Wilson Loops with Lagrangians: Large-Spin Operator Product Expansion and Cusp Anomalous Dimension Dictionary

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In the context of planar conformal gauge theory, we study five-point correlation functions between the interaction Lagrangian and four of the lightest single-trace, gauge-invariant scalar primaries. After performing two light-cone operator product expansions (OPEs), we express this correlator in terms of the three-point functions between two leading-twist spinning operators and the Lagrangian. For finite values of spin, we compute these structure constants in perturbation theory up to two loops in $\mathcal{N}=4$ super Yang-Mills theory. Large values of spin are captured by null polygon kinematics, where we use dualities with null polygon Wilson loops as well as factorization properties to bootstrap the universal behavior of the structure constants at all loops. We find explicit maps that relate the Lagrangian structure constants with the leading-twist anomalous dimension. From the large-spin map, we recover the cusp anomalous dimension at strong and weak coupling, including genus-one terms.

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Introduction—The operator product expansion (OPE) encodes the data of a conformal field theory (CFT) in its four-point correlation functions. Capturing all CFT data requires infinitely many four-point functions. Iterating the OPE, this infinity of data can in turn be packaged in higher-point functions of the simplest operators. This is the philosophy of the multipoint bootstrap [1–9], which trades an infinity of data for a larger functional complexity.

In null polygon limits, this complexity reduces, and the conformal bootstrap is enhanced by dualities with Wilson loops, both at four [10] and higher points [1,2]. While null squares and pentagons allow for no finite conformal cross ratios, null hexagons are complicated functions of three variables. Here, we consider a sweet spot: the null square limit of a five-point function, which has a single finite cross ratio.

We will focus on the correlation function of four single-trace lightest scalar operators and the interaction Lagrangian in planar conformal gauge theories. Such correlators yield integrands for scalar operators [11]. In the null square limit, they probe the quantum corrections to null-square Wilson loops [12–14], and in particular were used to compute the full four-loop cusp anomalous dimension for $\mathcal{N}=4$ super Yang-Mills (SYM) and quantum chromodynamics (QCD) [15].

By studying the Lagrangian correlation function via the conformal bootstrap, we translate all its properties to its OPE constituents: the three-point functions of two leading-twist spinning operators and the Lagrangian. At finite values of spin, we compute these structure constants at weak coupling and connect them, via conformal perturbation theory, to leading-twist anomalous dimensions. For large values of spin, we find an exact inversion formula that leads to direct maps between the null-square correlator to the structure constant (16). Using conformal perturbation theory at large spin, we obtain an even simpler map between these structure constants and the cusp anomalous dimension (30).

Perturbative data—We consider five-point functions of one primary scalar operator $\mathcal{O}(x)$ and four of the lightest scalar operators ϕ of the theory. For example, in $\mathcal{N}=4$ SYM these would be the $20'$ operators $\phi_j \propto \mathrm{Tr}(y_j\Phi(x_j))^2$. It is convenient to extract a space-time dependent prefactor of the five-point correlator

$$\langle\phi_1\ldots\phi_4\mathcal{O}(x_5)\rangle \equiv \left(\frac{1}{x_{12}^2x_{34}^2}\right)^{\Delta_\phi}\left(\frac{x_{14}^2}{x_{15}^2x_{45}^2}\right)^{\Delta_\mathcal{O}/2} \times\prod_{i=1}^{n_\mathcal{O}}(y_i\,y_{i+1})\times G_\mathcal{O}(u_i)+(\text{other}),\quad(1)$$

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where $n_{\mathcal{O}} = 4, 5$ depending on whether the fifth operator carries R charge or not, and (other) refers to other y contractions that will be subleading in all the limits we consider. In this way, $G_{\mathcal{O}}(u_i)$ becomes a function of five cross ratios

$$u_i = \frac{x_{i,i+1}^2 x_{i+2,i-1}^2}{x_{i,i+2}^2 x_{i+1,i-1}^2}, \qquad i = 1, \ldots, 5, \tag{2}$$

where we identify the points $(x_1, \ldots, x_5)$ periodically. Two particular correlators will be important for us: the correlation function of five light operators (G_ϕ), and the five-point function of four light correlators and one Lagrangian ($G_{\mathcal{L}}$).

