

Constraints on RG Flows from Protected Operators

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We consider protected operators with the same conformal dimensions in the ultraviolet and infrared fixed point. We derive a sum rule for the difference between the two-point function coefficient of these operators in the ultraviolet and infrared fixed point which depends on the two-point function of the scalar operator. In even dimensional conformal field theories, scalar operators with exactly integer conformal dimensions are associated with Type-B conformal anomalies. The sum rule, in these cases, computes differences between Type-B anomaly coefficients. We argue the positivity of this difference in cases in which the conformal manifold contains weakly coupled theories. The results are tested in free theories as well as in $\mathcal{N} = 2$ superconformal QCD, necklace quivers and holographic RG flows. We further derive sum rules for currents and stress tensor two-point functions.

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1 Introduction

Renormalization Group (RG) flows are one of the most important and fascinating topics in theoretical physics, providing both conceptual and concrete connections between the long-distance, macroscopic behavior of a physical system and its short-distance, microscopic description [1–5]. The concept of *irreversibility* of RG flows was first shown in two-dimensional Quantum Field Theory (QFT) by Zamolodchikov [6], where he demonstrated the existence of a function monotonically decreasing along RG flows. This so-called *C-function* can then be interpreted as a non-perturbative counting of degrees of freedom. Similar properties have been established in three [7] and four dimensions [8] and are referred to as the *F*- and *a*-theorems, respectively. Cardy further proposed that in even dimensions, the relevant *C*-function is related to the trace anomaly [9]. Attempts have since been made to generalise the two- and four-dimensional proofs to six dimensions and higher using background dilatons [10–13], but a general proof remains elusive.⁴

Further generalizations to non-unitary two-dimensional QFTs [21] or including defects [22–24], as well as alternative proofs highlighting connections with fascinating non-perturbative results [25–27] have also been discussed in the literature. These *C*-functions being deeply related to the stress tensor, it is natural to ask whether similar quantities can be associated with other particular operators that can be tracked along an RG flow, and the constraints they impose. Such questions have been explored for instance in two dimensions for flavor currents [28], and to study critical exponents [29].

In this work, we explore the evolution of certain scalar operators along RG flows, whose two-point function in the fixed points is constrained to take the form

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{UV}} = \frac{C_{\Delta}^{\text{UV}}}{x^{2\Delta}} \ , \qquad \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{IR}} = \frac{C_{\Delta}^{\text{IR}}}{x^{2\Delta}} \ , \qquad (1.1)$$

where Δ is the conformal dimension of the operator $\mathcal{O}$, and the numerators C_{UV} , C_{IR} are positive numbers.⁵ We stress that the fact that the conformal dimensions in the UV and the IR are the same is a consequence of the assumption that the operator is protected, as discussed in more detail in Section 2. The presence of such operators in a generic QFT is of course not guaranteed, and it is often the consequence of an unbroken

⁴It has however been established and checked for large classes of RG flows between superconformal field theories (SCFTs) [14–20].

⁵The positivity of the two-point functions at the fixed point is a consequence of unitarity. Furthermore, note that we are not excluding the case for which $C_{\Delta}^{\text{IR}} = 0$, occurring when the operator is completely integrated out in the deep IR.

symmetry. They are for instance quite common supersymmetric theories, where certain BPS conditions forbid mixing along the RG flow, the most obvious example being operators that are part of the chiral ring in $4d$ $\mathcal{N} = 1$ theories. In the appropriate choice of normalisation the coefficients C_{Δ}^{UV} and C_{Δ}^{IR} then encode information about the theory and capture part of the data of their chiral ring.

We will however assume the existence of these operators without making any particular requirement about the origin of the underlying protection mechanism. In particular, we will not assume any supersymmetry, our goal rather being to derive general properties of RG flows involving this type of operators. We will then be able to derive a sum rule for the difference δC_{Δ} of the numerators for any protected operator $\mathcal{O}$ whose scaling dimension Δ remains constant along the RG flow:

$$\delta C_{\Delta} = C_{\Delta}^{\text{UV}} - C_{\Delta}^{\text{IR}} = \int d^d x \mathcal{D}_{\Delta} \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle, \quad (1.2)$$

where the differential operator $\mathcal{D}_{\Delta}$ depends on the scaling dimension, and whose exact form is derived in Section 2.1. Similar sum rules have been crucial in deriving the c -theorem and its generalizations [26, 27, 30–33]. As we will see in our case the differential operators $\mathcal{D}_{\Delta}$ is not manifestly positive. Furthermore, the two-point function is in general very difficult to compute non-perturbatively off criticality.

In order to circumvent this issue, we will expand this correlator in terms of its *spectral decomposition* as a sum of form factors $\langle 0 | \mathcal{O} | \alpha \rangle$, where $|\alpha\rangle$ is a state in the Fock space defined at large distances. In particular, we show that the sum rule given in equation (1.2) is not sensitive to single-particle states. As a consequence, it receives contributions only from multi-particle states. We then explore cases in which in the IR theory the operator $\mathcal{O}$ vanishes, and comment on the general case by using perturbative expansions for the spectral decomposition around the high-energy limit.

Our results are particularly interesting for theories in even spacetime dimensions where the protected operators have integer conformal dimensions $\Delta \in \mathbb{N}$. There, the relevant two-point function coefficients are associated with conformal anomalies [34, 35], and have in particular been discussed in the context of the supersymmetric theories [36–42]. By combining our results with those of references [36, 37] we will argue that $\delta C_{\Delta} \geq 0$ in the case of Type-B anomalies associated with theories whose conformal manifold contains a free point. Our strategy will be to use the fact that C_{Δ} is covariantly constant over along conformal manifolds [38] to tackle the difference δC_{Δ} at weak coupling, where non-perturbative effects are suppressed.

The rest of this work is organized as follows:

- ★ In Section 2 we present a derivation of the sum rule resembling the case of the c -function [6, 30], with the crucial difference that a non-trivial differential operator is applied to the two-point function of the protected operators. We further adapt the derivation to study two-point functions of currents and the stress tensor.
- ★ In Section 3, after briefly reviewing the spectral decomposition of CFTs, we combine the sum rule with the spectral decomposition to conclude that single-particle states do not contribute. Restricting ourselves to even spacetime dimensions and integer conformal dimensions of the protected operators, we then discuss our result in relation to Type-B anomalies. In cases where those anomalies are associated to theories whose UV CFT admits a free-field limit, we argue that $\delta C_\Delta \geq 0$. We also comment on the connection between our results and the average null energy operator.
- ★ In Section 4 we apply the sum rule to explicit examples, namely free theories, supersymmetric flows from $\mathcal{N} = 2$ SQCD and quiver theories, and holographic RG flows.

We discuss open questions and give our conclusions in Section 5. In addition, Appendix A gives further details on the conformal perturbation theory expansion necessary to complete the proof and probe the convergence of the sum rule in Section 2, and Appendix B reviews some aspects of the spectral decomposition used in the work.

2 The Sum Rule

Let us derive a sum rule for the evolution of the coefficients of the two-point function of protected operators. Our setup is the following: we consider an ultraviolet (UV) conformal field theory (CFT) which is deformed by a relevant operator, triggering an RG flow. Even though we will not assume the existence of a Lagrangian description, it is useful to consider a formal action for the UV CFT, $\mathcal{A}_{UV}$, deformed by a set of relevant operators Φ^I :

$$\mathcal{A} = \mathcal{A}_{UV} + \int d^d x g_I \Phi^I(x) . \quad (2.1)$$

In order to avoid an old—but still active—discussion on the nature of the endpoints of RG flows [43–49], we assume that the deep infrared (IR) is described by either a trivial theory with no local degrees of freedom or another CFT. We further assume that the spectrum of the UV theory contains an operator, $\mathcal{O}$, that is protected along

the flow and does not mix with other operators, meaning that it has the same conformal dimension in the UV and IR fixed point. In practice, this means that along the RG flow parameterized by a scale Λ , the connected component of the two-point function of $\mathcal{O}$ at separated points is given by

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle = \frac{C_\Delta(\Lambda|x|)}{|x|^{2\Delta}}, \quad (2.2)$$

where Δ is the conformal dimension of the operator $\mathcal{O}$ at both the UV and IR fixed point. The fact that the operator is protected can be understood from the fact that the function C_Δ asymptotes to a constant in both the UV and IR:

$$C_\Delta^{\text{UV}} = \lim_{|x| \rightarrow 0} C_\Delta(\Lambda|x|), \quad C_\Delta^{\text{IR}} = \lim_{|x| \rightarrow \infty} C_\Delta(\Lambda|x|). \quad (2.3)$$

We stress again that this is not a generic situation: in the most general case the conformal dimension in the UV is different from the conformal dimension in the IR. This feature is common in supersymmetric theories, where these types of protected operator correspond to for instance chiral-ring operators. There, they are protected by BPS condition interpreted as null vectors in the Hilbert space, ensuring that their two-point function is of the form given in equation (2.2). In the sequel, we will however not restrict ourselves to supersymmetric theories, and only assume the existence of an operator with the properties described above.

Furthermore, note that the fact that the operator is encoded in the behavior of the numerator of its two-point function is also the case for conserved currents such as the stress tensor or flavor currents. In those cases, this is ensured by the conservation equation. The evolution of the equivalent of C_Δ has been extensively studied [6, 21, 25, 26, 28, 50], and we will derive similar sum rules for those cases as well.

