

Four-loop splitting functions in QCD – The quark-quark case –

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Abstract

We have computed the even- N moments $N \leq 20$ of the pure-singlet quark splitting function P_{ps} at the fourth order of perturbative QCD via the anomalous dimensions of off-shell flavour-singlet operator matrix elements. Our results, derived analytically for a general gauge group, agree with all results obtained for this function so far, in particular with the lowest six even moments obtained via physical cross sections. Using these results and all available endpoint constraints, we construct approximations for P_{ps} at four loops that should be sufficient for most collider-physics applications. Together with the known results for the non-singlet splitting function P_{ns}^+ at this order, this effectively completes the quark-quark contribution for the evolution of parton distribution at N³LO accuracy. Our new results thus provide a major step towards fully consistent N³LO calculations at the LHC and the reduction of the residual uncertainty in the parton evolution to the percent level.

Collinear factorization in Quantum Chromodynamics (QCD) is controlled by a set of universal functions, the splitting functions P , governing the scale dependence of the parton distribution functions (PDFs), which are essential ingredients in all theoretical predictions for scattering processes with initial-state hadrons [1]. The splitting functions are calculable in the perturbative approach to QCD. They have been known, for a long time, at three-loop accuracy [2, 3], which is the next-to-next-to-leading order (NNLO or N^2 LO) in the expansion in powers of the strong coupling α_s .

This theoretical accuracy, however, faces challenges due to the precision of the currently available experimental data collected at the Large Hadron Collider (LHC). Also, the expectations for the LHC's Run 3, the high-luminosity phase (HL-LHC) as well as the plans for the future Electron-Ion Collider (EIC) [4] indicate the need to increase the precision by one quantum loop to N^3 LO accuracy. This will reduce the uncertainty in the parton evolution to percent-level precision. It is also required for a fully consistent use of the available N^3 LO QCD predictions for hard scattering cross sections for key processes [5–8] in proton-proton collisions at the LHC as well as for structure functions in deep-inelastic electron-proton scattering (DIS) probed by the EIC [9–11].

The PDFs of quarks and antiquarks of flavour i and of the gluon, $q_i(x, \mu^2)$, $\bar{q}_i(x, \mu^2)$ and $g(x, \mu^2)$, are given by the respective number distributions in the fractional hadron momentum x at the factorization scale μ . For n_f flavours, the flavour-summed quark distributions

$$q_s(x, \mu^2) = \sum_{i=1}^{n_f} [q_i(x, \mu^2) + \bar{q}_i(x, \mu^2)] \quad (1)$$

define the singlet quark distribution q_s , which mixes under QCD evolution with the gluon distribution g through the matrix of splitting functions P_{ij} as

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix} \quad (2)$$

where \otimes stands for the Mellin convolution in the momentum variable x . The perturbative expansion in the strong coupling with $a_s \equiv \alpha_s(\mu^2)/(4\pi)$ can be written as

$$P_{ij}(x, \alpha_s) = \sum_{n=0} a_s^{n+1} P_{ij}^{(n)}(x), \quad (3)$$

such that the terms $P_{ij}^{(k)}$ correspond to the QCD predictions at N^k LO accuracy. Here and below we identify, without loss of information, the renormalization scale with the scale μ in eq. (1).

The quark-quark splitting function P_{qq} in eq. (2) is the sum $P_{qq} = P_{ns}^+ + P_{ps}$, where P_{ns}^+ denotes the splitting function for non-singlet combinations of the quark-antiquark sums $q_i + \bar{q}_i$, and P_{ps} is the pure-singlet contribution. The results for the four-loop (N^3 LO) non-singlet splitting functions $P_{ns}^{(3)}$ are complete in the planar limit, i.e., the large- n_c limit of a colour $SU(n_c)$ gauge group, and very good approximations for the remaining non-planar terms have been obtained [12, 13].

In this letter, we address the computation of the pure-singlet part $P_{ps}^{(3)}$ at N^3 LO. Together with the result for $P_{ns}^{(3)+}$, this provides the quark-quark splitting functions at four loops. We present

analytic results for the first 10 even Mellin moments $N \leq 20$ of $P_{\text{ps}}^{(3)}(x)$, valid for a general compact simple gauge group. These correspond (up to a negative sign, which is a standard convention) to the anomalous dimensions $\gamma_{\text{ps}}^{(3)}$,

$$\gamma_{\text{ij}}^{(n)}(N) = - \int_0^1 dx x^{N-1} P_{\text{ij}}^{(n)}(x). \quad (4)$$

This extends previous results on this function, which is already known in the limit of large numbers of flavours n_f [14]. Following the approach of refs. [2, 3] via physical quantities in inclusive DIS, also low moments ($N \leq 8$) of $P_{\text{ps}}^{(3)}$ have been presented before [15], which have been used in approximations [16] for applications in N³LO PDF fits. For the parts proportional to the quartic colour factor $d_R^{abcd} d_R^{abcd}$ (see below), the moments up to $N = 16$ have been derived [17] by computing anomalous dimensions of off-shell flavour-singlet operator matrix elements (OMEs).

Here we extend the method of OMEs further. The starting point is the standard set of the spin- N twist-two irreducible flavour-singlet quark and gluon operators, given by

$$\begin{aligned} O_{\text{q}}^{\{\mu_1, \dots, \mu_N\}} &= \frac{1}{2} \bar{\Psi} \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} \Psi + \text{traceless}, \\ O_{\text{g}}^{\{\mu_1, \dots, \mu_N\}} &= \frac{1}{2} F^{\nu\{\mu_1} D^{\mu_2} \dots D^{\mu_{N-1}} F^{\mu_N\}_{\nu}} + \text{traceless}, \end{aligned} \quad (5)$$

where Ψ represents the quark field, $F_{\mu\nu}$ the gluon field-strength tensor and $D_\mu = \partial_\mu - igA_\mu$ the covariant derivative with the coupling g , where $g^2/(4\pi) = \alpha_s$. The curly brackets $\{\dots\}$ denote symmetrization. Flavour non-singlet operators have been discussed in ref. [12].

Contraction of the Lorentz indices with N identical light-like ($\Delta \cdot \Delta = 0$) vectors Δ^μ allows for a compact notation in terms of the quantities

$$F^{\mu;a} = \Delta_\nu F^{\mu\nu;a}, \quad A^a = \Delta_\mu A^{\mu;a}, \quad D = \Delta_\mu D^\mu, \quad \partial = \Delta_\mu \partial^\mu, \quad (6)$$

with labels a in the adjoint representation of the colour gauge group. The projection of eq. (5) defines the scalar operators for even N (suppressing the index N here and below) as

$$O_{\text{q}} = \frac{1}{2} \bar{\Psi} \not{\Delta} D^{N-2} \Psi, \quad O_{\text{g}} = \frac{1}{2} F_\nu^a D_{ab}^{N-2} F^{\nu;b}. \quad (7)$$

These physical (gauge-invariant) operators, when evaluated in general Green's functions, mix under renormalization with non-physical, so-called alien operators, which also involve (anti-)ghost fields \bar{c} and c . The general theory of the renormalization of gauge-invariant operators has been worked out in a series of classical papers by Dixon and Taylor [18], Kluberg-Stern and Zuber [19, 20] and Joglekar and Lee [21–23]. These led the way to explicit computations at two loops in ref. [24], which solved an issue that had beset the pioneering calculation of ref. [25], and refs. [26, 27]. Recently, the complete three-loop renormalization via a direct calculation of the alien counter-terms has been published [28].