To study these correlators, we will consider two lightlike OPEs [16] between the lightest operators, as depicted on the right of Fig. 1. The leading behavior under this Lorentzian OPE is controlled by the exchange of leading-twist (twist-two) operators in the OPE decomposition:

$$G_{\mathcal{O}}(u_i) = \sum_{J_1, J_2, \ell} \mathcal{F}(u_i) \times C(J_1) C(J_2) C_{\mathcal{O}}(J_1, J_2, \ell), \tag{3}$$

where $C(J)$ are the structure constants of one leading-twist operator with spin J and two lightest scalars operators, while $C_{\mathcal{O}}(J_1, J_2, \ell)$ are the three-point functions of two leading-twist spinning operators and the operator $\mathcal{O}(x)$. The quantum number $\ell = 0, 1, 2, \ldots, \min(J_1, J_2)$ labels the tensor structures of three-point functions with two spinning operators [17]. Meanwhile, $\mathcal{F}$ is the theory-independent conformal block worked out in [1] and recalled in Supplemental Material, Appendix A [18].

In principle, using the integrability formalism for spinning operators [19,20], it is possible to compute the structure constants C_ϕ at any order in perturbation theory. However, the structure constants $C_{\mathcal{L}}$ are not on the same integrability footing: Despite some tree-level results [21], it is presently not clear how to systematically consider superdescendants like the Lagrangian in the integrability formalism.

In perturbation theory, we can explicitly evaluate the correlator $G_{\mathcal{L}}$ [11,22], and use it to extract the perturbative data $C_{\mathcal{L}}$. We extracted thousands of OPE coefficients up to two loops, contained in the attached *Mathematica* file.

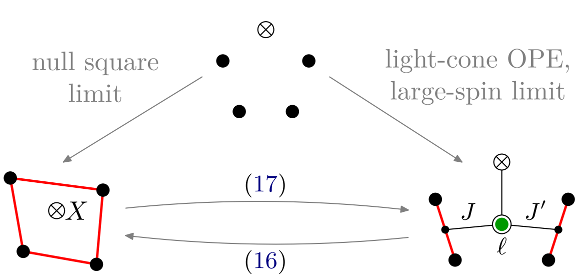

FIG. 1. Correlation function of four scalar operators (black dot) and one Lagrangian operator (crossed circle) in the null square limit, and the maps that we obtain.

This data could be useful to develop future integrability formulations. We were able to identify a pattern and write the tree-level data as

$$C_{\mathcal{L}}^{(0)}(J_1, J_2, \ell) = 2 \frac{2J_1!}{\sqrt{(2J_1)!}} \frac{2J_2!}{\sqrt{(2J_2)!}} \times \left[(-1)^{\ell} \binom{J_1 + J_2}{\ell - 1} + \sum_{m=0}^{\ell-1} \binom{J_1}{m} \binom{J_2}{m} \right]. \tag{4}$$

Since the Lagrangian is exactly marginal, conformal perturbation theory relates the two-point function of two operators with the three-point functions of the two operators and the Lagrangian in a differential equation [23]. For the case of spinning operators this was worked out in [24] to be

$$\frac{\partial \gamma(J)}{\partial \lambda} = \sum_{\ell=0}^{J} \frac{C_{\mathcal{L}}(J, J, \ell)}{1 + J - \ell}, \tag{5}$$

where $\Delta_J = 2 + J + \gamma(J)$ is the dimension of the leading-twist operator. A remarkable feature of this anomalous dimension is that in any planar gauge theory, it develops logarithmic scaling at large values of spin [25,26]:

$$\gamma(J) \simeq f(\lambda) \ln(J) + g(\lambda), \tag{6}$$

where $f(\lambda)$ and $g(\lambda)$ are the cusp and collinear anomalous dimensions, respectively, with $\lambda = g_{\mathrm{YM}}^2 N/(4\pi)^2$, where g_{YM} is the Yang-Mills coupling. Below we evaluate (5) at large values of spin, obtaining a map between the large-spin Lagrangian structure constants and the ubiquitous cusp anomalous dimension.