2.1 Derivation of the Sum Rule

Our strategy to derive the sum rule is similar to what was done in the most common derivation of the c -theorem in two dimensions [6, 30]. There, the derivative of the c -function, interpolating the central charges of UV and IR theories, was written in terms of the two-point function of the trace of the stress tensor. Integrating the resulting expression, one obtains a sum rule for the difference $\delta c = c^{\text{UV}} - c^{\text{IR}}$. We will follow similar procedure for protected operators as we are interested in the difference of the quantities defined in equation (2.3):

$$\delta C_\Delta = C_\Delta^{\text{UV}} - C_\Delta^{\text{IR}}. \quad (2.4)$$

We therefore start with the two-point function of a protected operator $\mathcal{O}$ defined at any point of the RG flow, as shown in equation (2.2). Applying the d'Alembertian to the correlator, for any spacetime dimension d we have the relation:

$$\left(x^2\Box - 4\Delta\left(\Delta - \frac{d-2}{2}\right)\right)\langle\mathcal{O}(x)\overline{\mathcal{O}}(0)\rangle = 2(d-4\Delta)\frac{C'_\Delta}{|x|^{2(\Delta-1)}} + 4\frac{C''_\Delta}{|x|^{2(\Delta-2)}} , \quad (2.5)$$

where primed quantities indicate partial derivation with respect to $|x|^2$: $C'_\Delta = \frac{\partial C_\Delta}{\partial |x|^2}$. This relation will enable us to extract a sum rule for δC_Δ . Indeed, focusing first on the right-hand side of equation (2.5), by integrating over $|x|^2$ we find that

$$\int_0^\infty d|x|^2 \left(2(d-4\Delta)C'_\Delta + 4|x|^2 C''_\Delta\right) = 8\left(\Delta - \frac{d-2}{4}\right)\delta C_\Delta + 4\left(|x|^2 \frac{\partial}{\partial |x|^2} C_\Delta\right)\Bigg|_{|x|^2=0}^{|x|^2=\infty} , \quad (2.6)$$

where we have used integration by parts and equation (2.3). The boundary term on the right-hand side of equation (2.6) can be shown to always vanish by using perturbation theory around both ends of the RG flow. Using the formal action given in equation (2.1), we have the following expansion around the UV fixed point

$$C_\Delta = C_\Delta^{\text{UV}} + \sum_I g_I c_1^I |x|^{d-\Delta_{\Phi^I}} + \dots . \quad (2.7)$$

A derivation of this expansion, including higher-order terms, is given in Appendix A. The coefficients c_1^I can be computed in terms of the UV conformal data, but their precise value will not be relevant for our purpose. As the deformation operators Φ^I are by assumption relevant in order to trigger a non-trivial RG flow, we have $d - \Delta_{\Phi^I} > 0$, from which we infer that the boundary contribution in the UV is trivial:

$$\lim_{|x|^2 \rightarrow 0} |x|^2 \frac{\partial C_\Delta}{\partial |x|^2} = \lim_{|x|^2 \rightarrow 0} \sum_I \left[\frac{d - \Delta_{\Phi^I}}{2} c_1^I |x|^{d-\Delta_{\Phi^I}} + \dots \right] = 0 , \quad (2.8)$$

A similar analysis can be performed around the deep infrared. At the end of the flow, we perturb the CFT with irrelevant operators Ψ^I with coupling constants λ_I . We then obtain a similar (conformal) perturbative expansion:

$$C_\Delta = C_\Delta^{\text{IR}} + \sum_I \tilde{c}_1^I \lambda_I |x|^{d-\Delta_{\Psi^I}} + \dots . \quad (2.9)$$

The operator at the end of the RG flow being irrelevant it has conformal dimension $d - \Delta_{\Psi^I} < 0$, and we therefore have

$$\lim_{|x|^2 \rightarrow \infty} |x|^2 \frac{\partial}{\partial |x|^2} C_\Delta = \lim_{|x|^2 \rightarrow \infty} \left[\frac{d - \Delta_{\Psi^I}}{2} \tilde{c}_1^I \lambda_I |x|^{d-\Delta_{\Psi^I}} + \dots \right] = 0 . \quad (2.10)$$

We defer to Appendix A for additional details. We can therefore conclude that the last term in equation (2.6) vanishes, so that δC_Δ can be expressed as the integral of a differential operator acting on the two-point function, as advertised around equation (1.2). Restoring the numerical factors coming from the change of variables to obtain an integral over $|x|^2$, we find:

$$\delta C_\Delta = \frac{\Gamma(\frac{d}{2})}{4\pi^{\frac{d}{2}}(2\Delta - \nu)} \int d^d x |x|^{2\Delta-d} (|x|^2 \square - 4\Delta(\Delta - \nu)) \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle, \quad (2.11)$$

where we defined $\nu = \frac{1}{2}(d - 2)$, the unitarity bound for the conformal dimension of scalar operators—saturated by free scalar fields.

The convergence of this sum rule is guaranteed by conformal perturbation theory. In fact, around the UV and the IR fixed points where singularities are in general present, the expansions in equations (2.7) and (2.9) ensure that the integral in the sum rule is well behaved, and that the sum rule is convergent.

In the next section, we will show that this sum rule can be used to find bounds on δC_Δ , and we check its validity explicitly in a variety of examples in Section 4.

2.2 Conserved Currents and the Stress Tensor

In the derivation of the sum rule given in equation (2.11), we have only made use of the functional form of the two-point function of the protected operator, see equation (2.2). While the focus of this work is on scalar operators, this nonetheless enable us to extend it to the case of other spinning operators that are under control under the RG flow, namely the stress tensor and conserved currents associated with unbroken flavor symmetries.

Conserved Currents: the conservation equation for unbroken symmetries can be used to track conserved currents and safely define their two-point function along the RG flow. We will focus here on the parity-even component of the correlator of currents associated with an Abelian symmetry. The non-Abelian case can similarly be obtained by choosing an orthogonal Killing basis for the generators of adjoint representation: $\text{Tr}(T^a T^b) \propto \delta^{ab}$.

A Lorentz-scalar correlator can then be obtained by contracting the indices of the two currents, in which case we obtain a two-point function similar to that of protected

operators [51].⁶

$$\langle J^\mu(x) J_\mu(0) \rangle = (d-2) \frac{C_J(\Lambda|x|)}{|x|^{2(d-1)}}. \quad (2.12)$$

At the two fixed points, the function $C_J(\Lambda|x|)$ reduces to the flavor central charge of the two CFTs, which is a positive number by unitarity:

$$C_J^{\text{UV}} = \lim_{|x| \rightarrow 0} C_J(\Lambda|x|), \quad C_J^{\text{IR}} = \lim_{|x| \rightarrow \infty} C_J(\Lambda|x|). \quad (2.13)$$

We will now assume $d > 2$ to find a sum rule for $\delta C_J = C_J^{\text{UV}} - C_J^{\text{IR}}$. As the functional form of equation (2.12) is the same as that of protected scalar operators, we can follow the same procedure to find:

$$\delta C_J = \frac{\Gamma(\frac{d}{2})}{8\pi^{\frac{d}{2}}(3d-2)(d-2)} \int d^d x |x|^{d-2} (|x|^2 \square - 2d(d-1)) \langle J^\mu(x) J_\mu(0) \rangle, \quad (2.14)$$

As for that of scalar operators, this sum rule is not manifestly positive definite. A similar sum rule for the flavor central charge was recently proposed in reference [51]. There, the sum rule is a consequence of the conservation equation. It would be interesting to understand the connections between these sum rules, and check if they differ or are simply equivalent. We leave a detailed study for future work. In the case of $d = 2$, positivity of δC_J was proven in reference [28].

The stress tensor: we can repeat the same reasoning to the parity-even sector of the two-point function of the stress tensor. Contracting the indices, we obtain [51]

$$\langle T^{\mu\nu}(x) T_{\mu\nu} \rangle = \frac{(d-1)(d-2)}{2} \frac{C_T(\Lambda|x|)}{|x|^{2d}}. \quad (2.15)$$

where we find once again a correlator of the form given in equation (2.12) for protected scalar operators. At the two fixed points we have:

$$C_T^{\text{UV}} = \lim_{|x| \rightarrow 0} C_T(\Lambda|x|), \quad C_T^{\text{IR}} = \lim_{|x| \rightarrow \infty} C_T(\Lambda|x|), \quad (2.16)$$

so that $C^T(\Lambda|x|)$ interpolates between the two central charges of the UV and IR CFTs. In two dimension $C_T(\Lambda|x|)$ is a C -function, but we cannot find a sum rule in that case

⁶The uncontracted two-point function $\langle J^\mu J^\nu \rangle$ depends generically on two independent functions, $C_J^{(1)}(\Lambda|x|)$ and $C_J^{(2)}(\Lambda|x|)$, which gives the flavor central charge C_J at the CFT points. We defer to Appendix A of reference [51] for a derivation of equation (2.12).

due to the factor $d - 2$ in equation (2.15). When $d > 2$, we can again use the procedure for scalar protected operators described above, and we conclude that:

$$\delta C_T = \frac{\Gamma(\frac{d}{2})}{4\pi^{\frac{d}{2}}(3d+2)(d-2)(d-1)} \int d^d x |x|^d (|x|^2 \square - 2d(d+2)) \langle T^{\mu\nu}(x) T_{\mu\nu}(0) \rangle . \quad (2.17)$$

This equation is very different from the sum rule given in reference [51] for the same quantity (see equations (2.31) and (2.32) therein). It would be interesting to compare the two sum rules, and check whether they are equivalent or encode different properties. We leave this analysis for future work. Furthermore notice that, as for flavor currents, a theorem on the positivity of δC_T was proven only in two dimensions [6]. The fact that the case $d = 2$ behaves differently than in higher dimension is consistent with the non-positivity of the integrand in the sum rule above. The study of the evolution of the central charge is nonetheless a current subject of research [32, 33, 52]. Even if a positivity theorem cannot be constructed as counterexamples are known [53, 54], it is however interesting to bound the difference δC_T and describe it in terms of physical quantities via sum rules.

All the sum rules derived above are not manifestly positive due to the presence of the differential operator applied to the two-point function, which is itself not manifestly positive definite. As shown in references [32, 33] it is still possible to study the resulting sum rule. In the following section we will do that for the scalar case by combining the sum rule with the spectral decomposition of the two-point function.

3 Spectral Decomposition and Constraints on RG Flows

The sum rule we have derived in equation (2.11) give constraints on the possible RG flows. They can however be difficult to extract, as the two-point function, in general, involves non-perturbative effects, and makes its evaluation away from the fixed points difficult, as conformal perturbation theory cannot be reliably used.

In order to nonetheless find constraints on the RG flows from the protected operators, we will combine the sum rule with the spectral decomposition of a scalar two-point function. The latter is an expansion in terms of form factors $|\langle 0 | \mathcal{O} | \alpha \rangle|$, where the set of states $|\alpha\rangle$ represent a basis of the asymptotic Fock basis. As briefly reviewed in Appendix B, applied to the two-point function of scalar operators, it takes the form:

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \int_0^\infty ds \, \rho(s) G_s(x) , \quad (3.1)$$

where $G_s(x)$ is the propagator of a free scalar field of mass $m^2 = s$ in d dimension

$$G_s(x) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ipx}}{p^2 + s} = \frac{1}{(2\pi)^{\nu+1}} \left(\frac{\sqrt{s}}{|x|} \right)^\nu K_\nu(\sqrt{s}x), \quad \nu = \frac{d-2}{2}. \quad (3.2)$$

which is written in terms of the modified Bessel function of the second kind $K_\alpha(x)$.

Even though the spectral density $\rho(s)$ is arguably the simplest non-perturbative quantity in Quantum Field Theory, its explicit expression is known in very few cases. Recent attempts have been made to constrain the spectral density at the non-perturbative level using modern methods of the conformal bootstrap [50, 51, 55]. We will here instead make use of only some of its analytical properties and asymptotic behavior.

3.1 The Spectral Decomposition in CFTs

Before combining the spectral decomposition with the sum rule we have found in the previous section to get new constraints on the RG flows, we will examine some of its intriguing properties in the case of CFTs, particularly its connection to conformal anomalies.

In the context of the CFTs, the spectral decomposition of the stress tensor has been discussed in reference [25] and more recently in references [50, 51]. We provide a brief overview of their results in Appendix B.1. By adapting their methods in the case of protected scalar operators, we will show that their spectral decomposition is sufficient to reproduce the structure of the associated conformal anomalies.

