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Field Redefinitions and Infinite Field Anomalous Dimensions

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Abstract: Field redefinitions are commonly used to reduce the number of operators in the Lagrangian by removing redundant operators and transforming to a minimal operator basis. We give a general argument that such field redefinitions, while leaving the S-matrix finite, lead to infinite Green's functions and infinite field anomalous dimensions γ_ϕ. These divergences cannot be removed by counterterms without reintroducing redundant operators.

Contents

1 Introduction

The S-matrix of quantum field theories is unchanged by field redefinitions [1–4], so that Lagrangians related by field redefinitions are equivalent, and give the same physical theory. While the S-matrix remains invariant under field redefinitions, Green's functions can (and do) change. Field redefinitions are often used to reduce the number of operators in the Lagrangian, and their couplings, to a minimal basis. In general, working in a minimal basis unavoidably leads to Green's functions and field anomalous dimensions which are infinite even after the addition of renormalization counterterms, even though the S-matrix is finite. A classic example of this phenomenon occurs due to penguin diagrams in the low-energy theory of weak interactions (see the discussion in [4, §6]). Green's functions cannot be made finite by a simple rescaling of the fields — any attempt to make them finite reintroduces the redundant operators which were eliminated to obtain a minimal basis. This observation is relevant for theories of inflation, where one computes fluctuations from correlation functions of quantum fields.

Start with an EFT Lagrangian including all allowed operators which contribute to the action, *i.e.* all operators that are not total derivatives. This is equivalent to reducing the set of operators by using only integration-by-parts identities. The resulting set of operators is referred to as a Green's basis in the literature. Only a subset of operators in the Green's basis is independent under field redefinitions. The choice of independent operators is arbitrary, but the number of them is not. The independent operators (in some convention) are referred to as "physical" operators $\mathcal{O}_i$, and the remaining ones are referred to as "redundant" operators $\mathcal{R}_j$. The Green's basis has both sets of operators $\{\mathcal{O}, \mathcal{R}\}$. The Lagrangian coefficients of the physical operators are denoted by C_i and of the redundant operators by D_j. Field redefinitions can remove the redundant operators from the Lagrangian and modify the coefficients from $\{C, D\} \to \{\overline{C}, 0\}$. The resulting Lagrangian and coefficients will be referred to as being in the physical basis.

The Lagrangian in the Green's basis is renormalized in the $\overline{\text{MS}}$ scheme. The Lagrangian has counterterms which depend on $\{C, D\}$, and Green's functions and S-matrices computed with the renormalized Lagrangian are finite. The β-functions and field anomalous dimension are finite,

$$\mu \frac{\mathrm{d}C_i}{\mathrm{d}\mu} = \beta_{C_i}(\{C, D\}) \,, \qquad \mu \frac{\mathrm{d}D_i}{\mathrm{d}\mu} = \beta_{D_i}(\{C, D\}) \,, \qquad \mu \frac{\mathrm{d}\phi}{\mathrm{d}\mu} = -\gamma_\phi(\{C, D\}) \,, \tag{1.1}$$

and depend on all the parameters in the Lagrangian. After a transformation to the physical basis, the β-functions and anomalous dimensions have the form

$$\mu \frac{\mathrm{d}\overline{C}_i}{\mathrm{d}\mu} = \beta_{\overline{C}_i}(\{\overline{C}\}) \,, \qquad \qquad \mu \frac{\mathrm{d}\phi}{\mathrm{d}\mu} = -\gamma_\phi(\{\overline{C}\}) \,, \tag{1.2}$$

and depend only on the physical couplings. The β-functions are finite, but Green's functions and the field anomalous dimension γ_ϕ are *infinite*, as was recently encountered in a specific case in ref. [5].

Infinite Green's functions and field anomalous dimensions are generic when field redefinitions are made. Start with the renormalized Lagrangian in the physical basis. Loop graphs computed with insertions of only the physical operators $\mathcal{O}$ can still lead to divergences which require counterterms with redundant operators $\mathcal{R}$. These divergences induce non-zero values for the redundant coefficients D which are $1/\epsilon^k$ poles and generate β-functions for the redundant couplings: $\mu \, \mathrm{d}D_i/\mathrm{d}\mu \neq 0$. These are, however, obscured because the theory is parametrized at the special point in theory space with $D(\bar{\mu}) = 0$. Nevertheless, the β-functions of the physical couplings and the field anomalous dimension depend on the counterterms for redundant operators. An additional field redefinition is required to remove the counterterms of the redundant operators, and thereby transform the Lagrangian back to the physical basis, such that $\overline{D}(\mu) = 0$ for all μ, and the bare coupling of redundant operators vanishes, $D_b(\mu) = 0$. This field redefinition is infinite, since it removes counterterm coefficients of redundant operators. Since the S-matrix is invariant under field redefinitions, and remains finite, this means that any resulting $\beta_{\overline{C}_i}$ is finite. However, Green's functions and the field anomalous dimension are modified by this field redefinition, and become infinite.

We now demonstrate the above results with an explicit computation in the $O(n)$ EFT to two-loop order.

