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Ill-posedness of time-dependent inverse problems in Lebesgue-Bochner spaces

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Abstract

We consider time-dependent inverse problems in a mathematical setting using Lebesgue-Bochner spaces. Such problems arise when one aims to recover parameters from given observations where the parameters or the data depend on time. There are various important applications being subject of current research that belong to this class of problems. Typically inverse problems are ill-posed in the sense that already small noise in the data causes tremendous errors in the solution. In this article we present two different concepts of ill-posedness: temporally (pointwise) ill-posedness and uniform ill-posedness with respect to the Lebesgue-Bochner setting. We investigate the two concepts by means of a typical setting consisting of a time-depending observation operator composed by a compact operator. Furthermore we develop regularization methods that are adapted to the respective class of ill-posedness.

Keywords: dynamic inverse problems, Lebesgue-Bochner spaces, ill-posedness, regularization methods, parameter identification

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1. Introduction

Time-dependent or dynamic inverse problems (DIPs) yield a large and versatile subfield of inverse problems, which has seen a growing interest in recent years. Considering an inverse problem, given by an operator equation

$$F(\vartheta) = y, \quad F : \mathcal{D}(F) \subseteq \mathbb{X} \to \mathbb{Y},$$

the dependence on time may be reflected in the source $\vartheta \in \mathbb{X}$, in the data $y \in \mathbb{Y}$, or in the forward operator F. The dependence results for example from changes in the underlying physical setup or from an intrinsic time-dependence of the respective system, which evolves in time.

We shall see that typically the operator F in dynamic inverse problems can be decomposed into two types of operators that preserve the causality. In order to compute the output $y(t)$ at a fixed time t we apply operators of the form A_t acting at fixed time or $B_{[0,t]}$ acting in a time interval. Again due to causality the output at time t only depends on the restriction $\vartheta|_{[0,t]}$ to the initial time interval. We thus see that we can approach such problems in a global or local setting.

Many time-dependent problems have been addressed and studied in relation to applications (see [27]), for instance, in dynamic computerized tomography [14, 15], magnetic resonance imaging [17], emission tomography [18], magnetic particle imaging [1, 3, 26, 31], or structural health monitoring [30, 32]. We state some recent examples for dynamic inverse problems in real-world applications in the following:

(a) *Dynamic Computerized Tomography (DCT)*
The aim of CT is to recover the interior ϑ of an object from x-ray measurements. If the object undergoes a motion (e.g. if the patient is moving), then the function ϑ to be recovered depends on time and the forward operator is given as

$$F[\vartheta(t)] = S(t) R[\vartheta(t)]. \tag{1.1}$$

Here R denotes the 2D Radon transform acting on the spatial variable of $\vartheta(t,x)$ and $S(t)$ describes the measurement geometry. The observation operator $S(t)$ depends on t, since in this way, changes in the measurement process in time are also included in the mathematical model. In the notation above we have

$$y(t) = A_t[\vartheta],$$

the problem is fully local in time.

In particular, we want to point out two approaches: The first one is to reconstruct the initial state of the object from data that has been collected while the object is changing in time, and taking into account the motion of the object (see, e.g. [13, 15]). In this case the motion of the object itself is not of interest. The second approach is to reconstruct the moving object, i.e. the object is to be reconstructed at a range of time points. Here, one usually has to deal with sparse data or limited angle problems [2, 8, 9]. Including motion, the Radon transform is now computed for a deformed version of ϑ over the whole time interval, which changes the setting to

$$y(t) = A_t B_{[0,t]}[\vartheta].$$

(b) *Dynamic Load Monitoring (DLM)*
DLM aims for computing loads in elastic structures D from time-dependent sensor data that are acquired at the structure's surface. If the displacements $u(t,x)$ are small, then the wave

propagation in an elastic material is described using Hooke's law by the Cauchy equation of motion

$$\rho\ddot{u}(t,x) - \nabla\cdot(\mathbb{C}:\varepsilon(u))(t,x) = f(t,x). \tag{1.2}$$

Here $\mathbb{C}(x)$ denotes the elasticity tensor in x, ρ is the mass density, $\varepsilon(u) = (\nabla u + \nabla u^\top)/2$ is the linearized Green strain tensor, and $f(t,x)$ is a volume body force. The hyperbolic PDE system (1.2) is uniquely solvable if appropriate initial and boundary values are given. *Dynamic load monitoring* means computing f (the load) from (partial) measurements of displacements u at the boundary of a structure. This is a linear inverse problem with forward operator

$$F(f) := S(t)[Lf],$$

and the operator L maps a source term f to the (unique, weak) solution $u =: Lf$ of (1.2) equipped with appropriate initial and boundary values, whereas $S(t)$ is the mathematical model of the data acquisition process. For example, $S(t)$ may correspond to the trace operator γ,

$$\gamma[g] = g_{|\Gamma},$$

with $\Gamma\subset\partial D$ or

$$S(t)[g] = \left(\int_\Gamma \langle g(x), \chi_j(x)\rangle\,\mathrm{d}s_x\right)_{j=1,\dots,J},$$

where χ_j is the characteristic function of sensor j. Note that in these examples S is actually independent of time, but can be extended (by identical copies) as an operator on a time-dependent g defined on $[0,T]\times\partial D$, yielding a trace on $\Gamma_T = [0,T]\times\Gamma$. Let us mention that L depends on the whole time interval, hence we effectively compute the data via

$$y(t) = A_t\left[B_{[0,t]}f\right].$$

(c) *Magnetic Particle Imaging (MPI)*
MPI is a relatively novel medical imaging technique, e.g. to monitor blood vessels in a patient. Magnetic nanoparticles are injected into the bloodstream and distribute inside the patient's body. A strong external magnetic field with a field free point (FFP) is applied such that the nanoparticles' magnetic moment vectors align with the field lines of this field. Only in the FFP, where the applied field is very weak, the particles can move freely. When the FFP moves around the field of view, these particles inside the FFP abruptly change their magnetization. The resulting change in the total magnetic field induces voltages in the receive coils at each time point t during the measurement, which serve as data (see also [31]).

The goal in MPI is to reconstruct the concentration c of the magnetic particles inside the body D from the measured voltages. The physical model is given by the K integral equations

$$v_k(t) = \int_0^t \tilde{a}_k(t-\tau)\int_D c(x)\,s_k(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \quad s_k(x,t) = p_k^{\mathrm{R}}(x)^T\,\partial_t\overline{m}(x,t),$$

where K is the number of receive coils, $\tilde{a}_k$, $k = 1,\dots,K$, are transfer functions, p_k^R the coil sensitivity of the kth coil, and $\overline{m}$ is the mean magnetization. The function s_k is called the system function and represents a kind of induction potential for the kth coil. In practice, the system function is measured in a time-consuming calibration process. This yields two types of inverse problems. First, the actual imaging problem is to reconstruct the concentration c from the time-dependent data. Secondly, to avoid the expensive calibration, the model-based reconstruction of the system function from data that was obtained for known particle concentrations yields an inverse problem where both the source and the data are time-depending. In both cases, the forward operator can be formulated as a composition

$$F[f](t) = SI(t)[f(t)],$$

where S represents the convolution in time and $I(t)$ the integral operator in space given a time-varying weight. Note that Lebesgue-Bochner spaces are a convenient choice for the source space of the inverse problem for the system function. The data space is given by the Lebesgue space $L^s([0,T])$. In this case we can write the problem in the form

$$y(t) = B_{[0,t]}A_t[f].$$

The concept of ill-posedness for inverse problems has been discussed in detail in the literature. Linear ill-posed problems have been classified by Hadamard [12]. For nonlinear forward operators, the definition has to be adapted to the local character of nonlinear operators, which has been done by Hofmann and Scherzer in [20, 22, 23].

Here, we address both linear and nonlinear problems, i.e. our definition of ill-posedness will be based on the definition by Hofmann and Scherzer.

Since the dependence on time of a quantity is of an entirely different nature than the dependence on space, it is not surprising that this difference is also reflected in the respective mathematical modelling. Regarding inverse problems, the choice of Lebesgue-Bochner spaces as source or data space yields an adequate setting, since they admit a different treatment of these two types of variables. A prominent example where this setting occurs naturally is parameter identification for parabolic linear partial differential equations. The differing roles of the temporal and spatial variables in this context are addressed in [42, chapter 23.1] for a classical initial boundary value problem for the heat equation,

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } D \times (0,T), \\ u &= 0 && \text{on } \partial D \times [0,T], \\ u(x,0) &= u_0(x) && \text{in } D. \end{aligned}$$

We summarize some of the properties from [42, chapter 23.1]:

(a) For a fixed time t, the mapping $x \mapsto u(x,t)$ is an element of a Sobolev space V and we denote this function by $u(t)$.
(b) If we now vary $t \in [0,T]$, we obtain the function $t \mapsto u(t)$, where $u(t)$ is the function defined in a). In this way, we obtain a function which has values in the Banach space V, in contrast to the real-valued function $(x,t) \mapsto u(x,t)$.

