Lagrangian correlators:
The Wilson loop – Large spin OPE – Cusp anomalous dimension dictionary

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In the context of planar conformal gauge theory, we study five-point correlation functions between the interaction Lagrangian and four of the lightest single-trace, gauge-invariant scalar primaries. After performing two light-cone OPEs, we express this correlator in terms of the three-point functions between two leading-twist spinning operators and the Lagrangian. For finite values of spin, we compute these structure constants in perturbation theory up to two loops in $\mathcal{N}=4$ Super Yang–Mills theory. Large values of spin are captured by null polygon kinematics, where we use dualities with null polygon Wilson loops as well as factorization properties to bootstrap the universal behavior of the structure constants at all loops. In both cases, we find explicit maps that relate the Lagrangian structure constants with the leading-twist anomalous dimension. From the large-spin map, we recover the cusp anomalous dimension at strong and weak coupling, including genus-one terms.

I. INTRODUCTION

The operator product expansion (OPE) encodes the data of a conformal field theory (CFT) in its four-point correlation functions. Capturing all CFT data requires infinitely many four-point functions. Iterating the OPE, this infinity of data can in turn be packaged in higher-point functions of the simplest operators. This is the philosophy of the multi-point bootstrap [1–9], which trades an infinity of data for a larger functional complexity.

In null polygon limits, this complexity reduces, and the conformal bootstrap is enhanced by dualities with Wilson loops, both at four [10] and higher points [1, 2]. While null squares and pentagons allow for no finite conformal cross ratios, null hexagons are complicated functions of three variables. Here, we consider a sweet spot: The null square limit of a five-point function, which has a single finite cross-ratio.

We will focus on the correlation function of four single-trace lightest scalar operators and the interaction Lagrangian in planar conformal gauge theories. Usually such correlators are considered in the context of Lagrangian insertions, where one uses Born-level approximation to obtain integrands for scalar operators [11]. Our goal here is different, we will consider this correlation function at quantum level, which is related to Wilson loops and the cusp anomalous dimension [12–14]. In fact, the full four-loop cusp anomalous dimension for $\mathcal{N}=4$ Super Yang-Mills and QCD were computed from this observable [15].

By studying the Lagrangian correlation function via the conformal bootstrap, we translate all its properties to its OPE constituents: the three-point functions of two leading-twist spinning operators and the Lagrangian. At finite values of spin, we compute these structure constants at weak coupling and connect them, via conformal perturbation theory, to leading-twist anomalous dimensions. For large values of spin, we obtain their universal behavior in terms of a simple map with Wilson loop expectation values (16). Using conformal perturbation theory at large spin, we obtain an even simpler map between these structure constants and the cusp anomalous dimension (30).

II. PERTURBATIVE DATA

We consider five-point functions of one primary scalar operator $\mathcal{O}(x)$ and four of the lightest scalar operators ϕ of the theory. For example, in $\mathcal{N}=4$ SYM these would be the $20'$ operators $\phi_j \propto \mathrm{Tr}(y_j \cdot \Phi(x_j))^2$. It is convenient to extract a space-time dependent prefactor of the five-point correlator

$$\langle \phi_1 \ldots \phi_4 \mathcal{O}(x_5)\rangle \equiv \left(\frac{1}{x_{12}^2 x_{34}^2}\right)^{\Delta_\phi} \left(\frac{x_{14}^2}{x_{15}^2 x_{45}^2}\right)^{\Delta_{\mathcal{O}}/2} \times \\ \times \prod_{i=1}^{n_{\mathcal{O}}} (y_i \cdot y_{i+1}) \times G_{\mathcal{O}}(u_i)\,, \quad (1)$$

where $n_{\mathcal{O}} = 4, 5$ depending on whether the fifth operator carries R-charge or not. In this way, $G_{\mathcal{O}}(u_i)$ becomes a function of five cross-ratios

$$u_i = \frac{x_{i,i+1}^2 x_{i+2,i-1}^2}{x_{i,i+2}^2 x_{i+1,i-1}^2}\,, \quad i = 1, \ldots, 5\,, \quad (2)$$

where we identify the points $(x_1, \ldots, x_5)$ periodically. Two particular correlators will be important for us: The correlation function of five light operators (G_ϕ), and the five-point function of four light correlators and one Lagrangian ($G_{\mathcal{L}}$).

To study these correlators, we will consider two light-like OPEs [16] between the lightest operators, as depicted

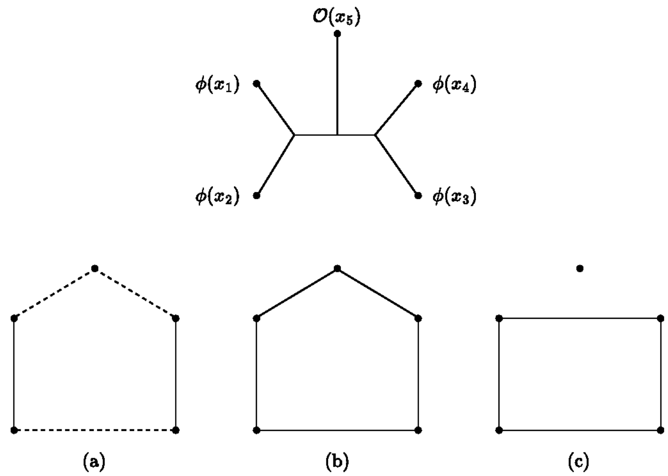

FIG. 1. *Top:* Five-point correlator $\langle\phi\phi\phi\phi\mathcal{O}\rangle$. *Bottom:* Several kinematical limits that we study. Solid lines represent null distances (Lorentzian OPE), while dashed lines connect points that approach each other (Euclidean OPE).

at the top of Figure 1. The leading behavior under this Lorentzian OPE is controlled by the exchange of leading-twist (twist-two) operators in the OPE decomposition:

$$G_{\mathcal{O}}(u_i) = \sum_{J_1,J_2,\ell} \mathcal{F}(u_i) \times C(J_1)C(J_2)C_{\mathcal{O}}(J_1,J_2,\ell)\,, \quad (3)$$

where $C(J)$ are the structure constants of one leading-twist operator with spin J and two lightest scalars operators, while $C_{\mathcal{O}}(J_1,J_2,\ell)$ are the three point functions of two leading-twist spinning operators and the operator $\mathcal{O}(x)$. The quantum number $\ell = 0,1,2,\ldots,\min(J_1,J_2)$ labels the tensor structures of three-point functions with two spinning operators [17]. Meanwhile, $\mathcal{F}$ is the theory-independent conformal block worked out in [1] and recalled in (A1).

In principle, using the integrability formalism for spinning operators [18, 19], it is possible to compute the structure constants C_ϕ at any order in perturbation theory. However, the structure constants $C_{\mathcal{L}}$ are not on the same integrability footing: Despite some tree-level results [20], it is presently not clear how to systematically consider super-descendants like the Lagrangian in the integrability formalism. Using the known integrand [11] and five-point integrals [21], we extract perturbative data for the structure constants $C_{\mathcal{L}}$. For their tree-level expression, we find

$$C^{(0)}_{\mathcal{L}}(J_1,J_2,\ell) = 2\frac{2\,J_1!}{\sqrt{(2J_1)!}}\frac{2\,J_2!}{\sqrt{(2J_2)!}}\times \\ \times\left[(-1)^\ell\binom{J_1+J_2}{\ell-1} + \sum_{m=0}^{\ell-1}\binom{J_1}{m}\binom{J_2}{m}\right]. \quad (4)$$

The one- and two-loop corrections can be found in the attached `Mathematica` file. This data could be useful to develop future integrability formulations.

Since the Lagrangian is exactly marginal, conformal perturbation theory relates the two-point function of two operators with the three-point functions of the two operators and the Lagrangian in a differential equation [22]. For the case of spinning operators this was worked out in [23] to be

$$\frac{\partial\gamma(J)}{\partial\lambda} = \sum_{\ell=0}^{J}\frac{C_{\mathcal{L}}(J,J,\ell)}{1+J-\ell}\,, \quad (5)$$

where $\Delta_J = 2+J+\gamma(J)$ is the dimension of the leading-twist operator. A remarkable feature of this anomalous dimension is that in any planar gauge theory, it develops logarithmic scaling at large values of spin [24, 25]:

$$\gamma(J) \simeq f(\lambda)\ln(J) + g(\lambda)\,, \quad (6)$$

where $f(\lambda)$ and $g(\lambda)$ are the cusp and collinear anomalous dimensions respectively. Below we evaluate (5) at large values of spin, obtaining a map between the large-spin Lagrangian structure constants and the ubiquitous cusp anomalous dimension.