2 Example

The example theory is the $O(n)$ EFT to dimension six with Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi_b \cdot \partial^\mu \phi_b) - \frac{1}{2} m_b^2 (\phi_b \cdot \phi_b) - \frac{1}{4}\lambda_b (\phi_b \cdot \phi_b)^2 + C_{4,b}\mathcal{O}_{4,b} + D_{4,b}\mathcal{R}_{4,b} + C_{6,b}\mathcal{O}_{6,b} + D_{2,b}\mathcal{R}_{2,b} \\ &= \frac{1}{2} Z_\phi (\partial_\mu \phi \cdot \partial^\mu \phi) - \frac{1}{2} Z_\phi Z_{m^2} m^2 (\phi \cdot \phi) - \frac{1}{4}\mu^{2\epsilon} Z_\phi^2 Z_\lambda \lambda (\phi \cdot \phi)^2 \\ &\quad + \mu^{2\epsilon} Z_\phi^2 Z_{C_4} C_4 \mathcal{O}_4 + \mu^{2\epsilon} Z_\phi^2 Z_{D_4} D_4 \mathcal{R}_4 + \mu^{4\epsilon} Z_\phi^3 Z_{C_6} C_6 \mathcal{O}_6 + Z_\phi Z_{D_2} D_2 \mathcal{R}_2 \,, \end{aligned} \tag{2.1}$$

where ϕ is an n-component real scalar field. The subscripts b refer to bare quantities. The dimension six terms are

$$\begin{aligned} \mathcal{O}_4 &= (\partial_\mu \phi \cdot \partial^\mu \phi)(\phi \cdot \phi) \,, & \mathcal{R}_4 &= (\phi \cdot \partial_\mu \phi)^2 \,, \\ \mathcal{O}_6 &= (\phi \cdot \phi)^3 \,, & \mathcal{R}_2 &= (\partial_\mu \partial^\mu \phi \cdot \partial_\nu \partial^\nu \phi) \,, \end{aligned} \tag{2.2}$$

where we have divided the dimension-six operators into "physical" operators $\mathcal{O}_{4,6}$ and "redundant" operators $\mathcal{R}_{4,2}$. The subscript denotes the number of fields in the operator. The $O(n)$ EFT has an expansion in a mass scale M, so the dimension-six coefficients C_4, C_6, D_4, D_2 are order $1/M^2$, and terms of higher order in $1/M$ are neglected in eq. (2.1). The physical operator coefficients are denoted collectively by $\{C\}$, and the redundant operator coefficients by $\{D\}$. We include the dimension-two mass term $(\phi \cdot \phi)$ and dimension-four $(\phi \cdot \phi)^2$ interaction in the physical operators, and m^2 and λ in the physical coefficients.

One can make a field redefinition in eq. (2.1) to eliminate two of the dimension-six operators. Our choice in this paper is to eliminate $\mathcal{R}_{4,2}$ and retain $\mathcal{O}_{4,6}$, so that the minimal basis of dimension-six operators is $\{\mathcal{O}_4, \mathcal{O}_6\}$. The choice of minimal operator basis is arbitrary, but the number of minimal operators is the same in any basis. All dimension-six operators $\{\mathcal{O}_4, \mathcal{O}_6, \mathcal{R}_4, \mathcal{R}_2\}$ are included in the Green's basis.

The Lagrangian eq. (2.1) in the Green's basis can be renormalized in dimensional regularization in the $\overline{\text{MS}}$ scheme. The counterterms to two-loop order and dimension-six are given in Appendix. A.1, and the β-functions and field anomalous dimension are given in Appendix. A.2. The field anomalous dimension and β-functions are all finite, and are functions of all the parameters in eq. (2.1). The 't Hooft consistency conditions for the counterterms given in [6, §6] are satisfied, which implies that the β-functions and anomalous dimensions are finite.

The field redefinition

$$\phi_b \to \phi_b + f\, \phi_b (\phi_b \cdot \phi_b) + g\, \partial^2 \phi_b \,, \tag{2.3}$$

can be used to eliminate redundant operators in the Lagrangian. Equation (2.3) is the most general field redefinition compatible with $O(n)$ invariance to order $1/M^2$. f

and g are functions of the bare couplings of order $1/M^2$, and independent of μ, so the field-redefinition eq. (2.3) preserves μ-independence of the Lagrangian. The field redefinition

$$\phi_b \to h\,\phi_b\,, \tag{2.4}$$

with h a function of the bare couplings corresponds to a simple rescaling of the bare field and will modify Z_ϕ while keeping Z of the couplings unchanged.

There are two independent terms in eq. (2.3), so we can eliminate two operators from the Lagrangian, which have been chosen to be $\mathcal{R}_4$ and $\mathcal{R}_2$. The general form for f and g must be compatible with the EFT power counting, so that the new Lagrangian retains the $1/M$ expansion. In addition, the dimensions of the terms must match in $4-2\epsilon$ dimensions, where coupling constant dimensions can be fractional, e.g. λ_b has dimension 2ϵ. The allowed redefinition is

$$\begin{aligned}\phi_b &= \widehat{\phi}_b + (a_1 D_{4,b} + a_2\lambda_b D_{2,b})\widehat{\phi}_b(\widehat{\phi}_b\cdot\widehat{\phi}_b) + a_3 D_{2,b}\,\partial^2\widehat{\phi}_b\,,\\ \phi &= \widehat{\phi} + (a_1 Z_{D_4} D_4 + a_2 Z_{D_2}\lambda_b D_2)Z_\phi\widehat{\phi}(\widehat{\phi}\cdot\widehat{\phi}) + a_3 Z_{D_2} D_2\,\partial^2\widehat{\phi}\,,\end{aligned} \tag{2.5}$$

where a_i are numbers, and the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[1+2a_3 m_b^2 D_{2,b}\right](\partial\widehat{\phi}_b\cdot\partial\widehat{\phi}_b) - \frac{1}{2}m_b^2(\widehat{\phi}_b\cdot\widehat{\phi}_b)\\ &\quad - \frac{1}{4}\left[\lambda_b + 4a_1 m_b^2 D_{4,b} + 4a_2 m_b^2\lambda_b D_{2,b})\right](\widehat{\phi}_b\cdot\widehat{\phi}_b)^2\\ &\quad + \left[C_{4,b} + a_1 D_{4,b} + (a_2+a_3)\lambda_b D_{2,b})\right](\partial\widehat{\phi}_b\cdot\partial\widehat{\phi}_b)(\widehat{\phi}_b\cdot\widehat{\phi}_b)\\ &\quad + \left[D_{4,b} + 2a_1 D_{4,b} + 2(a_2+a_3)\lambda_b D_{2,b})\right](\widehat{\phi}_b\cdot\partial\widehat{\phi}_b)^2\\ &\quad + \left[C_{6,b} - a_1\lambda_b D_{4,b} - a_2\lambda_b^2 D_{2,b})\right](\widehat{\phi}_b\cdot\widehat{\phi}_b)^3 + \left[1-a_3\right]D_{2,b}(\partial^2\widehat{\phi}_b\cdot\partial^2\widehat{\phi}_b)\,.\end{aligned} \tag{2.6}$$