Contrary to the case of the stress tensor, the absence of a Lorentz structure for scalars implies that only spin-zero states can contribute. Furthermore, imposing scale invariance, we are left with only two possible contributions for the spectral density $\rho(s)$:⁷

$$\text{a) } \rho(s) = \tilde{C}_\Delta s^{\Delta - \frac{d-2}{2}} \delta(s), \quad \text{or} \quad \text{b) } \rho(s) = \tilde{C}_\Delta s^{\Delta - \frac{d}{2}}. \quad (3.3)$$

The first possibility leads to a vanishing correlator, which is not physical since it should imply—in unitary theories—that $\mathcal{O}$ is the trivial operator as a consequence of the Reeh–Schlieder theorem [56, 57]. The only exception is $\Delta = \frac{1}{2}(d-2)$ which corresponds to an operator saturating the unitarity bound; in that case we have

$$\int_0^\infty ds \delta(s) \frac{\tilde{C}_{\frac{d-2}{2}}}{p^2 + s} = \frac{\tilde{C}_{\frac{d-2}{2}}}{p^2}, \quad (3.4)$$

⁷Observe that $\tilde{C}_\Delta \propto C_\Delta$, and the two constant differs only by factors depending on the spacetime dimension and conformal dimension of the operator.

which is nothing but the free-theory propagator, as expected from the fact that a scalar operator whose conformal dimension saturates the unitarity bound, $\Delta = \frac{1}{2}(d-2)$, satisfies the equations of motion of a free scalar field. As a consequence only a single-particle state contribute to the propagator between two fundamental free scalar.

Let us now turn to case b), which is the only non-trivial possibility when $\Delta > \frac{1}{2}(d-2)$. The integration is straightforward in position space, and we obtain the correct expression for the two-point function of a scalar operator of dimension Δ —up to the case of the free scalar, discussed above⁸

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle = \int_0^\infty ds \tilde{C}_\Delta s^{\Delta-\frac{d}{2}} G_s(x-y) \propto \frac{\tilde{C}_\Delta}{|x-y|^{2\Delta}}. \quad (3.5)$$

As we recover the expression of the two-point function, this shows that the spectral density at the CFT point—up to the free scalar field—is the correct expression of a two-point function in CFT, confirming that the correct spectral density for a CFT is given by $\rho(s) \propto s^{\Delta-d/2}$.

It is instructive to study the same integral in momentum space, as it will make the connection with conformal anomalies clear. By direct integration we have

$$\langle \mathcal{O}(p) \overline{\mathcal{O}}(-p) \rangle = \int_0^\infty ds s^{\Delta-\frac{d}{2}} \frac{\tilde{C}_\Delta}{p^2+s} \propto \tilde{C}_\Delta p^{2\Delta-d} \csc\left(\frac{\pi}{2}(d-2\Delta)\right), \quad (3.6)$$

Among all the factors a special role is played by the cosecant function. Indeed, the convergence of the integral is, strictly speaking, only ensured when $d > 2\Delta$. However it is straightforward to define the analytic continuation of the integral above so that equation (3.6) holds for any $\Delta \notin \frac{d}{2} + \mathbb{N}$. In the latter cases, due to the cosecant function, the two-point function remains divergent, and needs to be regularized. Using dimensional regularization in $d-\epsilon$ dimensions, we can obtain a finite result in the limit $\epsilon \rightarrow 0$ by subtracting the divergent term. To see this, let us consider the indefinite integral

$$\int ds \frac{s^{\Delta-\frac{d-\epsilon}{2}}}{p^2+s} = \frac{2s^{\Delta-\frac{d-2-\epsilon}{2}} {}_2F_1\left(1, \Delta-\frac{d-\epsilon-2}{2}, 1+\Delta-\frac{d-\epsilon-2}{2}; -\frac{s}{p^2}\right)}{p^2(2\Delta-(d-\epsilon-2))}. \quad (3.7)$$

The evaluation of the above expression in the regime $s \rightarrow 0$ does not give any divergence, however for $s \rightarrow \infty$ we have a divergent term of the form

$$\left(\int_0^\infty ds \frac{s^{\Delta-\frac{d-\epsilon}{2}}}{p^2+s} \right)_{\text{div.}} \propto \frac{p^{2\Delta-d}}{\epsilon}. \quad (3.8)$$

⁸The free propagator is usually normalized differently, since we are interested in the functional dependence on the spacetime coordinates, we omit here those factors which can be thought as included in the definition of $\tilde{C}_\Delta$.

This divergent term needs to be subtracted with a proper counterterm: this procedure will introduce a choice of the regularization scheme which will lead to a constant $\tilde{c}$ in the two-point function and introduce a scale μ . After the subtracting the divergence, we can safely take $\epsilon \rightarrow 0$ and the regularized two-point function reads

$$\langle \mathcal{O}(p) \overline{\mathcal{O}}(-p) \rangle = p^{2\Delta-d} \left(\tilde{C}_\Delta \log \frac{p^2}{\mu^2} + \tilde{c} \right). \quad (3.9)$$

This coincides with the expected result derived in references [34, 35]. Furthermore, in even spacetime dimensions, the operators for which the regularization above is required have integer conformal dimension. For these, the presence of the logarithm $\ln \mu$ is the hallmark of a type-B conformal anomaly [58], as we explain further in Section 3.3.

3.2 Protected Operators Along RG Flows

Having reviewed the spectral decomposition for CFTs, let us now apply it to the sum rule to extract constraints on RG flows. In position space, using equations (3.1) and (3.2), the sum rule can be expressed as

$$\delta C_\Delta = \frac{1}{4\Delta - 2\nu} \int dx x^{2\Delta-1} \int ds (x^2 s - 4\Delta(\Delta - \nu)) \rho(s) \left(\frac{\sqrt{s}}{x} \right)^\nu K_\nu(\sqrt{s}x), \quad (3.10)$$

where we used $\nu = \frac{1}{2}(d-2)$ for convenience and we used the properties of $K_\nu(x)$ to trade the d'Alembertian for powers of s . It is now crucial to know the analytic structure of the spectral density in the complex s plane. As depicted schematically in Figure 1, it can be decomposed into single- and multi-particle states. The single-particle states contributions is given by

$$\rho_{\text{sp}}(s) = \sum_i c_i \delta(s - m_i^2), \quad (3.11)$$

where the sum runs over all single-particle states contributing to the two-point functions. For multi-particle states, we instead have a contribution given by

$$\rho_{\text{mp}}(s) = \sigma(s) \Theta(s - s_{\text{th}}), \quad (3.12)$$

where s_{th} is the threshold energy corresponding to a multiple of the mass of the lightest exchanged particle and $\sigma(s)$ is a theory-dependent function.

Coming back to equation (3.10), we observe that single-particle states do not contribute to the sum rule. This can be seen from the following identity:

$$\int dx x^{2\Delta-1} (x^2 m_i^2 - 4\Delta(\Delta - \nu)) \left(\frac{m_i}{x} \right)^\nu K_\nu(m_i x) = 0, \quad (3.13)$$

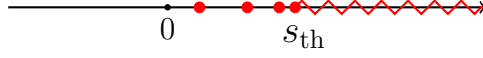


Figure 1: Schematic analytic structure of the spectral density. The red dots indicate poles corresponding to single-particle contributions and their location is given by the square mass of the particle exchanged. The zigzag line denotes the branch cut corresponding to multi-particle contributions.

for any Δ and ν , and using the properties of the modified Bessel function of the second kind $K_\nu(x)$. This is a remarkable feature of the sum rule, and is a consequence of its specific functional form.

Note that this assumes that the spectral decomposition can be separated in terms of single- and multi-particle contributions, where the latter contributes in the CFT limit at both high- and low-energy. There are indeed examples for which this assumption is not correct, for instance in the planar-limit of Yang–Mills theory. This is however believed to be a large- N artifact, as multi-particle states are $1/N$ -suppressed and therefore absent in the planar limit, and not a feature of physical theories.⁹

As there are no contributions from single-particle states, the sum rule can be rewritten as

$$\delta C_\Delta = \frac{1}{4\Delta - 2\nu} \int dx \, x^{2\Delta-1} \int_{s_{\text{th}}}^\infty ds \, \sigma(s) (x^2 s - 4\Delta(\Delta - \nu)) \left(\frac{\sqrt{s}}{x} \right)^\nu K_\nu(\sqrt{s}x), \quad (3.14)$$

To proceed, we will use the asymptotic expansion of the spectral decomposition. The UV divergence of the sum rule indeed fixes the leading behavior of the spectral density to be

$$\rho(s) \sim C_\Delta^{\text{UV}} \frac{s^{\Delta-\frac{d}{2}}}{\Gamma(\Delta + 2\nu)}. \quad (3.15)$$

The derivation of the above asymptotic is given in Appendix B.2. Its physical interpretation is that the spectral density at high energy is dominated by the UV contribution of case b) in equation (3.3). Assuming analyticity of the spectral density, we find the following expansion

$$\rho(s) = C_\Delta^{\text{UV}} \frac{s^{\Delta-\frac{d}{2}}}{\Gamma(\Delta + 2\nu)} \left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots \right). \quad (3.16)$$

⁹We thank Zohar Komargodski for bringing this point to our attention.

This expansion is correct in free theories and can be extended to perturbation theory, see Appendix B.2 for a detailed explanations. Since one can think of the expansion above as a (conformal) perturbation around the UV fixed point for highly non-perturbative theories (such as RG flows starting and ending at isolated fixed points which are far away in the theory space) the expansion above can be trusted only around the UV fixed point since non-perturbative effects will arise along the RG flow.

This leads us to distinguish between two possibilities:

- i) The branch cut associated with multi-particle states starts at $s = s_{\text{th}} > 0$. In the infrared, the protected operator $\mathcal{O}$ vanishes, meaning that it has a zero two-point function.¹⁰
- ii) The branch cut instead starts from $s = s_{\text{th}} = 0$. This is the case for which the protected operator $\mathcal{O}$ is non-trivial in the infrared.

As these two cases lead to different physical behaviors, we will discuss them separately.

Case i) The branch cut starts away from the origin, $s_{\text{th}} > 0$. The expansion in equation (3.16) does not account for non-perturbative effects, however, since single-particle states do not contribute we can use it to approximate the multi-particle contribution around the uv fixed point. The sum rule in the form given in equation (3.14) is therefore

$$\delta C_{\Delta} = \frac{1}{4\Delta - 2\nu} \int_0^{\infty} dx x^{2\Delta-1} \int_{s_{\text{th}}}^{\infty} ds \sigma(s) (x^2 s - 4\Delta(\Delta - \nu)) \left(\frac{\sqrt{s}}{x}\right)^{\nu} K_{\nu}(\sqrt{s}x) . \quad (3.17)$$

It is then easy to show by direct computation that the leading term in the expansion of equation (3.16) is given by

$$\delta C_{\Delta} = C_{\Delta}^{\text{UV}} . \quad (3.18)$$

Moreover, it is possible to check that any other contribution will vanish. This can be done in full generality in $d = 3$, as the Bessel function simplifies to an exponential, but is more arduous to show in higher dimensions. We have checked this is indeed the case in various dimensions for the first few corrections.