Multiplication with a test function $v \in V$ and integration over D yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)_H + a(u(t),v) = (f(t),v)_H \quad \text{in } (0,T) \text{ for all } v \in V, \quad u(0) = u_0 \in H \tag{1.3}$$

with

$$\begin{aligned} a(w,v) &:= \int_D \sum_{i=1}^N \partial_{x_i} w_i(x)\, \partial_{x_i} v(x)\,\mathrm{d}x, \\ (f(t),v)_H &:= \int_D f(x,t)\, v(x)\,\mathrm{d}x. \end{aligned}$$

This variational formulation shows the necessity to use two function spaces V and H:

(c) The time-derivative yields the space H, whereas V is obtained from the spatial derivative $-\Delta$ and the boundary condition. In this example for the heat equation, we have $V \subseteq H$ and V is dense in H. In this context, we use the *Gelfand* or *evolution triple*

$$V \subseteq H \subseteq V^*.$$

(d) The time-derivative $\dot{u} := \frac{\mathrm{d}}{\mathrm{d}t}$ is interpreted as a generalized derivative, i.e. the variational equation (1.3) has to be satisfied only for almost all $t \in (0,T)$.
(e) These considerations result in the choice

$$u \in W_2^1(0,T;V,H) = H^1(0,T;V,H) := \left\{u \in L^2(0,T;V), \dot{u} \in L^2(0,T;V^*)\right\}$$

as the state space for (1.3).

For further reading on Lebesgue-Bochner spaces and related analytic results, we also refer to [24].

Mathematical frameworks, particularly suited for parameter identification for partial differential equations, have been presented in [19, 25, 29]. In [25, 29], the focus is on time-depentent PDEs and Lebesgue-Bochner spaces are used to formulate the respective mathematical problems.

Outline. Section 2 provides the reader with all necessary preliminaries about Lebesgue-Bochner spaces. We particularly discuss criteria for relatively compact subsets in such spaces. In section 3 we develop the aforementioned concepts for pointwise and uniform ill-posedness of time-dependent inverse problems. As an example we consider time-dependent observations of a compact operator which applies, e.g. to dynamic CT. Corresponding to these concepts, we construct two types of regularization concepts in section 4. For both methodologies we present examples: variational tracking, classical variational techniques and Kaczmarz-based regularization. The article ends with an outlook to future research in the field.

2. Some preliminaries about Lebesgue-Bochner spaces

We start with a short introduction to the theory of Lebesgue-Bochner spaces (see also [24, 42]). We use the notation $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and let $1 \leqslant p, p^* < \infty$ with $\frac{1}{p} + \frac{1}{p^*} = 1$. If not otherwise specified, let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$. Its dual is denoted by $\mathcal{X}^*$. Furthermore, let (Ω,μ) denote a finite measure space with measure μ.

Recall that a function $u : \Omega \to \mathcal{X}$ is called *strongly μ-measurable*, if there is a sequence $\{u_n : \Omega \to \mathcal{X}\}$ of μ-simple functions such that u is its pointwise limit, i.e. $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \in \Omega$ a.e.

For a finite measure space (Ω,μ) with a measure μ we define the space

$$L^p(\Omega;\mathcal{X}) := \{u : \Omega \to \mathcal{X} \,|\, u \,\text{is strongly}\, \mu - \text{measurable and}\, \|u\|_{p,\mathcal{X}} < \infty\},$$

where

$$\|u\|_{p,\mathcal{X}} := \left(\int_\Omega \|u\|_{\mathcal{X}}^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}. \tag{2.1}$$

Note that we identify functions that are equal μ-almost everywhere in Ω. The space $L^p(\Omega;\mathcal{X})$ is called a *Lebesgue-Bochner space* or just *Bochner space* and is a Banach space if equipped with the norm $\|\cdot\|_{p,\mathcal{X}}$. For the special case $\mathcal{X} = \mathbb{R}$ we briefly write $L^p(\Omega)$.

Particularly in applications, the measure space is often given as the time interval $\Omega = [0,T] \subseteq \mathbb{R}$, $T > 0$, with the Lebesgue measure. The respective Bochner spaces

$$L^p(0,T;\mathcal{X}) := \{u : (0,T) \to \mathcal{X} \,|\, u \text{is strongly} \, \mu - \text{measurable and} \|u\|_{p,\mathcal{X}} < \infty\} \tag{2.2}$$

play an important role in the theory of nonlinear operator equations or evolution equations. We thus set $\Omega := [0,T]$ from now on.

The following statement is taken from [42].

Proposition 2.1. *Let $\widetilde{\mathcal{X}}$ be a second Banach space over $\mathbb{K}$.*

(i) The space $C^m([0,T];\mathcal{X})$, $m = 0,1,\ldots$, of all continuous functions $u : [0,T] \to \mathcal{X}$ with continuous derivatives up to order m on $[0,T]$ with the norm

$$\|u\| := \sum_{i=0}^{m} \max_{t \in [0,T]} |D^{(i)} u(t)|,$$

where $D^{(i)}$ denotes the ith derivative, is a Banach space over $\mathbb{K}$.

(ii) The set of all step functions $\sigma : [0,T] \to \mathcal{X}$ is dense in $L^p(0,T;\mathcal{X})$.

(iii) The space $C([0,T];\mathcal{X})$ is dense in $L^p(0,T;\mathcal{X})$ and the respective embedding is continuous.

(iv) $L^p(0,T;\mathcal{X})$ is
- *separable, if $\mathcal{X}$ is separable,*
- *uniformly convex, if $1 < p < \infty$ and $\mathcal{X}$ is uniformly convex.*

(v) If $X \hookrightarrow \widetilde{\mathcal{X}}$ is continuous, then

$$L^q(0,T;\mathcal{X}) \hookrightarrow L^r\left(0,T;\widetilde{\mathcal{X}}\right), \quad 1 \leqslant q \leqslant r \leqslant \infty,$$

is also continuous.

The Hölder inequality transfers from Lebesgue to Bochner spaces and is used to derive the dual space of a Bochner space, see, e.g. [42]. For $1 < p < \infty$ and $u \in L^p(0,T;\mathcal{X})$, $v \in L^{p^*}(0,T;\mathcal{X}^*)$, the Hölder inequality reads as

$$\int_0^T \left|\langle v(t), u(t)\rangle_{\mathcal{X}^* \times \mathcal{X}}\right| \mathrm{d}t \leqslant \left(\int_0^T \|v(t)\|_{\mathcal{X}^*}^{p^*} \mathrm{d}t\right) \cdot \left(\int_0^T \|u(t)\|_{\mathcal{X}}^{p} \mathrm{d}t\right)$$

and we have

$$(L^p(0,T;\mathcal{X}))^* = L^{p^*}(0,T;\mathcal{X}^*)$$

if $\mathcal{X}$ fulfils the Radon–Nikodym property (which is the case if, e.g. $\mathcal{X}$ is reflexive and separable, see [24]).

Regarding ill-posed problems, compact operators are of special interest since their inverse is only continuous if the operator has finite dimensional range. This is why the characterization of relatively compact sets in Lebesgue-Bochner spaces is very important. Recall that an operator $T : \mathcal{X} \to \mathcal{Y}$ between Banach spaces is compact, if it maps compact subsets in $\mathcal{X}$ to relatively compact subsets of $\mathcal{Y}$. Relatively compact subsets in Lebesgue-Bochner spaces have been characterized by Diaz and Mayoral [10] as well as recently in the two articles [40, 41] by van Nerveen. To present the main result from [41] we need some further definitions.

A set $V \subset L^p(0,T;\mathcal{X})$ is called *uniformly L^p-integrable* if

$$\lim_{r \to \infty} \sup_{f \in V} \|\chi_{\|f(t)\| > r} f\|_p = 0, \tag{2.3}$$

where $\chi_{\|f(t)\| > r}$ is the characteristic function of the set $\{t : \|f(t)\| > r\}$. The set V is called *uniformly tight*, if for all $\varepsilon > 0$ there exists a compact set $\mathcal{K} \subset \mathcal{X}$ such that

$$\sup_{f \in V} \mu(\{t : f(t) \notin \mathcal{K}\}) \leqslant \varepsilon, \tag{2.4}$$

and it is called *scalarly relatively compact* if for all $x^* \in \mathcal{X}^*$ the set $\{t \mapsto \langle f(t), x^* \rangle : f \in V\}$ is relatively compact in $L^p([0,T])$.