In terms of renormalized couplings and fields, the Lagrangian is eq. (2.6) with $C\to Z_C C$, $D\to Z_D D$ and $\widehat{\phi}_b \to \sqrt{Z_\phi}\,\widehat{\phi}$, so that $Z_{\hat{\phi}} = (1+2a_3 Z_{m^2}Z_{D_2}m^2 D_2)Z_\phi$.

Note that the kinetic term is no longer canonically normalized, not even at tree level. The additional rescaling

$$\widetilde{\phi}_b = \left[1-a_3 m_b^2 D_{2,b}\right]\widehat{\phi}_b\,, \qquad\qquad \widetilde{\phi} = \left[1-a_3 m_b^2 Z_{D_2} D_2\right]\widehat{\phi}\,, \tag{2.7}$$

transforms the Lagrangian to

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial\widetilde{\phi}_b\cdot\partial\widetilde{\phi}_b) - \frac{1}{2}m_b^2\left[1-2a_3 m_b^2 D_{2,b}\right](\widetilde{\phi}_b\cdot\widetilde{\phi}_b)\\ &\quad - \frac{1}{4}\left[\lambda_b + 4a_1 m_b^2 D_{4,b} + 4(a_2-a_3)m_b^2\lambda_b D_{2,b}\right](\widetilde{\phi}_b\cdot\widetilde{\phi}_b)^2\end{aligned}$$

$$
\begin{aligned}
&+ [C_{4,b} + a_1 D_{4,b} + (a_2+a_3)\lambda_b D_{2,b}]\,(\partial\widetilde{\phi}_b \cdot \partial\widetilde{\phi}_b)(\widetilde{\phi}_b \cdot \widetilde{\phi}_b) \\
&+ [D_{4,b} + 2a_1 D_{4,b} + 2(a_2+a_3)\lambda_b D_{2,b}]\,(\widetilde{\phi}_b \cdot \partial\widetilde{\phi}_b)^2 \\
&+ \left[C_{6,b} - a_1\lambda_b D_{4,b} - a_2\lambda_b^2 D_{2,b}\right](\widetilde{\phi}_b \cdot \widetilde{\phi}_b)^3 + [1-a_3]\,D_{2,b}(\partial^2\widetilde{\phi}_b \cdot \partial^2\widetilde{\phi}_b)\,,
\end{aligned}
\tag{2.8}
$$

restoring canonical normalization of the kinetic energy term, and gives $Z_{\tilde{\phi}} = Z_\phi$.

Comparing with the original form eq. (2.1) gives the transformed coefficients $(\widetilde{C}, \widetilde{D})$

$$
\begin{aligned}
\widetilde{m}_b^2 &= m_b^2\left[1 - 2a_3 m_b^2 D_{2,b}\right]\,, \\
\widetilde{\lambda}_b &= \lambda_b + 4a_1 m_b^2 D_{4,b} + 4(a_2-a_3) m_b^2 \lambda_b D_{2,b}\,, \\
\widetilde{C}_{4,b} &= C_{4,b} + a_1 D_{4,b} + (a_2+a_3)\lambda_b D_{2,b}\,, \\
\widetilde{C}_{6,b} &= C_{6,b} - a_1\lambda_b D_{4,b} - a_2\lambda_b^2 D_{2,b}\,, \\
\widetilde{D}_{4,b} &= D_{4,b} + 2a_1 D_{4,b} + 2(a_2+a_3)\lambda_b D_{2,b}\,, \\
\widetilde{D}_{2,b} &= (1-a_3)\,D_{2,b}\,,
\end{aligned}
\tag{2.9}
$$

which are functions of the original couplings and a_i. The choice $a_1 = -1/2$, $a_2 = -1$, $a_3 = 1$ gives $\widetilde{D}_{4,b} = 0$, $\widetilde{D}_{2,b} = 0$, so that the redundant operators are eliminated. The new bare couplings in the physical basis $(\overline{C}, \overline{D})$ are functions of the original bare couplings,

$$
\begin{aligned}
\overline{m}_b^2 &= m_b^2\left[1 - 2m_b^2 D_{2,b}\right]\,, & \overline{\lambda}_b &= \lambda_b - 2m_b^2 D_{4,b} - 8m_b^2\lambda_b D_{2,b}\,, \\
\overline{C}_{4,b} &= C_{4,b} - \frac{1}{2}D_{4,b}\,, & \overline{C}_{6,b} &= C_{6,b} + \frac{1}{2}\lambda_b D_{4,b} + \lambda_b^2 D_{2,b}\,, \\
\overline{D}_{4,b} &= 0\,, & \overline{D}_{2,b} &= 0\,.
\end{aligned}
\tag{2.10}
$$

and are the values of $(\widetilde{C}, \widetilde{D})$ at $a_1 = -1/2$, $a_2 = -1$, $a_3 = 1$.