Physically, our result is clear. The IR regime is encoded in the behavior of the spectral decomposition around $s = 0$. By assumption $s_{\text{th}} > 0$ and single-particle state do not contribute, and the operator therefore vanishes in the infrared so that $C_{\Delta}^{\text{IR}} = 0$, and we obtain $\delta C_{\Delta} = C_{\Delta}^{\text{UV}}$.

¹⁰In fact, the IR coefficient C_{Δ}^{IR} of the two-point function coefficient is associated with the discontinuity of the branch cut at low energy, i.e. $s \sim 0$.

Case ii) The branch cut now starts at $s_{\text{th}} = 0$, and the sum rule is given by

$$\delta C_{\Delta} = \frac{1}{4\Delta - 2\nu} \int_0^{\infty} dx x^{2\Delta-1} \int_0^{\infty} ds \sigma(s) (x^2 s - 4\Delta(\Delta - \nu)) \left(\frac{\sqrt{s}}{x}\right)^{\nu} K_{\nu}(\sqrt{s}x). \quad (3.19)$$

The first order in the expansion given in equation (3.16) can be found using the following identity

$$\int_0^{\infty} ds s^{\frac{\alpha}{2}-1} K_{\nu}(\sqrt{s}x) = 2^{\alpha-1} x^{-\alpha} \Gamma\left(\frac{\alpha-\nu}{2}\right) \Gamma\left(\frac{\alpha+\nu}{2}\right), \quad (3.20)$$

with $\alpha > \nu$, and we obtain that at leading order the sum rule vanishes, that is, $C_{\Delta}^{\text{UV}} \sim C_{\Delta}^{\text{IR}}$. However, one can check that the higher-order terms diverge, although physically we expect the difference δC_{Δ} to be finite. This means that non-perturbative must be taken into account in order to resum the expansion given in equation (3.16), and that δC_{Δ} is dominated by non-perturbative effects.

Let us now summarise the two different behaviors and their physical interpretation. At a scale Λ along the RG flow, high-energy contributions in the sum rule are dominated by those for which $s \gg \Lambda$. Conversely, low-energy contributions are dominated by $s \ll \Lambda$. In case i) the low-energy contributions are given by single-particle states, as we expect s_{th} to be at least of the same of magnitude of Λ . However we have found around equation (3.13) that such single-particle states do not contribute to the sum rule and we conclude that $C_{\Delta}^{\text{IR}} = 0$ and $\delta C_{\Delta} = C_{\Delta}^{\text{UV}}$.

On the other hand for case ii), massless states participate to the sum rule as the branch cut starts at $s_{\text{th}} = 0$, and IR contributions are non-trivial. We have found that in that case, at leading order $C_{\Delta}^{\text{IR}} = C_{\Delta}^{\text{UV}}$. However non-perturbative effect must be taken into account, and we cannot conclude $\delta C_{\Delta} \geq 0$. Despite this, when the protected operators are associated with type-B anomalies, there is strong evidence that this is indeed the case, as we will argue in the next paragraph.

3.3 Type-B Conformal Anomalies

The constraints above have a natural interpretation in terms of anomaly coefficients in even spacetime dimensions. We will now review the connection between conformal anomalies and the two-point function discussed above explicitly. The standard procedure is to consider a CFT coupled to a curved spacetime background with metric $\gamma_{\mu\nu}(x)$, and spacetime-dependent sources $J^I(x)$ for a collection of scalar operators $\mathcal{O}_I$. The resulting quantum effective action $W[\gamma_{\mu\nu}, J^I] = \log Z[\gamma_{\mu\nu}, J^I]$ then acts as a generating functional for connected correlator involving the stress tensor $T_{\mu\nu}$ and the

operators $\mathcal{O}_I$. We will take the scalar operators to have dimensions $\Delta - d/2 \in \mathbb{N}$. It is well known that conformal anomalies are neatly encoded into a local anomaly obtained from a variation of the quantum effective action W under generalised Weyl transformations [59, 60]

$$\begin{aligned}\delta_\sigma W[\gamma_{\mu\nu}, J^I] &= \int d^d x \sqrt{\gamma} \sigma(x) \mathcal{A}(\gamma_{\mu\nu}, J^I), \\ \delta_\sigma \gamma_{\mu\nu} &= 2\sigma(x) \gamma_{\mu\nu}, \quad \delta_\sigma J^I(x) = -\Delta \sigma(x) J^I(x).\end{aligned}\tag{3.21}$$

An important point is that the anomaly $\mathcal{A}$ must be a *local* function of the sources and their derivatives. Up to scheme-dependent local counterterms, one then distinguishes between two types of anomalies [58]: those that vanish when integrated over spacetime at constant σ , called type A, and those that do not, called type B. An example of type-A anomalies is a terms of the form $\sigma a \sqrt{\gamma} E_d$ where E_d is the Euler density, which can be written as a total derivative, and its coefficient, a , is then the quantity that is relevant for the a -theorem [8]. On the other hand, type-B anomalies can also be shown to equivalently arise through an explicit $\log \mu$ dependence in the effective action, and by extension in the associated correlators.¹¹

In the case of protected operators with integer conformal dimension $\Delta = n - d/2$, the anomaly contains a term

$$\delta_\sigma W \supset \int d^d x \sqrt{\gamma} \sigma(x) C_{IJ} J^I \Delta_c \bar{J}^J, \quad \Delta_c = \square^n + \text{curvature terms}, \tag{3.22}$$

which does not integrate to zero for constant σ and is therefore associated with a type-B anomaly. One then finds that the anomaly coefficient is the numerator of the two-point function by functional derivation with respect to the sources:

$$C_{IJ} = \langle \mathcal{O}_I(1) \bar{\mathcal{O}}_J(0) \rangle \propto \frac{\delta^2 W}{\delta J^I(x) \delta \bar{J}^J(y)} \bigg|_{\substack{x=1, y=0 \\ J=\text{const}}} . \tag{3.23}$$

As we have seen in the beginning of this section, for a single operator, in momentum space this correlator involves a term proportional to $\log \mu$, see equation (3.9), confirming the presence of a type-B conformal anomaly.

Type-B anomalies along the conformal manifold Among Type-B anomaly coefficients, a special role is played by those related to marginal operators, i.e. operators

¹¹Strictly speaking, type-A anomalies can also give rise to a $\log \mu$ dependence, but only when the partition function is computed on a space with a non-trivial topology such as the d -sphere, see e.g. [39, 61] and references therein.

for which $\Delta = d$. The corresponding Type-B anomaly coefficient $\chi_{ij} = \langle \mathcal{O}_i(1) \overline{\mathcal{O}}_j(0) \rangle$ define the Zamolodchikov metric of the conformal manifold. The Zamolodchikov metric is related to other type-B anomaly coefficients through an intricate web of Weyl consistency conditions [36, 38, 47, 59, 60, 62]. In particular, it can be shown that type-B anomalies are covariantly constant with respect to the connection ∇_χ defined by the Zamolodchikov metric [36, 38]:

$$\nabla_\chi C_\Delta = 0 . \quad (3.24)$$

As a corollary, it means that computing a coefficient at a single point of the conformal manifold, one can in principle then find its value at any point using equation (3.24) [36, 37]. In particular, if it admits a free point¹², computations can be made significantly easier. Note however that while these points have a special role in the structure of conformal manifolds [63, 64], they can only occur if it is non-compact [65], and their presence is therefore not guaranteed. This is trivially the case for instance with isolated CFTs, since they do not have any marginal deformations.

When such points exist, this is very useful to study the sum rule. Indeed, starting from a generic point of the conformal where the spectral decomposition on the associated RG flow is difficult to compute, we can then use equation (3.24) to go close to a free point. There the expansion given in equation (3.16) is now a good approximations, since non-perturbative effects are expected to be suppressed. Through the arguments above, we therefore expect that for protected operators associated with type-B anomalies,

$$\delta C_\Delta = C_\Delta^{\text{UV}} - C_\Delta^{\text{IR}} \geq 0 . \quad (3.25)$$

Using the covariance of C_Δ over the conformal manifold, see equation (3.24), we can in principle use this result to compute the difference between RG flows starting and ending at any point of the conformal manifold. This provides strong hints that protected operators associated with type-B anomalies indeed satisfy $\delta C_\Delta \geq 0$.

Note that generically in the free-field limit we can construct protected operators as combinations of fields which are either massless or massive, and the corresponding particle states play a role in the sum rule. A simple example is the case of two scalar fields ϕ_1, ϕ_2 in four dimensions, where the first is massive while the other is massless. It is then straightforward to see that the combination $\phi_1^2 + \phi_2^2$ then has $\delta C_{\Delta=2} = \frac{1}{2} C_{\Delta=2}^{\text{UV}}$, since the massive scalar will be integrated out, while the massless field will survive in the IR. This is consistent with our result via a direct evaluation of the sum rule, as we

¹²Depending of the particular RG flow it can be enough to assume the presence of a point of the conformal manifold in which only some of the coupling constants are small.

will see in Section 4. We will see explicit examples of such weak-coupling regimes in the following section.

Connection with ANEC? The generalization of the c -theorem in four dimensions, the a -theorem, cannot be proved by considering only two-point functions of stress energy tensor. This is due to the fact that the a -coefficient itself does not appear in the two-point function, contrary to its two-dimensional cousin. Recently a sum rule was found for the difference $\delta a = a^{\text{UV}} - a^{\text{IR}}$ involving the three-point function of the stress tensor. Positivity of δa was then shown to be a consequence of that of the expectation value of the average null energy (ANE) operator in any state [27]. The same approach also provided an alternative proof of the two-dimensional c -theorem [26], unifying conceptually these two theorems into a similar framework. It is then natural to ask whether the same can be also done in the case of Type-B anomaly coefficients.

To answer this question, let us consider the three-point function between protected operators and the stress tensor in four spacetime dimensions [36–38]:

$$\langle T^{\mu\nu}(p_1) \mathcal{O}(p_2) \overline{\mathcal{O}}(p_3) \rangle \sim \tilde{C}_\Delta p_2^{\Delta-2} p_3^{\Delta-2} \left(\delta^{\mu\nu} - \frac{p_1^\mu p_1^\nu}{p_1^2} \right) + \dots \quad (3.26)$$

The two scalar operators can be used to define the state¹³

$$|\psi(p)\rangle \simeq \int d^4x \, e^{ip \cdot x} \mathcal{O}(x) |0\rangle \quad (3.27)$$

and the stress tensor can be used to construct the ANE operator contracting with a null vector u (i.e. such that $u \cdot u = 0$) and integrating as

$$\mathcal{E}(v=0, \vec{x}=0) = \int_{-\infty}^{\infty} du \, T_{uu}(u, v=0, \vec{x}=0) = \int \frac{dp_\nu d^2\vec{p}}{\pi(2\pi)^2} T_{uu}(p_u=0, p_\nu, \vec{p}) \quad (3.28)$$

Here we use the convention $u = -(y + i\tau)$ and $v = y - i\tau$, where $(\tau, y, \vec{x})$ are the coordinates in Euclidean signature. From the equation above it is clear that the Type-B anomaly coefficient is proportional to a vanishing term when we interpret the three-point function in equation (3.26) as a vacuum expectation value of the ANE operator: the tensor structure involving the momentum associated with the stress tensor p_1 contains either a term proportional to δ_{uu} , which vanishes by definition of a null vector, or contains $p_{1u} = 0$.