III. NULL SQUARE

We approach the null square limit of the five-point function with the Lagrangian (Figure 1c) by first taking $x_{12}^2, x_{34}^2 \to 0$ (or $u_1,u_3 \to 0$), projecting into leading-twist operators. Next, we take $x_{23}^2 \to 0$ (or $u_2 \to 0$), which we find projects both to large spin J_i and large polarization ℓ. Finally, we take $x_{14}^2 \to 0$, which makes the two values of spin approach each other, $J_2 \to J_1$.

The intuition is that once we create a null square inside a five-point function, the OPE decomposition starts developing four-point-like features. Four-point functions have only one spinning operator flowing in the OPE channel, and this is exactly what the leading term of the five-point function reproduces. We make this precise in Appendix A, via the so-called "Casimir trick" introduced in [26–28], and systematized for higher-point functions in [5]. In the end, the five-point block in the null square limit becomes a simple Bessel-Clifford function

$$\mathcal{F}(u_i) = (u_1u_3)^{1+\gamma(J)}\,(u_2u_4u_5)^{\frac{\Delta_{\mathcal{L}}}{2}}\,2^{2J-2+\gamma(J)+\frac{\Delta_{\mathcal{L}}}{2}}\times \\ \times\pi^{-1/2}J^{\frac{1+\Delta_{\mathcal{L}}}{2}}\mathcal{K}_{\Delta_{\mathcal{L}}/2}\left(Ju_2(J+j_1u_4+j_2u_5)\right)\,, \quad (7)$$

where $\mathcal{K}_n(z) = z^{-n/2}K_n(2\sqrt{z})$, and we introduced the variables

$$J^2 = \frac{J_1^2+J_2^2}{2}\,, \quad j_1 = J_1-\ell\,, \quad j_2 = J_2-\ell\,. \quad (8)$$

The null square limit is described by all these variables being large ($J, j_1, j_2 \to \infty$) with $J \gg j_1, j_2$, while the ratio $r = j_2/j_1$ is finite. This single finite quantum number is associated with the single cross-ratio x that remains finite in the null square limit:

$$x = \frac{u_4}{u_5} = \frac{x_{13}^2 x_{25}^2 x_{45}^2}{x_{15}^2 x_{24}^2 x_{35}^2} \,. \tag{9}$$

From here onward we will consider Born-level normalized quantities, which we denote by $\hat{G} = G/G^{(0)}$, in order to make our statements universal and independent of the prefactor choices such as (1).

Conformal symmetry implies that null square correlators must factorize into two terms [12],

$$\hat{G}_{\mathcal{L}}(u_1, u_2, u_3, u_4, u_5) = \hat{G}_4(u, v) \times \hat{F}(x) \,, \tag{10}$$

which are invariant under cyclic permutations of the null square, $(x_1, x_2, x_3, x_4) \to (x_2, x_3, x_4, x_1)$ with x_5 fixed. This imposes

$$\hat{G}_4(u, v) = \hat{G}_4(v, u) \quad \text{and} \quad \hat{F}(x) = \hat{F}\left(\frac{1}{x}\right) , \tag{11}$$

where $u = u_1 u_3$ and $v = u_2$ are four-point cross ratios.

The first term $\hat{G}_4(u, v)$ is the null four-point function of the lightest operators, which captures all the divergences of the Lagrangian correlator, and depends on the four-point cross-ratios u and v. The second term $\hat{F}(x)$ is a finite function of the remaining finite cross-ratio.

Thus our bootstrap problem is: Can we fix the universal behavior of the structure constants such that the Lagrangian correlator factorizes into the square symmetric functions (10)? To start answering this question, we use the explicit expression for the conformal blocks (7) to write the null square correlator as

$$\begin{aligned}\hat{G}_{\mathcal{L}} = \left(u_2^3 u_4 u_5\right) \int dJ\, dj_1\, dj_2\, (u_1 u_3)^{\frac{\gamma}{2}} 2^{2+\gamma} J^3 \hat{C}(J)^2 \times \\ \times \hat{C}_{\mathcal{L}}(J_1, J_2, \ell)\, \mathcal{K}_2(J u_2 (J + j_1 u_4 + j_2 u_5)) \,, \end{aligned} \tag{12}$$

where we used the tree-level normalized quantities

$$\begin{aligned} C(J_1) = C(J_2) = 2^{-J} \pi^{1/4} J^{1/4} \times \hat{C}(J) \,, \\ C_{\mathcal{L}}(J_1, J_2, \ell) \simeq 8 \times \hat{C}_{\mathcal{L}}(J_1, J_2, \ell) \,. \end{aligned} \tag{13}$$

The tree-level behavior (13) shows the physics of these structure constants. $\hat{C}(J)$ is large and captures the divergent part $\hat{G}_4$ of the correlator in the null square limit. On the other hand, the structure constant $\hat{C}_{\mathcal{L}}(J_1, J_2, \ell)$ is finite and controls the finite part of the correlator $\hat{F}(x)$. We expect that it only depends on the finite ratio r,

$$\hat{C}_{\mathcal{L}}(J_1, J_2, \ell) = \hat{C}_{\mathcal{L}}\left(\frac{J_2 - \ell}{J_1 - \ell}\right) \equiv \hat{C}_{\mathcal{L}}(r) \,. \tag{14}$$

Indeed, we can prove this to be true, using a five-point null square inversion formula (see Appendix B).

Assuming the simple dependence (14) allows us to integrate (12) in one of the two variables j_i, resulting in the following factorized expression for the null square correlator:

$$\begin{aligned}\hat{G}_{\mathcal{L}}(u_i) = \underbrace{\int_0^\infty dJ\, 2^{2+\gamma} J \hat{C}(J)^2 u^{\gamma/2} v K_0(2J\sqrt{v})}_{\hat{G}_4(u,v)} \times \\ \times \underbrace{\int_0^\infty dr\, \frac{x}{(r+x)^2} \hat{C}_{\mathcal{L}}(r)}_{\hat{F}(x)} \,. \end{aligned} \tag{15}$$

The first term is *exactly* the same as the null square four-point function of lightest operators considered in [10] and therefore automatically obeys the cyclicity (11). The invariance under $x \to 1/x$ of the function $\hat{F}(x)$ is also automatically satisfied, provided that $\hat{C}_{\mathcal{L}}(r) = \hat{C}_{\mathcal{L}}(1/r)$. Physical structure constants must have this property, since inverting the ratio r is the same as swapping the spins $J_1 \leftrightarrow J_2$. Thus, the map between $\hat{F}(x)$ and the Lagrangian structure constants is simply

$$\boxed{\hat{F}(x) = x \int_0^\infty dr\, \frac{\hat{C}_{\mathcal{L}}(r)}{(x+r)^2} \,.} \tag{16}$$

We can invert this map by noticing that the right hand side is the derivative of the Cauchy kernel, whose inversion is well understood in terms of its discontinuities. Therefore, one can write the structure constants in terms of discontinuities of $F(x)$:

$$\boxed{r \frac{d}{dr} \hat{C}_{\mathcal{L}}(r) \Big|_{r \geq 0} = \frac{\text{Disc}}{2\pi i} \hat{F}(-r) \,,} \tag{17}$$

where we used the fact that physical structure constants $\hat{C}_{\mathcal{L}}(r)$ must be regular at physical values of spins and polarization ($r \geq 0$).