In general, we have coefficients

$$
\widetilde{C}_{i,b} = F_i(\{a\}, \{C_b\}, \{D_b\})\,, \qquad \widetilde{D}_{i,b} = G_i(\{a\}, \{C_b\}, \{D_b\})\,, \tag{2.11}
$$

which are functions of the field redefinition parameters $\{a\}$ and the original coefficients $\{C_b\}, \{D_b\}$. The parameters $\{a\}$ are chosen to set $\widetilde{D}_{i,b} = 0$ giving

$$
\overline{C}_{i,b} = F_i(\{C_b\}, \{D_b\})\,, \qquad \overline{D}_{i,b} = 0\,, \tag{2.12}
$$

substituting the values for the parameters which set $\widetilde{D} = 0$ back in eq. (2.11). The transformation is shown schematically in figure 1.

The original theory was parameterized by C and D. We can use eq. (2.12) to determine $\overline{C}$, and parameterize the theory instead by $\overline{C}$ and D. We still need to retain

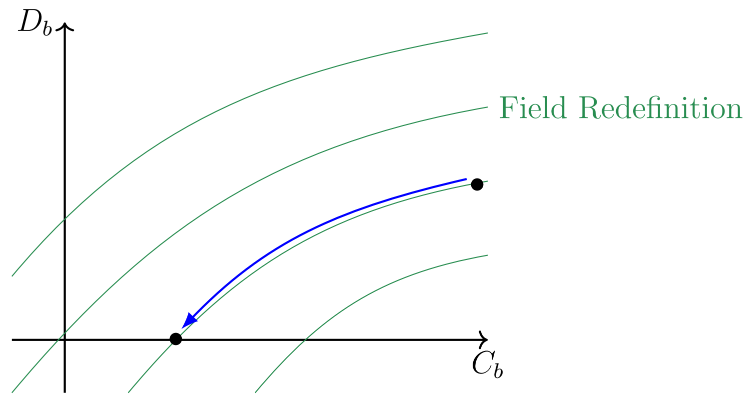

Figure 1. Field redefinitions lead to a set of equivalent theories with the same (finite) S-matrix, shown by the green curves in the space of bare couplings. The coefficients $\widetilde{C}_b$ and $\widetilde{D}_b$ vary along the field redefinition curves as the parameters $\{a\}$ in the field redefinition are varied. The coefficients $\overline{C}_b$ are the values of $\widetilde{C}_b$ when the green curve intersects the C_b-axis and the redundant couplings D_b vanish. The bare couplings are infinite.

D so that eq. (2.12) can be inverted to obtain the original couplings C and D from $\overline{C}$ and D.

The new renormalized couplings are given by the same functions F_i of the original renormalized couplings

$$\overline{C}_i(\mu) = F_i(\{C(\mu)\}, \{D(\mu)\}) , \qquad\qquad \overline{D}_i(\mu) = 0 , \tag{2.13}$$

obtained by dropping the $1/\epsilon$ terms in eq. (2.12). In the $O(n)$ example, the relations are

$$\begin{aligned} \overline{m}^2(\mu) &= m^2(\mu)\left[1 - 2m^2(\mu)D_2(\mu)\right] , \\ \overline{\lambda}(\mu) &= \lambda(\mu) - 2m^2(\mu)D_4(\mu) - 8m^2(\mu)\lambda(\mu)D_2(\mu) , \\ \overline{C}_4(\mu) &= C_4(\mu) - \frac{1}{2}D_4(\mu) , \\ \overline{C}_{6,b} &= C_{6,b} + \frac{1}{2}\lambda(\mu)D_4(\mu) + \lambda^2(\mu)D_2(\mu) , \\ \overline{D}_4(\mu) &= 0 , \\ \overline{D}_2(\mu) &= 0 . \end{aligned} \tag{2.14}$$

The field renormalization for the field in eq. (2.8) at $a_1 = -1/2$, $a_2 = -1$, $a_3 = 1$, denoted by $\overline{\phi}$ is

$$\overline{\phi}_b = Z_{\overline{\phi}}\overline{\phi} \qquad\qquad Z_{\overline{\phi}} = Z_\phi . \tag{2.15}$$

The overall field redefinition is a combination of eq. (2.5) and eq. (2.7), and is infinite, leading to infinite Green's functions. The only way to restore finite Green's functions is to undo the field redefinition and reintroduce the redundant operators. A simple rescaling of ϕ does not make the Green's functions finite, since the transformation eq. (2.5) is non-linear, and cannot be compensated for by a rescaling.

The new bare and and renormalized couplings are related by

$$\overline{C}_{i,b}\,\mu^{-f_i\epsilon} = \overline{C}_i(\mu) + \overline{C}_{i,\text{c.t.}}(\mu) = Z_{\overline{C}_i}\overline{C}_i(\mu)\,, \tag{2.16}$$

where $f_i = F_i - 2$, and F_i is the number of fields in $\mathcal{O}_i$, which determines the fractional dimension of the operator in $4-2\epsilon$ dimensions. The counterterms for $\overline{C}_i$ to two-loop order are given in Appendix. B.1, and the β-functions and field anomalous dimension are given in Appendix. B.2. The β-functions for $\overline{C}_i$ depend only on the physical couplings $\overline{m}^2$, $\overline{\lambda}$, $\overline{C}_4$ and $\overline{C}_6$, and are finite. The field renormalization $Z_{\bar{\phi}}$ is a function of the physical couplings $\overline{C}$ as well as the original redundant couplings D before the field redefinition. The field anomalous dimension $\gamma_{\bar{\phi}}$ computed from the logarithmic derivative of $Z_{\bar{\phi}}$,

$$\gamma_{\bar{\phi}} \equiv \frac{1}{2}Z_{\bar{\phi}}^{-1}\dot{Z}_{\bar{\phi}} = \frac{1}{2}Z_{\bar{\phi}}^{-1}\left(\sum_i \frac{\partial Z_{\bar{\phi}}}{\partial \overline{C}_i}\dot{C}_i + \sum_i \frac{\partial Z_{\bar{\phi}}}{\partial D_i}\dot{D}_i\right)\,, \tag{2.17}$$

where $\dot{C} \equiv \mu\,\mathrm{d}C/\mathrm{d}\mu$, is given in eq. (B.8). It is finite, provided one includes both the C and D terms in eq. (2.17).