Proposition 2.2. *([41, theorem 1]). Let $1 \leqslant p < \infty$. A subset $V \subset L^p(0,T;\mathcal{X})$ is relatively compact if and only if it is uniformly L^p-integrable, uniformly tight, and scalarly relatively compact.*

Bounded sequences $\{f_n\} \subset L^p(0,T;\mathcal{X})$ fulfill the first two conditions in proposition 2.2:

Lemma 2.3. *If $\{f_n\} \subset L^p(0,T;\mathcal{X})$ is a bounded sequence, then $\{f_n\}$ is uniformly L^p-integrable and uniformly tight.*

Proof. See appendix. □

Lemma 2.3 is important especially to prove compactness of operators between spaces as $L^p(0,T;\mathcal{X})$. Another characterization of relatively compact sets in $L^p(0,T;\mathcal{X})$, where μ is the Lebesgue measure, is given in [11, 38]. For $z \in (0,T)$ and $f \in L^p(0,T;\mathcal{X})$ we define the translation operator

$$\tau_z : [-z, T-z] \to \mathcal{X}, \qquad (\tau_z f)(t) := f(t+z).$$

Theorem 2.4. *([38, theorem 1]). Let $p \in [1,+\infty)$ and $V \subset L^p(0,T;\mathcal{X})$. Then V is relatively compact, if and only if*

(a) for any subset $(a,b) \subset [0,T]$ the set $\left\{ \int_a^b f(t)\,\mathrm{d}(t) : f \in V \right\}$ is relatively compact in $\mathcal{X}$,
(b) for z with $0 \leqslant z < T$ it holds

$$\sup_{f \in V} \|\tau_z f - f\|_{L^p([-z,T-z],\mathcal{X})} = 0 \qquad \textit{for } z \to 0.$$

Theorem 2.4 is an analogue of the well-known Arzelà–Ascoli Theorem, which characterizes relatively compact sets in $\mathcal{C}([0,T],\mathcal{X})$. Note that $p = +\infty$ is not possible: for example, the set $\{f\}$ for a discontinuous $f \in L^\infty(0,T;\mathcal{X})$ is compact, but condition (b) of theorem 2.4 is not satisfied. As a consequence from lemma 2.3 and theorem 2.4 we immediately get

Corollary 2.5. *A bounded sequence $\{f_n\} \subset L^p(0,T;\mathcal{X})$ is relatively compact in $L^p(0,T;\mathcal{X})$, if for z with $0 \leqslant z < T$ and $x^* \in \mathcal{X}^*$ it holds*

$$\sup_{n \in \mathbb{N}} \|\tau_z g_n - g_n\|_{L^p([-z,T-z])} = 0 \qquad \textit{for } z \to 0, \tag{2.5}$$

where $g_n(t) := \langle f_n(t), x^* \rangle$.

Since parameter identification problems for partial differential equations play a crucial role in view of many real-world applications, compact embedding theorems for Sobolev spaces are of utmost importance. For completeness we recapitulate some of them to finish this section.

Theorem 2.6 (The Rellich–Kondrachov theorem). *The Sobolev space embedding $W^{m,p}(D) \hookrightarrow L^q(D)$, D a bounded domain in $\mathbb{R}^n$, is compact, if any of these conditions holds true:*

(a) $n - mp > 0$ *and* $1 \leqslant q < np/(n - mp)$*,*
(b) $n = mp$ *and* $1 \leqslant q < +\infty$*,*
(c) $mp > n$ *and* $1 \leqslant q \leqslant +\infty$*.*

3. Ill-posedness in Lebesgue-Bochner spaces

We at first introduce a setting in Lebesgue-Bochner spaces for time-dependent inverse problems. Consider Banach spaces $\mathcal{X}$, $\mathcal{Y}$, and an operator equation

$$F(\vartheta) = y, \quad F : \mathcal{D}(F) \subseteq \mathbb{X} \to \mathbb{Y}, \tag{3.1}$$

where we set

$$\mathbb{X} := L^p(0, T; \mathcal{X}), \quad \mathbb{Y} := L^q(0, T; \mathcal{Y})$$

for any $1 \leqslant p, q < \infty$. We assume that only noisy data, denoted by y^δ, are available with a noise level $\delta > 0$, fulfilling

$$\left\| y - y^\delta \right\|_{\mathbb{Y}} \leqslant \delta.$$

This is a fairly general mathematical model for time-depending problems, since in many applications, the Banach space $\mathcal{X}$ is a suitable function space such as $L^r(D)$ or $H^1(D)$ on some open subset $D \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$. As an example, weak solutions of hyperbolic and parabolic equations often are elements of spaces such as $L^p(0, T; H^1(D))$ or subspaces thereof. It is important to emphasize the difference in the physical meaning of the temporal variable $t \in [0, T]$ and spatial variables $x = (x_1, \ldots, x_N)^T \in D$, on which both the source function ϑ as well as the data y may depend. The principle of *causality* describes the nature of time in contrast to space: it is usually possible to move freely in space, but one can only advance in time. In the context of evolution equations, which describe the evolution of a system in time, Bochner spaces allow us to encode the specific role of time: At a fixed time t_0, the system is described by a function $u(t_0) \in \mathcal{X}$ where we can encode, e.g. regularity properties with respect to the spatial variable.

The question that arises at this point is how the nature of the temporal variable can be reflected in the notion of ill-posedness in a suitable manner. To this end, we introduce two concepts of ill-posedness with respect to time, beginning with the one that translates directly from the standard definition for ill-posedness of nonlinear inverse problems.

Definition 3.1. The inverse problem (3.1) is called *uniformly (locally) ill-posed* in ϑ^+, if for each $\rho > 0$ there is a sequence $\{\vartheta_k^{(\rho)}\}_{k \in \mathbb{N}} \subseteq B_\rho(\vartheta^+) \cap \mathcal{D}(F)$ with

$$\vartheta_k^{(\rho)} \nrightarrow \vartheta^+, \text{ but } F\left(\vartheta_k^{(\rho)}\right) \to F\left(\vartheta^+\right)$$

for $k \to \infty$, i.e.

$$\lim_{k \to \infty} \left\| F\left(\vartheta_k^{(\rho)}\right) - F\left(\vartheta^+\right) \right\|_{\mathbb{Y}}^q = \lim_{k \to \infty} \int_0^T \left\| F\left(\vartheta_k^{(\rho)}(t)\right) - F\left(\vartheta^+(t)\right) \right\|_{\mathcal{Y}}^q \mathrm{d}t = 0.$$

Definition 3.1 corresponds to the classical concept of local ill-posedness for nonlinear inverse problems in Hilbert and Banach spaces transferred to Lebesgue-Bochner spaces. Note that the statement $\vartheta_k^{(\rho)} \nrightarrow \vartheta^+$ for $k \to \infty$ translates to

$$\int_0^T \left\| \vartheta_k^{(\rho)}(t) - \vartheta^+(t) \right\|_{\mathcal{X}}^p \mathrm{d}t \nrightarrow 0,$$

which means that there is a subset $I \subseteq [0,T]$ with Lebesgue measure $\mu(I) > 0$ such that

$$\int_I \left\| \vartheta_k^{(\rho)}(t) - \vartheta^+(t) \right\|_{\mathcal{X}}^p \mathrm{d}t > 0, \tag{3.2}$$

and thus

$$\left\| \vartheta_k^{(\rho)} - \vartheta^+ \right\|_{\mathcal{X}} > 0$$

on I.

However, the above definition can also be applied point-wise in time, if the respective DIP allows for point-evaluations in time:

Definition 3.2. The inverse problem (3.1) with $F = F_t$ for all t is called *temporally (locally) ill-posed*, if there is a set of positive measure $\Sigma \subset (0,T)$ such that for almost every $t_0 \in \Sigma$ the operator equation

$$F_{t_0}\left(\tilde{\vartheta}\right) := F(\vartheta(t_0)) = \tilde{y}, \quad F_{t_0} : \mathcal{D}(F_{t_0}) \subseteq \mathcal{X} \to \mathcal{Y}$$

is (locally) ill-posed. Here,

$$\mathcal{D}(F_{t_0}) := \{\vartheta(t_0) \in \mathcal{X} : \vartheta \in (C([0,T];\mathcal{X}) \cap \mathcal{D}(F))\}$$

and $\tilde{y} \in F_{t_0}(\mathcal{D}(F_{t_0}))$. This means that for all $\rho > 0$ there is a sequence $\{\tilde{\vartheta}_k^{(\rho)}\}_{k\in\mathbb{N}} \subseteq B_\rho^{\mathcal{X}}(\tilde{\vartheta}^+) \cap \mathcal{D}(F_{t_0})$ with

$$\lim_{k\to\infty} \|\tilde{\vartheta}_k^{(\rho)} - \tilde{\vartheta}^+\|_{\mathcal{X}} \neq 0, \text{ but } \lim_{k\to\infty} \|f_{t_0}\left(\tilde{\vartheta}_k^{(\rho)}\right) - F_{t_0}\left(\tilde{\vartheta}^+\right)\|_{\mathcal{Y}} = 0.$$

Let us mention that an obvious example for a locally ill-posed problem on Bochner spaces is the dynamically sampled Radon transform discussed in the introduction. If the sampling operator $S(t)$ does not map to a finite dimensional space, the compactness of the Radon transform immediately implies the ill-posedness of the concatenation.

Remark 3.3. The regularity of continuity in $[0,T]$ in definition 3.2 can be weakened to functionals $\vartheta(t_0)$ being bounded in $\mathcal{X}$ for all $t_0 \in [0,T]$.