¹³Another smearing factor may be necessary to properly normalize the sum rule [26, 27]. However as it is not crucial for our discussion, we will omit this issue.

dim	anomaly	Type	ANE	sum rule via 2-pt func.
2	c	A	[26]	[6, 30]
4	a	A	[27]	—
4	c	B	[32]	This work, [33]
4	C_Δ	B	—	This work

Figure 2: Summary of the different anomaly coefficients in $d = 2, 4$, their type of conformal anomaly, and whether it appears in the two-point function. We further indicate whether a sum rule using the ANE operator is known.

Note that the average null energy condition (ANEC) is a suitable tool to describe anomaly coefficients of type A. Type-B anomaly coefficients as the c -function in four dimensions and C_Δ discussed in this work are instead more naturally defined by two-point functions at the fixed points. A sum rule for the c -function in four dimensions—as opposed to the a -function discussed above—was derived recently using the ANE operator in [32]. However, contrary to the cases discussed above, the ANE operator is considered between two different states and therefore positivity is not guaranteed. We summarize those observations in Table 2. It would be interesting to also explore the connection between ANEC and the anomaly coefficient C_Δ to the uniformize the results of this work with the sum rules derived in references [26, 27, 32].

4 Examples

In this section, we apply the sum rule given in equation (2.11) to various examples, such as free fields and supersymmetric field theories, where the constraints on type-B conformal anomalies we have obtained in the previous section can be directly checked.

4.1 Free Scalar Theory

The simplest example we can consider is the free real scalar theory in $d > 2$ of mass m . Its action is given by

$$\mathcal{A} = \int d^d x \left[\frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 \right], \quad (4.1)$$

and the two-point between two scalars takes the simple form:

$$\langle \varphi(x)\varphi(0) \rangle = \mathcal{N}_d \left(\frac{m}{|x|} \right)^\nu K_\nu(m|x|), \quad (4.2)$$

where $\nu = \frac{1}{2}(d-2)$ and $\mathcal{N}_d$ is a normalization constant we choose so that in the massless limit $m \rightarrow 0$, $\langle \varphi(x)\varphi(0) \rangle x^{d-2} \rightarrow 1$ and such that in the ultraviolet, the two-point function is normalized to one. One can easily check that e.g. $\mathcal{N}_3 = \mathcal{N}_5 = \sqrt{\frac{2}{\pi}}$, $\mathcal{N}_4 = 1$, $\mathcal{N}_6 = \frac{1}{2}$, $\dots$

The operators we are interested in are powers of the free field φ . They satisfy our definition of protected operators as we are in a free-field theory. As discussed above, single-particle states do not contribute to the sum rule. As a consequence we expect no contribution from the sum rule when we substitute the free massive propagator in equation (4.2) and $\Delta = 2\nu$, since only a single-particle state contributes to the spectral density. In fact one can check that this is the case by direct integration. Let us check the sum rule for the cases φ^n , with $n = 2, 3, 4$. The correlators are then easily computed using Wick's theorem to reduce any such correlator in terms of the two-point function given in equation (4.2):

$$\langle \varphi^n(x)\varphi^n(0) \rangle = w_n \langle \varphi(x)\varphi(0) \rangle^n, \quad (4.3)$$

where w_n is the combinatorial factor given by Wick contractions. For the cases we are interested we have $w_2 = 2$, $w_3 = 18$ and $w_4 = 72$. The sum rule given in equation (2.11) can be now explicitly used to compute the difference $\delta C_{n\nu}$

$$\delta C_{n\nu} = \frac{2w_n \mathcal{N}_d^n}{\nu(2n-1)} \int d|x|^2 |x|^{2\Delta-d} (|x|^2 \square - 4\nu^2 n(n-1)) \left[\left(\frac{m}{|x|} \right)^\nu K_\nu(m|x|) \right]^n. \quad (4.4)$$

Since the infrared is a trivial theory, we expect that $\delta C_{n\nu} = C_{n\nu}^{\text{UV}} = w_d$. For $n = 2, 3, 4$, this is checked in a straightforward fashion by direct computation of the integral above.

4.1.1 The Spectral Decomposition

In free theory, the spectral decomposition can be computed exactly since only a certain number of multi-particle states will contribute. For the operator φ^2 , which has $\Delta = 2\nu$, the spectral density is given by [50]:

$$\rho(s) = \frac{\Theta(s-4m^2)}{N_d} |\langle 0|\varphi^2|n=2 \rangle|^2, \quad N_d = (2\pi)^{d-4} 2^{d-1} \sqrt{s}(s-4m^2)^{\frac{3-d}{2}}. \quad (4.5)$$

The form factor can be computed straightforwardly using an oscillator representation, and one finds

$$\langle 0|\varphi^2|n=2 \rangle = \int \frac{d^{d-1}k_1}{(2\pi)^{d-1}} \frac{1}{\omega_{\vec{k}_1}} \int \frac{d^{d-1}k_2}{(2\pi)^{d-1}} \frac{1}{\omega_{\vec{k}_2}} \langle 0|\mathbf{a}_{p_1}\mathbf{a}_{p_2}\mathbf{a}_{k_1}^\dagger\mathbf{a}_{k_2}^\dagger|0 \rangle = 2. \quad (4.6)$$

Plugging back the form factor into equation (4.5), we conclude that the spectral decomposition is given by

$$\begin{aligned}\rho(s) &= \frac{4\Theta(s-4m^2)}{(2\pi)^{d-4}2^{d-1}\sqrt{s}(s-4m^2)^{\frac{3-d}{2}}} \\ &= \Theta(s-4m^2)2^{7-2d}\pi^{4-d}\left(\frac{1}{s}\right)^{3/2}\left(1+2\frac{m^2}{s}+\mathcal{O}\left(\frac{m^4}{s^2}\right)\right).\end{aligned}\tag{4.7}$$

This expression can be used to recover the well-known expression of the two-point function of the protected operator φ^2 at any point of the RG flow. For instance, as a crosscheck in four dimension, using equation (3.1), evaluating the integral we have

$$\langle\varphi^2(x)\varphi^2(0)\rangle=\int_0^\infty ds\,\rho(s)\frac{\sqrt{s}}{x}K_1(\sqrt{s}x)=2\left(\frac{m}{x}K_1(mx)\right)^2,\tag{4.8}$$

which is the correct expression in a free-field theory.

4.1.2 Evolution Along the RG Flow

The existence of a simple sum rule for δC_Δ in free-field theory enables us to understand which energy scales contribute most by inserting an IR cutoff ϵ . For ease of exposition, we will consider the free scalar field in $d=3$ and the operator $:\varphi^2:$ for which $\Delta=1$ where the sum rule takes a very simple form. We can then define the progressive contribution to the sum rule as a function of the cutoff scale ϵ :

$$\delta C_1(\epsilon)=\frac{2}{3}\int_{1/\epsilon}^\infty dx\,x(x^2\Box-2)\frac{e^{-2mx}}{x^2}=\frac{e^{-\frac{2m}{\epsilon}}(4m+6\epsilon)}{3\epsilon}.\tag{4.9}$$

The limit $\epsilon\rightarrow\infty$ corresponds to the full sum rule, i.e. the case where the entire RG flow—from the UV to the deep IR—is considered. The function above is depicted in Figure 3. One can see that most of the information about δC_Δ is not encoded close to the endpoint, but rather during the bulk of the RG flow. For generic theories, we therefore also expect this data to be contained in the non-perturbative regime.

In this case, the most important contributions occur as we approach the IR fixed point ($m/\epsilon\gg 1$). However, an expansion around this point is not enough to have a good estimate of δC_Δ , and demonstrates the deeply non-perturbative nature of the sum rule. This is not unexpected, as there is an analogous behavior in the two-dimensional c -theorem [30].

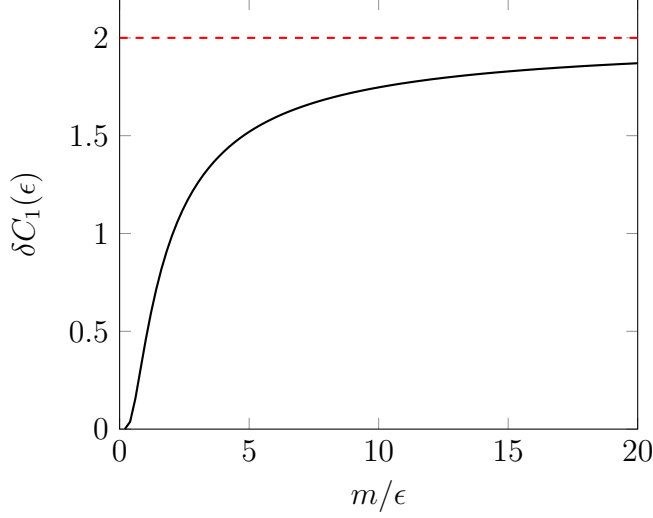


Figure 3: Evolution of $\delta C_1(\epsilon)$ along the RG flow for the case of $3d$ free scalar massive theory. The quantity $\delta C_1(\epsilon)$ represents the progressive contribution to the difference δC_1 as a function of the energy scale $\Lambda = \frac{m}{\epsilon}$.

4.2 Free Majorana Fermion in Two Dimensions

Another natural check to perform is in the case of a two-dimensional Majorana fermion which deformed through a mass term:

$$\mathcal{A} = \int dz d\bar{z} \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + im \bar{\psi} \psi \right), \quad (4.10)$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. The two-point functions are

$$\begin{aligned} \langle \psi(z, \bar{z}) \psi(0, 0) \rangle &= -\frac{m}{2\pi} \left(\frac{\bar{z}}{z} \right)^{\frac{1}{2}} K_1(m|z|), & \langle \bar{\psi}(z, \bar{z}) \psi(0, 0) \rangle &= i \frac{m}{2\pi} K_0(m|z|), \\ \langle \bar{\psi}(z, \bar{z}) \bar{\psi}(0, 0) \rangle &= -\frac{m}{2\pi} \left(\frac{z}{\bar{z}} \right)^{\frac{1}{2}} K_1(m|z|). \end{aligned} \quad (4.11)$$

We could in principle consider the operator $\bar{\psi} \psi$. However this operator is not related to a type-B conformal anomaly, but rather one of type A. It can in fact be used to compute the central charge of the theory, which was first performed by Cardy [30] and revisited recently in reference [26]. Instead, we will consider the operator $(\bar{\psi} \psi)^2$ which is indeed related to a type-B conformal anomaly and whose two-point correlator is given by

$$\langle (\bar{\psi} \psi)^2(z, \bar{z}) (\bar{\psi} \psi)^2(0, 0) \rangle = 2m^4 \left(K_0(mr)^2 - K_1(mr)^2 \right)^2. \quad (4.12)$$

This operator has a conformal dimension $\Delta = 2$. It is straightforward to compute the sum rule given in equation (2.11) by applying the differential operator, for which we find

$$\delta C_2 = \frac{1}{8\pi} \int d^2x \, x^2 (x^2 \square - 16) \langle (\bar{\psi}\psi)^2(z, \bar{z}) (\bar{\psi}\psi)^2(0, 0) \rangle = 2. \quad (4.13)$$

This is the expected result, as using Wick's theory to compute the anomaly coefficients in the UV, we find $C_2^{\text{UV}} = 2$; in the IR, the operator is integrated out, so that $C_2^{\text{IR}} = 0$. Similar checks can be also performed for higher powers of $\bar{\psi}\psi$.