Null square—We approach the null square limit of the five-point function with the Lagrangian by first taking $x_{12}^2, x_{34}^2 \to 0$ (or u_1, $u_3 \to 0$), projecting into leading-twist operators. Next, we take $x_{23}^2 \to 0$ (or $u_2 \to 0$), which we find projects both to large-spin J_i and large polarization ℓ. Finally, we take $x_{14}^2 \to 0$, which makes the two values of spin approach each other, $J_2 \to J_1$.

The intuition is that once we create a null square inside a five-point function, the OPE decomposition starts developing four-point-like features. Four-point functions have only one spinning operator flowing in the OPE channel, and this is exactly what the leading term of the five-point function reproduces. We make this precise in Supplemental Material, Appendix A [18], via the so-called "Casimir trick" introduced in [27–29], and systematized for higher-point functions in [5]. In the end, the five-point block in the null square limit becomes a simple Bessel-Clifford function,

$$\mathcal{F}(u_i) = (u_1 u_3)^{1+\gamma(J)} (u_2 u_4 u_5)^{\frac{\Delta_{\mathcal{L}}}{2}} 2^{2J-2+\gamma(J)+\frac{\Delta_{\mathcal{L}}}{2}} \times \pi^{-1/2} J^{\frac{1+\Delta_{\mathcal{L}}}{2}} \mathcal{K}_{\Delta_{\mathcal{L}}/2}(J u_2 (J + j_1 u_4 + j_2 u_5)), \tag{7}$$

where $\mathcal{K}_n(z) = z^{-n/2} K_n(2\sqrt{z})$, and we introduced the variables

$$J^2 = \frac{J_1^2 + J_2^2}{2}, \qquad j_1 = J_1 - \ell, \qquad j_2 = J_2 - \ell. \tag{8}$$

The null square limit is described by all these variables being large ($J, j_1, j_2 \to \infty$) with $J \gg j_1, j_2$, while the ratio $r = j_2/j_1$ is finite. This single finite quantum number is associated with the single cross-ratio x that remains finite in the null square limit:

$$x = \frac{u_4}{u_5} = \frac{x_{13}^2 x_{25}^2 x_{45}^2}{x_{15}^2 x_{24}^2 x_{35}^2}. \tag{9}$$

From here onward we will consider Born-level normalized quantities, which we denote by $\hat{G} = G/G^{(0)}$, in order to make our statements universal and independent of the prefactor choices such as (1).

Conformal symmetry implies that null square correlators must factorize into two terms [12],

$$\hat{G}_{\mathcal{L}}(u_1, u_2, u_3, u_4, u_5) = \hat{G}_4(u, v) \times \hat{F}(x), \tag{10}$$

which are invariant under cyclic permutations of the null square, $(x_1, x_2, x_3, x_4) \to (x_2, x_3, x_4, x_1)$ with x_5 fixed. This imposes

$$\hat{G}_4(u, v) = \hat{G}_4(v, u) \quad \text{and} \quad \hat{F}(x) = \hat{F}\left(\frac{1}{x}\right), \tag{11}$$

where $u = u_1 u_3$ and $v = u_2$ are four-point cross ratios.

The first term $\hat{G}_4(u, v)$ is the null four-point function of the lightest operators, which captures all the divergences of the Lagrangian correlator, and depends on the four-point cross-ratios u and v. The second term $\hat{F}(x)$ is a finite function of the remaining finite cross ratio.