IV. WEAK AND STRONG COUPLING

Both weak and strong coupling results for the function $\hat{F}(x)$ have been computed in $\mathcal{N} = 4$ SYM. We can use these results together with our map (17) to compute the structure constants $\hat{C}_{\mathcal{L}}$ in these regimes. At weak coupling, the first orders of $\hat{F}(x)$ were computed in [12–15]:

$$\begin{aligned} \hat{F}^{(0)}(x) &= 1 \,, \\ \hat{F}^{(1)}(x) &= -6\zeta_2 - 2H_{00} \,, \\ \hat{F}^{(2)}(x) &= 24\zeta_2 H_{-1-1} - 12\zeta_2 H_{-10} + 24\zeta_2 H_{00} \\ &\quad + 8H_{-1-100} - 4H_{-1000} + 12H_{0000} - 4H_{-200} \\ &\quad - 12\zeta_2 H_{-2} + 8\zeta_3 H_{-1} - 4\zeta_3 H_0 + 107\zeta_4 \,, \end{aligned} \tag{18}$$

where $H_a \equiv H_a(x)$ are harmonic polylogarithms [29], recalled in Appendix C, where we also collect the three-loop and genus-one contributions of $\hat{F}(x)$.[1]

The discontinuities of the harmonic polylogarithms appearing in the perturbative expansion of $\hat{F}(x)$ can be easily evaluated using the `HPL` package [30] for `Mathematica`, resulting in the following expression for the weak coupling structure constants:

$$\begin{aligned}
\hat{C}^{(0)}_{\mathcal{L}}(r) &= 1\,,\\
\hat{C}^{(1)}_{\mathcal{L}}(r) &= -4\zeta_2 - 2H_{00}\,,\\
\hat{C}^{(2)}_{\mathcal{L}}(r) &= 56\zeta_4 - 4\zeta_3 H_0 + 8\zeta_2 H_2 + 12\zeta_2 H_{00}\\
&\quad + 8H_{210} + 4H_{200} + 4H_{30} + 12H_{0000}\,, \qquad (19)
\end{aligned}$$

where $H_a \equiv H_a(r)$, and the three-loop and genus-one corrections are written in Appendix C. In practice, the discontinuity fixes all but the constant term. This in turn can be determined by performing the explicit integration in (16), and matching with the $\hat{F}(x)$ expansion (18).[2]

Even though the individual harmonic polylogarithms have a branch point at $r = 1$, the particular combination appearing in the weak-coupling expansion of $\hat{C}_{\mathcal{L}}(r)$ is real and single-valued for physical values of spins and polarizations ($r > 0$). This is not true for the unphysical region $r < 0$, where $\hat{C}_{\mathcal{L}}(r)$ has a logarithmic branch cut.

At strong coupling, the leading behavior of the function $\hat{F}(x)$ is known [12],

$$\hat{F}(x) = \frac{x}{(x-1)^2}\left(\frac{(x+1)}{(x-1)}\frac{\log x}{2} - 1\right)\sqrt{\lambda} + \dots\,. \qquad (20)$$

Using the inversion formula (17), we can compute the leading term of the structure constant at strong coupling,

$$\hat{C}_{\mathcal{L}}(r) = \frac{r}{2(1+r)^2}\sqrt{\lambda} + \dots\,. \qquad (21)$$

V. WILSON LOOPS AND AMPLITUDES

In $\mathcal{N} = 4$ SYM, n-point correlation functions of $20'$ operators in the limit where their insertions approach the cusp of a null polygon are dual to both null polygonal Wilson loops and MHV gluon scattering amplitudes [31, 32]. In particular, in the five-point null pentagon limit:

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_\phi = (\widehat{\mathrm{MHV}_5})^2\,, \qquad (22)$$

By promoting this relation to super correlation functions and super amplitudes, one obtains that the correlation function of four $20'$ correlators and one Lagrangian, when the points approach the cusps of a null pentagon is dual to (the top component of) the NMHV scattering amplitude [33, 34]

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_{\mathcal{L}} = (\widehat{\mathrm{NMHV}_5})^2\,, \qquad (23)$$

For five points, the NMHV amplitude is the parity conjugate of the MHV amplitude,[3] thus in the null pentagon limit both correlators are identical

$$\lim_{x^2_{i,i+1}\to 0} \hat{G}_\phi = \lim_{x^2_{i,i+1}\to 0} \hat{G}_{\mathcal{L}} = \langle \widehat{W_5}\rangle\,, \qquad (24)$$

which immediately implies $\hat{C}_{\mathcal{L}} = \hat{C}_\phi$, that is[4]

$$\begin{aligned}
&\hat{C}_{\mathcal{L}}(J_1, J_2, \ell) =\\
&\qquad \mathcal{N}(\lambda)\, e^{-\frac{f(\lambda)}{4}(\log \ell^2 + 2\log 2 \log(J_1 J_2)) - \frac{g(\lambda)}{2}\log \ell}\,. \qquad (25)
\end{aligned}$$

The story is completely different when we consider the null square limit of these five-point correlators. As pointed out in [12], the duality with Wilson loops continues to hold even if one adds an extra operator at finite distance to the null square configuration

$$\lim_{x^2_{1,2}, x^2_{2,3}, x^2_{3,4}, x^2_{1,4}\to 0} \hat{G}_{\mathcal{L}} = \langle \widehat{W_4 \mathcal{L}}\rangle\,. \qquad (26)$$

One can recast this duality as an equation for $\hat{F}(x)$ by using that Lagrangian correlators are obtained from a derivative with respect to the coupling,

$$\frac{\partial}{\partial\lambda}\log\langle \hat{W}_4\rangle = 8\int dx_5\, \frac{x^2_{13}x^2_{24}}{x^2_{15}x^2_{25}x^2_{35}x^2_{45}}\hat{F}(x)\,. \qquad (27)$$

where the space-time prefactor arises from the Born-level ratio $\langle \phi_1 \dots \phi_4 \mathcal{L}(x_5)\rangle^{(0)}/\langle \phi_1 \dots \phi_4\rangle^{(0)}$.

VI. CUSP ANOMALOUS DIMENSION

The UV cusp divergences of the Wilson loop are controlled by the cusp anomalous dimension. In principle, one can match the divergences appearing on both sides of the relation (27) to compute this quantity. In practice,

[1] The higher-genus contributions to the cusp anomalous dimension start at four loops, but due to its derivative relation with this quantity, $\hat{F}(x)$ features genus-one terms already at three loops.

[2] The integrals of harmonic polylogarithms can also be trivially done using the package `HPL` [30].

[3] This implies that one is the complex conjugate of the other. Parity-odd terms (imaginary) are important to establish the duality at integrand level, however they stem from a total derivative and integrate to zero.

[4] As the null square, the null pentagon limit is also governed by the regime of large spin and large polarization. The difference is that for the pentagon there are neither finite cross-ratios, nor finite ratios among the quantum numbers J_i, ℓ, see appendix Appendix D.

this is done with the help of the functional $\mathcal{I}$ formulated in [14] and recalled below,[5]

$$\frac{\partial f(\lambda)}{\partial \lambda} = \mathcal{I}[8\hat{F}(x)]\,, \tag{28}$$

where one is instructed to first expand the function $\hat{F}(x)$ around small values of x,[6] and then act with the linear functional on individual terms as

$$\mathcal{I}[x^p] = \frac{\sin \pi p}{\pi p}\,. \tag{29}$$

Starting from the conformal perturbation theory relation (5), we propose an alternative and more explicit map. It relates the three point function $\hat{C}_{\mathcal{L}}$ with the cusp anomalous dimension simply as

$$\boxed{\frac{\partial f(\lambda)}{\partial \lambda} = 8\hat{C}_{\mathcal{L}}(1)\,.} \tag{30}$$

The large-spin limit of the sum (5) is dominated by the region where spins and polarizations are of the same order. Therefore, we can trade the sum over polarizations by an integral and replace the structure constants by their large spin and polarization behavior (14). Since the sum runs over structure constants of identical spins, the ratio r becomes one, and $\hat{C}_{\mathcal{L}}(1)$ becomes a constant that can be factored out of the integral. The integral is then trivial and evaluates to $\log J$. Matching the log-divergent terms on both sides of the equation (6) yields the map (30).