4.3 $\mathcal{N} = 2$ SQCD and its Higgs Phase

The $\mathcal{N} = 2$ superconformal quantum chromodynamics (SQCD) is defined by the quiver diagram in Figure 4. The gauge group $SU(N)$ of the $\mathcal{N} = 2$ super-Yang-Mills theory is denoted by the central circular node. The $U(N) \times U(N)$ subgroup of the full $U(2N)$ flavor group is represented by square nodes. Each arrow is associated with an $\mathcal{N} = 1$ hypermultiplets transforming in the bifundamental representation of the adjacent nodes. The circular node denotes an adjoint $\mathcal{N} = 2$ vector multiplet with a scalar φ and gaugino λ .

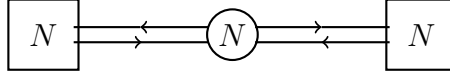


Figure 4: Quiver diagram of $\mathcal{N} = 2$ SQCD. The circular node denotes an adjoint $\mathcal{N} = 2$ vector multiplet.

We will package the two scalars in the $\mathcal{N} = 1$ hypermultiplets associated with arrows flowing from left to right as q_1 , while those associated with arrows flowing in the other direction are denoted by q_2 . We follow the notation of reference [36]. They both transform in the (anti)-fundamental representation of the flavor symmetry and the adjoint of gauge node $SU(N)$. This enables us to RG flow by considering the specific direction in which q_1 acquires a vacuum expectation value (vev), and q_2 does not. The full Lagrangian of SQCD is then given by

$$\begin{aligned} \mathcal{L} = -\text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\lambda}_I \bar{\sigma}^\mu D_\mu \lambda^I + D^\mu \varphi \overline{D_\mu \varphi} + ig\sqrt{2} (\epsilon_{IJ} \lambda^I \lambda^J \varphi - \epsilon^{IJ} \bar{\lambda}_I \bar{\lambda}_J \varphi) + \frac{g^2}{2} [\varphi, \bar{\varphi}]^2 \right] + \\ + \left[D^\mu \bar{q}^I D_\mu q_I + i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi + i \tilde{\psi} \bar{\sigma}^\mu D_\mu \tilde{\psi} + i\sqrt{2}g (\epsilon^{IJ} \bar{\psi} \bar{\lambda}_I q_J - \epsilon_{IJ} \tilde{q}^I \lambda^J \psi) + \right. \\ \left. + f \tilde{\psi} \lambda^I q_I - g \bar{q}^I \bar{\lambda}_I \tilde{\psi} + g \tilde{\psi} \varphi \psi - g \bar{\psi} \bar{\varphi} \tilde{\psi} + g^2 \bar{q}_I (\bar{\varphi} \varphi + \varphi \bar{\varphi}) q^I + g^2 V(q) \right], \end{aligned} \quad (4.14)$$

where $q_I = (q_1, q_2)$ is a shorthand for the scalar of the hypermultiplets, and $V(q)$ is the scalar potential for q_I , whose exact form will not be relevant for our purpose.

One can use the Lagrangian above to show that $\mathcal{N} = 2$ SQCD is a superconformal theory, which we take as the UV theory. We can then trigger a specific RG flow by giving a vacuum expectation value (vev) to the scalar q_1 :

$$\langle (q_1)_i^a \rangle = v \delta_i^a, \quad \langle (q_2)_i^a \rangle = 0, \quad (4.15)$$

where $i = 1, \dots, 2N$, $a = 1, \dots, N$ correspond to flavor and gauge indices, respectively.

The (UV) CFT phase: We will focus the anomaly coefficient associated to the operator made out of traces of the scalar in the $\mathcal{N} = 2$ vector multiplet,

$$\mathcal{O} = \text{Tr } \varphi^2, \quad (4.16)$$

whose dimension is $\Delta = 2$. As the UV theory is conformal, the form of its two-point function is fixed by symmetry; furthermore, it is a so-called Coulomb-branch operator, and is protected by a BPS condition. A straightforward computation using the Lagrangian in equation (4.14) above shows that the two-point function takes the form

$$\langle \text{Tr } [\varphi^2] (x) \text{Tr } [\varphi^2] (0) \rangle = \frac{2(N^2 - 1)}{(2\pi)^4} \frac{1}{|x|^4}. \quad (4.17)$$

Its Fourier transform, after regularization of the divergence due to the fact that the conformal dimension of the operator is integer [34–36], is given by

$$\langle \text{Tr } [\varphi^2] (p) \text{Tr } [\varphi^2] (-p) \rangle = -\frac{2(N^2 - 1)}{(4\pi)^2} \log \left(\frac{p^2}{\mu^2} \right) + \tilde{c}, \quad (4.18)$$

where $\tilde{c}$ depends on the choice of the regularization scheme, and the presence of a logarithm $\log \mu$ confirms that we indeed have a type-B anomaly.

The Higgs phase: In the Higgs phase we can redefine the scalar q_1 to take into account the vev of the field:

$$(q_1)_i^a \rightarrow v \delta_i^a + (q_1)_i^a, \quad (4.19)$$

leaving q_2 unchanged. The vev furthermore induces a mass for the scalar field φ in the vector multiplet, $m^2 = 2g^2 v^2$, and new interaction terms which are studied in detail in reference [36, 37]. One of those terms can be either be computed directly or by observing that the classical expression of the stress tensor involves the following term

$$-\frac{v}{3} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \square) \text{Tr } [Q_1 + \overline{Q}_1], \quad (4.20)$$

where Q_1 is the scalar associated with the top-left arrow in Figure 4. The cubic interaction between the dilaton, φ , and $\overline{\varphi}$ can also be obtained directly from the Lagrangian

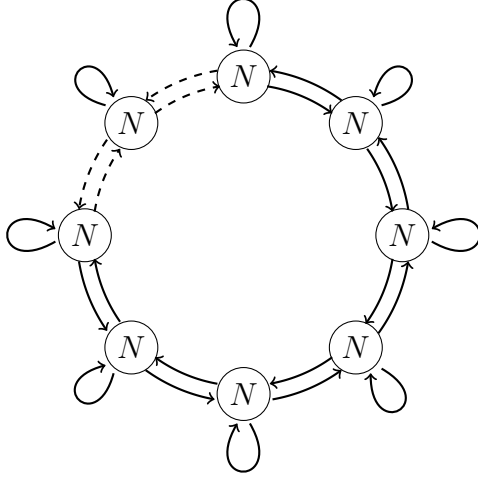


Figure 5: $\mathcal{N} = 2$ necklace quiver. Each node corresponds to a $\mathcal{N} = 2$ vector multiplet in the adjoint representation of $SU(N)$, while arrows correspond to bifundamental $\mathcal{N} = 1$ hypermultiplets.

by observing that after taking into account the redefinition given in equation (4.19), we have

$$\mathcal{L} \supset -g^2 v^2 \text{Tr} [(\bar{\varphi}\varphi + \varphi\bar{\varphi})(Q_1 + \bar{Q}_1)] , \quad (4.21)$$

where the mass term for the scalar field φ is now explicit. We now have all the tools needed to discuss the leading contribution to the Type-B anomaly coefficient in the Higgs phase.

This is a particular example of the case of type-B anomalies discussed in Section 3.3, where we start near the free-field point of the conformal manifold. Having access to perturbation theory it is easy to devise the fate of the anomaly coefficients of $\mathcal{O} = \text{Tr } \varphi^2$. For the purposes of the sum rule derived in Section 2, since φ obtains a mass in the Higgs phase proportional to the vev v of q_1 , the protected operator φ^2 is completely integrated out in the IR fixed point. From the arguments presented in Section 3 we expect

$$\delta C_2 = C_2^{\text{UV}} = 2 \frac{N^2 - 1}{(2\pi)^4} \geq 0 . \quad (4.22)$$

4.4 Necklace Quivers

The quiver above can be generalized straightforwardly to various types of 4d $\mathcal{N} = 2$ quiver theories. Among them, we consider the so-called necklace quivers depicted in

Figure 5. This theory is furthermore superconformal and we take it to be the UV CFT. There are various ways to deform this theory while preserving $\mathcal{N} = 2$ supersymmetry.

In particular, we can similarly to the case of SQCD discussed above construct Coulomb-branch operators that are protected by BPS conditions. The i -th node of the quiver is associated with a $\mathcal{N} = 2$ vector multiplet, which contains a scalar φ_i . Gauge-invariant powers of this scalar are $\mathcal{O}_i = \text{Tr } \varphi^n$, as well as linear combinations thereof are then Coulomb-branch operators forming the chiral ring of the theory.

By giving a vev to the scalar in the hypermultiplets, we can again trigger a Higgs-branch RG flow. For cases where at least part of the chiral ring is preserved along the flow, we can track the evolution of their two-point function coefficients. Each of the gauge couplings can be interpreted as a coordinate of the conformal manifold, and as a result we can always use the covariance of C_Δ along the conformal manifold to reach a free-field limit of the necklace quiver and perform a computation there [36, 37]. It can then be shown that only operators of the so-called untwisted sector survive to the infrared as they remain massless, and their coefficients is therefore the same at both endpoints: $\delta C_\Delta = 0$. On the other hand, the fields of the twisted sector become massive due to the vev of the hypermultiplets, and are therefore integrated out in the deep IR so that $C_\Delta^{\text{IR}} = 0$. We therefore conclude that as expected from the discussion in Section 3.3, in both cases $\delta C_\Delta = C_\Delta^{\text{UV}} \geq 0$. Similar arguments can be made for other ($\mathcal{N} = 2$)-preserving deformations, such as those involving Higgs-branch operators.