Thus our bootstrap problem is this: can we fix the universal behavior of the structure constants such that the Lagrangian correlator factorizes into the square symmetric functions (10)? To start answering this question, we use the explicit expression for the conformal blocks (7) to write the null square correlator as

$$\hat{G}_{\mathcal{L}} = (u_2^3 u_4 u_5) \int dJ dj_1 dj_2 (u_1 u_3)^{\frac{\gamma}{2}} 2^{2+\gamma} J^3 \hat{C}(J)^2 \times \hat{C}_{\mathcal{L}}(J_1, J_2, \ell) \mathcal{K}_2(J u_2 (J + j_1 u_4 + j_2 u_5)), \tag{12}$$

where we factored out the tree-level large-spin scaling of the structure constants:

$$C(J_1) = C(J_2) \simeq 2^{-J} \pi^{1/4} J^{1/4} \times \hat{C}(J), \tag{13a}$$

$$C_{\mathcal{L}}(J_1, J_2, \ell) \simeq 8 \times \hat{C}_{\mathcal{L}}(J_1, J_2, \ell). \tag{13b}$$

The tree-level behavior (13) shows the physics of these structure constants: $\hat{C}(J)$ is large and captures the divergent part $\hat{G}_4$ of the correlator in the null square limit. On the other hand, the structure constant $\hat{C}_{\mathcal{L}}(J_1, J_2, \ell)$ is finite and controls the finite part of the correlator $\hat{F}(x)$. We expect that it only depends on the finite ratio r,

$$\hat{C}_{\mathcal{L}}(J_1, J_2, \ell) = \hat{C}_{\mathcal{L}}\left(\frac{J_2 - \ell}{J_1 - \ell}\right) \equiv \hat{C}_{\mathcal{L}}(r). \tag{14}$$

Indeed, we can prove this to be true, using a five-point null square inversion formula; see Supplemental Material, Appendix B [18].

Assuming the simple dependence (14) allows us to integrate (12) in one of the two variables j_i, resulting in the following factorized expression for the null square correlator:

$$\hat{G}_{\mathcal{L}}(u_i) = \underbrace{\int_0^\infty dJ 2^{2+\gamma} J \hat{C}(J)^2 u^{\gamma/2} v K_0(2J\sqrt{v})}_{\hat{G}_4(u,v)} \times \underbrace{\int_0^\infty dr \frac{x}{(r+x)^2} \hat{C}_{\mathcal{L}}(r)}_{\hat{F}(x)}. \tag{15}$$

The first term is *exactly* the same as the null square four-point function of lightest operators considered in [10] and therefore automatically obeys the cyclicity (11). The invariance under $x \to 1/x$ of the function $\hat{F}(x)$ is also automatically satisfied, provided that $\hat{C}_{\mathcal{L}}(r) = \hat{C}_{\mathcal{L}}(1/r)$. Physical structure constants must have this property, since inverting the ratio r is the same as swapping the spins $J_1 \leftrightarrow J_2$. Thus, the map between $\hat{F}(x)$ and the Lagrangian structure constants is simply

$$\hat{F}(x) = x \int_0^\infty dr \frac{\hat{C}_{\mathcal{L}}(r)}{(x+r)^2}. \tag{16}$$

We can invert this map by noticing that the right hand side is the derivative of the Cauchy kernel, whose inversion is well understood in terms of its discontinuities. Therefore, one can write the structure constants in terms of discontinuities of $F(x)$:

$$r \frac{d}{dr} \hat{C}_{\mathcal{L}}(r)\Big|_{r \geq 0} = \frac{\text{Disc}}{2\pi i} \hat{F}(-r), \tag{17}$$

where we used the fact that physical structure constants $\hat{C}_{\mathcal{L}}(r)$ must be regular at physical values of spins and polarization ($r \geq 0$).