3.1. An example: time-dependent observations of compact operators on $\mathcal{X}$

We consider the linear operator equation

$$F(\vartheta) = y, \tag{3.3}$$

where $\vartheta \in \mathbb{X} := L^p(0,T;\mathcal{X})$ and $y \in \mathbb{Y} := L^q(0,T;\mathcal{Y})$ and the spaces $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. We assume that the operator F has a representation

$$F[\vartheta(t)] = S(t)K[\vartheta(t)] \tag{3.4}$$

with a compact linear operator

$$K : \mathcal{X} \to \mathcal{Y}$$

and operators $S(t) \in \mathcal{L}(\mathcal{Y})$ that are linear and bounded for every $t \in [0,T]$. We furthermore suppose that the family of operator norms $\{\|S(t)\|\}$ is uniformly bounded,

$$\sup_{t\in(0,T)} \|S(t)\| \leqslant c_S, \tag{3.5}$$

for a constant $c_S > 0$. A setting such as (3.4) is very important regarding practical applications. For instance in dynamic CT, the function $\vartheta \in \mathbb{X}$ represents a moving object, the operator K is the Radon transform, and $S(t)$ models the time-depending measurement process, see (1.1).

Note that the operator $F_{t_0} : \mathcal{X} \to \mathcal{Y},\ F_{t_0}(\vartheta(t_0)) = S(t_0)K[\vartheta(t_0)]$ is compact for each fixed $t_0 \in [0,T]$ and $\vartheta \in C([0,T];\mathcal{X})$ since it is a composition of a compact and a bounded operator. This directly allows us to formulate the following proposition:

Proposition 3.4. *If* $\dim(F_{t_0}(\mathcal{X})) = \infty$ *for all* $t_0 \in [0,T]$*, then the inverse problem (3.3) is temporally ill-posed for all* $t_0 \in [0,T]$.

As a specific example we consider the inverse problem of DLM and show its uniform ill-posedness in an appropriate Lebesgue-Bochner setting. At first we need a decent mathematical setup, which we recapitulate from [5]. We state Cauchy's equation of motion for a linear elastic body $D \subset \mathbb{R}^3$,

$$\rho(x)\ddot{u}(t,x) - \nabla \cdot (\mathbb{C}(x) : \varepsilon(u))(t,x) = f(t,x), \qquad (t,x) \in [0,T] \times D \tag{3.6}$$

with the fourth order elasticity tensor $\mathbb{C} \in H^1(\Omega, \mathbb{R}^{3\times3\times3\times3})$, the mass density $\rho \in L^\infty(D)$, the linearized Green strain tensor $\varepsilon(u) = (\nabla u + \nabla u^\top)/2$, and the vector field $f(t,x)$ representing the dynamic load acting on x at time t. Under the additional assumptions that

$$0 < \rho_{\min} < \rho(x) < \rho_{\max} < \infty \tag{3.7}$$

and

$$\sup_{x\in\overline{\Omega}} (X, \mathbb{C}(x) : X)_F \geqslant \alpha \|X\|_F^2 \qquad \text{for all } X \in \mathbb{R}^{3\times3} \tag{3.8}$$

with $\alpha > 0$ we get the following existence and uniqueness result for a weak solution of an initial boundary value problem associated with (3.6).

Proposition 3.5. *Let* D *be a bounded domain with Lipschitz continuous boundary,* $u_0 \in H^1(D)^3$, $u_1 \in L^2(D)^3$, $f \in L^2(0,T;L^2(D)^3)$ *and the assumptions (3.7) and (3.8) hold true. Then, the elastic wave equation (3.6) equipped with the initial and boundary values*

$$[\mathbb{C}(x) : \varepsilon(u)] \cdot \nu = 0 \qquad \text{on } [0,T] \times \partial D \tag{3.9}$$

$$u(0,x) = u_0(x) \qquad \text{in } D \tag{3.10}$$

$$\dot{u}(0,x) = u_1(x) \qquad \text{in } D \tag{3.11}$$

has a unique weak solution $u \in L^2(0,T;H^1(D)^3)$ *with* $\dot{u} \in L^2(0,T;L^2(D)^3)$*. Moreover we even have that* $u \in \mathcal{C}(0,T;H^1(D)^3)$ *with* $\dot{u} \in \mathcal{C}(0,T;L^2(D)^3)$ *and for* u_0*,* u_1 *fixed the solution operator* $L : L^2(0,T;L^2(D)^3) \to L^2(0,T;H^1(D)^3)$, $L(f) := u$ *is continuous. If* $u_0 = u_1 = 0$*, then* L *even is a linear operator.*

The Neumann condition (3.9), where ν is the outer unit normal vector field, means that the structure D is traction-free at the boundary. A proof that is based on results from Lions [34] and the second inequality of Korn can be found in [5]. We introduce the notations $V := H^1(D)^3$, $H := L^2(D)^3$ and

$$W^{1,q,r}(0,T;V,H) := \{u \in L^q(0,T;V) : \dot{u} \in L^r(0,T;H)\}$$

for $1 \leqslant q, r \leqslant +\infty$.

A key ingredient to prove the uniform ill-posedness of the DLM-problem is the Lemma of Aubin–Lions, see [33].

Theorem 3.6 (Lemma of Aubin–Lions). *If $q < +\infty$, then the embedding*

$$W^{1,q,r}(0,T;V,H) \hookrightarrow L^q(0,T;H)$$

is compact.

In theorem 3.6 we interpret H as a subset of the dual space V^*.

Let functions $\chi_j \in H^{1/2}(\partial D)^3$, $j =, 1, \ldots, J$, with small supports on ∂D be given which define the sensor characteristics, e.g. regarding size, sensitivity, etc. Then, a mathematical model for DLM is represented by the forward operator

$$F[f] := S(t)\gamma\iota L[f], \tag{3.12}$$

where $L : L^2(0,T;H) \to W^{1,2,2}(0,T;V,H)$, $L[f] := u$ maps the dynamic load to the unique weak solution of the IBVP (3.6), (3.9)–(3.11), $\iota : W^{1,2,2}(0,T;V,H) \hookrightarrow L^2(0,T;H)$ is the embedding being compact due to the Lemma of Aubin–Lions, $\gamma : L^2(0,T;H) \to L^2(0,T;H^{-1/2}(\partial D)^3)$ is the trace operator, which is continuous, and

$$S(t)[g] := \int_{\partial D} \langle g(t,x), \chi_j(x)\rangle \, \mathrm{d}s_x, \qquad g \in L^2\left(0,T;H^{-1/2}(\partial D)^3\right), \, j =, 1 \ldots, J, \tag{3.13}$$

is the observation operator being linear and continuous as a mapping from

$$L^2\left(0,T;H^{-1/2}(\partial D)^3\right) \to L^2\left(0,T;\mathbb{R}^J\right).$$

In (3.13), $\langle \cdot,\cdot\rangle$ is to be understood as the dual pairing in $H^{-1/2}(\partial D)^3 \times H^{1/2}(\partial D)^3$. With the notations $\vartheta := f$, $K := \gamma\iota L$, $\mathbb{X} := L^2(0,T;H)$ and $\mathbb{Y} := L^2(0,T;\mathbb{R}^J)$ (i.e. $\mathcal{X} := H$, $\mathcal{Y} := \mathbb{R}^J$) we immediately obtain the following result.

Theorem 3.7. *The operator $F = S(t)K : \mathbb{X} \to \mathbb{Y}$ is compact.*

Proof. Since $\iota : W^{1,2,2}(0,T;V,H) \hookrightarrow L^2(0,T;H)$ is compact, we have that K is compact and so is F as a composition of a continuous and a compact operator. □

Corollary 3.8. *The inverse problem of DLM, i.e. $(F,\mathbb{X},\mathbb{Y})$, is uniformly ill-posed.*

Proof. This follows immediately from the compactness of F. □

If we choose a fixed time t_0 and neglect time-dependence, then the DLM problem can be characterized by the elliptic problem

$$-\nabla\cdot(\mathbb{C}(x) : \varepsilon(u))(x) = f(x), \qquad x \in D, \tag{3.14}$$

with traction-free boundary conditions

$$[\mathbb{C}(x) : \varepsilon(u)]\cdot\nu = 0 \qquad \text{on } \partial D. \tag{3.15}$$

The weak formulation of the Neumann problem (3.14) and (3.15) is given as

$$\int_D (\varepsilon(u), [\mathbb{C}(x) : \varepsilon(v)])_F \, \mathrm{d}x = \int_D f(x)\cdot v(x)\,\mathrm{d}x \tag{3.16}$$

for all $v \in V$. Using again (3.8), the Poincaré inequality as well as the Lax-Milgram theorem, we have that the equation $A(u) = f$ has a unique solution which depends continuously on f. Here, $A : V \to V^*$ is the operator induced by the symmetric bilinear form in (3.16), i.e.

$$A(u)[v] := \int_D (\varepsilon(u), [\mathbb{C}(x) : \varepsilon(v)])_F \, \mathrm{d}x, \quad v \in V.$$

If we consider the inverse problem of computing the source term f from full field data $u(x)$, $x \in D$, then we immediately get the following result.