A general procedure to construct the basis of alien operators was formulated by two of us in ref. [29] and was used to construct an explicit basis up to four loops for any fixed spin N . Building

on this work, we here consider the four-loop renormalization of singlet quark operators. For this we need two sets of alien operators, $O_A^i = O_q^i + O_g^i + O_c^i$ with $i = I, II$, which read

$$\begin{aligned} O_q^I &= \eta g \bar{\Psi} \not{t}^a \Psi (\partial^{N-2} A_a), \\ O_g^I &= \eta (D.F)^a (\partial^{N-2} A_a), \\ O_c^I &= -\eta (\partial \bar{c}^a) (\partial^{N-1} c_a) \end{aligned} \quad (8)$$

with a coupling η which is a function of N and α_s , and

$$\begin{aligned} O_q^{II} &= g^2 \bar{\Psi} \not{t}_a \Psi \sum_{i+j=N-3} \kappa_{ij} f^{abc} (\partial^i A_b) (\partial^j A_c), \\ O_g^{II} &= g (D.F)_a \sum_{i+j=N-3} \kappa_{ij} f^{abc} (\partial^i A_b) (\partial^j A_c), \\ O_c^{II} &= -g \sum_{i+j=N-3} \eta_{ij} f^{abc} (\partial \bar{c}_a) (\partial^i A_b) (\partial^{j+1} c), \end{aligned} \quad (9)$$

where t^a (f^{abc}) denote the fundamental (adjoint) colour-group generators. The couplings η_{ij} and κ_{ij} obey constraints due to the (anti-)BRST symmetry of the alien operators [29], viz

$$\eta_{ij} = - \sum_{s=0}^i (-1)^{s+j} \binom{s+j}{s} \eta_{(i-s)(j+s)} = 2 \kappa_{ij} + \eta \binom{i+j+1}{i}. \quad (10)$$

Hence the κ_{ij} are dependent on the η_{ij} for which we obtain a compact expression in terms of binomial coefficients

$$\eta_{ij} = \eta \left[(-1)^i - 3 \binom{N-2}{i} - \binom{N-2}{i+1} \right], \quad (11)$$

where, in the present case, the coupling $\eta = \eta(\alpha_s, N)$ from eq. (8) is found to factorize to the loop-order required.

The physical operators in eq. (7) mix under renormalization with the alien ones in eqs. (8) and (9). The latter are summarized from now on collectively as O_A and we denote renormalized operators as $[O]_i$, so that

$$\begin{pmatrix} O_q \\ O_g \\ O_A \end{pmatrix} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ Z_{Aq} & Z_{Ag} & Z_{AA} \end{pmatrix} \begin{pmatrix} [O]_q \\ [O]_g \\ [O]_A \end{pmatrix}, \quad (12)$$

where the Z -factors are determined in terms of the anomalous dimensions γ_{ij} in eq. (4) (and the corresponding ones γ_{qA} , γ_{gA} etc. including the alien operators) through the standard renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} Z_{ij} = \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma_3 \xi \frac{\partial}{\partial \xi} \right) Z_{ij} = -\gamma_{ik} Z_{kj}. \quad (13)$$

Here β is the QCD β -function and γ_3 the gluon anomalous dimension, all known to a more than sufficiently high order [30–33]. The Z_{ij} involving the alien operators can be gauge dependent, ξ is the gauge parameter with $\xi = 1$ for Feynman gauge. Moreover, since the alien operators cannot mix into the set of physical operators [21–23], the entries Z_{Aq} and Z_{Ag} in eq. (12) have to vanish.

The setup of our OME computations follows previous work [12, 17]. The necessary Feynman rules are determined from eqs. (7) – (11), which are sufficient for the present computations. The diagrams for the OMEs $A_{ij} = \langle j(p) | O_i | j(p) \rangle$ with (physical or alien) spin- N twist-two operators O_i inserted in Green's functions with off-shell quarks, gluons or ghosts have been generated using QGRAF [34] and then processed, see ref. [35], by a FORM [36–38] program which collects self-energy insertions, determines the colour factors [39] and classifies the topologies according to the conventions of the FORCER package [40]. For computational efficiency, diagrams with the same colour factor and topology are merged into meta-diagrams.

An optimized in-house version of FORCER, briefly discussed in ref. [13], is employed to perform the integral reductions for fixed integer values of N . In practice, the range in N is limited by the occurrence of high powers of scalar products in the loop integrals for high values of N , which lead to large-size expressions and long computing times in the topology transformations and parametric reductions encoded in FORCER. The divergences in the OMEs A_{ij} are treated in dimensional regularization with $D = 4 - 2\epsilon$ dimensions, hence the Z -factors in eq. (12) are simple Laurent series in ϵ and the anomalous dimensions γ_{ij} can be read off from their single poles $1/\epsilon$.

For the quark-quark splitting functions, the physical OMEs A_{qq} have been obtained at even $N \leq 20$ up to four loops. This includes both the flavour non-singlet parts [13] and the pure-singlet contributions addressed in the present letter. The physical OMEs A_{qg} , A_{gq} and A_{gg} and those with the alien operators inserted into a quark two-point function, A_{Aq} , have been computed up to three loops. All others were needed at two loops only for the extraction of $P_{ps}^{(3)}$ at four loops using eq. (13). This leads to the following results for the N^3 LO contributions to the pure-singlet anomalous dimensions in eq. (4) for QCD, i.e., the gauge group $SU(n_c = 3)$,

$$\begin{aligned}
\gamma_{ps}^{(3)}(N=2) &= -691.5937093 n_f + 84.77398149 n_f^2 + 4.466956849 n_f^3, \\
\gamma_{ps}^{(3)}(N=4) &= -109.3302335 n_f + 8.776885259 n_f^2 + 0.306077137 n_f^3, \\
\gamma_{ps}^{(3)}(N=6) &= -46.03061374 n_f + 4.744075766 n_f^2 + 0.042548957 n_f^3, \\
\gamma_{ps}^{(3)}(N=8) &= -24.01455020 n_f + 3.235193483 n_f^2 - 0.007889256 n_f^3, \\
\gamma_{ps}^{(3)}(N=10) &= -13.73039387 n_f + 2.375018759 n_f^2 - 0.021029241 n_f^3, \\
\gamma_{ps}^{(3)}(N=12) &= -8.152592251 n_f + 1.819958178 n_f^2 - 0.024330231 n_f^3, \\
\gamma_{ps}^{(3)}(N=14) &= -4.840447180 n_f + 1.438327380 n_f^2 - 0.024479943 n_f^3, \\
\gamma_{ps}^{(3)}(N=16) &= -2.751136330 n_f + 1.164299642 n_f^2 - 0.023546009 n_f^3, \\
\gamma_{ps}^{(3)}(N=18) &= -1.375969240 n_f + 0.960873318 n_f^2 - 0.022264393 n_f^3, \\
\gamma_{ps}^{(3)}(N=20) &= -0.442681568 n_f + 0.805745333 n_f^2 - 0.020918264 n_f^3. \tag{14}
\end{aligned}$$