In order to remove the dependence of $Z_{\bar{\phi}}$ on D, we can perform an additional rescaling of the field

$$\overline{\phi} = [1 + a_4 D_2 + a_5 D_4]\,\check{\phi}\,, \tag{2.18}$$

$Z_{\check{\phi}} = Z_{\bar{\phi}}\,[1 + 2a_4 D_2 + 2a_5 D_4]$ with[1]

$$\begin{aligned} a_4 &= (n+2)\bar{\lambda}^2\bar{m}^2 \left\{\frac{1}{\epsilon}\right\}_2 \\ a_5 &= -\frac{1}{2}(n+2)\bar{m}^2\left\{\frac{1}{\epsilon}\right\}_1 + \frac{7}{4}(n+2)\bar{m}^2\bar{\lambda}\left\{\frac{1}{\epsilon}\right\}_2 - \frac{1}{2}(n+2)(n+5)\bar{m}^2\bar{\lambda}\left\{\frac{1}{\epsilon^2}\right\}_2 \end{aligned} \tag{2.19}$$

which completely removes all D_2 and D_4 dependence from the Lagrangian. This shows that when working in the physical basis, *i.e.* setting $D_2(\mu) = 0$ and $D_4(\mu) = 0$, we are implicitly making the infinite field redefinition eq. (2.18) and using the field $\check{\phi}$. The anomalous dimension of $\check{\phi}$ given in eq. (B.8) is infinite, because the logarithmic

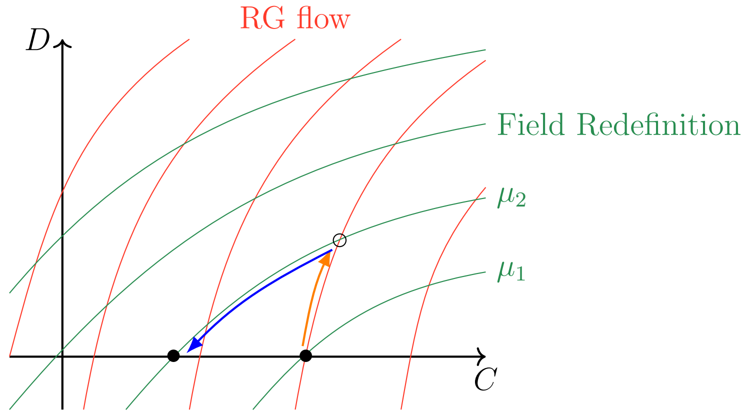

Figure 2. Field redefinitions lead to a set of equivalent theories with the same (finite) S-matrix, shown by the green curves in the space of renormalized couplings. The coefficients $\widetilde{C}$ and $\widetilde{D}$ vary along the field redefinition curves as the parameters $\{a\}$ in the field redefinition are varied. The coefficients $\overline{C}$ are the values of $\widetilde{C}$ when the green curve intersects the C-axis. The renormalized couplings are finite. Starting from vanishing redundant couplings $D(\mu_1) = 0$ can still lead to non-zero redundant couplings $D(\mu_2) \neq 0$ at a different value of μ.

derivative of $1 + a_4 D_2 + a_5 D_4$ does not vanish even at $D_{2,4} = 0$ since the β-functions of $D_{2,4}$ do not vanish at that point, and a_4 and a_5 are infinite.

Figure 2 shows the renormalized coupling constant space for the theory (figure 1 showed the bare coupling constant space). In the space of renormalized couplings, we have two different flows. There is a flow due to field redefinitions, analogous to that in figure 1. Along these flow lines, the S matrix is invariant. In addition, we have a flow due to a change in μ which also leaves the S-matrix invariant.[2] Since the S-matrix is invariant under both flows, the two are compatible. A point on the field-redefinition curve at $\mu = \mu_1$ flows to some point on the field redefinition curve at $\mu = \mu_2$.

Suppose we compute loop corrections in the Green's basis starting with a renormalized Lagrangian with $D(\mu) = 0$ at $\mu = \mu_1$. Even though the renormalized coupling vanishes, $D(\mu) = 0$, loop corrections can generate counterterms for $D(\mu)$ which depend on the non-zero physical couplings $C(\mu)$. Thus RG evolution induces non-zero couplings $D(\mu)$ as μ evolves from μ_1 to μ_2. This flow is shown by the orange arrow in figure 2. One then needs to do a field redefinition at μ_2 to make $D(\mu_2) = 0$, shown

[1]The notation $\{\}_{1,2}$ denote the one and two-loop terms, and must be multiplied by $1/(16\pi^2)$ and $1/(16\pi^2)^2$, respectively.