Weak and strong coupling—Both weak and strong coupling results for the function $\hat{F}(x)$ have been computed in $\mathcal{N} = 4$ SYM. We can use these results together with our map (17) to compute the structure constants $\hat{C}_{\mathcal{L}}$ in these regimes. At weak coupling, the first orders of $\hat{F}(x)$ were computed in [12–15]

$$\hat{F}^{(0)}(x) = 1,$$
$$\hat{F}^{(1)}(x) = -6\zeta_2 - 2H_{00},$$
$$\begin{aligned}\hat{F}^{(2)}(x) = {} & 24\zeta_2 H_{-1-1} - 12\zeta_2 H_{-10} + 24\zeta_2 H_{00} \\ & + 8H_{-1-100} - 4H_{-1000} + 12H_{0000} - 4H_{-200} \\ & - 12\zeta_2 H_{-2} + 8\zeta_3 H_{-1} - 4\zeta_3 H_0 + 107\zeta_4, \end{aligned} \quad (18)$$

where $H_a \equiv H_a(x)$ are harmonic polylogarithms [30], recalled in Supplemental Material, Appendix C [18], where we also collect the three-loop and genus-one contributions [31] of $\hat{F}(x)$.

The discontinuities of the harmonic polylogarithms appearing in the perturbative expansion of $\hat{F}(x)$ can be easily evaluated using the HPL package [32] for Mathematica, resulting in the following expression for the weak coupling structure constants:

$$\hat{C}^{(0)}_{\mathcal{L}}(r) = 1,$$
$$\hat{C}^{(1)}_{\mathcal{L}}(r) = -4\zeta_2 - 2H_{00},$$
$$\begin{aligned}\hat{C}^{(2)}_{\mathcal{L}}(r) = {} & 56\zeta_4 - 4\zeta_3 H_0 + 8\zeta_2 H_2 + 12\zeta_2 H_{00} \\ & + 8H_{210} + 4H_{200} + 4H_{30} + 12H_{0000}, \end{aligned} \quad (19)$$

where $H_a \equiv H_a(r)$, and the three-loop and genus-one corrections are written in the Supplemental Material, Appendix C [18]. In practice, the discontinuity fixes all but the constant term. This in turn can be determined by performing the explicit integration in (16), and matching with the $\hat{F}(x)$ expansion (18)—which is trivial to do with the package HPL [32].

Even though the individual harmonic polylogarithms have a branch point at $r = 1$, the particular combination appearing in the weak-coupling expansion of $\hat{C}_{\mathcal{L}}(r)$ is real and single-valued for physical values of spins and polarizations ($r > 0$). This is not true for the unphysical region $r < 0$, where $\hat{C}_{\mathcal{L}}(r)$ has a logarithmic branch cut.

At strong coupling, the leading behavior of the function $\hat{F}(x)$ is known [12]:

$$\hat{F}(x) = \frac{x}{(x-1)^2}\left(\frac{(x+1)}{(x-1)}\frac{\log x}{2} - 1\right)\sqrt{\lambda} + \dots. \quad (20)$$

Using the inversion formula (17), we can compute the leading term of the structure constant at strong coupling,

$$\hat{C}_{\mathcal{L}}(r) = \frac{r}{2(1+r)^2}\sqrt{\lambda} + \dots. \quad (21)$$

Wilson loops and amplitudes—In $\mathcal{N} = 4$ SYM, n-point correlation functions of $20'$ operators in the limit where their insertions approach the cusp of a null polygon are dual to both null polygonal Wilson loops and MHV gluon scattering amplitudes [33,34]. In particular, in the five-point null pentagon limit,

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_\phi = (\widehat{\mathrm{MHV}}_5)^2. \quad (22)$$

By promoting this relation to supercorrelation functions and superamplitudes, one obtains that the correlation function of four $20'$ correlators and one Lagrangian, when the points approach the cusps of a null pentagon is dual to (the top component of) the NMHV scattering amplitude [35,36]

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_{\mathcal{L}} = \widehat{\mathrm{MHV}}_5 \times \widehat{\mathrm{NMHV}}_5. \quad (23)$$