4.5 Holographic Renormalization

Even though most of the results from holography are derived in the context of AdS/CFT, it is possible to describe RG flows by considering holographic renormalization techniques [66–68]. Since we only focus here on two-point functions of scalar operators let us consider the toy model defined by the action

$$\mathcal{A} = \frac{1}{4\pi G} \int d^{d+1}x \sqrt{g} \left[-\frac{1}{4}R + \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right], \quad (4.23)$$

where we demand the potential to both a maximum and a minimum. If an RG flow is triggered on CFT living on the boundary of AdS, it corresponds in the bulk to a flow from the maximum of the potential to its minimum. Since the two fixed points are described by CFTs on the boundary, we need to require that at the two stationary points the potential takes the form

$$V(\phi_{\text{UV/IR}}) = -\frac{d(d-1)}{4L_{\text{UV/IR}}^2}, \quad (4.24)$$

where $\phi_{\text{UV/IR}}$ are the value of the field ϕ at either or the two stationary points, and $L_{\text{UV/IR}}$ is the AdS radius at these points. Because of Poincaré invariance, at the boundary the bulk metric is given by the *domain-wall Ansatz*:

$$ds^2 = e^{2A(r)} dx_i dx^i + dr^2, \quad \phi = \phi(r). \quad (4.25)$$

The null-energy condition and its connections with the boundary c -theorems is well known [69], and we now briefly review it. Using Einstein's equation for the action defined in equation (4.23), the warp factor is related to the null-energy condition:

$$A'' = \frac{2}{d-1} (T_i^i - T_D^D) < 0, \quad (4.26)$$

where primed quantities are derivatives with respect to r . It is then possible to define the monotonically decreasing c -function [70]

$$c(r) = \frac{\pi^{\frac{d}{2}}}{G \Gamma(d/2)} \frac{1}{(A')^{d-1}}, \quad (4.27)$$

which matches the c -coefficients in the UV and IR CFT points since

$$c_{\text{UV/IR}} = \frac{\pi^{\frac{d}{2}}}{\Gamma(d/2)G} L_{\text{UV/IR}}^{d-1}. \quad (4.28)$$

The c -theorems are therefore a consequence of the fact that $L_{\text{UV}} \geq L_{\text{IR}}$.

We have focused in this work on two-point functions coefficients. To compute them in the bulk, we consider the quadratic expansion of the potential close to the fixed points of the boundary theory, that is close to the stationary points of the potential:

$$V(\phi) \sim V(\phi_{\text{UV/IR}}) + \frac{1}{2} \frac{m_{\text{UV/IR}}^2}{L_{\text{UV/IR}}^2} (\phi - \phi_{\text{UV/IR}})^2, \quad (4.29)$$

where

$$m_{\text{UV/IR}}^2 = L_{\text{UV/IR}}^2 V''(\phi_{\text{UV/IR}}). \quad (4.30)$$

The boundary dual of h is an operator whose dimension is related to the mass m_i via the standard relation

$$\Delta = \frac{d + \sqrt{d^2 + 4m_{\text{UV/IR}}^2 L_{\text{UV/IR}}^2}}{2}. \quad (4.31)$$

For non-marginal operators, the fact that the operator is protected is ensured by the condition $m_{\text{UV}} L_{\text{UV}} = m_{\text{IR}} L_{\text{IR}} \neq 0$, since this implies that the conformal dimension in the UV is equal to the conformal dimension in the IR. Finally let us evaluate the

two-point function in the two critical points of the boundary theory. In the AdS/CFT correspondence, since the supergravity action is proportional to L^{d-1} we have that

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle \propto \frac{L_{UV/IR}^{d-1}}{|x|^{2\Delta}}, \quad (4.32)$$

implying, by definition of C_Δ , that

$$C_\Delta \propto c \propto L^{d-1}. \quad (4.33)$$

Therefore in this toy model, the positivity of $\delta C_\Delta = C_\Delta^{UV} - C_\Delta^{IR}$ is a consequence of the classical null energy condition. The argument above follows the procedure of [69] and makes use of the classical null energy condition.

Note that it was pointed out in [53] that at the quantum level the null energy condition can be violated, and the c -function must be modified in order to be monotonically decreasing. Nonetheless here we only require only a weaker version: $C_\Delta^{UV} > C_\Delta^{IR}$. We therefore expect this result to hold even though suitable modifications could be required in order to define C_Δ along the RG flow.

5 Conclusions

In this work we have studied the evolution of two-point functions of protected operators along RG flows. Our main results consists of a sum rule for the two-point function coefficients $\delta C_\Delta = C_\Delta^{UV} - C_\Delta^{IR}$. Even if the examples we have discussed are free fields and supersymmetric models, the sum rule does not rely on any specific hypothesis for the symmetries of the theory, apart from the existence of protected operators whose conformal dimensions in the UV and IR fixed point are the same. Our derivation was also adapted to derive a sum rule for the evolution of the central charges associated with flavor currents the stress tensor in $d > 2$.

We then combined the sum rule with the spectral decomposition of the two-point function, showing that single-particle states do not contribute. Furthermore we have explicitly shown that if multi-particle states start contributing at non-zero energies, the infrared value of the two-point function is zero, which is expected physically. We have also explored the case in which certain multi-particle states are massless and the associated branch cut in the spectral decomposition starts at zero. This paper mainly use perturbative expansions around the UV fixed point. It would be interesting to further investigate this case by including non-perturbative effects in specific examples in the future.

We have checked our results in examples in various spacetime dimensions for different values of the conformal dimensions of the protected operators. We have in particular focused our attention on even spacetime dimensions and integer conformal dimensions, as there the coefficients are related to type-B conformal anomalies. Despite previous results for the positivity of type-A anomalies—the c -theorem in two dimensions [6] and the a -theorem in four dimensions [8]—similar results on the positivity of type-B anomaly coefficients have not been proved to date. Here, we have argued that combining the sum rule we have derived, the spectral density of the two-point functions, and Ward identities derived in references [36–38], if the UV theory has a conformal manifold containing a free-field limit, then the quantity δC_Δ is positive. We stress that in even dimensions greater than two, although the c -coefficient and C_Δ are both type-B anomalies, there are important differences. In particular, in four dimensions there are counterexamples to a possible c -theorem. For instance, this is the case of $\mathcal{N} = 1$ SQCD [54]. However this is not in contradiction with our results, as in $\mathcal{N} = 1$ theories the stress tensor supermultiplet does not contain any protected scalar operators.

There are many directions that are worth exploring. As discussed in Section 3, there are possible connections with average null-energy (ANE) operators used the average null energy condition (ANEC). The latter offering a unified proof of the a - and c -theorem, [26, 27]. As we have shown, a direct connection with type-B conformal anomalies seems more arduous as the analogue of the simplest correlators used there vanish upon insertion of an ANE operator. However, it was shown that in four dimension the c -coefficient can be written in terms of a non-diagonal matrix element of the ANE operator, and it would therefore be interesting to check whether similar arguments can be made with the coefficients C_Δ .

Another direction would also be to explore the holographic implications of the sum rule. Recently, there has been renewed interest in defining a notion of distance between AdS vacua, especially in the context of string compactifications [71–74]. As there is strong evidence that δC_Δ decreases along the RG flow, it provides a coarse notion distance between two CFTs, and it would be interesting to tackle the question from the other side of the holographic duality through more complicated examples, particularly given that chiral-ring operators have a natural interpretation in string theory.

Matching for type-B conformal anomalies at the endpoints of the RG flow is furthermore an open subject of discussion in the literature [31, 33, 36, 37, 42]. It would be worth further studying this aspect in light of our results, in particular the question of dilaton contributions to conformal anomalies to give an independent proof of the main results in this work. It would also be interesting to make a connection with the tools

developed in reference [42].

The applications and examples presented in this work are the simplest and most studied in literature; a natural direction would be to investigate and validate our results in more involved examples. A possible direction could be an exploration of short RG flows where conformal perturbation theory is a good approximation [52]. Furthermore our derivation is general enough to also be used in the case of defect RG flows. When the bulk is fixed to be a CFT, and the theory defined on the defect is free to flow, the defect UV and IR theories are conformal but it is not along the flow. Interesting examples of such flows have been discussed in the literature [22, 75]. Finally, it would be interesting to further study the sum rules for flavor currents and the stress tensor we have derived.

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A Conformal perturbation theory and off-critical two-point function

In this appendix, we shortly review aspects of conformal perturbation theory that are needed to derive the sum rule in Section 2. These ideas trace back to references [76–78],

and utilise that in the vicinity of a critical point of an RG flow, correlators admit an expansion in terms of the conformal data.

We start from the ultraviolet fixed point: close to the CFT, we can write a formal action defining the deformation in terms of relevant operators Φ^I :

$$\mathcal{A} = \mathcal{A}_{\text{UV CFT}} + \int d^d x g_I \Phi^I(x) , \quad (\text{A.1})$$

Close the to fixed point, the expansion for the two-point function then takes the form

$$\begin{aligned} \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{off-crit.}} &\sim \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{UV}} + \sum_I g_I \int d^d y \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \Phi^I(y) \rangle_{\text{UV}} + \\ &+ \sum_{I,J} \frac{g_I g_J}{2} \int d^d y_1 \int d^d y_2 \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \Phi^J(y_1) \Phi^I(y_2) \rangle_{\text{UV}} + \dots . \end{aligned} \quad (\text{A.2})$$

Observe that this expansion is not expected to be convergent at any value of g_I , but its use is justified by the fact that here we are only interested in the behavior of the correlation function around the critical point. We further assumed that the operator $\mathcal{O}$ is protected in the sense defined in Section 2 and therefore by simple dimensional analysis, we have that

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{QFT}} = \frac{C_\Delta(\chi_I)}{x^{2\Delta}} , \quad (\text{A.3})$$

where C_Δ is a function of the dimensionless quantities

$$\chi_I = g_I |x|^{d-\Delta_{\Phi^I}} . \quad (\text{A.4})$$

As the RG flow is triggered by the presence of the relevant field Φ , implying that $\Delta_\Phi < d$, we can combine it with the expansion in equation (A.2) to obtain:

$$\begin{aligned} C_\Delta &= C_\Delta^{\text{UV}} + \sum_I c_1^I g_I |x|^{d-\Delta_{\Phi^I}} + \sum_{I,J} c_2^{IJ} g_I g_J |x|^{2d-\Delta_{\Phi^I}-\Delta_{\Phi^J}} + \dots \\ &= C_\Delta^{\text{UV}} + \sum_I c_1^I \chi_I + \sum_{IJ} c_2^{IJ} \chi_I \chi_J + \dots , \end{aligned} \quad (\text{A.5})$$

where the coefficients c_1^I and c_2^{IJ} can be fixed in terms of UV CFT data from the expansion above. Note that the assumption that the operator $\mathcal{O}$ is protected ensures that absence of any logarithmic terms, and only powers of χ_I appears in the expansion

A similar expansion can be performed *mutatis mutandis* around the infrared fixed point. There, we have the formal expansion

$$\mathcal{A} = \mathcal{A}_{\text{IR CFT}} + \int d^d x \lambda_I \Psi^I(x) + \dots , \quad (\text{A.6})$$

where the deformation operators are now irrelevant, as in the IR those are the types of operators we are interested in for our purpose. The two-point function then has a similar expansion in terms of the IR conformal data:

$$\begin{aligned} \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{off-crit.}} &\sim \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle_{\text{IR}} + \sum_I \lambda_I \int d^d y \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \Psi^I(y) \rangle_{\text{ir}} + \\ &+ \sum_{IJ} \frac{\lambda_I \lambda_J}{2} \int d^d y_1 \int d^d y_2 \langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \Psi^I(y_1) \Psi^J(y_2) \rangle_{\text{ir}} + \dots \quad (\text{A.7}) \end{aligned}$$

Combining this expansion with the fact that the operator is protected implies that

$$C_\Delta = C_\Delta^{\text{IR}} + \sum_I \tilde{c}_1^I \lambda_I |x|^{d-\Delta_{\Psi^I}} + \sum_{I,J} \tilde{c}_2^{IJ} \lambda_I \lambda_J |x|^{2d-\Delta_{\Psi^I}-\Delta_{\Psi^J}} + \dots \quad (\text{A.8})$$

The main difference with the expression given in equation (A.5) is that now $\Delta_{\Psi^I} > d$ as Ψ^I are irrelevant operators. Note that in general the number of irrelevant perturbations can be infinite, however this will not be important for the purposes of this work.