[2]The renormalized couplings change, but the S-matrix remains invariant because the coupling constant dependence is canceled by $\log \mu^2/s$ terms in the formulæ for S-matrix elements, where s is a kinematic invariant with dimension of mass-squared.

by the blue arrow in figure 2. RG evolution in the EFT with only physical couplings is equivalent to a combination of RG evolution and field redefinitions in the Green's basis. Performing the field redefinition to all orders in the counterterms is the same as the tranformation using bare couplings discussed earlier. During all the transformations, the S-matrix is invariant, and remains finite. The S-matrix is determined by the physical couplings (and vice-versa), so they remain finite as well, and the RG flow for the physical couplings is finite. The evolution of the physical couplings (the black dot in figure 2) is determined by the intersection of the field-redefinition invariance curve with the C axis; it does not depend on the starting point on the curve, *i.e.* the field-redefinition curves flow to other field-redefinition curves under a change in μ. As a result, the β-functions $\overline{C}(\mu)$ are finite, and only depend on $\overline{C}(\mu)$. The key point is that they do not depend on D.

In the above analysis, we have made use of (a) the invariance of the S-matrix under field redefinitions (b) a one-to-one relation between the physical couplings and the S-matrix. These conditions do not apply to the quantum field ϕ, and Green's functions are not invariant under field redefinitions, and the field anomalous dimension is generally infinite. The infinity arises due to the additional rescaling eq. 2.18 to remove D dependence in Z_ϕ, or equivalently, dropping the D derivative in eq. (2.17).

If one wants to keep Green's functions and field anomalous dimensions finite, redundant operators cannot be ignored and one has to use the full Green's basis at all steps in the computations.

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A Green's basis

The counterterms, β-functions and field anomalous dimension of the $O(n)$ theory eq. (2.1) to two-loop order in the Green's basis are listed below. The β-functions are defined by

$$\mu \frac{\mathrm{d}C_i}{\mathrm{d}\mu} \equiv -\epsilon f_i C_i + \beta_{C_i} \tag{A.1}$$

where f_i is defined below eq. (2.16).

A.1 Counterterms

$$Z_\phi = 1 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & -\frac{1}{2}(n+2)\lambda^2 & 0 \\ 2m^2 n & -(n+2)\lambda m^2 & 2(n+1)(n+2)\lambda m^2 \\ 0 & 0 & 0 \\ 2m^2 & -(n+2)\lambda m^2 & 4(n+2)\lambda m^2 \\ 0 & 6(n+2)\lambda^2 m^2 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.2}$$

$$Z_{m^2} = 1 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} (n+2)\lambda & -\frac{5}{2}(n+2)\lambda^2 & (n+2)(n+5)\lambda^2 \\ -4m^2 n & 7(n+2)\lambda m^2 & -2(n+2)(7n+6)\lambda m^2 \\ 0 & 0 & -6(n+2)(n+4)m^2 \\ -4m^2 & 7(n+2)\lambda m^2 & -26(n+2)\lambda m^2 \\ -6(n+2)\lambda m^2 & 36(n+2)\lambda^2 m^2 & -18(n+2)(n+5)\lambda^2 m^2 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.3}$$

$$Z_\lambda \lambda = \lambda + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} (n+8)\lambda^2 & -3(3n+14)\lambda^3 & (n+8)^2\lambda^3 \\ -4(3n+4)\lambda m^2 & 2(29n+154)\lambda^2 m^2 & -12(n+3)(3n+14)\lambda^2 m^2 \\ -12(n+4)m^2 & 144(n+4)\lambda m^2 & -36(n+4)(n+10)\lambda m^2 \\ -28\lambda m^2 & 2(37n+146)\lambda^2 m^2 & -48(3n+14)\lambda^2 m^2 \\ -12(n+8)\lambda^2 m^2 & 4(67n+302)\lambda^3 m^2 & -12(3n^2+52n+188)\lambda^3 m^2 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.4}$$

$$Z_{C_4} C_4 = C_4 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & 0 & 0 \\ (n+6)\lambda & -(9n+16)\lambda^2 & \left(n^2+12n+44\right)\lambda^2 \\ 0 & 6(n+4)\lambda & 0 \\ \lambda & \frac{1}{2}(5n-4)\lambda^2 & 2(n+5)\lambda^2 \\ 0 & (5n+22)\lambda^3 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.5}$$

$$Z_{C_6} C_6 = C_6 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & 0 & 0 \\ (n+8)\lambda^2 & -3(5n+58)\lambda^3 & \left(3n^2+47n+274\right)\lambda^3 \\ 3(n+14)\lambda & -\frac{3}{2}(53n+394)\lambda^2 & 3(n+14)(2n+25)\lambda^2 \\ 9\lambda^2 & -(23n+166)\lambda^3 & 4(8n+73)\lambda^3 \\ (n+26)\lambda^3 & -(61n+506)\lambda^4 & 3(n+11)(n+26)\lambda^4 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.6}$$

$$Z_{D_4}D_4 = D_4 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & 0 & 0 \\ -2(n-2)\lambda & -(n-2)\lambda^2 & -4(n-2)(n+5)\lambda^2 \\ 0 & 12(n+4)\lambda & 0 \\ 2(n+3)\lambda & -\frac{7}{2}(n+6)\lambda^2 & (3n^2+22n+44)\,\lambda^2 \\ 0 & 2(5n+22)\lambda^3 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.7}$$

$$Z_{D_2}D_2 = D_2 + \begin{bmatrix} 1 \\ C_4 \\ C_6 \\ D_4 \\ D_2 \end{bmatrix}^\intercal \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{6}(n+2)\lambda & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{6}(n+2)\lambda & 0 \\ 0 & \frac{1}{2}(n+2)\lambda^2 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{A.8}$$