We verify this result by recovering the known values of the cusp anomalous dimension at strong and weak coupling, including genus-one terms: At strong coupling, replacing $r = 1$ in (21) and using the map (30) yields the leading term of the cusp anomalous dimension: $f(\lambda) \simeq 8\sqrt{\lambda}$. Similarly at weak coupling, by evaluating (19) and (C3) at $r = 1$, we recover the four-loop anomalous dimension [15]:

$$\begin{aligned} 8\hat{C}_{\mathcal{L}}(1) = 8 - 32\zeta_2\lambda + 528\zeta_4\lambda^2 - \Big(64\zeta_3^2 + 1752\zeta_6 + \\ + \frac{1}{N^2}(1152\zeta_3^2 + 2976\zeta_6)\Big)\lambda^3\,. \end{aligned} \tag{31}$$

The map between three point functions and the cusp anomalous dimension (30) is simpler than the map (28) previously considered in the literature. However since the structure constants and the function $F(x)$ are also related to each other via (16) we must have the following consistency condition for the structure constant:

$$\mathcal{I}\left[x\int_0^\infty dr\,\frac{\hat{C}_{\mathcal{L}}(r)}{(x+r)^2}\right] = \hat{C}_{\mathcal{L}}(1)\,. \tag{32}$$

Unfortunately, this is **not** a bootstrap equation for $\hat{C}_{\mathcal{L}}(r)$. One simple way to see this, is to expand this function as a power series and note that the relation (32) acts trivially on each polynomial term

$$\mathcal{I}\left[x\int_0^\infty dr\,\frac{r^p}{(x+r)^2}\right] = \frac{\pi p}{\sin \pi p}\mathcal{I}\left[x^p\right] = 1\,, \tag{33}$$

and therefore (32) is trivially satisfied for any function $\hat{C}_{\mathcal{L}}(r)$. One might be worried that the expression above is only valid for $|p| < 1$, and that $\hat{C}_{\mathcal{L}}(r)$ has no regular expansion around $r = 0$. However, using the physical properties of the structure constants, i. e. invariance under swapping the spins $\hat{C}_{\mathcal{L}}(r) = \hat{C}_{\mathcal{L}}(1/r)$ and regularity around $r = 1$ (where we recover the cusp anomalous dimension) we can analytically continue this result for any p, see Appendix E.

VII. CONCLUSION

Multi-point conformal correlation functions organize the CFT data in non-trivial functions of conformal cross ratios. These functions have, generically, a complex analytic structure that does not follow from a single exchange of a physical operator. Instead, it is often the case that the intricate structure only emerges after summing the contributions of an infinite number of operators [1, 2, 9, 10, 35].

Using the conformal bootstrap, we analyzed the five-point correlation function of one Lagrangian and four lightest scalar operators, in terms of the three-point functions of two leading-twist spinning operators and the interaction Lagrangian. We computed these structure constants for finite and large values of spin, connecting them with anomalous dimensions (5), null pentagon Wilson loops (25), null square Wilson loops with insertions (16), and the cusp anomalous dimension (30).

In $\mathcal{N} = 4$ SYM, there are several distinct integrability frameworks developed to study the different observables listed above. Three-point correlation functions are described by integrable hexagon form factors [18], null polygonal Wilson loops can be constructed out of integrable pentagons [36], and anomalous dimensions can be computed via the Quantum Spectral curve [37]. The sharp maps that we derived here connect all these quantities and could be a great laboratory for developing a unifying integrability description of $\mathcal{N} = 4$ SYM.

It would be interesting to study the expectation value of the square Wilson loop with other types of insertions

[5] The factor 8 arises from the fact that all our quantities are Born-level normalized except for the cusp anomalous dimension. We can drop this factor by considering $\hat{f}(\lambda)$, but we refer from doing that to avoid confusion.

[6] The cusp singularities will arise when x_5 approaches the cusp points x_i, which correspond to $x \to 0$ or $x \to \infty$. Due to the symmetry $x \to 1/x$ of this function, both regimes map to the small x asymptotic.

using the techniques developed here. It should also be possible and very interesting to generalize our analysis to other physical observables, for example null square Wilson loops with two operator insertion, or null pentagon Wilson loops with a single operator insertion [38], and to connect these quantities with conformal manifold constraints [39, 40] and integrability.

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Appendix A: Conformal Blocks in the Null Square Limit

In this appendix, we construct the null square limit of conformal blocks $\mathcal{F}(u_1,\dots,u_5)$ for the scalar five-point function $\langle\phi\phi\phi\phi\mathcal{O}\rangle$ that arises from two $\phi\times\phi$ OPEs [1]

$$\begin{aligned}\mathcal{F}(u_i) = &\int_0^1 dt_1 dt_2\, \frac{\Gamma(2J_1+\tau_1)}{\Gamma\left(J_1+\frac{\tau_1}{2}\right)^2}\frac{\Gamma(2J_2+\tau_2)}{\Gamma\left(J_2+\frac{\tau}{2}\right)^2}\times \\ &\times (t_1(1-t_1))^{J_1+\frac{\tau_1-2}{2}}(t_2(1-t_2))^{J_2+\frac{\tau_2-2}{2}}\times \\ &\times \frac{(1-t_1u_4-u_2u_4+t_1u_2u_4)^{J_2-\ell}}{(1-t_1+t_1u_5)^{J_1-\ell+\frac{\Delta_{\mathcal{O}}+\tau_1-\tau_2}{2}}} u_1^{\frac{\tau_1}{2}}u_5^{\frac{\Delta_{\mathcal{O}}}{2}}\times \\ &\times \frac{(1-t_2u_5-u_2u_5+t_2u_2u_5)^{J_1-\ell}}{(1-t_2+t_2u_4)^{J_2-\ell+\frac{\Delta_{\mathcal{O}}+\tau_2-\tau_1}{2}}} u_3^{\frac{\tau_2}{2}}u_4^{\frac{\Delta_{\mathcal{O}}}{2}}\times \\ &\times ((t_1+t_2-t_1t_2)(1-u_2)+u_2)^{-J_1-J_2+\frac{\Delta_{\mathcal{O}}-\tau_1-\tau_2}{2}}\,.\end{aligned} \tag{A1}$$

This block is labeled by the twists τ_1,τ_2 and spins J_1,J_2 of the two exchanged fields $\mathcal{O}_1,\mathcal{O}_2$, along with the integer $\ell=0,\dots,\min(J_1,J_2)$. The latter denotes a basis of tensor structures $H_{12}^{\ell}V_{1,23}^{J_1-\ell}V_{2,31}^{J_2-\ell}$ for the spinning three-point function $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}\rangle$. Throughout the appendix, we assume that the two exchanged fields have equal twist: $\tau_1=2h=\tau_2$.

The ordered null square limit $\mathrm{NS}_<$ is

$$x_{12}^2,x_{23}^2,x_{34}^2,x_{41}^2\to 0,\quad x_{12}^2,x_{34}^2\ll x_{23}^2\ll x_{41}^2. \tag{A2}$$

For bookkeeping purposes, we define the above lightcone limits via infinitesimal rescalings $x_{ij}^2\to\epsilon_{ij}x_{ij}^2$, $\epsilon_{ij}\to 0$. In this notation, the five cross-ratios scale as

$$u_1=O(\epsilon_{12}),\quad u_2=O(\epsilon_{41}\epsilon_{23}),\quad u_3=O(\epsilon_{34}), \tag{A3}$$
$$u_4=O(\epsilon_{41}^{-1}),\quad u_5=O(\epsilon_{41}^{-1}). \tag{A4}$$

In particular, note that the ratio $x=u_4/u_5$ is finite in this limit. At leading order, the first two limits $x_{12}^2,x_{34}^2\to 0$ restrict the sum over descendants of $\mathcal{O}_1,\mathcal{O}_2$ to those with minimal twist $\Delta-J=2h$. As a result, the leading asymptotics in these two limits is

$$\mathcal{F}(u_i)\sim(u_1u_3)^h(u_4u_5)^{h_{\mathcal{O}}}\tilde{\mathcal{F}}(u_2,u_4,u_5), \tag{A5}$$

where $\tilde{\mathcal{F}}(u_2,u_4,u_5)$ is a leading-twist block. To simplify future calculations, we stripped off a u_4u_5-dependent prefactor and introduced the notation $2h_{\mathcal{O}}:=\Delta_{\mathcal{O}}$ for the half-twist of the external scalar $\mathcal{O}$. For the remaining two limits, the asymptotics of the leading-twist blocks $\tilde{\mathcal{F}}(u_2,u_4,u_5)$ are derived in two steps: first, we derive an integral representation for the most general solution to the Casimir equations. Next, we identify a basis of solutions consistent with this integral representation by analyzing a power series representation of the leading-twist blocks.