Theorem 3.9. *The inverse problem $\widetilde{F}: H \to V$, $\widetilde{F}(f) := u$, where u is the unique weak solution of (3.14) and (3.15), is well-posed.*

Proof. The well-posedness follows from the fact that $\widetilde{F}$ is continuously invertible as outlined above. □

Of course, if we use the compact embedding $V \hookrightarrow H$, the trace operator $\gamma : L^2(D)^3 \to H^{-1/2}(\partial D)$ and a (static) observation operator similar to S, then we obtain again a linear, ill-posed (static) inverse problem.

The examples above lead to the following general statements.

Proposition 3.10. *Let $V_1 \subset \mathcal{X} \subset V_2$ be Banach spaces with compact embedding $V_1 \subset \mathcal{X}$ and continuous embedding $\mathcal{X} \subset V_2$. Furthermore, let $K : L^p(0,T;\mathcal{X}) \to W^{1,q,r}(0,T;V_1,V_2)$ with $1 \leqslant q < +\infty$, $1 \leqslant r \leqslant +\infty$, and $S(t) : \mathcal{X} \to \mathcal{Y}$ is a family of uniformly bounded operators in $\mathcal{L}(\mathcal{X},\mathcal{Y})$.*

Then $F : L^p(0,T;\mathcal{X}) \to L^q(0,T,\mathcal{Y})$, $F[\vartheta(t)] := S(t)K[\vartheta(t)]$ is compact and the dynamic inverse problem $(F, L^p(0,T;\mathcal{X}), L^q(0,T,\mathcal{Y}))$ is uniformly ill-posed. In case we have for any fixed $t = t_0$ that $K : \mathcal{X} \to V_1$ is continuous, then $\widetilde{F} : \mathcal{X} \to \mathcal{Y}$, $\widetilde{F}[\vartheta] := S(t_0)K[\vartheta]$, is compact and $(F, L^p(0,T;\mathcal{X}), L^q(0,T,\mathcal{Y}))$ is also temporally ill-posed. If for any t_0 the operator $S(t_0)K$ is continuously invertible, then the problem $(F, L^p(0,T;\mathcal{X}), L^q(0,T,\mathcal{Y}))$ is temporally well-posed.

Proof. The proof essentially relies on the compact embedding $V_1 \subset \mathcal{X}$ as well as on the Lemma of Aubin–Lions which states that the embedding $W^{1,q,r}(0,T;V_1,V_2) \hookrightarrow L^q(0,T;\mathcal{X})$ is compact. □

It is now interesting to consider situations where the operator equation (3.1) is temporally ill-posed but not uniformly ill-posed. This is important to show that these are in fact different concepts with each of it having a justification of its own. As long as K is compact, $S(t)$ is linear and bounded for all $t \in [0,T]$ and the $f_n = F\vartheta_n$ are strongly measurable, parts (b) and (c) of the proof will remain valid. Example 3.11 shows a situation where the condition of uniform L^q-integrability of $\{f_n\}$ fails.

Example 3.11. We assume that the parameter ϑ to be recovered does not depend on time, i.e. $\vartheta(t) = \vartheta \in \mathcal{X}$ and use the embedding $\iota : \mathcal{X} \hookrightarrow \mathbb{X} = L^p(0,T;\mathcal{X})$ which is defined by $x \mapsto f_x$ with $f_x(t) = x$. The range of this embedding consists of all functions in $\mathbb{X}$ that are constant in time and we write $\iota(X) =: L^p_c(0,T,\mathcal{X}) \subset L^p(0,T;\mathcal{X})$. The forward operator $F : \iota(X) \subset \mathbb{X} \to \mathbb{Y}$ is supposed to be given as

$$[F\vartheta](t) := S(t)K[\vartheta], \qquad \vartheta \in L^p_c(0,T;\mathcal{X}) \tag{3.17}$$

for a linear, compact operator $K : \mathcal{X} \to \mathcal{Y}$ and a family of linear, bounded mappings $\{S(t) : \mathcal{Y} \to \mathcal{Y} : t \in (0,T)\}$. Setting (3.17) represents the important situation that we have to reconstruct a parameter that has only a spatial variable from time-dependent data. Such a situation is, e.g. given in seismology, see [28], where wave speed and mass density are computed from the full waveform. As a simple example we define $S(t) = \frac{1}{t}I$ yielding $[F\vartheta] = \frac{1}{t}K[\vartheta]$ for $\vartheta \in L^p_c(0,T,\mathcal{X})$, $t \in (0,T)$, and compact K. Let $\{\vartheta_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p_c(0,T,\mathcal{X})$. Then, the sequence $\{f_n := F\vartheta_n\}_{n\in\mathbb{N}}$ is not uniformly L^q-integrable and hence $F : \iota(X) \subset \mathbb{X} \to \mathbb{Y}$ is not compact. This follows immediately from

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \int_{\{t:\|f_n(t)\|\geqslant k\}} \|f_n(t)\|^q \,\mathrm{d}t \geqslant \lim_{k\to\infty} \int_{\{t:\|f_{n^*}(t)\|\geqslant k\}} \|f_{n^*}(t)\|^q \,\mathrm{d}t$$

$$= \lim_{k\to\infty} \|K[\vartheta_{n^*}]\|^q \int_0^{\|K[\vartheta_{n^*}]\|/k} \frac{1}{t^q} \,\mathrm{d}t = +\infty$$

for some $n^* \in \mathbb{N}$ fixed. This means that $F\vartheta = y$ is not uniformly ill-posed provided that K has an inverse which is bounded on $K(\iota(X)) \subset \mathbb{Y}$. But obviously, for fixed $t \in (0,T)$, $S(t)K[\vartheta]$ is compact as an operator from $\mathcal{X}$ to $\mathcal{Y}$ and hence $F\vartheta = y$ is temporally ill-posed.

As a consequence of the considerations in example 3.11 we obtain

Corollary 3.12. *The subset $\iota(\mathcal{X}) \subset \mathbb{X}$ is not compact.*

A simple application of definition (2.3) furthermore shows that $\iota(X)$ is not relatively compact in $\mathbb{X}$.

Let us provide another example of a temporally ill-posed DIP based on imaging time-dependent sources in a wall with Compton cameras.

Example 3.13. We assume that the source is confined in the wall represented by the planar region $P_0 = \{0\} \times D$ with $D \subset \mathbb{R}^2$. We image with a line of Compton detectors in distance d from the wall, located at $x_1 = 0$ and $x_2 = a(t)$, i.e. we move the line horizontally keeping it parallel to the wall. By registering coincidences in neighbouring detectors and their energies, we can use the Compton formula to determine the angle photons arrive from, i.e. we can localize them in specific cones. Assuming that the motion of the detectors and the temporal change of the unknown $\vartheta(t)$ happen on slower time scales than the detection of photons, we can effectively assume to measure all integrals

$$y(x_3,\phi,t) = \int_{P_0 \cap C_\phi((0,a(t),x_3)} \vartheta(x_2,x_3,t)\, \mathrm{d}\sigma(x_2,x_3), \tag{3.18}$$

where $C_\phi(z)$ denotes the cone starting at $z \in \mathbb{R}^3$ with opening angle ϕ and axis orthogonal to the (x_2,x_3)-plane. If b is the length of the line of detectors we can use a coordinate system such that $x_3 \in (0,b)$ without restriction of generality.

Now we can define the forward operator

$$F_t : L^p(D) \to L^p((0,b) \times [0,\pi)), \quad \vartheta \mapsto y(\cdot,t)$$

where y is given by (3.18). For $p > 1$ it is easy to see that F_t has no continuous inverse for any t. For simplicity assume that there exists a rectangle R_ϵ of width ϵ and length L that is inside D for ϵ sufficiently small. Now define $\vartheta_\epsilon(t)$ zero outside this rectangle and equal to $(\epsilon L)^{-1/p}$, i.e. $\vartheta_\epsilon(t)$ has norm one in $L^p(D)$. Since in our measurement setup the curvature of the cone sections is strictly bounded away from zero, there exists a constant A such that the length of the intersection of the cone-section with R_ϵ is bounded by $A\epsilon$. Hence, we see that $y_\epsilon = F_t\vartheta$ satisfies

$$|y_\epsilon(x_3,\phi,t)| \leqslant A\epsilon(\epsilon L)^{-1/p} = A\epsilon^{(p-1)/p}L^{-1/p}$$

for almost all (x_3,ϕ). This implies that $F_t\vartheta_\epsilon(t) \to 0$ even in the supremum norm and thus also in $L^p((0,b) \times [0,\pi))$.