The results for $N \leq 8$ agree with those obtained via cross sections for inclusive DIS in ref. [15]. As a further check, we have extended those DIS computations of $P_{ps}^{(3)}$ to $N = 10$ and $N = 12$, their results also agree with eq. (14). The large- n_f parts agree with all- N results of ref. [14]. In addition, the renormalization constants involving alien operators agree to the loop order required here with those recently published in ref. [28].

The analytic expressions for $\gamma_{\text{ps}}^{(3)}$ for a general gauge group are given in app. A in eqs. (A.3) – (A.12). They contain rational numbers and Riemann- ζ values, i.e. ζ_n with $n = 3, 4, 5$. New all- N results for $\gamma_{\text{ps}}^{(3)}$ based on eqs. (A.3) – (A.12) and, in cases, even higher fixed- N OME computations, have been derived for specific colour factors or terms proportional to certain Riemann- ζ values. They can be expressed in terms harmonic sums [41, 42], which are recursively defined by

$$S_{\pm m_1, m_2, \dots, m_d}(N) = \sum_{i=1}^N \frac{(\pm 1)^i}{i^{m_1}} S_{m_2, \dots, m_d}(i), \quad (15)$$

and their weight w is the sum of the absolute values of the indices m_d . In the results for the n -loop anomalous dimensions, quantities up to $w = 2n - 1$ occur, which can be composed of harmonic sums, Riemann- ζ_m values ($m \geq 3$), and simple denominators, $D_a^k = (N + a)^{-k}$. The latter count as objects of weight m and k , respectively. Due to (partial) conformal symmetry, especially the sums with the highest weights often arise with two specific (reciprocity respecting, cf. the discussion in ref. [12]) combinations of simple denominators,

$$\eta \equiv \frac{1}{N} - \frac{1}{N+1} \equiv D_0 - D_1 = \frac{1}{N(N+1)}, \quad (16)$$

$$\nu \equiv \frac{1}{N-1} - \frac{1}{N+2} \equiv D_{-1} - D_2 = \frac{3}{(N-1)(N+2)}. \quad (17)$$

With this notation (suppressing the argument N of the harmonic sums for brevity), we can summarize the known and new all- N results for $\gamma_{\text{ps}}^{(3)}$. The leading large- n_f contribution has been published in eq. (3.10) of ref. [14],

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N)|_{n_f^3} = 16/9 C_F \Big\{ & 2/3 \left(2S_{1,1,1} - 3\zeta_3 \right) \left(9\eta + 6\eta^2 - 4\nu \right) + 4/3 S_{1,1} \left(11D_0 - 13D_0^2 \right. \\ & + 6D_0^3 - 17D_1 - 4D_1^2 + 12D_1^3 + 2D_2 + 8D_2^2 + 4D_{-1} \Big) - 2/9 S_1 \left(94D_0 - 98D_0^2 + 87D_0^3 \right. \\ & - 18D_0^4 - 226D_1 + 100D_1^2 + 111D_1^3 - 90D_1^4 + 128D_2 + 88D_2^2 - 48D_2^3 + 4D_{-1} \Big) \\ & + 1/9 \left(52D_0 - 118D_0^2 + 146D_0^3 - 87D_0^4 + 18D_0^5 - 412D_1 + 430D_1^2 - 54D_1^3 - 309D_1^4 \right. \\ & \left. \left. + 198D_1^5 + 364D_2 + 72D_2^2 - 176D_2^3 + 96D_2^4 - 4D_{-1} \right) \Big\}. \end{aligned} \quad (18)$$

Additional all- N information can be obtained for the terms involving Riemann- ζ values, due to the reduced complexity of the harmonic sums. The reconstruction of the all- N expressions follows the same approach as in the nonsinglet case in ref. [12]. The ζ_3 dependence of the quartic colour factor $d_R^{abcd} d_R^{abcd} / n_c$ is given by (see app. A for the definition of all colour factors),

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N) \Big|_{\zeta_3 n_f d_R^{abcd} d_R^{abcd} / n_c} = 256 \Big\{ & \left(S_3 + 4S_{-2,1} - 2S_{-3} \right) \left(3 - 6\eta \right) \\ & + \left(2S_{1,1} - S_2 \right) \left(27\eta + 22\eta^2 - 32/3\nu \right) + S_{-2} \left(-1/3 + 92/3\eta + 20\eta^2 - 32/3\nu \right) \\ & + S_1 \left(-70\eta - 68\eta^2 - 24\eta^3 + 64/3\nu \right) + 8/3\eta - 3\eta^2 \Big\}. \end{aligned} \quad (19)$$

This is a new result based on computing the relevant OMEs for $N \leq 22$. Likewise, for the colour factor $n_f^2 C_F^2$ the term proportional to ζ_3 reads

$$\gamma_{\text{ps}}^{(3)}(N) \Big|_{\zeta_3 n_f^2 C_F^2} = 32/9 \left\{ 2S_1 (9\eta + 6\eta^2 - 4\nu) - 335D_0 + 145D_0^2 - 66D_0^3 \right. \\ \left. + 287D_1 + 187D_1^2 + 90D_1^3 + 36D_2 + 8D_2^2 + 12D_{-1} - 16D_{-1}^2 \right\}, \quad (20)$$

which is new as well. This expression has been derived from and verified using moments up to $N = 52$, which are part of the computations of the n_f^2 QED contributions to $\gamma_{ij}^{(3)}$ to very high N [43].