a. General solution to Casimir equations. The Casimir equations in cross-ratio space take the form

$$\tilde{\mathcal{D}}_a^2\tilde{\mathcal{F}}=\left(J_a^2+O(J_i)\right)\tilde{\mathcal{F}},\quad a=1,2, \tag{A6}$$

where at leading order in the limit $\mathrm{NS}_<$, the differential operators $\tilde{\mathcal{D}}_a^2$ reduce to

$$\begin{aligned}\mathcal{D}_a^2 =&\epsilon_{23}^{-1}\epsilon_{41}^{-1}\partial_2\left(\vartheta_2-\vartheta_4-\vartheta_5-h_{\mathcal{O}}\right)\\ &+\epsilon_{23}^{-1}\partial_2\partial_{6-a}+O(\epsilon_{23}^0).\end{aligned} \tag{A7}$$

Here, we introduced the compact notation $\partial_i:=\partial_{u_i}$ and $\vartheta_i:=u_i\partial_{u_i}$ for $i=2,4,5$. At leading order, we thereby obtain a simple system of two differential equations in three variables:

$$\partial_2\left(\vartheta_2-\vartheta_4-\vartheta_5-\frac{\Delta_{\mathcal{O}}}{2}\right)\tilde{\mathcal{F}}=\mathbf{J}^2\tilde{\mathcal{F}} \tag{A8}$$
$$\partial_2(\partial_4-\partial_5)\tilde{\mathcal{F}}=2\mathbf{j}^2\tilde{\mathcal{F}}. \tag{A9}$$

After applying the Laplace transform with respect to u_2, it is easy to express the solutions to this system as integrals of a one-variable function:

$$\tilde{\mathcal{F}}(u_2,u_4,u_5)=u_2^{h_{\mathcal{O}}}\int_0^\infty\frac{dt}{t^{1+h_{\mathcal{O}}}}e^{-t-\frac{\mathbf{J}^2u_2}{t}-\mathbf{j}^2u_2\frac{u_4-u_5}{t}}$$

$$\times \hat{f}\left(u_2 \frac{u_4+u_5}{t}\right). \qquad \text{(A10)}$$

We have thereby reduced the problem to identifying a basis of functions $\hat{f}(Y)$ corresponding to the basis of tensor structures $\ell = 0, \ldots, \min(J_1, J_2)$.

b. Null square limit of leading-twist blocks. Define $(u_2, u_4, u_5) := (1-Z, v_2, v_1)$ and $\bar{h}_i := h + J_i$. For blocks in the ℓ-basis of tensor structures, the following integral representation for blocks was derived in [5]:

$$\tilde{\mathcal{F}} = Z^\ell \prod_{1\leq a\neq b\leq 2} \frac{\Gamma(2\bar{h}_a)}{\Gamma(\bar{h}_a)^2} \int_0^1 dt_a \frac{(t_a(1-t_a))^{\bar{h}_a-1}}{(1-(1-v_a)t_a)^{J_a-\ell+h_\mathcal{O}}} \times \frac{(1-(1-Z)v_a - Zv_a t_b)^{J_a-\ell}}{(1-Z(1-t_1)(1-t_2))^{\bar{h}_{12;\mathcal{O}}}}, \qquad \text{(A11)}$$

where $\bar{h}_{12;\mathcal{O}} := \bar{h}_1 + \bar{h}_2 - h_\mathcal{O}$. The first limit $x_{23}^2 \to 0$ corresponds to $Z \to 1^-$. We analyze it by expanding the integrand around $Z = 0$, resulting in a power series expansion of $\tilde{\mathcal{F}}$:

$$\tilde{\mathcal{F}} = \sum_{k=\ell}^{\infty} \frac{(\bar{h}_{12;\mathcal{O}})_k}{k!} \prod_{a=1}^{2} \sum_{m_a=0}^{J_a-\ell} \binom{J_a-\ell-m_a}{m_a} \times \qquad \text{(A12)}$$
$$\times f_{k,m_1,m_2}(v_1,v_2)\, Z^{k+m_1+m_2}.$$

The functions $f_{k,m_1,m_2}(v_1,v_2)$ can be determined explicitly in terms of a product of two Gauss hypergeometric functions with arguments $1-v_1, 1-v_2$. Now, for $Z = 1 + O(\epsilon_{23})$, the derivative operator acts as $\partial_Z \tilde{\mathcal{F}} = O(\epsilon_{23}^{-1})\tilde{\mathcal{F}}$. At the same time, the action of this derivative on each summand is $(k+m_1+m_2)/Z$. We deduce that the sum is dominated by the regime $k+m_1+m_2 = O(\epsilon_{23}^{-1})$. Moreover, since $0 \leq m_1, m_2 \leq \max(J_1, J_2) = O(\epsilon_{23}^{-1/2})$, the powers $Z^{m_1+m_2} = 1 + O(\epsilon_{23}^{1/2})$ are trivial at leading order. This allows us to resum over m_1, m_2 and approximate the power series by

$$\tilde{\mathcal{F}} \sim \sum_{k=\ell}^{\infty} \frac{(\bar{h}_{12;\mathcal{O}})_k}{k!} Z^k \prod_{1\leq a\neq b\leq 2} \frac{(\bar{h}_a)_k}{(2\bar{h}_a)_k} \times \qquad \text{(A13)}$$
$$\times F_1\left(\bar{h}_a; J_a-\ell-h_\mathcal{O}, \ell-J_b; 2\bar{h}_a+k; 1-v_a, v_b\right),$$

where F_1 denotes the Appell function of the first kind:

$$F_1(b; a_1, a_2; c; z_1, z_2) = \prod_{a=1}^{2} \sum_{n_a=0}^{\infty} \frac{(a_a)_{n_a}}{n_a!} z_a^{n_a} \frac{(b)_{n_1+n_2}}{(c)_{n_1+n_2}}.$$

Using this explicit expression for the power series, we can now approximate the region $k = O(\epsilon_{23}^{-1})$ that dominates the sum in the lightcone limit by an integral. In this case, the integrand admits an expansion near $\epsilon_{23} = 0$ with $J_1^2, J_2^2 = O(\epsilon_{23}^{-1})$ and $0 \leq \ell \leq \min(J_1, J_2)$. At leading order, the block therefore reduces to

$$\tilde{\mathcal{F}} \sim \mathcal{N}_{J_1 J_2}^{h,h_\mathcal{O}} \int_0^\infty \frac{dk}{k^{1+h_\mathcal{O}}} e^{-ku_2 - \frac{(J_1+J_2)^2}{2k}} \times \prod_{1\leq a\neq b\leq 2} e^{-\frac{J_a}{k}\left(\frac{3J_a}{2} + (J_a-\ell)(1-v_a) - (J_b-\ell)v_b\right)}, \qquad \text{(A14)}$$

where

$$\mathcal{N}_{J_1 J_2}^{h,h_\mathcal{O}} := \frac{1}{\Gamma(J_1+J_2+2h-h_\mathcal{O})} \prod_{a=1}^{2} \frac{\Gamma(2J_a+2h)}{\Gamma(J_a+h)}. \qquad \text{(A15)}$$

The following approximation of blocks was based solely on the lightcone limit $x_{23}^2 \to 0$, where the cross-ratios $u_4, u_5 = v_2, v_1$ remained finite. In the final lightcone limit $x_{41}^2 \to 0$, the latter scale as $v_a = O(\epsilon_{41}^{-1})$. We would like to identify the blocks in this limit with the integral representation (A10) of solutions to the Casimir equations by changing variables to $t := ku_2$. To obtain a basis of blocks that is consistent with the functional form of eq. (A10), we assume that the ℓ-dependent terms in the exponential remain as leading contributions,

$$J_a v_b u_2 (J_b - \ell) = O(1) \iff J_b - \ell = O\big(\epsilon_{23}^{-1/2}\epsilon_{41}^{1/2}\big). \qquad \text{(A16)}$$