For five points, the NMHV amplitude is the parity conjugate of the MHV amplitude [37], thus in the null pentagon limit both correlators are identical:

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_\phi = \lim_{x^2_{i,i+1}\to 0} \hat{G}_{\mathcal{L}} = \langle \hat{W}_5 \rangle, \quad (24)$$

which immediately implies $\hat{C}_{\mathcal{L}} = \hat{C}_\phi$, that is [38],

$$\hat{C}_{\mathcal{L}}(J_1, J_2, \ell) = \mathcal{N}(\lambda) e^{-\frac{f(\lambda)}{4}(\log \ell^2 + 2\log 2 \log(J_1 J_2)) - \frac{g(\lambda)}{2}\log \ell}. \quad (25)$$

The story is completely different when we consider the null square limit of these five-point correlators. As pointed out in [12], the duality with Wilson loops continues to hold even if one adds an extra operator at finite distance to the null square configuration

$$\lim_{x^2_{1,2}, x^2_{2,3}, x^2_{3,4}, x^2_{1,4}\to 0} \hat{G}_{\mathcal{L}} = \langle \widehat{W_4 \mathcal{L}} \rangle. \quad (26)$$

One can recast this duality as an equation for $\hat{F}(x)$ by using Lagrangian correlators obtained from a derivative with respect to the coupling,

$$\frac{\partial}{\partial\lambda}\log\langle \hat{W}_4 \rangle = 8\int dx_5 \frac{x^2_{13}x^2_{24}}{x^2_{15}x^2_{25}x^2_{35}x^2_{45}}\hat{F}(x), \quad (27)$$

where the space-time prefactor arises from the Born-level ratio $\langle \phi_1 \dots \phi_4 \mathcal{L}(x_5)\rangle^{(0)} / \langle \phi_1 \dots \phi_4 \rangle^{(0)}$.

Cusp anomalous dimension—The UV cusp divergences of the Wilson loop are controlled by the cusp anomalous dimension. In principle, one can match the divergences appearing on both sides of the relation (27) to compute this quantity. In practice, this is done with the help of the functional $\mathcal{I}$ formulated in [14] and recalled below [39],

$$\frac{\partial f(\lambda)}{\partial \lambda} = \mathcal{I}[8\hat{F}(x)], \quad (28)$$

where one is instructed to first expand the function $\hat{F}(x)$ around small values of x [40], and then act with the linear functional on individual terms as

$$\mathcal{I}[x^p] = \frac{\sin \pi p}{\pi p}. \tag{29}$$

Starting from the conformal perturbation theory relation (5), we propose an alternative and more explicit map. It relates the three point function $\hat{C}_{\mathcal{L}}$ with the cusp anomalous dimension simply as

$$\frac{\partial f(\lambda)}{\partial \lambda} = 8\hat{C}_{\mathcal{L}}(1). \tag{30}$$

The large-spin limit of the sum (5) is dominated by the region where spins and polarizations are of the same order. Therefore, we can trade the sum over polarizations by an integral and replace the structure constants by their large-spin and polarization behavior (14). Since the sum runs over structure constants of identical spins, the ratio r becomes one, and $\hat{C}_{\mathcal{L}}(1)$ becomes a constant that can be factored out of the integral. The integral is then trivial and evaluates to $\log J$. Matching the log-divergent terms on both sides of Eq. (6) yields the map (30).