Next, the ζ_4 part in $\gamma_{\text{ps}}^{(3)}(N)$ can be derived with the help of the no- π^2 conjecture/theorem [44, 45]. This has been done in ref. [46], eq. (9).¹ Eqs. (A.3) – (A.12) agree with this result,

$$\gamma_{\text{ps}}^{(3)}(N) \Big|_{\zeta_4} = 16C_F(C_A - C_F) \left\{ C_F n_f \left(46\nu - 8\nu^2 - 117\eta - 87\eta^2 - 18\eta^3 \right) \right. \\ \left. + (C_A - C_F) n_f \left(38\nu - 8\nu^2 - 195/2\eta - 69\eta^2 - 12\eta^3 + S_1 (8\nu - 18\eta - 12\eta^2) \right) \right. \\ \left. + n_f^2 \left(15\eta + 10\eta^2 - 20/3\nu \right) \right\}. \quad (21)$$

Finally, with the moments up to $N = 20$ the terms with ζ_5 can be readily determined and verified for all colour factors, thus extending the result in eq. (3.13) ref. [17], where the all- N ζ_5 contributions to the quartic colour factors have been given,

$$\gamma_{\text{ps}}^{(3)}(N) \Big|_{\zeta_5} = 160n_f C_F^3 \left(9\eta + 6\eta^2 - 4\nu \right) + 80/3 n_f C_A C_F^2 \left(-9\eta - 6\eta^2 + 4\nu \right) \\ + 40/9 n_f C_A^2 C_F \left(-1 - 214\eta - 144\eta^2 + 104\nu \right) \\ + 320/3 n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(-1 + 56\eta + 36\eta^2 - 16\nu \right). \quad (22)$$

Expressions in x -space for the leading large- n_f part of eq. (18) have been presented in ref. [14], eq. (4.21). On the other hand, the N -space terms with Riemann- ζ values do not correspond to the x -space contributions with Riemann- ζ values, as the inverse Mellin transformation generates additional terms with ζ_n . Similarly, it is not possible to read off the coefficients of ζ_n in the limit $N \rightarrow \infty$ from eqs. (19) – (22), as non- ζ harmonic sums contribute to these.

For phenomenology applications, the moments in eq. (14) can be used to construct approximate representations for the n_f^1 and n_f^2 parts of $P_{\text{ps}}^{(3)}(x)$, subject to the constraints imposed by the known terms in the limits $x \rightarrow 0, 1$. At small x , the coefficient of the leading logarithm $(\ln^2 x)/x$ is known since long [47], as well as those of the highest three sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$, see ref. [48]. At large x , the leading terms are of the form $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$. The coefficients for $k = 4, 3$ are known [49] for all j . With the 10 Mellin moments $N \leq 20$ in eq. (14), the coefficients of all remaining unknown small- x and large- x (for $j = 1$) terms can be ‘fitted’ together with a three-parameter interpolating function. Thus all approximations include

¹The coefficients proportional to ν in eqs. (9) and (12) in ref. [46] need to be replaced as follows: $\nu \rightarrow \nu/3$. Note further that the definition of ν in ref. [14] differs from that in ref. [17] and the present paper by this factor of 3.

- the next-to-leading and next-to-next-to-leading small- x terms: $(\ln x)/x$ and $1/x$,
- the remaining three sub-dominant small- x logarithms: $\ln^k x$ with $k = 3, 2, 1$,
- the two remaining $j = 1$ large- x terms: $(1-x)\ln^k(1-x)$ with $k = 2, 1$.

Choosing 10 two-parameter polynomials together one function that includes $\ln^k(1-x)$ with $k = 1$ or 2 or the dilogarithm $\text{Li}_2(x)$ (suitably suppressed as $x \rightarrow 1$), we have thus build 80 trial functions that fulfil all known constraints. These functions are shown $n_f = 4$ by the black-dotted curves in the right part of fig. 1; other values of n_f show qualitatively the same behaviour. As a check, exactly the same procedure has been applied to the NNLO splitting function $P_{\text{ps}}^{(2)}(x)$, where it can be compared with the exact result [3]; this comparison is shown in the left part of the figure.

Since our approximation procedure passes this test (and others), we have selected two representatives for each physically relevant value of n_f that indicate the remaining uncertainty in $P_{\text{ps}}^{(3)}(x)$. The chosen approximation, shown in red in fig. 1, are, with $x_1 = 1-x$, $L_1 = \ln(1-x)$ and $L_0 = \ln x$,

$$\begin{aligned}
P_{\text{ps,A}}^{(3)}(n_f=3, x) &= p_{\text{ps,0}}^{(n_f=3)}(x) + 67731 x_1 L_0/x + 274100 x_1/x + 40006 L_0^3 + 10620 L_0^2 \\
&\quad + 353656 x_1 L_0 - 2365.1 x_1 L_1^2 - 7412.1 x_1 L_1 + 1533.0 x_1^2 L_1^2 - 104493 x_1(1+2x) + 34403 x_1 x^2, \\
P_{\text{ps,B}}^{(3)}(n_f=3, x) &= p_{\text{ps,0}}^{(n_f=3)}(x) + 54593 x_1 L_0/x + 179748 x_1/x - 2758.3 L_0^3 - 103604 L_0^2 \\
&\quad + 4700.0 x_1 L_0 - 1986.9 x_1 L_1^2 - 2801.2 x_1 L_1 - 6005.9 x_1^2 L_1^2 - 195263 x_1 + 12789 x_1 x(1+x),
\end{aligned} \tag{23}$$

$$\begin{aligned}
P_{\text{ps,A}}^{(3)}(n_f=4, x) &= p_{\text{ps,0}}^{(n_f=4)}(x) + 90154 x_1 L_0/x + 359084 x_1/x + 52525 L_0^3 + 13869 L_0^2 \\
&\quad + 461167 x_1 L_0 - 2491.5 x_1 L_1^2 - 7498.2 x_1 L_1 + 1727.2 x_1^2 L_1^2 - 136319 x_1(1+2x) + 45379 x_1 x^2, \\
P_{\text{ps,B}}^{(3)}(n_f=4, x) &= p_{\text{ps,0}}^{(n_f=4)}(x) + 72987 x_1 L_0/x + 235802 x_1/x - 3350.9 L_0^3 - 135378 L_0^2 \\
&\quad + 5212.9 x_1 L_0 - 1997.2 x_1 L_1^2 - 1472.7 x_1 L_1 - 8123.3 x_1^2 L_1^2 - 254921 x_1 + 17138 x_1 x(1+x),
\end{aligned} \tag{24}$$

$$\begin{aligned}
P_{\text{ps,A}}^{(3)}(n_f=5, x) &= p_{\text{ps,0}}^{(n_f=5)}(x) + 112481 x_1 L_0/x + 440555 x_1/x + 64577 L_0^3 + 16882 L_0^2 \\
&\quad + 562992 x_1 L_0 - 2365.7 x_1 L_1^2 - 6570.1 x_1 L_1 + 1761.7 x_1^2 L_1^2 - 166581 x_1(1+2x) + 56087 x_1 x^2, \\
P_{\text{ps,B}}^{(3)}(n_f=5, x) &= p_{\text{ps,0}}^{(n_f=5)}(x) + 91468 x_1 L_0/x + 289658 x_1/x - 3814.9 L_0^3 - 165795 L_0^2 \\
&\quad + 4908.9 x_1 L_0 - 1760.8 x_1 L_1^2 + 804.5 x_1 L_1 - 10295 x_1^2 L_1^2 - 311749 x_1 + 21521 x_1 x(1+x),
\end{aligned} \tag{25}$$