Changing variables according to (8) from (J_1, J_2, ℓ) to (J, j_1, j_2), allows us to parameterize the large-spin limit as

$$J^2 = O(\epsilon_{23}^{-1}\epsilon_{41}^{-1}), \quad j_a^2 = O(\epsilon_{23}^{-1}\epsilon_{41}). \qquad \text{(A17)}$$

After expanding the Gamma functions of $\mathcal{N}_{J_1 J_2}^{h,h_\mathcal{O}}$ in eq. (A15), we finally obtain

$$\tilde{\mathcal{F}} = 2^{2h+2J+h_\mathcal{O}-1}\pi^{-1/2} J^{1/2+h_\mathcal{O}} u_2^{h_\mathcal{O}} \times \int_0^\infty \frac{dt}{t^{1+h_\mathcal{O}}} e^{-t-\frac{J^2 u_2}{t} - \frac{J j_1 u_2 u_4}{t} - \frac{J j_2 u_2 u_5}{t}}. \qquad \text{(A18)}$$

This expression coincides with (7) after integrating over t, in addition to setting $2h = 2+\gamma$ for the twist of the exchanged fields and $2h_\mathcal{O} = \Delta_\mathcal{L}$ for the twist (scaling dimension) of the fifth external scalar.

Appendix B: Null Square Inversion Formula

This appendix is divided into two parts: first, we invert the conformal block decomposition of the five-point function in the ordered null-square limit $NS_<$ of eq. (A2). Next, by specializing this inversion formula to five-point functions that factorize in the null square limit, we demonstrate that $C_\mathcal{O}(J_1, J_2, \ell)$ reduces to a one-variable function of the ratio $r = (J_2-\ell)/(J_1-\ell)$, thereby proving the uniqueness of eq. (15).

a. Derivation of the inversion formula. The derivation is based on the observation that the null square blocks in eq. (A10) are the integral transform of a simple power-law-times-exponential function. After the change of variables $t := ku_2$, we can identify this integral as a straightforward generalization of the Laplace transform, which we denote by

$$G(u_2,u_4,u_5) =: \int_0^\infty dk\, e^{-ku_2}\mathbf{L}^{-1}[G]\left(k,\frac{u_4}{k},\frac{u_5}{k}\right). \tag{B1}$$

In this case, we can express the inverse Laplace transform of the five-point blocks in the ordered null-square limit as

$$\mathbf{L}^{-1}[\tilde{\mathcal{F}}](k,w_4,w_5) = \frac{\mathcal{N}_5(J)}{k^{1+h_{\mathcal{O}}}}e^{-J\left(\frac{J}{k}+j_1w_4+j_2w_5\right)}, \tag{B2}$$

where $w_4 = u_4/k$, $w_5 = u_5/k$, $h_{\mathcal{O}} = \Delta_{\mathcal{O}}/2$, and

$$\mathcal{N}_5(J) := 2^{2h+2J+\Delta_{\mathcal{O}}/2-1}J^{1/2+\Delta_{\mathcal{O}}/2}\pi^{-1/2}. \tag{B3}$$

Given that the five-point function reduces to leading-twist exchange $G_{\mathcal{O}} \sim (u_1u_3)^h(u_4u_5)^{h_{\mathcal{O}}}\tilde{G}_{\mathcal{O}}(u_2,u_4,u_5)$ in the limit $u_1,u_3\to 0$, we can then express the null square conformal block decomposition as

$$\begin{aligned}&\mathbf{L}^{-1}[\tilde{G}_{\mathcal{O}}](k,w_4,w_5) = \\ &\int_0^\infty d(J^2)e^{-J^2k^{-1}}\int_0^\infty d(Jj_1)e^{-Jj_1w_4}\int_0^\infty d(Jj_2)e^{-Jj_2w_5} \\ &\times\frac{\mathcal{N}_5(J)}{8J^3k^{1+h_{\mathcal{O}}}}C(J)^2C_{\mathcal{O}}(J,j_1,j_2),\end{aligned}$$

where the original measure is $dJ_1dJ_2d\ell/4$, with a factor of four to account for even spin exchange in the two OPEs. Now, in this k-space, the conformal block decomposition itself reduces to another series of Laplace transforms with respect to (J^2, Jj_1, Jj_2). After applying their inverse transforms, we obtain

$$\begin{aligned}C_{\mathcal{O}}(J,j_1,j_2) = \frac{8J^3}{\mathcal{N}_5(J)C(J)^2}\int_{(c+i\mathbb{R})^3}\frac{dk}{k^{1-h_{\mathcal{O}}}}dw_4dw_5 \\ \times e^{J^2/k+Jj_1w_4+Jj_2w_5}\,\mathbf{L}^{-1}[\tilde{G}_{\mathcal{O}}](k,w_4,w_5)\,.\end{aligned} \tag{B4}$$

It is now straightforward to write down an inversion formula for the position space correlator by inserting the formula for the inverse Laplace transform with respect to k:

$$\mathbf{L}^{-1}[\tilde{G}_{\mathcal{O}}](k,w_4,w_5) = \int_{c+i\mathbb{R}} du_2 e^{ku_2}\tilde{G}_{\mathcal{O}}(u_2,kw_4,kw_5). \tag{B5}$$

Here, following the standard definition of the inverse Laplace transform, $c>0$ is a constant shift of the contours of integration to the right of all poles and branch cuts of the integrand in the complex plane. As a result, we obtain the following inversion formula for the five-point function in the ordered null square limit:

$$\boxed{\begin{aligned}C_{\mathcal{O}}(J,j_1,j_2) &= \frac{2^{4-2h-\Delta_{\mathcal{O}}/2}J^{\frac{5-\Delta_{\mathcal{O}}}{2}}}{\hat{C}(J)^2} \\ &\times\int\limits_{(c+i\mathbb{R})^4}\frac{dk\,du_2\,dw_4\,dw_5}{k^{1+\Delta_{\mathcal{O}}/2}}\,e^{ku_2+J^2/k+Jj_1w_4+Jj_2w_5} \\ &\times\frac{G_{\mathcal{O}}(u_1,u_2,u_3,u_4=kw_4,u_5=kw_5)}{(u_1u_3)^h(w_4w_5)^{\Delta_{\mathcal{O}}/2}}\,.\end{aligned}} \tag{B6}$$

b. Consequences of factorization for OPE coefficients. We now consider five-point functions with the factorization property

$$G_{\mathcal{O}}(u_1,\ldots,u_5) \sim G_4(u_1u_3,u_2)(u_4u_5)^{h_{\mathcal{O}}}f_{\mathcal{O}}(u_4,u_5), \tag{B7}$$

where $G_4(u,v)$ is the four-point function in the null square limit, while $f_{\mathcal{O}}$ is a symmetric function of two variables that is homogeneous of degree $-h_{\mathcal{O}}$:

$$f_{\mathcal{O}}(\lambda u_5,\lambda u_4) = f_{\mathcal{O}}(\lambda u_4,\lambda u_5) = \lambda^{-h_{\mathcal{O}}}f_{\mathcal{O}}(u_4,u_5). \tag{B8}$$

Given $G_4(u,v) \sim u^hg_4(v)$ in the ordered null square limit, the inverse Laplace transform (B5) of a factorized function (B7) will itself factorize as well:

$$\mathbf{L}^{-1}[g_4\,f_{\mathcal{O}}](k,w_4,w_5) = k^{-h_{\mathcal{O}}}\mathbf{L}^{-1}[g_4](k)\,f_{\mathcal{O}}(w_4,w_5).$$