4. Regularization of time-dependent inverse problems

In this section we define problem-adapted classes of regularization methods for DIPs that address the two different sorts of ill-posedness. Again we consider the inverse problem (3.1) and aim for a stable solution of

$$F(\vartheta) = y^\delta, \tag{4.1}$$

where $y^\delta \in \mathbb{Y}$ denotes a noise contaminated version of the exact data y, i.e.

$$\|y - y^\delta\|_{\mathbb{Y}} < \delta \tag{4.2}$$

for a (small) positive noise level $\delta > 0$. We assume that

$$y^{\delta}(t) \in F_t(\mathcal{D}(F_t)) \qquad \text{for all } t \in [0,T] \tag{4.3}$$

and that there exists a solution ϑ^+ of (3.1) with

$$\vartheta^+ \in (C([0,T];\mathcal{X}) \cap \mathcal{D}(F)), \tag{4.4}$$

which implies that also $y(t)$ is well-defined and

$$y(t) \in F_t(\mathcal{D}(F_t)) \qquad \text{for all } t \in [0,T].$$

We note that condition (4.3) is not an essential restriction with respect to applications. Usually data are acquired for discrete time instances $t_k \in [0,T]$, $k = 0,1,\ldots$, only. Hence, data $y^{\delta}(t)$ that are continuous in time can be obtained by simple interpolation, e.g. using piecewise linear spline functions. Condition (4.4) can be justified by the fact that $C([0,T];\mathcal{X}) \cap \mathcal{D}(F)$ is dense in $\mathcal{D}(F)$ and the fact that in applications the temporal development of the exact solution mostly is continuous in time (at least this is not an essential confinement).

Definition 4.1. A *temporal (pointwise) regularization method* for (4.1) is a family of mappings $\widetilde{R} : [0,T] \times \mathcal{Y} \times [0,+\infty) \to \mathcal{X}$ that satisfies the following condition: For $x^{\delta}_{t,\alpha} := \widetilde{R}(t, y^{\delta}(t), \alpha)$ there exists a *parameter choice* $\alpha : [0,T] \times [0,+\infty) \times \mathcal{Y} \to [0,\bar{\alpha})$, $0 < \bar{\alpha} \leqslant +\infty$, such that

$$\lim_{\delta \to 0} \|\vartheta^+(t) - x^{\delta}_{t,\alpha(t,\delta,y^{\delta}(t))}\|_{\mathcal{X}} = 0 \qquad \text{for all } t \in [0,T].$$

Definition 4.1 reflects the fact that dynamic inverse problems can be solved by defining a partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0,T]$ and using a stationary regularization method for $F_{t_k} : \mathcal{D}(F_{t_k}) \subset \mathcal{X} \to \mathcal{Y}$ for each t_k. This procedure is called *tracking*.

Remark 4.2. Obviously a temporal regularization yields an element $x^{\delta}_{t,\alpha} \in \mathcal{X}$ for $t \in [0,T]$ fixed. A stable regularization of (4.1), however, demands for a solution in the Lebesgue-Bochner space $\mathbb{X}$. But this can easily we achieved by simple interpolation techniques. Again assume that we have $x^{\delta}_{t_k,\alpha} \in \mathcal{X}$ given for $t_k \in \Delta$, $k = 0,\ldots,N$. Define $I\{x^{\delta}_{t_k,\alpha}\}$ as the piecewise constant interpolation which is defined as

$$I\{x^{\delta}_{t_k,\alpha}\}(t) = x^{\delta}_{t_k,\alpha} \qquad \text{for } t \in [t_k, t_{k+1}), \qquad k = 0,\ldots,N-1.$$

Because of

$$\begin{aligned}\int_0^T \|I\{x^{\delta}_{t_k,\alpha}\}(t)\|^p_{\mathcal{X}}\,\mathrm{d}t &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k-1}} \|x^{\delta}_{t_k,\alpha}\|^p_{\mathcal{X}}\,\mathrm{d}t \\ &\leqslant T \max\left\{\|x^{\delta}_{t_k,\alpha}\|^p_{\mathcal{X}} : k = 0,\ldots,N-1\right\} < +\infty\end{aligned}$$

we see that $I\{x^{\delta}_{t_k,\alpha}\} \in \mathbb{X}$. Of course other interpolation methods, such as piecewise linear interpolation, can be used to obtain solutions that are smooth with respect to t. We emphasize that we see temporal regularization not as a regularization method in the Lebesgue-Bochner spaces $\mathbb{X}$, $\mathbb{Y}$, rather than as regularization in $\mathcal{X}$, $\mathcal{Y}$ for fixed $t \in [0,T]$, which is the core idea of tracking methods.

In contrast to tracking, problem (3.1) can also be solved uniformly in t.

Definition 4.3. A *full (uniform) regularization method* for (4.1) is a family of mappings $R : \mathbb{Y} \times [0,+\infty) \to \mathbb{X}$ that satisfies the following condition: For $x^{\delta}_{\alpha} := R(y^{\delta}, \alpha)$ there exists a *parameter choice* $\alpha : \mathbb{Y} \times [0,+\infty) \to [0,\bar{\alpha})$, $0 < \bar{\alpha} \leqslant +\infty$, such that

$$\lim_{\delta \to 0} \|\vartheta^+ - x^{\delta}_{\alpha(y^{\delta},\alpha)}\|_{\mathbb{X}} = 0.$$

4.1. Example 1: variational tracking by Tikhonov functionals

We define the functional $\mathcal{J}^{\delta}_{t,\alpha} : \mathcal{D}(F_t) \subset \mathcal{X} \to \mathbb{R}$ by

$$\mathcal{J}^{\delta}_{t,\alpha}(x) := S_t\left(F_t(x), y^{\delta}(t)\right) + \alpha_t Q_t(x), \qquad t \in [0,T], \quad x \in \mathcal{X}, \tag{4.5}$$

where $S_t : \mathcal{Y} \times \mathcal{Y} \to [0,+\infty)$ is a functional defining the data fitting term, $Q_t : \mathcal{X} \to [0,+\infty]$ is a penalty term to stabilize the reconstruction process and $\alpha_t > 0$ acts as the regularization parameter. Note that we allow for the data fitting term, the penalty term, and the regularization parameter to depend on t.

Based on the well-established theory for regularization methods in Banach spaces, see [4, 7, 21, 37], we formulate assumptions on F_t, $\mathcal{D}(F_t)$, and Q_t that the minimization of (4.5) yields a temporal regularization for the special case that

$$S_t(\tilde{y}_1, \tilde{y}_2) := \frac{1}{r}\|\tilde{y}_1 - \tilde{y}_2\|^r_{\mathcal{Y}}, \qquad \tilde{y}_1, \tilde{y}_2 \in \mathcal{Y} \tag{4.6}$$

with $1 < r < \infty$.

Proposition 4.4. *Under the assumptions that $\mathcal{X}$, $\mathcal{Y}$ are reflexive Banach spaces, $C([0,T];\mathcal{X}) \cap \mathcal{D}(F)$ is dense in $\mathcal{D}(F)$, and that for fixed $t \in [0,T]$ we have that $F_t : \mathcal{D}(F_t) \subset \mathcal{X} \to \mathcal{Y}$ is weak-to-weak sequentially continuous, $Q_t : \mathcal{X} \to [0,+\infty]$ is proper, convex and lower semi-continuous, $\mathcal{D}(F_t) \cap \mathcal{D}(Q_t) \neq \emptyset$ and the level sets $\mathcal{M}_{Q_t} := \{x \in \mathcal{X} : Q_t \leqslant c\}$ are weakly sequentially pre-compact. Furthermore we choose an index function $\alpha_t : [0,\infty) \to [0,\infty)$, i.e. α_t is strictly increasing and continuous with $\alpha_t(0) = 0$, which has the asymptotic behavior*

$$\alpha_t(\delta) \to 0 \qquad \textit{and} \qquad \frac{\delta^r}{\alpha_t(\delta)} \to 0 \qquad \textit{as} \qquad \delta \to 0.$$

Then the Tikhonov functional (4.5) with data fitting term (4.6) has a minimizer $x^{\delta}_{t,\alpha} \in \mathcal{D}(F_t)$ and $\widetilde{R}(t, y^{\delta}(t), \alpha) := x^{\delta}_{t,\alpha}$ is a temporal regularization method in the sense that, if $\{\delta_n\} \subset (0,\infty)$ is a sequence with $\delta_n \to 0$ as $n \to \infty$, the sequence $\{x^{\delta_n}_{t,\alpha(\delta_n)}\}$ has a weakly converging subsequence whose weak limit is a Q_t-minimizing solution $\vartheta^+(t) \in \mathcal{X}$ of $F_t(\vartheta(t)) = y(t)$.

Since for fixed t the operator $F_t[\vartheta(t)] := S(t)K[\vartheta(t)]$ from (3.4) is linear and compact, we immediately get

Corollary 4.5. *If $C([0,T];\mathcal{X}) \cap \mathcal{D}(F)$ is dense in $\mathcal{D}(F)$ and Q_t satisfies the assumptions in proposition 4.4, then the Tikhonov regularization (4.5) and (4.6) yields a temporal regularization method for (3.4) in the sense of proposition 4.4.*

Remark 4.6. Variational tracking as in Proposition (4.4) naturally leads to *dynamic algorithms* according to Osipov *et al* [36], i.e. if two data sets y^{δ}_1, y^{δ}_2 coincide at a given time interval, $y^{\delta}_1(t) = y^{\delta}_2(t)$ for all $t \in [0,t_0]$ for given $t_0 \in (0,T]$, then the algorithm's output coincides on this time interval as well. In our setting it is quite obvious that under this assumption we have $\widetilde{R}(t, y^{\delta}_1(t), \alpha) = \widetilde{R}(t, y^{\delta}_2(t), \alpha)$ for all $t \in [0,t_0]$. The reason is that variational tracking just means to compute a temporal frame of stationary solutions. Dynamic algorithms in the sense of Osipov *et al* inherently show causality since, if $y^{\delta}_1(t) \neq y^{\delta}_2(t)$ for a $t > t_0$, this does not affect the output in the interval $[0,t_0]$.