with the known endpoint contribution, with coefficients rounded to seven significant figures,

$$\begin{aligned}
p_{\text{ps,0}}^{(n_f)}(x) &= n_f \left\{ 1749.227 L_0^2/x - (7.506173 - 0.7901235 n_f) L_0^6 \right. \\
&\quad + (28.54979 + 3.792593 n_f) L_0^5 - (854.8001 - 77.36626 n_f + 0.1975309 n_f^2) L_0^4 \\
&\quad - (199.1111 - 13.69547 n_f) x_1^2 L_1^3 - 13.16872 x_1^2 L_1^4 - (247.5505 - 40.55967 n_f \\
&\quad \left. + 1.580247 n_f^2) x_1 L_1^3 - (56.46091 - 3.621399 n_f) x_1 L_1^4 \right\}.
\end{aligned} \tag{26}$$

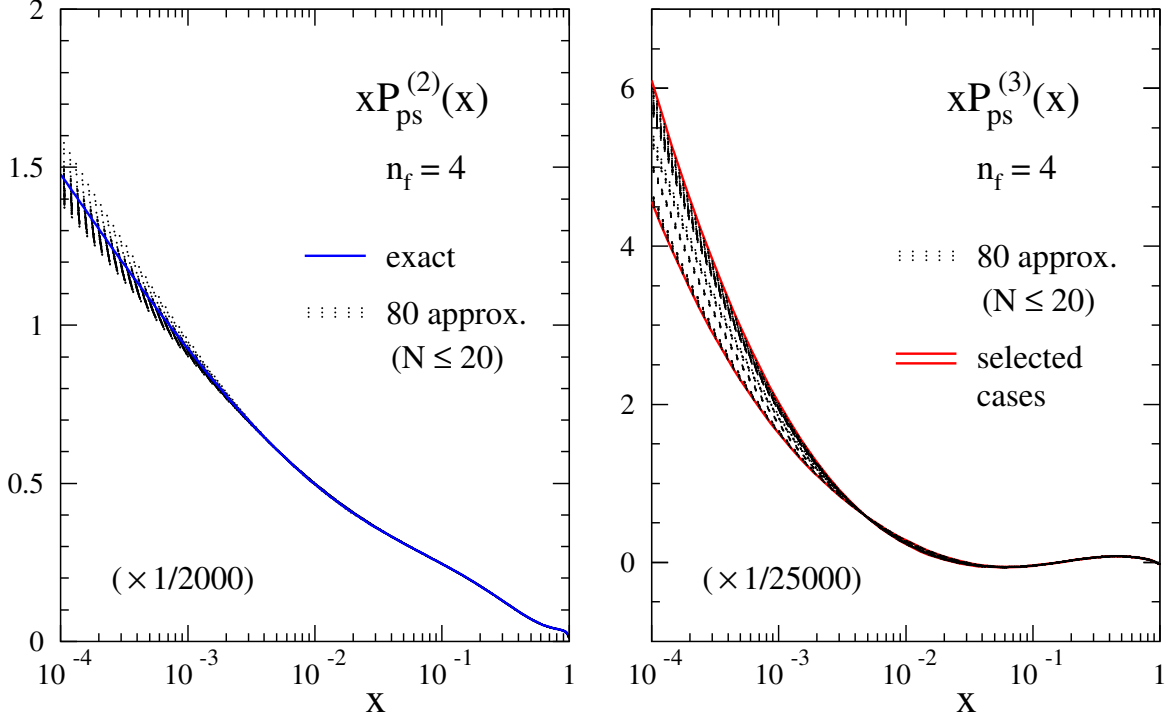


Figure 1: 80 trial functions, constructed as described in the text, for the splitting functions $P_{ps}^{(n)}(x)$ at $n_f = 4$. At $n = 2$ (left panel) the known exact result is shown by the solid (blue) line. At $n = 3$ (right panel) two functions, shown by the solid (red) lines, are chosen to represent the remaining uncertainty.

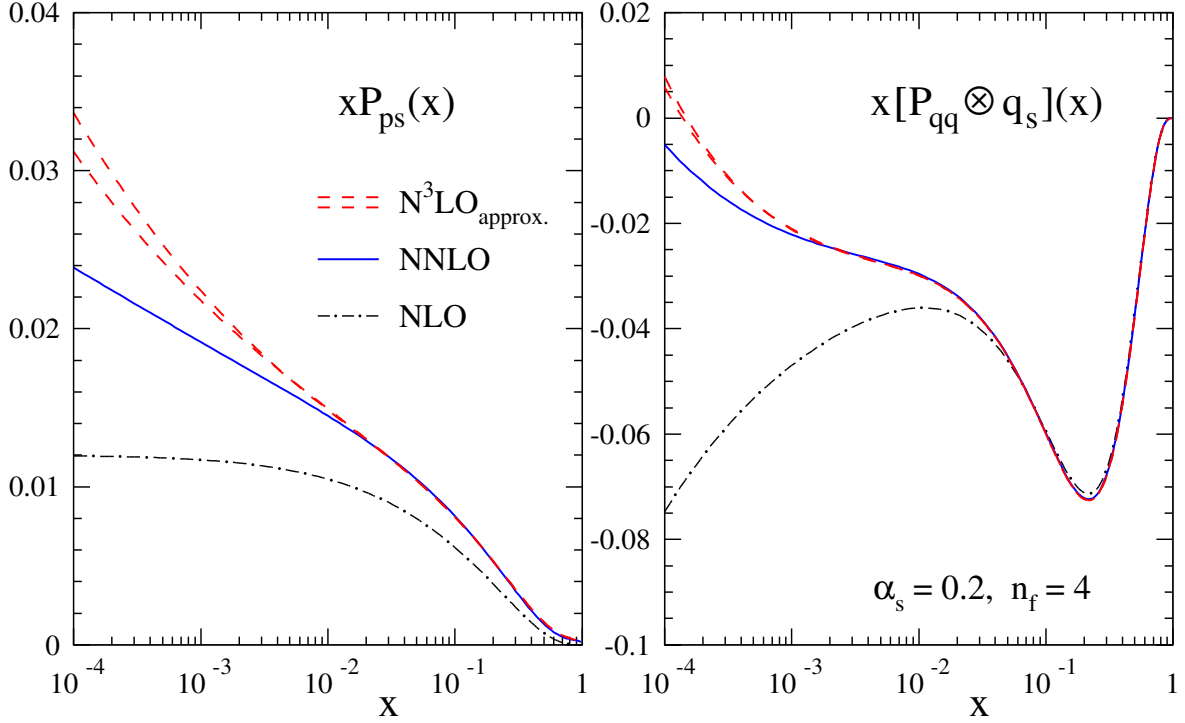


Figure 2: Left: The NLO, NNLO and $N^3\text{LO}$ approximations for $P_{ps}(x)$ for a fixed value $\alpha_s(\mu_0^2) = 0.2$ of the strong coupling and $n_f = 4$. Right: The resulting perturbative expansion of the contribution of P_{qq} to the scale derivative of the singlet quark PDF q_s in eq. (2) for the initial distribution (28).