We can therefore separate the integral over k from the integrals over w_4,w_5 in the inversion formula (B5). In doing so, we identify the inversion formula for the four-point OPE coefficients $C(J)^2$:

$$C(J)^2 = \frac{4J}{\mathcal{N}_4(J)}\int_{c+i\mathbb{R}}\frac{dk}{k}e^{\frac{J^2}{k}}\mathbf{L}^{-1}[g_4](k), \tag{B9}$$

where $\mathcal{N}_4(J) = 4^{J+h}J^{1/2}\pi^{-1/2}$, which follows from the Laplace transform with respect to v of the Bessel function $K_0(2J\sqrt{v})$ in four-point conformal blocks. Given $\mathcal{N}_5(J) = 2^{h_{\mathcal{O}}-1}J^{h_{\mathcal{O}}}\mathcal{N}_4(J)$, the inversion formula therefore reduces to

$$\begin{aligned}\left(\frac{J}{2}\right)^{h_{\mathcal{O}}-2}&C_{\mathcal{O}}(J,j_1,j_2) = \\ &\int_{(c+i\mathbb{R})^2}dw_4dw_5e^{Jj_1w_4+Jj_2w_5}f_{\mathcal{O}}(w_4,w_5).\end{aligned} \tag{B10}$$

The RHS of this equation is the inverse Laplace transform of $f_{\mathcal{O}}$ with respect to each of its arguments w_4,w_5. Since the latter function is homogeneous of degree $-h_{\mathcal{O}}$, then the LHS (i.e. the Laplace transform of $f_{\mathcal{O}}$) must be a homogeneous function of degree $h_{\mathcal{O}}-2$ in (Jj_1,Jj_2). As a result, factorization in the null square limit implies that OPE coefficients take the most general form

$$C_{\mathcal{O}}(J,j_1,j_2) = j_1^{h_{\mathcal{O}}-2}C_{\mathcal{O}}(r),\;\; r = j_2/j_1, \tag{B11}$$

in agreement with eq. (14) for the OPE coefficients $C_{\mathcal{L}}$ normalized by their tree-level value $C_{\mathcal{L}}^{(0)} = 8$. Finally, after re-inverting the relation between $C_{\mathcal{O}}$ and $f_{\mathcal{O}}$ to

$$f_{\mathcal{O}}(u_4, u_5) = \int_{\mathbb{R}_+^2} dj_1 dj_2 e^{-(j_1 u_4 + j_2 u_5)} \left(\frac{j_1}{2}\right)^{h_{\mathcal{O}}-2} C_{\mathcal{O}}(r), \tag{B12}$$

we can explicitly integrate over r by parameterizing the homogeneous function as

$$f_{\mathcal{O}}(u_4, u_5) = (u_4 u_5)^{-h_{\mathcal{O}}/2} F_{\mathcal{O}}(x), \quad x = u_4/u_5. \tag{B13}$$

As a result, the null square conformal block decomposition reduces to

$$F_{\mathcal{O}}(x) = \frac{\Gamma(h_{\mathcal{O}} - 1)}{2^{h_{\mathcal{O}}-2}} x^{h_{\mathcal{O}}/2} \int_0^\infty \frac{dx}{(x+r)^{h_{\mathcal{O}}}} C_{\mathcal{O}}(r), \tag{B14}$$

in agreement with eq. (16) for $\mathcal{O} = \mathcal{L}$.

Appendix C: Three-Loop Results

The weak-coupling expressions for the finite function $\hat{F}(x)$ and the structure constant $\hat{C}_{\mathcal{L}}$ are given in terms of harmonic polylogarithms (HPLs). These functions are defined recursively, via

$$H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x \frac{dz}{z - a_1} H_{a_2,\ldots,a_n}(z) \tag{C1}$$

with the seed $H(x) = 1$ and $a_i \in \{-1, 0, 1\}$. We use the compact HPL notation introduced in [29], in which a string of $n-1$ zero indices followed by ± 1 is replaced by $\pm n$, i. e. $H_{3,0} = H_{0,0,1,0}(x)$.

The three-loop contribution to $\hat{F}(x)$ is given by

$$\begin{aligned}
\hat{F}^{(3)}(x) =& 16\zeta_3 H_{-2-1} + 32\zeta_3 H_{-20} + 16\zeta_3 H_{-1-2} - \\
&-32\zeta_3 H_{-1-1-1} + 16\zeta_3 H_{-1-10} + 16\zeta_3 H_{-100} - \\
&-8\zeta_3 H_{000} - 144\zeta_2 H_{-3-1} + 88\zeta_2 H_{-30} - \\
&-96\zeta_2 H_{-2-2} - 96\zeta_2 H_{-1-3} - 516\zeta_4 H_{-1-1} + \\
&+360\zeta_4 H_{-10} - 646\zeta_4 H_{00} + 48\zeta_2 H_{-2-1-1} - \\
&-32\zeta_2 H_{-2-10} + 88\zeta_2 H_{-200} + 48\zeta_2 H_{-1-2-1} - \\
&-48\zeta_2 H_{-1-20} + 48\zeta_2 H_{-1-1-2} - \\
&-96\zeta_2 H_{-1-1-1-1} + 48\zeta_2 H_{-1-1-10} - \\
&-96\zeta_2 H_{-1-100} + 96\zeta_2 H_{-1000} - 216\zeta_2 H_{0000} + \\
&+48H_{-400} - 48H_{-3-100} + 40H_{-3000} - \\
&-32H_{-2-200} - 32H_{-1-300} + 16H_{-2-1-100} - \\
&-16H_{-2-1000} + 40H_{-20000} + 16H_{-1-2-100} - \\
&-32H_{-1-2000} + 16H_{-1-1-200} - \\
&-32H_{-1-1-1-100} + 16H_{-1-1-1000} - \\
&-48H_{-1-10000} + 48H_{-100000} - 120H_{000000} - \\
&-40\zeta_3^2 - 48\zeta_3 H_{-3} - 8\zeta_2\zeta_3 H_0 + 144\zeta_2 H_{-4} + \\
&+394\zeta_4 H_{-2} - 112\zeta_5 H_{-1} + 32\zeta_5 H_0 - \\
&-\frac{3085}{2}\zeta_6 + \frac{1}{N^2}\Big(- 96\zeta_3 H_{-20} - 96\zeta_3 H_{-100} + \\
&+192\zeta_3 H_{000} + 48\zeta_2 H_{-30} + 144\zeta_2 H_{-2-2} + \\
&+144\zeta_2 H_{-1-3} - 1620\zeta_4 H_{-1-1} + 312\zeta_4 H_{-10} + \\
&+300\zeta_4 H_{00} - 48\zeta_2 H_{-1-20} - 432\zeta_2 H_{-1-100} + \\
&+96\zeta_2 H_{-1000} + 48\zeta_2 H_{0000} + 48H_{-3000} + \\
&+48H_{-2-200} + 48H_{-1-300} + 96H_{-20000} - \\
&-48H_{-1-2000} - 432H_{-1-10000} + \\
&+96H_{-100000} + 144H_{000000} - 528\zeta_3^2 - 231\zeta_6 + \\
&+384\zeta_2\zeta_3 H_{-1} + 192\zeta_2\zeta_3 H_0 - 216\zeta_4 H_{-2} + \\
&+48\zeta_5 H_{-1} + 288\zeta_5 H_0 + \\
&+\frac{1}{1+x}\Big(- 48\zeta_2 H_{-30} + 288\zeta_2 H_{-200} + \\
&+144\zeta_2 H_{-1000} - 48\zeta_2 H_{0000} + 180\zeta_4 H_{-10} - \\
&-300\zeta_4 H_{00} - 192\zeta_3 H_{000} - 96H_{-400} - \\
&-48H_{-3000} + 192H_{-20000} + 240H_{-100000} - \\
&-144H_{000000} - 2217\zeta_6 - 288\zeta_2 H_{-4} - \\
&-480\zeta_3\zeta_2 H_0 + 1440\zeta_4 H_{-2} - 432\zeta_5 H_0\Big)\Big).
\end{aligned} \tag{C2}$$