A.2 β-Functions and γ_ϕ

The β-functions and field-anomalous dimensions computed from the counterterms are

$$\begin{aligned}
\beta_{m^2} = \ & \Big\{2(n+2)\lambda m^2 - 8m^4 nC_4 - 8m^4 D_4 - 12(n+2)\lambda m^4 D_2\Big\}_1 \\
& + \Big\{-10(n+2)\lambda^2 m^2 + 28(n+2)\lambda m^4 C_4 + 28(n+2)\lambda m^4 D_4 + 144(n+2)\lambda^2 m^4 D_2\Big\}_2 \\
\beta_\lambda = \ & \Big\{2(n+8)\lambda^2 - 24(n+4)m^2 C_6 - 8(3n+4)\lambda m^2 C_4 - 56\lambda m^2 D_4 \\
& \quad - 24(n+8)\lambda^2 m^2 D_2\Big\}_1 \\
& + \Big\{-12(3n+14)\lambda^3 + 576(n+4)\lambda m^2 C_6 + 8(29n+154)\lambda^2 m^2 C_4 \\
& \quad + 8(37n+146)\lambda^2 m^2 D_4 + 16(67n+302)\lambda^3 m^2 D_2\Big\}_2 \\
\beta_{C_4} = \ & \Big\{2(n+6)\lambda C_4 + 2\lambda D_4\Big\}_1 \\
& + \Big\{24(n+4)\lambda C_6 - 4(9n+16)\lambda^2 C_4 + 2(5n-4)\lambda^2 D_4 + 4(5n+22)\lambda^3 D_2\Big\}_2 \\
\beta_{C_6} = \ & \Big\{6(n+14)\lambda C_6 + 2(n+8)\lambda^2 C_4 + 18\lambda^2 D_4 + 2(n+26)\lambda^3 D_2\Big\}_1 \\
& - \Big\{6(53n+394)\lambda^2 C_6 + 12(5n+58)\lambda^3 C_4 + 4(23n+166)\lambda^3 D_4 + 4(61n+506)\lambda^4 D_2\Big\}_2 \\
\beta_{D_4} = \ & \Big\{-4(n-2)\lambda C_4 + 4(n+3)\lambda D_4\Big\}_1 \\
& + \Big\{48(n+4)\lambda C_6 - 4(n-2)\lambda^2 C_4 - 14(n+6)\lambda^2 D_4 + 8(5n+22)\lambda^3 D_2\Big\}_2 \\
\beta_{D_2} = \ & \Big\{\frac{2}{3}(n+2)\lambda C_4 + \frac{2}{3}(n+2)\lambda D_4 + 2(n+2)\lambda^2 D_2\Big\}_2
\end{aligned} \tag{A.9}$$

$$\gamma_\phi = \Big\{-2m^2nC_4 - 2m^2D_4\Big\}_1 + \Big\{(n+2)\lambda^2 + 2(n+2)\lambda m^2C_4 + 2(n+2)\lambda m^2D_4 - 12(n+2)\lambda^2m^2D_2\Big\}_2 \tag{A.10}$$

B Physical basis

The counterterms, β-functions, and field anomalous dimension in the physical basis with physical couplings $\overline{C}$ and redundant couplings $\overline{D}=0$ are listed below. Note that the wavefunction renormalization $Z_{\bar\phi}$ depends on the redundant couplings D *before* the field redefinition.

B.1 Counterterms

$$Z_{\bar\phi} = 1 + \begin{bmatrix}1\\ \bar C_4\\ \bar C_6\\ D_4\\ D_2\end{bmatrix}^{\intercal} \begin{bmatrix}0 & -\frac{1}{2}(n+2)\bar\lambda^2 & 0\\ 2n\bar m^2 & -(n+2)\bar\lambda\bar m^2 & 2(n+1)(n+2)\bar\lambda\bar m^2\\ 0 & 0 & 0\\ (n+2)\bar m^2 & -\frac{7}{2}(n+2)\bar\lambda\bar m^2 & (n+2)(n+5)\bar\lambda\bar m^2\\ 0 & -2(n+2)\bar\lambda^2\bar m^2 & 0\end{bmatrix} \begin{bmatrix}\left\{\frac{1}{\epsilon}\right\}_1\\ \left\{\frac{1}{\epsilon}\right\}_2\\ \left\{\frac{1}{\epsilon^2}\right\}_2\end{bmatrix} \tag{B.1}$$

$$Z_{\bar m^2} = 1 + \begin{bmatrix}1\\ \bar C_4\\ \bar C_6\\ D_4\\ D_2\end{bmatrix}^{\intercal} \begin{bmatrix}(n+2)\bar\lambda & -\frac{5}{2}(n+2)\bar\lambda^2 & (n+2)(n+5)\bar\lambda^2\\ -4n\bar m^2 & \frac{20}{3}(n+2)\bar\lambda\bar m^2 & -2(n+2)(7n+6)\bar\lambda\bar m^2\\ 0 & 0 & -6(n+2)(n+4)\bar m^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}\left\{\frac{1}{\epsilon}\right\}_1\\ \left\{\frac{1}{\epsilon}\right\}_2\\ \left\{\frac{1}{\epsilon^2}\right\}_2\end{bmatrix} \tag{B.2}$$

$$Z_{\bar\lambda}\bar\lambda = \bar\lambda + \begin{bmatrix}1\\ \bar C_4\\ \bar C_6\\ D_4\\ D_2\end{bmatrix}^{\intercal} \begin{bmatrix}(n+8)\bar\lambda^2 & -3(3n+14)\bar\lambda^3 & (n+8)^2\bar\lambda^3\\ -8(n+3)\bar\lambda\bar m^2 & \frac{8}{3}(22n+113)\bar\lambda^2\bar m^2 & -12\left(2n^2+21n+50\right)\bar\lambda^2\bar m^2\\ -12(n+4)\bar m^2 & 120(n+4)\bar\lambda\bar m^2 & -36(n+4)(n+10)\bar\lambda\bar m^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \begin{bmatrix}\left\{\frac{1}{\epsilon}\right\}_1\\ \left\{\frac{1}{\epsilon}\right\}_2\\ \left\{\frac{1}{\epsilon^2}\right\}_2\end{bmatrix} \tag{B.3}$$