Specifically, we note that eqs. (23) – (25) include a numerical prediction, with a precision of $\pm 10\%$, of the so far uncalculated coefficient of the next-to-leading small- x logarithm $(\ln x)/x$ of $P_{\text{ps}}^{(3)}(x)$ for $n_f = 3, 4, 5$. The above uncertainty bands also lead to the following predictions for the $\gamma_{\text{ps}}^{(3)}(N)$ at $N = 22$ (with brackets indicating the error on the last digits),

$$\gamma_{\text{ps}}^{(3)}(N=22) = 6.2478570(6), \quad 10.5202730(8), \quad 15.6913948(10) \quad \text{for } n_f = 3, 4, 5. \quad (27)$$

The numerical implications for the evolution of q_s due to the quark-quark splitting functions in eq. (24) are illustrated in fig. 2 for our default value $n_f = 4$. We show the perturbative expansion for P_{ps} alone and for the convolution $P_{\text{qq}} \otimes q_s$ through N³LO, in the latter case using the same schematic (order-independent) model input [3] for the singlet quark PDF in eq. (1),

$$xq_s(x, \mu_0^2) = 0.6x^{-0.3}(1-x)^{3.5} (1 + 5.0x^{0.8}), \quad (28)$$

together with $\alpha_s(\mu_0^2) = 0.2$. The latter value of the strong coupling corresponds to scales in the range $\mu_0^2 \simeq 25 \dots 50 \text{ GeV}^2$ for $\alpha_s(M_Z^2) = 0.114 \dots 0.120$ beyond the leading order. This fixed input facilitates a direct comparison of effects of the various perturbative order for the splitting function $P_{\text{qq}} = P_{\text{ns}}^+ + P_{\text{ps}}$, where the four-loop results for $P_{\text{ns}}^{(3)+}$ have been taken from refs. [12, 13].

Fig. 2 shows that the convolution of the splitting functions with q_s dampens the residual small- x uncertainties induced by the approximations A and B in eq. (24). The uncertainties in these convolutions are practically negligible down to $x \lesssim 10^{-3}$ and, even if our error band were to underestimate the uncertainty by, say, a factor of 2, perfectly tolerable even at x as small as $x \approx 10^{-4}$. Further phenomenological studies, such as the scale stability of the parton evolution in eq. (2) under the variation of the renormalization scales require additional information at four loops on the other splitting functions P_{qg} , P_{gq} and P_{gg} . These will be subject of forthcoming publications.

With the four-loop results for the pure-singlet quark splitting function P_{ps} presented here, we have provided a major step towards improving the accuracy of the flavour-singlet parton evolution by one perturbative order beyond the current state of the art. The evolution of PDFs at N³LO is expected to achieve percent-level precision, and a brief phenomenological study of our results is consistent with this expectation. We have shown that the knowledge of 10 Mellin moments of $P_{\text{ps}}^{(3)}(x)$, together with the present knowledge of the behavior at the endpoints $x \rightarrow 1$ and $x \rightarrow 0$, is sufficient for the construction of approximations that display negligible residual uncertainties in a wide kinematic range of parton momentum fractions x probed at current and future colliders.

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A Mellin moments of $P_{\text{ps}}^{(3)}$

Here we provide the exact results for the four-loop pure-singlet anomalous dimensions $\gamma_{\text{ps}}^{(3)}(N)$ at even $N \leq 20$ for a general compact simple gauge group. The numerical values in QCD, i.e., $\text{SU}(n_c = 3)$ have been given in eq. (14). The colour factors are $C_A = n_c$ and $C_F = (n_c^2 - 1)/(2n_c)$ for the quadratic Casimir invariants in $\text{SU}(n_c)$. The relevant quartic colour factor $d_R^{abcd} d_R^{abcd}/n_c$ is obtained from the symmetrized trace of four generators T_r^a , see, e.g., [12, 17, 31],

$$d_r^{abcd} = \frac{1}{6} \text{Tr} (T_r^a T_r^b T_r^c T_r^d + \text{five } bcd \text{ permutations}) . \quad (\text{A.1})$$

with

$$\frac{d_R^{abcd} d_R^{abcd}}{n_c} = \frac{1}{96 n_c^3} (n_c^2 - 1)(n_c^4 - 6n_c^2 + 18) \quad (\text{A.2})$$

for the fundamental representation. In QCD: $C_A = 3$, $C_F = 4/3$ and $d_R^{abcd} d_R^{abcd}/n_c = 5/36$.