B The Spectral Decomposition

We review here the derivation of the Källén–Lehmann spectral decomposition based on the one used in Appendix A of reference [25]. The idea is based on the following steps:

- ★ find a basis of the Hilbert space;
- ★ use the basis of the Hilbert space to construct a resolution of the identity;
- ★ insert the resolution of the identity in the two-point function.

Let us consider the Fock basis of the theory spanned the single- and multi-particle states of the theory. In an interacting theory the existence of this basis is guaranteed at least asymptotically by the Haag–Ruelle theorem [79] assuming the existence of a mass gap. Although we only need the existence of a basis of the Hilbert space of the theory diagonalizing the Laplace operator—or equivalently the momentum operator $\mathbf{P}_\mu$ —we will make this assumption. We will denote those states by $|\alpha\rangle$. Assuming this is a complete basis of the Hilbert space, we can write the resolution of the identity

$$\mathbf{1} = \int |\alpha\rangle \langle \alpha| \, d\alpha. \quad (\text{B.1})$$

We now consider a two-point function

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle = \langle \mathcal{O}(x) \mathbf{1} \overline{\mathcal{O}}(0) \rangle . \quad (\text{B.2})$$

Using equation (B.1), we can rewrite the two-point function in terms of the basis of the Hilbert space $\{|\alpha\rangle\}$:

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle = \int e^{-ip_\alpha \cdot x} |\langle 0 | \mathcal{O} | \alpha \rangle|^2 d\alpha , \quad (\text{B.3})$$

where we made use of the assumption that we have chosen an eigenbasis of the momentum operator, and used $\langle 0 | \mathcal{O} | \alpha \rangle = \langle \alpha | \overline{\mathcal{O}} | 0 \rangle$. The spectral density $\rho(p^2)$ is then defined as:

$$\rho(p^2) (2\pi)^{-d} = \int \delta(p - p_\alpha) |\langle 0 | \mathcal{O} | \alpha \rangle|^2 d\alpha . \quad (\text{B.4})$$

The two-point function can then be written as

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle = \int \frac{d^d p}{(2\pi)^d} \int_0^\infty ds \rho(s) e^{-ip \cdot x} \delta(p^2 - s) . \quad (\text{B.5})$$

Note that in the previous step, it is crucial to use the fact that $p^2 > 0$. Assuming the convergence of the two integrals, we recognize the propagator $G_s(x)$ of a free scalar field of mass $\sqrt{s}$, and we conclude that

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(0) \rangle = \int_0^\infty ds \rho(s) G_s(x) , \quad (\text{B.6})$$

Equivalently, in momentum space we have the decomposition

$$\langle \mathcal{O}(p) \overline{\mathcal{O}}(-p) \rangle = \int_0^\infty ds \frac{\rho(s)}{p^2 + s} . \quad (\text{B.7})$$

The physical interpretation is quite natural: the two-point function is expanded in terms of contribution of the element of the Fock space $|\alpha\rangle$. Each element contributes with a free propagator weighted by a form factor $|\langle 0 | \mathcal{O} | \alpha \rangle|^2$. The single-particle contributions are expected to appear in a discrete sum defining the spectral density ρ from $M^2 < s < 4M^2 = s_{\text{th}}$, where M is the mass gap of the theory

¹⁴ Each of these contributions gives a pole in the complex momentum plane. Beyond $s_{\text{th}} = 4M^2$ multi-particle states are expected to appear and form a continuum. A branch cut is therefore expected beyond this threshold value of the momentum and we

¹⁴More precisely M is not the mass gap of the theory but it is the energy of the first single particle excitation with non-vanishing form factor.

obtain the pole structure depicted in Figure 1.¹⁵ There is also the possibility in which multi-particle states contribute from $s = 0$, for instance because of the presence of single-particle states. This is maybe the most interesting case since it is the case in which the operator $\mathcal{O}$ does not vanish in the infrared. In the latter case around $s = 0$ the branch-cut is dominated by the IR fixed point as the limit $s \rightarrow \infty$ is dominated by the UV fixed point.

B.1 The Stress Tensor in CFT

We review here the application of the spectral decomposition to the stress tensor in CFT. In general the spectral decomposition for tensors have to be split in two components, since there are two possible Lorentz structures corresponding to spin-0 and spin-2 states, respectively [25]:

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \sum_{J=0,2} \int_0^\infty ds \, \rho^{(J)}(s) \Pi_{\mu\nu\rho\sigma}^{(J)} G_s(x) , \quad (\text{B.8})$$

where

$$\Pi_{\mu\nu\rho\sigma}^{(0)} = \frac{S_{\mu\nu} S_{\rho\sigma}}{\Gamma(d)} , \quad \Pi_{\mu\nu\rho\sigma}^{(2)} = \frac{1}{\Gamma(d-1)} \left[\frac{d-1}{2} S_{\mu(\rho} S_{\nu\sigma)} - S_{\mu\nu} S_{\rho\sigma} \right] , \quad (\text{B.9})$$

$G_{m^2}(x)$ is the free massive propagator and $S_{\mu\nu} = \partial_\mu \partial_\nu - \delta_{\mu\nu} \square$. Note that the case $d = 2$ is simpler since the term corresponding to spin-2 states is trivial and only spin-0 states contribute. In that case, scale invariance implies only two possibility for the spectral function

$$\text{i) } \rho^{(0)}(s) = \tilde{c} \frac{\delta(s)}{s} , \quad \text{ii) } \rho^{(0)}(s) = \tilde{c} \frac{1}{s} . \quad (\text{B.10})$$

Observe that the first possibility is actually non-zero and non-divergent. In fact, we can compute the correlation of the trace of the stress tensor and we get

$$\langle T_\mu^\mu(x) T_\rho^\rho(0) \rangle \propto -\tilde{c} \square \delta^{(2)}(x) . \quad (\text{B.11})$$

The possibility ii) gives a divergence in the correlator in position space which is not expected and we therefore conclude that ii) is unphysical. The conclusion is that, since the correlation function above is ultralocal, by invoking the Reeh–Schlieder theorem [56, 57] we have that $T_\mu^\mu = 0$ and therefore scale invariance, together with locality (existence

¹⁵In some cases, two-particle states cannot contribute due to symmetries setting the associated form factors to zero. The first multi-particle state is then a three-particle state appearing at $m^2 = 9M^2$.

of the stress tensor) and unitarity, implies conformal invariance in two-dimensions. Above two dimensions we have that

$$\text{i) } \rho^{(J)}(s) = \tilde{c} s^{\frac{d}{2}-1} \delta(s) , \quad \text{ii) } \rho^{(J)}(s) = \tilde{c} s^{\frac{d}{2}-2} . \quad (\text{B.12})$$

The case i) gives zero correlation functions for a CFT. While it is possible to give a meaning of this type of behavior, see the discussion in Section 3 of [25], on the other hand the case ii) above two dimensions is not divergent and therefore we can not exclude it. Nonetheless, the momentum space of the two-point function of T_μ^μ is naively divergent, and after regularization a logarithmic term in momentum space appears. In even spacetime dimensions this matches the expectation for type-B conformal anomalies. Conversely, a type-A anomalies can appear in the two-point function of T_μ^μ only in two dimensions, since it is the only case for which the two-point function is ultralocal.

In higher dimensions conformal invariance is not a consequences of unitarity, scale invariance, and locality as in two dimension—at least not the way we have argued for the $d = 2$ case.

B.2 Asymptotic of the Spectral Density

A way to recover the leading term of the spectral density is to use Tauberian theory. In the small- x limit, we recover the two-point function of the UV theory

$$\int_0^\infty ds \, \rho(s) \left(\frac{\sqrt{s}}{x} \right)^\nu K_\nu(\sqrt{s}x) \stackrel{x \rightarrow 0}{\sim} \frac{C_{\text{UV}}}{x^{2\Delta}} , \quad (\text{B.13})$$

where $\nu = (d - 2)/2$. Using that for large values of x

$$K_\nu(\sqrt{s}x) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-\sqrt{s}x}}{\sqrt{x}} , \quad (\text{B.14})$$

one can use an inverse Laplace transform to obtain

$$\rho(s) \sim \frac{C_{\text{UV}}}{2\Gamma(2\Delta - 2\nu)} s^{\Delta - \frac{d}{2}} , \quad (\text{B.15})$$

which matches the expectation that at high energy the spectral density reproduces the high-energy behavior of the correlator. To make the statement mathematically precise one should invoke Tauberian theorems, see references [80] for a review and [81–88] for examples of applications in physics. The main obstacle in doing this is the presence of the Bessel function. In three spacetime dimensions the Bessel function reduces to a simple exponential and the Tauberain theorem follows. In $d \neq 3$ however the proof

require some modification. In $d = 2$ for instance the kernel is simply given by K_0 which corresponds to the asymptotic of the one-dimensional conformal blocks. In the latter case a Tauberian theorem was proved in reference [85]. It would be interesting to adapt the proof to any spacetime dimension. If we add the assumption that the spectral density is an analytic function we can simply recover the expansion

$$\rho(s) = \frac{C_{uv}}{2\Gamma(2\Delta - 2\nu)} s^{\Delta - \frac{d}{2}} \left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \dots \right) . \quad (\text{B.16})$$

The motivation for such an assumption in a generic, non-Lagrangian theory is unfortunately not provided. It is however possible to show that this is correct in free theories and perturbation theory. In fact one can show that the spectral decomposition is related with the imaginary part of the correlator or its discontinuity

$$\rho(s) \sim \text{Im } \tilde{G}(k) \sim \text{disc } \tilde{G}(k) , \quad (\text{B.17})$$

which satisfy the analyticity condition above in perturbative examples. Clearly non-perturbative effects are not completely captured by the expansion above.

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