4.2. Example 2: variational regularization on Lebesgue-Bochner spaces

Temporal regularization completely neglects topology, regularity and geometry of the corresponding time-space functional. To obtain a holistic regularization it is more convenient to develop regularization methods for (4.1) in $\mathbb{X}$ and $\mathbb{Y}$. One possibility is to use variational regularization techniques. To this end we define the Tikhonov functional $\mathcal{J}_\alpha^\delta : \mathcal{D}(F) \subset \mathbb{X} \to \mathbb{R}$ by

$$\mathcal{J}_\alpha^\delta(\vartheta) := S\left(F(\vartheta), y^\delta\right) + \alpha Q(\vartheta), \qquad \vartheta \in \mathbb{X}, \tag{4.7}$$

where, again, $S : \mathbb{Y} \times \mathbb{Y} \to [0, +\infty)$ is an error functional denoting the data fitting term and $Q : \mathbb{X} \to [0, +\infty]$ is a penalty term whose influence is controlled by the parameter $\alpha > 0$. Of course the most popular choice for the data fitting term is again a power of the norm residual

$$S(y_1, y_2) := \frac{1}{r} \|y_1 - y_2\|_{\mathbb{Y}}^r, \qquad y_1, y_2 \in \mathbb{Y} \tag{4.8}$$

with $1 < r < \infty$.

Accordingly, we can use established results on regularization theory (see again [4, 7, 21, 37]) to state the following result:

Proposition 4.7. *Let us assume that $\mathcal{X}$, $\mathcal{Y}$ are reflexive Banach spaces with $1 < p < \infty$. We assume that the forward operator $F : \mathcal{D}(F) \subset \mathbb{X} \to \mathbb{Y}$ is weak-to-weak sequentially continuous, $Q : \mathbb{X} \to [0, +\infty]$ is proper, convex and lower semi-continuous, $\mathcal{D}(F) \cap \mathcal{D}(Q) \neq \emptyset$ and the level sets $\mathcal{M}_Q := \{x \in \mathcal{X} : Q \leqslant c\}$ are weakly sequentially pre-compact. Furthermore we choose an index function $\alpha : [0, \infty) \to [0, \infty)$, i.e. α is strictly increasing and continuous with $\alpha(0) = 0$, which has the asymptotic behavior*

$$\alpha(\delta) \to 0 \qquad \text{and} \qquad \frac{\delta^r}{\alpha(\delta)} \to 0 \qquad \text{as} \qquad \delta \to 0. \tag{4.9}$$

Then the Tikhonov functional (4.7) with data fitting term (4.8) has a minimizer $x_\alpha^\delta \in \mathcal{D}(F)$ and $R(y^\delta, \alpha) := x_\alpha^\delta$ is a full (uniform) regularization method in the sense that, if $\{\delta_n\} \subset (0, \infty)$ is a sequence with $\delta_n \to 0$ as $n \to \infty$, then the sequence $\{x_{\alpha(\delta_n)}^{\delta_n}\}$ has a weakly converging subsequence whose weak limit is a Q-minimizing solution $\vartheta^+ \in \mathbb{X}$ of $F(\vartheta) = y$.

It can be shown that for the specific setting (3.4) and $Q(\vartheta)$ being a power of the norm in $\mathbb{X}$, the Tikhonov method (4.7) with data fitting term (4.8) yields a full regularization method for (4.1).

Theorem 4.8. *Let $\mathcal{X}$, $\mathcal{Y}$ be reflexive Banach spaces, $1 < p < \infty$, and $F : \mathbb{X} \to \mathbb{Y}$ be defined as in (3.4). Furthermore let the penalty term Q be defined as*

$$Q(\vartheta) := \frac{1}{q} \|\vartheta\|_{\mathbb{X}}^q, \qquad \vartheta \in \mathbb{X}.$$

Then $R : \mathbb{Y} \times (0, +\infty) \to \mathbb{X}$, where $R(y^\delta, \alpha) := x_\alpha^\delta$ is the minimizer of the Tikhonov functional (4.7) with data fitting term (4.8), and a priori parameter choice $\alpha(\delta)$ as in (4.9) is a full regularization method for (4.1) in the sense of proposition 4.7.

Proof. Since every bounded set in a Lebesgue-Bochner space $\mathbb{X}$ has a weakly converging subsequence, it immediately follows that the level sets $\mathcal{M}_c(Q)$ are weakly sequentially pre-compact.

It remains to show that F is weak-to-weak sequentially continuous. Let $\{\vartheta_n\} \subset \mathbb{X}$ be a sequence with $\vartheta_n \rightharpoonup \vartheta$ weakly as $n \to \infty$ to a limit $\vartheta \in \mathbb{X}$. Since $\mathbb{X}^* \cong L^{p^*}(0, T; \mathcal{X}^*)$ and $\mathbb{Y}^* \cong L^{q^*}(0, T, \mathcal{Y}^*)$ we have for every $y^* \in L^{q^*}(0, T; \mathcal{Y}^*)$ that

$$\langle y^*, F(\vartheta_n)\rangle_{\mathbb{Y}^*\times\mathbb{Y}} = \int_0^T \langle y^*(t), S(t)K[\vartheta_n(t)]\rangle_{\mathcal{Y}^*\times\mathcal{Y}}\,\mathrm{d}t$$
$$= \int_0^T \langle K^*S(t)^*y^*(t), \vartheta_n(t)\rangle_{\mathcal{X}^*\times\mathcal{X}}\,\mathrm{d}t$$

converges to

$$\int_0^T \langle K^*S(t)^*y^*(t), \vartheta(t)\rangle_{\mathcal{X}^*\times\mathcal{X}} = \langle y^*, F(\vartheta)\rangle_{\mathbb{Y}^*\times\mathbb{Y}} \tag{4.10}$$

as $n\to\infty$ due to the weak convergence of $\vartheta_n \rightharpoonup \vartheta$. Equation (4.10) proves that $K^*S(t)^*y^*(t)\in\mathbb{X}^*$ and hence can be represented by a function from $L^{p^*}(0,T;\mathcal{X}^*)$. This shows the weak sequential continuity of F and the assertion follows from proposition 4.7. □

4.3. Example 3: variational regularization of time variations

In the following let us provide some other examples of variational regularization naturally defined in Bochner spaces. For this purpose we first revisit example 3.11, where we considered the embedding of a time-independent ϑ into a Lebesgue-Bochner space by constant extension in time. An obvious candidate for a regularization functional is thus

$$Q(\vartheta) := \frac{1}{q}\int_0^T \|\vartheta\|_{\mathbb{X}}^q\,\mathrm{d}t + \frac{\lambda}{r}\int_0^T \|\vartheta\|_{\mathbb{X}'}^r\,\mathrm{d}t$$

with some space $\mathbb{X}'\supset\mathbb{X}$. Choosing λ large will lead to an optimal ϑ being close to a time-independent function. Moreover, the Aubin–Lions lemma can be used again for compactness properties.

Another canonical choice is to penalize a time variance or generalization thereof via

$$Q(\vartheta) := \frac{1}{q}\int_0^T \|\vartheta(t)-\overline{\vartheta}\|_{\mathbb{X}}^q\,\mathrm{d}t$$

with

$$\overline{\vartheta} = \frac{1}{T}\int_0^T \vartheta(t)\,\mathrm{d}t.$$

Again, the variational model in the Lebesgue-Bochner space $L^p(0,T;\mathbb{X})$ can be useful for the modelling of time-invariant solutions ϑ. A particular advantage could be the regularization of problems where the unknown is actually time-invariant, but there are additional perturbations arising. An example can be dust drifting through a sample during image acquisiton, which would actually add a dynamic component to an image ϑ.

4.4. Kaczmarz-based regularization for problems with static source

We furthermore want to emphasize that inverse problems with time-dependent data and/or time-dependent forward operator are often formulated in a semi-discrete setting

$$F_i(x) = y_i,\quad i = 0,\ldots,N-1, \tag{4.11}$$

where the indices i refer to discrete time points $t_i\in[0,T]$ or sections $I_i := [t_i, t_{i+1}]$ of the time interval $[0,T]$ at which the measurements are taken. The parameter that is to be identified is static in this setting. These problems are usually temporally or locally ill-posed and can be regularized using Kaczmarz's method, possibly in combination with other iterative methods such as the Landweber technique [35] or sequential subspace optimization [6]. We introduce three scenarios that have been addressed in the literature and which fit into this framework.