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N=2) = & n_f C_F^3 \left(\frac{227938}{2187} + \frac{1952}{81} \zeta_3 + \frac{256}{9} \zeta_4 - \frac{640}{3} \zeta_5 \right) \\ & + n_f C_A C_F^2 \left(-\frac{162658}{6561} + \frac{8048}{27} \zeta_3 - \frac{1664}{9} \zeta_4 + \frac{320}{9} \zeta_5 \right) \\ & + n_f C_A^2 C_F \left(-\frac{410299}{6561} - \frac{26896}{81} \zeta_3 + \frac{1408}{9} \zeta_4 + \frac{4480}{27} \zeta_5 \right) \\ & + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{1024}{9} + \frac{256}{9} \zeta_3 - \frac{2560}{9} \zeta_5 \right) + n_f^2 C_F^2 \left(-\frac{73772}{6561} - \frac{5248}{81} \zeta_3 + \frac{320}{9} \zeta_4 \right) \\ & + n_f^2 C_A C_F \left(\frac{160648}{6561} + 48 \zeta_3 - \frac{320}{9} \zeta_4 \right) + n_f^3 C_F \left(-\frac{1712}{729} + \frac{128}{27} \zeta_3 \right) , \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N=4) = & n_f C_F^3 \left(\frac{1995890620891}{52488000000} - \frac{897403}{202500} \zeta_3 + \frac{18997}{2250} \zeta_4 - \frac{484}{15} \zeta_5 \right) \\ & + n_f C_A C_F^2 \left(\frac{209865827521}{26244000000} + \frac{6743539}{202500} \zeta_3 - \frac{29161}{750} \zeta_4 + \frac{242}{45} \zeta_5 \right) \\ & + n_f C_A^2 C_F \left(-\frac{55187654921}{3280500000} - \frac{3104267}{67500} \zeta_3 + \frac{34243}{1125} \zeta_4 + \frac{3164}{135} \zeta_5 \right) \\ & + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{172231}{675} - \frac{5368}{25} \zeta_3 - \frac{3728}{45} \zeta_5 \right) + n_f^2 C_F^2 \left(-\frac{141522185707}{26244000000} - \frac{1207}{135} \zeta_3 + \frac{242}{45} \zeta_4 \right) \\ & + n_f^2 C_A C_F \left(\frac{9398360351}{1640250000} + \frac{57877}{10125} \zeta_3 - \frac{242}{45} \zeta_4 \right) + n_f^3 C_F \left(-\frac{46099151}{72900000} + \frac{484}{675} \zeta_3 \right) , \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=6) = & n_f C_F^3 \left(\frac{140565274663259489}{5403265623000000} - \frac{62727544}{24310125} \zeta_3 + \frac{343156}{77175} \zeta_4 - \frac{1936}{147} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(\frac{336481838777617}{360217708200000} + \frac{2111992}{324135} \zeta_3 - \frac{1389806}{77175} \zeta_4 + \frac{968}{441} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{6194882229735067}{864522499680000} - \frac{2396237}{165375} \zeta_3 + \frac{41866}{3087} \zeta_4 + \frac{9544}{1323} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{64697569}{330750} - \frac{426976}{3675} \zeta_3 - \frac{39808}{441} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{812984663253277}{270163281150000} - \frac{2594876}{694575} \zeta_3 + \frac{968}{441} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{3092531515013}{964868861250} + \frac{217432}{99225} \zeta_3 - \frac{968}{441} \zeta_4 \right) + n_f^3 C_F \left(-\frac{19597073837}{61261515000} + \frac{1936}{6615} \zeta_3 \right), \quad (\text{A.5})
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=8) = & n_f C_F^3 \left(\frac{3960340604223955458923}{192072198786048000000} - \frac{34718701049}{18003384000} \zeta_3 + \frac{13529827}{4762800} \zeta_4 - \frac{1369}{189} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{43838488788848637899}{13719442770432000000} + \frac{10167760657}{18003384000} \zeta_3 - \frac{10211371}{952560} \zeta_4 + \frac{1369}{1134} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{8552512702477166383}{2939880593664000000} - \frac{97528710971}{18003384000} \zeta_3 + \frac{1340251}{170100} \zeta_4 + \frac{128}{63} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{1183211180737}{7715736000} - \frac{18321694}{297675} \zeta_3 - \frac{18164}{189} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{5115927245667479753}{2743888554086400000} - \frac{15129691}{7144200} \zeta_3 + \frac{1369}{1134} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{15301312238130101}{7349701484160000} + \frac{8397097}{7144200} \zeta_3 - \frac{1369}{1134} \zeta_4 \right) \\
& + n_f^3 C_F \left(-\frac{162840799744061}{816633498240000} + \frac{1369}{8505} \zeta_3 \right), \quad (\text{A.6})
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=10) = & n_f C_F^3 \left(\frac{19206657411733877390649313}{1118944450162341495000000} - \frac{45224548192}{28017383625} \zeta_3 + \frac{1080128}{539055} \zeta_4 - \frac{25088}{5445} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{1538138456874500390560463}{298385186709957732000000} - \frac{31074715888}{28017383625} \zeta_3 - \frac{97295744}{13476375} \zeta_4 + \frac{12544}{16335} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(-\frac{202179113304531644762417}{284176368295197840000000} - \frac{192321673117627}{109828143810000} \zeta_3 + \frac{23430848}{4492125} \zeta_4 - \frac{14912}{49005} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{1240606813603}{9901861200} - \frac{182828576543}{6303268125} \zeta_3 - \frac{1624576}{16335} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{367710354086746558213}{296017050307497750000} + \frac{12544}{16335} \zeta_4 - \frac{1243744}{898425} \zeta_3 \right) \\
& + n_f^2 C_A C_F \left(\frac{314242565140920849001}{215285127496362000000} - \frac{12544}{16335} \zeta_4 + \frac{89550464}{121287375} \zeta_3 \right) \\
& + n_f^3 C_F \left(-\frac{2205751150439}{15885856515375} + \frac{25088}{245025} \zeta_3 \right), \quad (\text{A.7})
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=12) = & n_f C_F^3 \left(\frac{88961716829219432715740321165467}{6065706909620362869213148800000} - \frac{671898176890091}{475272972023850} \zeta_3 \right. \\
& \left. + \frac{2642589184}{1758511755} \zeta_4 - \frac{124820}{39039} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{116621076523257514706541796876157}{19410262110785161181482076160000} - \frac{1989733300788683}{1267394592063600} \zeta_3 \right. \\
& \left. - \frac{18419693641}{3517023510} \zeta_4 + \frac{62410}{117117} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(\frac{108542242435054542124290045599}{210067771761744168630758400000} + \frac{7458806358343849}{228131026571448000} \zeta_3 \right. \\
& \left. + \frac{108549713}{29066310} \zeta_4 - \frac{549260}{351351} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{13789024918875535939}{130167745507800000} - \frac{261789233833}{33202669500} \zeta_3 - \frac{11883280}{117117} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{2797424774494087428631891051}{3209368735248869242969920000} - \frac{15566147588}{15826605795} \zeta_3 + \frac{62410}{117117} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{17400519563132679535658867}{16159059366288012971596800} + \frac{536686847}{1055107053} \zeta_3 - \frac{62410}{117117} \zeta_4 \right) \\
& + n_f^3 C_F \left(-\frac{127821768039445576087}{1233139451029305019200} + \frac{24964}{351351} \zeta_3 \right), \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=14) = & n_f C_F^3 \left(\frac{378715964141637885273854172708551}{29692271585554223835309120000000} - \frac{367639947406454}{290813881483125} \zeta_3 \right. \\
& \left. + \frac{505959889}{430404975} \zeta_4 - \frac{22472}{9555} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{158391038218926832370900571697207}{25124229803161266322184640000000} - \frac{14044584522181}{8616707599500} \zeta_3 \right. \\
& \left. - \frac{17190217637}{4304049750} \zeta_4 + \frac{11236}{28665} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(+\frac{212618793832045564739311832977}{171789605491700966305536000000} + \frac{77579129461513987}{76774864711545000} \zeta_3 \right. \\
& \left. + \frac{933124519}{331080750} \zeta_4 - \frac{15356}{6615} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(+\frac{19073114986773056430079}{207292134721171500000} + \frac{1655601644872528}{246073284331875} \zeta_3 - \frac{2945344}{28665} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{4032068581057610850590942023}{6344502475545774323784000000} - \frac{14300095639}{19368223875} \zeta_3 + \frac{11236}{28665} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(+\frac{15677247599879616342540623}{19014492933703619352000000} + \frac{72862649}{195638625} \zeta_3 - \frac{11236}{28665} \zeta_4 \right) \\
& + n_f^3 C_F \left(-\frac{1225131890207918292167}{150908674077012852000000} + \frac{22472}{429975} \zeta_3 \right), \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=16) = & n_f C_F^3 \left(\frac{52845922593469053066814397892836049514811}{4700647014054801534593682386898124800000} - \frac{1576051471675106357}{1378077594461568000} \zeta_3 \right. \\
& \left. + \frac{1209091491169}{1274723049600} \zeta_4 - \frac{18769}{10404} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{272131316618720437180274758003365964006181}{43089264295502347400442088546566144000000} - \frac{186362119838618569}{120140097978700800} \zeta_3 \right. \\
& \left. - \frac{805069021181}{254944609920} \zeta_4 + \frac{18769}{62424} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(\frac{108178133162924948555733173122528196346341}{64633896443253521100663132819849216000000} + \frac{12453233313645527413979}{7816915474984167552000} \zeta_3 \right. \\
& \left. + \frac{19557316769}{8852243400} \zeta_4 - \frac{131747}{46818} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{194754027746301317663486903}{2385646868143957017600000} + \frac{488331702547711013}{28175156700490800} \zeta_3 - \frac{808523}{7803} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{27940032477586559318231174397709681}{58566283098537309493828740710400000} - \frac{220452248921}{382416914880} \zeta_3 + \frac{18769}{62424} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{5139790280893055367044834414063191}{7912907734820757808449287577600000} + \frac{1631966041}{5730523200} \zeta_3 - \frac{18769}{62424} \zeta_4 \right) \\
& + n_f^3 C_F \left(-\frac{13608819731912112034987483}{206666738093821415406796800} + \frac{18769}{468180} \zeta_3 \right), \tag{A.10}
\end{aligned}$$

