} + m_3^2s_{23}s_{15} - 2s_{12}s_{23}s_{15} - m_1^2s_{34}s_{15} + s_{23}s_{34}s_{15} + m_2^2s_{45}s_{15} \\
& - 2m_3^2s_{45}s_{15} + s_{23}s_{45}s_{15} + s_{34}s_{45}s_{15} + s_{12}s_{23}^2 - m_1^2m_3^2s_{23} - s_{23}^2s_{34} + m_1^2s_{23}s_{34} - m_2^2s_{23}s_{45} + s_{23}s_{34}s_{45}, \\
R_{53} = {} & -s_{45}m_2^4 + m_1^2m_3^2m_2^2 + m_3^2s_{15}m_2^2 - s_{12}s_{15}m_2^2 - 2m_3^2s_{23}m_2^2 + s_{12}s_{23}m_2^2 - m_1^2s_{34}m_2^2 + s_{23}s_{34}m_2^2 \\
& + m_3^2s_{45}m_2^2 + s_{15}s_{45}m_2^2 + s_{23}s_{45}m_2^2 + s_{34}s_{45}m_2^2 - m_1^2m_3^4 - s_{12}s_{23}^2 - m_3^4s_{15} - 2m_1^2m_3^2s_{15} \\
& + m_3^2s_{12}s_{15} + m_1^2m_3^2s_{23} + m_3^2s_{12}s_{23} + m_3^2s_{15}s_{23} + s_{12}s_{15}s_{23} - s_{23}^2s_{34} + m_1^2m_3^2s_{34} + m_1^2s_{23}s_{34} \\
& + m_3^2s_{23}s_{34} - 2s_{12}s_{23}s_{34} + m_3^2s_{15}s_{45} - s_{15}s_{23}s_{45} - m_3^2s_{34}s_{45} + s_{23}s_{34}s_{45}, \\
R_{54} = {} & s_{45}m_2^4 - m_1^2m_3^2m_2^2 + m_3^2s_{15}m_2^2 + s_{12}s_{15}m_2^2 - s_{12}s_{23}m_2^2 + m_1^2s_{34}m_2^2 - 2s_{15}s_{34}m_2^2 + s_{23}s_{34}m_2^2 \\
& - 2s_{15}s_{45}m_2^2 - 2s_{34}s_{45}m_2^2 - m_3^2s_{15}^2 - s_{12}s_{15}^2 - m_1^2s_{34}^2 - s_{23}s_{34}^2 + m_1^2m_3^2s_{15} - 2m_3^2s_{12}s_{15} + s_{12}s_{15}s_{23} \\
& + m_1^2m_3^2s_{34} + m_1^2s_{15}s_{34} + m_3^2s_{15}s_{34} + s_{12}s_{15}s_{34} - 2m_1^2s_{23}s_{34} + s_{12}s_{23}s_{34} + s_{15}s_{23}s_{34} + s_{15}^2s_{45} \\
& + s_{34}^2s_{45} - 2s_{15}s_{34}s_{45}, \\
R_{55} = {} & -m_1^2m_3^4 + s_{15}m_3^4 + m_1^2s_{12}m_3^2 - 2s_{12}s_{15}m_3^2 + s_{12}s_{23}m_3^2 + m_1^2s_{34}m_3^2 - s_{23}s_{34}m_3^2 + m_1^2s_{45}m_3^2 + m_2^2s_{45}m_3^2
\end{aligned}
$$

$$
\begin{aligned}
& - 2s_{12}s_{45}m_3^2 - 2s_{15}s_{45}m_3^2 + s_{34}s_{45}m_3^2 - m_2^2s_{45}^2 + s_{15}s_{45}^2 - s_{34}s_{45}^2 + s_{12}^2s_{15} - s_{12}^2s_{23} - m_1^2s_{12}s_{34} \\
& + s_{12}s_{23}s_{34} + m_2^2s_{12}s_{45} - 2s_{12}s_{15}s_{45} + s_{12}s_{23}s_{45} + m_1^2s_{34}s_{45} - 2m_2^2s_{34}s_{45} + s_{12}s_{34}s_{45} + s_{23}s_{34}s_{45}, \\
R_{56} = {} & s_{34}m_1^4 + m_3^2s_{15}m_1^2 - s_{12}s_{15}m_1^2 + m_3^2s_{23}m_1^2 + s_{12}s_{23}m_1^2 - 2s_{23}s_{34}m_1^2 + m_2^2s_{45}m_1^2 + m_3^2s_{45}m_1^2 + s_{15}s_{45}m_1^2 \\
& - 2s_{23}s_{45}m_1^2 - 2s_{34}s_{45}m_1^2 - s_{12}s_{23}^2 - m_2^2s_{45}^2 - s_{15}s_{45}^2 + s_{34}s_{45}^2 - m_1^4m_3^2 - m_3^2s_{15}s_{23} + s_{12}s_{15}s_{23} \\
& + s_{23}^2s_{34} - 2m_2^2s_{15}s_{45} + m_3^2s_{15}s_{45} + s_{12}s_{15}s_{45} + m_2^2s_{23}s_{45} + s_{12}s_{23}s_{45} + s_{15}s_{23}s_{45} - 2s_{23}s_{34}s_{45}, \\
R_{57} = {} & m_3^4m_1^4 - m_3^2s_{34}m_1^4 - m_3^4s_{15}m_1^2 + m_3^2s_{12}s_{15}m_1^2 - 2m_3^2s_{12}s_{23}m_1^2 + m_3^2s_{23}s_{34}m_1^2 + s_{12}s_{23}s_{34}m_1^2 \\
& - 2m_2^2m_3^2s_{45}m_1^2 + m_3^2s_{15}s_{45}m_1^2 + m_2^2s_{34}s_{45}m_1^2 + m_3^2s_{34}s_{45}m_1^2 - 2s_{23}s_{34}s_{45}m_1^2 + s_{12}^2s_{23}^2 + m_2^4s_{45}^2 \\
& - m_2^2s_{15}s_{45}^2 - m_2^2s_{34}s_{45}^2 - s_{12}^2s_{15}s_{23} + m_3^2s_{12}s_{15}s_{23} - s_{12}s_{23}^2s_{34} + m_2^2m_3^2s_{15}s_{45} + m_2^2s_{12}s_{15}s_{45} \\
& -2m_3^2s_{12}s_{15}s_{45} - 2m_2^2s_{12}s_{23}s_{45} + s_{12}s_{15}s_{23}s_{45} + m_2^2s_{23}s_{34}s_{45} + s_{12}s_{23}s_{34}s_{45}.
\end{aligned} \tag{B.1}
$$

C Supplementary material

Attached to the arxiv version of this article are the following auxiliary files in `Mathematica` syntax.

- `pent3mAB.m`: Alphabet and square roots.
- `UTMatrix.m`: Matrix of canonical differential equations.
- `kinvecReplace2.m`: Transformation between kinematic variables and $\{x_1, \ldots, x_8\}$.
- `sDpentReplace2.m`: Rationalized expressions of square roots: $\Delta_1, \ldots, \Delta_7$.
- `x1Replace.m`: Mapping between x_1 and $t_1, ..., t_4$.
- `FeynIntegralwithBoundaryweight1-3.m`: The results of 19 canonical integrals up to weight-3.
- `FeynIntegralwithBoundaryweight4.m`: Pentagon integral of weight-4.
- `lettertPentvector2.m`: Arguments corresponding to variables $\{x_1, t_1, \ldots, t_4\}$ in the MPLs.
- `letterkinvecPent2.m`: Arguments corresponding to variables $x_2, \ldots, x_8$ in the MPLs.

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