While the three-loop contribution to the structure constant $\hat{C}_{\mathcal{L}}$ is a much simpler function, given by

$$\begin{aligned}
\hat{C}_{\mathcal{L}}^{(3)}(r) =& 16\zeta_3 H_{20} - 16\zeta_3 H_{21} + 8\zeta_3 H_{000} + 196\zeta_4 H_{00} + \\
&+ 16\zeta_2 H_{22} + 48\zeta_2 H_{30} + 16\zeta_2 H_{31} + 48\zeta_2 H_{200} + \\
&+ 48\zeta_2 H_{210} + 32\zeta_2 H_{211} + 96\zeta_2 H_{0000} + 48H_{50} + \\
&+ 32H_{230} + 32H_{320} + 40H_{400} + 48H_{410} + \\
&+ 16H_{2120} + 32H_{2200} + 16H_{2210} + 40H_{3000} + \\
&+ 16H_{3100} + 16H_{3110} + 48H_{20000} + 48H_{21000} + \\
&+ 16H_{21100} + 32H_{21110} + 120H_{000000} - 24\zeta_3^2 - \\
&- 1079\zeta_6 + 32\zeta_3 H_3 + 48\zeta_2 H_4 + 156\zeta_4 H_2 - \\
&- 32\zeta_5 H_0 + \frac{1}{N^2}\Big(- 96\zeta_3 H_{20} + 192\zeta_3 H_{100} + \\
&+ 288\zeta_4 H_{10} + 96\zeta_2 H_{120} - 96\zeta_2 H_{200} - \\
&- 96\zeta_2 H_{1000} - 96\zeta_2 H_{1100} - 96H_{50} - 96H_{140} - \\
&- 48H_{230} - 48H_{320} - 48H_{1300} + 48H_{2200} + \\
&+ 288H_{3000} + 192H_{12000} + 336H_{20000} + \\
&+ 432H_{21000} + 144H_{100000} + 240H_{110000} - \\
&- 624\zeta_3^2 - 528\zeta_6 - 96\zeta_3 H_3 + 288\zeta_2\zeta_3 H_0 + \\
&+ 288\zeta_2\zeta_3 H_1 + 144\zeta_4 H_2 + 144\zeta_5 H_0 + 432\zeta_5 H_1\Big).
\end{aligned} \tag{C3}$$

Note that the $\hat{F}^{(3)}(x)$ contains terms with a rational prefactor multiplying the HPLs, while the structure constant is simply a linear combination of HPLs. This happens because in the inversion formula (17), one divides the discontinuity by x and then integrates. Once we integrate the terms with x or $1+x$ in the denominator times a HPL, via the very definition of these functions (C1) we get another HPL but with different weight.

Appendix D: Null Pentagon Limit

The null pentagon limit can be achieved by first taking $x_{12}^2, x_{34}^2 \to 0$ (or $u_1, u_3 \to 0$), projecting to leading-twist operators in the OPE. Further taking $x_{45}^2, x_{15}^2 \to 0$ (or $u_4, u_5 \to 0$), large-spin operators dominate. At this stage, the polarization ℓ is still finite, but by taking the last distance to become null, $x_{23}^2 \to 0$ (or $u_2 \to 0$), we project also to large ℓ. The conformal block in the pentagon limit ($u_i \to 0$) simplifies dramatically, and is given by a simple exponential [1]:

$$\mathcal{F}(u_i) = 2^{3+\gamma_1+\gamma_2} J_1^{1-\frac{\gamma_2}{2}} J_2^{1-\frac{\gamma_1}{2}} \ell^{-2+\gamma_1+\gamma_2} \times \times u_1^{\frac{2+\gamma_1}{2}} u_3^{\frac{2+\gamma_2}{2}} u_4^2 u_5^2 \, e^{-\ell u_2 - \frac{J_2^2 u_4}{\ell} - \frac{J_1^2 u_5}{\ell}} \,, \tag{D1}$$

where $\gamma_i = \gamma(J_i)$ are the anomalous dimensions of the two exchanged operators.

Notice that this limit is very different than the null square limit considered in the main text. Once we take the five neighboring distances to become null separated, the null pentagon correlation function has no finite cross-ratios. In terms of the quantum numbers, this limit is approached by first taking the spins J_i to be large, and then the polarization ℓ, hence there are also no finite ratios of the quantum numbers in the null pentagon limit.

The conformal block in this limit is independent of the fifth external operator, so any difference in the correlation functions must come from the different three-point functions of the block decomposition (3). Conversely, the equality between the correlators (24) implies that their tree-level normalized structure constants must also be identical:

$$\hat{C}_\phi(J_1, J_2, \ell) = \hat{C}_\mathcal{L}(J_1, J_2, \ell) \,. \tag{D2}$$

The null pentagon correlator $\hat{G}_\phi$ (or $\hat{G}_\mathcal{L}$) must be cyclically symmetric (i. e. invariant under $u_i \to u_{i+1}$). By demanding this symmetry of the correlator, one can bootstrap the universal behavior of the structure constants. This was done for $\hat{C}_\phi$ in [1], which, due to (D2), immediately gives the following result: The three-point function of two leading-twist large-spin operators and the Lagrangian in the limit of large ℓ are

$$\hat{C}_\mathcal{L}(J_1, J_2, \ell) = \mathcal{N}(\lambda) \, e^{-\frac{f(\lambda)}{4}(\log \ell^2 + 2\log 2 \log(J_1 J_2)) - \frac{g(\lambda)}{2}\log \ell} \,, \tag{D3}$$

where $\mathcal{N}(\lambda)$ is a coupling-dependent but spin-independent factor that bootstrap arguments cannot fix.

Appendix E: Trivial Relation

In the following, we want to show that

$$\mathcal{I}\left[x \int_0^\infty dr \, \frac{\hat{C}_\mathcal{L}(r)}{(x+r)^2}\right] = \hat{C}_\mathcal{L}(1) \tag{E1}$$

is trivially satisfied for any physical structure constant.

The first step is to use the fact that $\hat{C}_\mathcal{L}(r)$ is invariant under the inversion $r \to 1/r$ to write the single integral above as the sum of two integrals

$$\int_0^\infty dr \, \frac{\hat{C}_\mathcal{L}(r)}{(x+r)^2} = \int_0^1 dr \left(\frac{\hat{C}_\mathcal{L}(r)}{(x+r)^2} + \frac{\hat{C}_\mathcal{L}(r)}{(1+xr)^2}\right) . \tag{E2}$$

The advantage of this step is that now it is clear that the integral of any polynomial in r is convergent.

To complete our derivation, we note that the structure constant $\hat{C}_\mathcal{L}(r)$ better be regular around $r = 1$, since at this value it is equal to the cusp anomalous dimension (30). Therefore, we can Taylor expand the structure constant around this point

$$\hat{C}_\mathcal{L}(r) = \sum_{n=0}^\infty c_n (r-1)^n \tag{E3}$$

and plug into the initial relation (E1) to obtain

$$\sum_{n=0}^\infty c_n \mathcal{I}\left[x \int_0^1 dr \left(\frac{(r-1)^n}{(x+r)^2} + \frac{(r-1)^n}{(1+xr)^2}\right)\right] = c_0 \,. \tag{E4}$$

Performing the integral and applying the functional for the first term of the sum $n = 0$ allows us to simplify the relation above into the sum rule

$$\sum_{n=1}^\infty c_n \mathcal{I}\left[x \int_0^1 dr \left(\frac{(r-1)^n}{(x+r)^2} + \frac{(r-1)^n}{(1+xr)^2}\right)\right] = 0 \,. \tag{E5}$$

Therefore, the relation (E1) will be trivially satisfied if each term of the sum (E5) is identically zero. It turns out that the integrals and the functional are simple enough to check this explicitly:

$$\begin{aligned} &\mathcal{I}\left[x \int_0^1 dr \left(\frac{(r-1)^n}{(x+r)^2} + \frac{(r-1)^n}{(1+xr)^2}\right)\right] \\ &\quad = (-1)^n \left(1 + \sum_{k=0}^n k \binom{n}{k} \mathcal{I}[x^k \log x]\right) \\ &\quad = (-1)^n \left(1 + \sum_{k=0}^n k \binom{n}{k} \frac{(-1)^k}{k}\right) = 0 \,, \end{aligned} \tag{E6}$$

where in the last line we used the fact that

$$\begin{aligned} \mathcal{I}[x^p \log^q(x)] &= \lim_{\epsilon \to 0} \frac{\partial^q}{\partial \epsilon^q} \mathcal{I}[x^{p+\epsilon}] = \\ &= \frac{q!}{p^q} \sum_{k=0}^q \frac{(-1)^{q+p-1}(\pi p)^{k-1}}{k!} \sin\left(\frac{\pi k}{2}\right) . \end{aligned} \tag{E7}$$