We verify this result by recovering the known values of the cusp anomalous dimension at strong and weak coupling, including genus-one terms: at strong coupling, replacing $r = 1$ in (21) and using the map (30) yields the leading term of the cusp anomalous dimension: $f(\lambda) \simeq 8\sqrt{\lambda}$. Similarly at weak coupling, we recover the four-loop anomalous dimension [15]:

$$8\hat{C}_{\mathcal{L}}(1) = 8 - 32\zeta_2\lambda + 528\zeta_4\lambda^2 - \left(64\zeta_3^2 + 1752\zeta_6 + \frac{1}{N^2}(1152\zeta_3^2 + 2976\zeta_6)\right)\lambda^3. \tag{31}$$

The map between three-point functions and the cusp anomalous dimension (30) is simpler than the map (28) previously considered in the literature. However since the structure constants and the function $F(x)$ are also related to each other via (16) we must have the following consistency condition for the structure constant:

$$\mathcal{I}\left[x\int_0^\infty dr \frac{\hat{C}_{\mathcal{L}}(r)}{(x+r)^2}\right] = \hat{C}_{\mathcal{L}}(1). \tag{32}$$

Unfortunately, this is *not* a bootstrap equation for $\hat{C}_{\mathcal{L}}(r)$. One simple way to see this is to expand this function as a power series and note that the relation (32) acts trivially on each polynomial term

$$\mathcal{I}\left[x\int_0^\infty dr \frac{r^p}{(x+r)^2}\right] = \frac{\pi p}{\sin \pi p}\mathcal{I}[x^p] = 1 \tag{33}$$

and therefore (32) is trivially satisfied for any function $\hat{C}_{\mathcal{L}}(r)$. One might be worried that the expression above is only valid for $|p| < 1$, and that $\hat{C}_{\mathcal{L}}(r)$ has no regular expansion around $r = 0$. However, using the physical properties of the structure constants, i. e.xspace, invariance under swapping the spins $\hat{C}_{\mathcal{L}}(r) = \hat{C}_{\mathcal{L}}(1/r)$ and regularity around $r = 1$ (where we recover the cusp anomalous dimension) we can analytically continue this result for any p, see Supplemental Material, Appendix E [18].

Conclusion—Multipoint conformal correlation functions organize the CFT data in nontrivial functions of conformal cross ratios. These functions have, generically, a complex analytic structure that does not follow from a single exchange of a physical operator. Instead, it is often the case that the intricate structure only emerges after summing the contributions of an infinite number of operators [1,2,9,10,41].

Using the conformal bootstrap, we analyzed the five-point correlation function of one Lagrangian and four lightest scalar operators, in terms of the three-point functions of two leading-twist spinning operators and the interaction Lagrangian. We computed these structure constants for finite and large values of spin, connecting them with anomalous dimensions (5), null pentagon Wilson loops (25), null square Wilson loops with insertions (16), and the cusp anomalous dimension (30).

In $\mathcal{N} = 4$ SYM, there are several distinct integrability frameworks developed to study the different observables listed above. Three-point correlation functions are described by integrable hexagon form factors [19], null polygonal Wilson loops can be constructed out of integrable pentagons [42], and anomalous dimensions can be computed via the quantum spectral curve [43]. The sharp maps that we derived here connect all these quantities and could be a great laboratory for developing a unifying integrability description of $\mathcal{N} = 4$ SYM.

It would be interesting to study the expectation value of the square Wilson loop with other types of insertions using the techniques developed here. It should also be possible and very interesting to generalize our analysis to other physical observables, for example, null square Wilson loops with two operator insertion, or null pentagon, Wilson loops with a single operator insertion [44], and to connect these quantities with conformal manifold constraints [45,46] and integrability.

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[38] As the null square, the null pentagon limit is also governed by the regime of large spin and large polarization. The difference is that for the pentagon there are neither finite cross-ratios, nor finite ratios among the quantum numbers J_i, ℓ, see Supplemental Material, Appendix D [18].

[39] The factor 8 arises from the fact that all our quantities are Born-level normalized except for the cusp anomalous dimension. We can drop this factor by considering a Born-level normalized cusp anomalous dimension $\hat{f}(\lambda)$, but we refrain from doing that to avoid confusion.

[40] The cusp singularities will arise when x_5 approaches the cusp points x_i, which correspond to $x \to 0$ or $x \to \infty$. Due to the symmetry $x \to 1/x$ of this function, both regimes map to the small x asymptotic.

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