$$\begin{aligned}
\gamma_{\text{ps}}^{(3)}(N=18) = & n_f C_F^3 \left(\frac{7253359571892497953990576741964828731260622709}{724111130666071269938038292783254417075200000} \right. \\
& \left. - \frac{109721806416706447798}{105229951281929298825} \zeta_3 + \frac{9447656272424}{12054216999825} \zeta_4 - \frac{236672}{165699} \zeta_5 \right) \\
& + n_f C_A C_F^2 \left(-\frac{28959853818114889791771647893787915791563566723}{4685424963133402334893188953303410934016000000} \right. \\
& \left. - \frac{751673700996836590582}{526149756409646494125} \zeta_3 - \frac{30932168351824}{12054216999825} \zeta_4 + \frac{118336}{497097} \zeta_5 \right) \\
& + n_f C_A^2 C_F \left(\frac{89162440521267641872038820315362221737605493}{45935538854249042498952832875523636608000000} \right. \\
& \left. + \frac{132300696845690213098985759}{67414515989255185999248000} \zeta_3 + \frac{50551793128}{28362863529} \zeta_4 - \frac{522368}{165699} \zeta_5 \right) \\
& + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{269658033224423391160920092267}{3660178707449866054252800000} + \frac{231728400575045385953}{9154108411989460920} \zeta_3 \right. \\
& \left. - \frac{17270272}{165699} \zeta_5 \right) \\
& + n_f^2 C_F^2 \left(-\frac{510320588931393559068451118322848217263}{1393078636581692811620875836456225600000} - \frac{10059181640656}{21697590599685} \zeta_3 + \frac{118336}{497097} \zeta_4 \right) \\
& + n_f^2 C_A C_F \left(\frac{5746360771534685866990135393992059327}{10978384977283253229773296769145600000} + \frac{24382094512}{108379573425} \zeta_3 - \frac{118336}{497097} \zeta_4 \right)
\end{aligned}$$

$$+ n_f^3 C_F \left(-\frac{12093315480521279173973861137}{220470747728360428912290678000} + \frac{236672}{7456455} \zeta_3 \right), \quad (\text{A.11})$$

$$\begin{aligned} \gamma_{\text{ps}}^{(3)}(N=20) = & n_f C_F^3 \left(\frac{2128032487727689123396891103081423002879945894061}{236298858959429600796016734112798923362304000000} \right. \\ & \left. - \frac{7463032385600125416449}{7804464223042296810000} \zeta_3 + \frac{9834028074797}{14900178793500} \zeta_4 - \frac{178084}{153615} \zeta_5 \right) \\ & + n_f C_A C_F^2 \left(-\frac{8442281731349030891500282315883757515259615913}{1413270687556397133947468505459323704320000000} \right. \\ & \left. - \frac{13512345934144930064021}{10405952297389729080000} \zeta_3 - \frac{7936779238702}{3725044698375} \zeta_4 + \frac{89042}{460845} \zeta_5 \right) \\ & + n_f C_A^2 C_F \left(\frac{250450109018215553669333751863263807123028219}{119012268425801863911365768880785154048000000} \right. \\ & \left. + \frac{40625424437896114995230699}{18397723661785041013440000} \zeta_3 + \frac{164760066767}{112031419500} \zeta_4 - \frac{4694036}{1382535} \zeta_5 \right) \\ & + n_f \frac{d_R^{abcd} d_R^{abcd}}{n_c} \left(\frac{124046988016629781809318499469746921}{1840118243383345660115052672000000} + \frac{34660205433264885994007}{1100342324269440252000} \zeta_3 \right. \\ & \left. - \frac{48237328}{460845} \zeta_5 \right) \\ & + n_f^2 C_F^2 \left(-\frac{20553091730130297702276618606953655791}{71772053747957053386934630643251200000} - \frac{2842660003013}{7450089396750} \zeta_3 + \frac{89042}{460845} \zeta_4 \right) \\ & + n_f^2 C_A C_F \left(\frac{688560020231378646396927215051130832957}{1602729320537086079392450022635008000000} + \frac{1316792611}{7223745375} \zeta_3 - \frac{89042}{460845} \zeta_4 \right) \\ & + n_f^3 C_F \left(-\frac{46235817346069201871585241841}{990993385042051234188945600000} + \frac{178084}{6912675} \zeta_3 \right). \quad (\text{A.12}) \end{aligned}$$

The results for $\gamma_{\text{ps}}^{(3)}$ at $N \leq 8$ in eqs. (A.3) – (A.6) have been published before in ref. [15], and the values at $N \leq 16$ of the terms proportional to the quartic group invariant $d_R^{abcd} d_R^{abcd}/n_c$ in eqs. (A.3) – (A.10) have already been obtained in ref. [17]. The n_f^3 contributions are known at all N [11].

A FORM file with our results for $\gamma_{\text{ps}}(N)$ at even $N \leq 20$ and all partial all- N expressions in the main text, and a FORTRAN subroutine of our approximations for the splitting function $P_{\text{ps}}^{(3)}(x)$ can be obtained from the preprint server <http://arXiv.org> by downloading the source. Furthermore they are available from the authors upon request.