Example 4.9. We consider magnetic particle imaging. If the concentration c of magnetic particles inside a body D is static, i.e. independent of time, the problem of reconstructing c from measurements of the induced voltages v_k, $k = 1,\dots,K$, is formulated as

$$v_k(t) = S_k(t) K_k[c](t), \quad k = 1,\dots,K,$$

with

$$K_k[c](t) = \int_D c(x) s_k(x,t)\,\mathrm{d}x$$

and

$$S_k(t) = \int_0^T \widetilde{a}_k(t-\tau) \int_D c(x) s_k(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

The measurements are taken at time instances $t_i \in [0,T]$, $i = 1,\dots,N$. We thus have a time-dependent forward operator

$$F = (S_k K_k)_{k=1,\dots,K} : L^2(D) \to L^2\left(0,T;\mathbb{R}^3\right)$$

and time-dependent data $v \in L^2(0,T;\mathbb{R}^3)$.

Example 4.10. In [25, 35], initial boundary value problems of the form

$$\begin{aligned}
\partial_t u &= f(t,u(t),\vartheta) && \text{in } (0,T)\times D,\\
u &= 0 && \text{on } (0,T)\times \partial D,\\
u(0) &= u_0 && \text{in } \{0\}\times D,\\
y_i &= C_i u, && i = 1,\dots,N,
\end{aligned}$$

are considered, where the parameter $\vartheta \in \mathcal{X}$ is to be identified from measurements $y_i = C_i u := (Cu(t_i))$ at time instances $t_i \in (0,T)$ of the state function

$$u \in W^{1,p,p^*}(0,T;V,V^*) = \left\{ v \in L^p(0,T;V) : \partial_t v \in L^{p^*}(0,T;V^*) \right\} \subseteq C(0,T;H).$$

In particular, the parameter $\vartheta \in \mathcal{X}$ is assumed to be independent of time. In a Hilbert space setting ($p = p^* = 2$), we may choose, e.g. $\mathcal{X} = L^2(D)$.

In [35], the case $f(t,u(t),\vartheta) = \Delta u + \Phi(u) + \vartheta$ with a nonlinear function Φ is addressed. The respective initial boundary value problems arise in several applications. For instance, the choice $\Phi(u) = u(1-u^2)$ is related to superconductivity, see [35].

Example 4.11. It is also possible to include the time-dependence, for example a motion or a deformation, in the mathematical model while the parameter that is to be reconstructed is considered static. This is the case in dynamic CT, where a known deformation of the investigated object can be incorporated in the forward operator, yielding a Radon transform along curves, see, e.g. [13].

As already mentioned, the respective semi-discrete problems (4.11) can be solved iteratively, for example by a combination of Kaczmarz' method with the Landweber iteration, see, e.g. [16, 25, 35]. The iteration reads

$$\vartheta_{n+1} = \vartheta_n - \left(F'_{[n]}(\vartheta_n)\right)^* \left(F_{[n]}(\vartheta_n) - y_{[n]}\right)$$

and is stopped at n_* with $[n_*] = [0]$ when the adapted discrepancy principle

$$\left\| F_{[n]}(\vartheta_n) - y_{[n]} \right\| \leqslant \tau_{[n]} \delta_{[n]}$$

is fulfilled for all $n = n_* - N, \ldots, n_* - 1$. Instead of using only one search direction, it is also possible to use multiple search directions

$$\vartheta_{n+1} = \vartheta_n - \sum_{k \in I_n} t_{n,k} \left(F'_{[n]}(\vartheta_k) \right)^* \left(F_{[n]}(\vartheta_k) - y_{[n]} \right),$$

which has been proposed in [6]. In this sense, one loop through all time-instances corresponds to one full iteration, in which the static source ϑ is reconstructed.

5. Conclusion and outlook

In this article we discuss the nature of time-dependence in inverse problems and introduce two novel concepts for ill-posedness and regularization of time-dependent inverse problems, i.e. the stable computation of time-dependent parameters from data varying in time. For such problems a mathematical setup in Lebesgue-Bochner spaces is convenient since, e.g. solutions of hyperbolic or parabolic PDEs show different regularities in time and space. But classical treatments of inverse problems usually rely on static Hilbert and Banach spaces. For both concepts, the pointwise (temporal) and uniform ill-posedness and regularization, we gave examples such as temporal observations of compact operators, variational tracking, Tikhonov and Kaczmarz-based methods. Future research will further extend these theoretical findings to more general Bochner spaces and aims at an integrated treatment of time-dependent inverse problems in the linear and nonlinear regimes. Studying the behavior of singular values for linear inverse problems in Bochner spaces with respect to their decay and their temporal behavior represents another intriguing subject. Last but not least the theoretical fundaments have to be supported by applications.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Proof of lemma 2.3

Let $c_f > 0$ be a constant with $\|f_n\|_{L^p(\mu;\mathcal{X})} \leqslant c_f$ for all $n \in \mathbb{N}$.

(a) We prove that

$$\lim_{r\to\infty} \sup_{n\in\mathbb{N}} \mu\left(\{t : \|f_n(t)\|_{\mathcal{X}} > r\}\right) = 0. \tag{A.1}$$

We assume the contrary. Let $\{r_k\}_{k\in\mathbb{N}}$ be a sequence with $\lim_{k\to\infty} r_k = +\infty$ such that

$$\sup_{n\in\mathbb{N}} \mu(\{t : \|f_n(t)\|_{\mathcal{X}} > r_k\}) > R > 0 \text{ for all } k \in \mathbb{N}$$

with some constant $R > 0$. Then for each $k \in \mathbb{N}$ there exists an index $n_k \in \mathbb{N}$ with

$$\mu(\{t : \|f_{n_k}(t)\|_{\mathcal{X}} > r_k\}) > R.$$

This implies the existence of values $t_1(k), t_2(k) \in [0,T]$ such that $\mu([t_1(k), t_2(k)]) > R$ and

$$\int_{t_1(k)}^{t_2(k)} \|f_{n_k}(t)\|_{\mathcal{X}} \, d\mu(t) > \mu([t_1(k), t_2(k)]) r_k > R r_k.$$

But this immediately leads to

$$\lim_{k\to\infty} \|f_{n_k}\|_{L^p(0,T;\mathcal{X})} = \lim_{k\to\infty} \int_0^T \|f_{n_k}(t)\|_{\mathcal{X}} \, d\mu(t) > R r_k = +\infty,$$

which contradicts the boundedness of $\{f_n\}$. Combining $\|f_n\| \leqslant c_f$ and (A.1) we finally obtain

$$\begin{aligned} \lim_{r\to\infty} \sup_{n\in\mathbb{N}} \|\chi_{\|f_n\|>r} f_n\|^p_{L^p(0,T;\mathcal{X})} &= \lim_{r\to\infty} \sup_{n\in\mathbb{N}} \int_0^T \|\chi_{\|f_n\|>r} f_n(t)\|^p_{\mathcal{X}} \, d\mu(t) \\ &\leqslant \lim_{r\to\infty} \sup_{n\in\mathbb{N}} \mu(\{t : \|f_n(t)\|_{\mathcal{X}} > r\}) \sup_{n\in\mathbb{N}} \|f_n\|^p_{L^p(0,T;\mathcal{X})} = 0, \end{aligned}$$

proving that $\{f_n\}$ is uniformly L^p-integrable.

(b) Next we prove that $\{f_n\}$ is uniformly tight. To this end we use arguments similar to those in the proof of theorem 1 from [41]. Let $\varepsilon > 0$ be given arbitrarily. Fix $n \in \mathbb{N}$. Since f_n is strongly measurable, its distribution on the Borel sets $\mathcal{B}(\mathcal{X})$ is tight. According to Ulam's Theorem (see, e.g. [39, theorem 3.1]) it follows that to each n there exists a compact set $\mathcal{K}_n \subseteq \mathcal{X}$ such that

$$\mu(\{t : f_n(t) \notin \mathcal{K}_n\}) \leqslant 2^{-n}. \tag{A.2}$$

Let $n_0 \in \mathbb{N}$ be sufficiently large that $2^{1-n_0} < \varepsilon$. Define further the sets

$$L_n := \{x \in \mathcal{X} : d(x, \mathcal{K}_n) < 2^{-n}\}$$

and

$$\mathcal{K} := \overline{\bigcap_{n\geqslant n_0} L_n} \subseteq \mathcal{X}.$$

We prove that $\mathcal{K}$ is totally bounded. Since $\mathcal{K}_n$ is compact, there are finitely many open balls $B(x_i, 2^{-n})$, $x_i \in \mathcal{X}$, that cover $\mathcal{K}_n$. Then the finitely many open balls $B(x_i, 2^{1-n})$ cover $\overline{L_n}$. Hence, $\mathcal{K}$ is totally bounded and as a closed set also compact. Furthermore we have by (A.2)

$$\begin{aligned} \mu(\{t : f_n(t) \notin \mathcal{K}\}) &\leqslant \sum_{n\geqslant n_0} \mu(\{t : f_n(t) \notin L_n\}) \leqslant \sum_{n\geqslant n_0} \mu(\{t : f_n(t) \notin \mathcal{K}_n\}) \\ &\leqslant \sum_{n\geqslant n_0} 2^{-n} \leqslant 2^{1-n_0} < \varepsilon. \end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary we have

$$\sup_{n\in\mathbb{N}} \mu(\{t : f_n(t) \notin \mathcal{K}\}) < \varepsilon,$$

showing that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly tight.

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