$$Z_{\bar{C}_4}\bar{C}_4 = \bar{C}_4 + \begin{bmatrix} 1 \\ \bar{C}_4 \\ \bar{C}_6 \\ D_4 \\ D_2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 0 & 0 \\ 2(n+2)\bar{\lambda} & -\frac{17}{2}(n+2)\bar{\lambda}^2 & 3(n+2)(n+4)\bar{\lambda}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{B.4}$$

$$Z_{\bar{C}_6}\bar{C}_6 = \bar{C}_6 + \begin{bmatrix} 1 \\ \bar{C}_4 \\ \bar{C}_6 \\ D_4 \\ D_2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 0 & 0 \\ 10\bar{\lambda}^2 & -\frac{2}{3}(23n+259)\bar{\lambda}^3 & 5(7n+62)\bar{\lambda}^3 \\ 3(n+14)\bar{\lambda} & -\frac{21}{2}(7n+54)\bar{\lambda}^2 & 3(n+14)(2n+25)\bar{\lambda}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \left\{\frac{1}{\epsilon}\right\}_1 \\ \left\{\frac{1}{\epsilon}\right\}_2 \\ \left\{\frac{1}{\epsilon^2}\right\}_2 \end{bmatrix} \tag{B.5}$$

B.2 β-Functions and γ_ϕ

$$\begin{aligned} \beta_{\bar{m}^2} &= \Big\{2(n+2)\bar{\lambda}\bar{m}^2 - 8n\bar{m}^4\bar{C}_4\Big\}_1 + \Big\{-10(n+2)\bar{\lambda}^2\bar{m}^2 + \frac{80}{3}(n+2)\bar{\lambda}\bar{m}^4\bar{C}_4\Big\}_2 \\ \beta_{\bar{\lambda}} &= \Big\{2(n+8)\bar{\lambda}^2 - 24(n+4)\bar{m}^2\bar{C}_6 - 16(n+3)\bar{\lambda}\bar{m}^2\bar{C}_4\Big\}_1 \\ &\quad + \Big\{-12(3n+14)\bar{\lambda}^3 + 480(n+4)\bar{\lambda}\bar{m}^2\bar{C}_6 + \frac{32}{3}(22n+113)\bar{\lambda}^2\bar{m}^2\bar{C}_4\Big\}_2 \\ \beta_{\bar{C}_4} &= \Big\{4(n+2)\bar{\lambda}\bar{C}_4\Big\}_1 + \Big\{-34(n+2)\bar{\lambda}^2\bar{C}_4\Big\}_2 \\ \beta_{\bar{C}_6} &= \Big\{6(n+14)\bar{\lambda}\bar{C}_6 + 20\bar{\lambda}^2\bar{C}_4\Big\}_1 - \Big\{42(7n+54)\bar{\lambda}^2\bar{C}_6 + \frac{8}{3}(23n+259)\bar{\lambda}^3\bar{C}_4\Big\}_2 \end{aligned} \tag{B.6}$$

The field anomalous dimension $\gamma_{\bar{\phi}}$ from eq. (2.15) and eq. (2.17) including the derivatives w.r.t. D_4 and D_2 is

$$\begin{aligned} \gamma_{\bar{\phi}} = &\Big\{-2n\bar{m}^2\bar{C}_4 - (n+2)\bar{m}^2 D_4\Big\}_1 \\ &+ \Big\{(n+2)\bar{\lambda}^2 + 2(n+2)\bar{\lambda}\bar{m}^2\bar{C}_4 + 7(n+2)\bar{\lambda}\bar{m}^2 D_4 + 4(n+2)\bar{\lambda}^2\bar{m}^2 D_2\Big\}_2 \end{aligned} . \tag{B.7}$$

The field anomalous dimension of $\check{\phi}$ in eq. (2.18) is

$$\gamma_{\check{\phi}} = \Big\{-2n\bar{m}^2\bar{C}_4\Big\}_1 + \Big\{(n+2)\bar{\lambda}^2 + 2(n+2)\bar{\lambda}\bar{m}^2\bar{C}_4\Big\}_2 + \frac{1}{\epsilon}\Big\{2\left(n^2-4\right)\bar{\lambda}\bar{m}^2\bar{C}_4\Big\}_2 \tag{B.8}$$

and is infinite. This is the same result as computing $\gamma_{\bar{\phi}}$ using eq. (2.17) but omitting the D term.

References

[1] J. S. R. Chisholm, *Change of variables in quantum field theories*, *Nucl. Phys.* **26** (1961) 469.

[2] H. D. Politzer, *Power Corrections at Short Distances*, *Nucl. Phys. B* **172** (1980) 349.

[3] C. Arzt, *Reduced effective Lagrangians*, *Phys. Lett. B* **342** (1995) 189 [hep-ph/9304230].

[4] A. V. Manohar, *Introduction to Effective Field Theories*, 1804.05863.

[5] E. E. Jenkins, A. V. Manohar, L. Naterop and J. Pagès, *Two Loop Renormalization of Scalar Theories using a Geometric Approach*, 2310.19883.

[6] E. E. Jenkins, A. V. Manohar, L. Naterop and J. Pagès, *An Algebraic Formula for Two Loop Renormalization of Scalar Quantum Field Theory*, 2308.06315.

[7] F. Herzog and B. Ruijl, *The R^*-operation for Feynman graphs with generic numerators*, *JHEP* **05** (2017) 037 [1703.03776].

[8] W. Cao, F. Herzog, T. Melia and J. R. Nepveu, *Renormalization and non-renormalization of scalar EFTs at higher orders*, *JHEP* **09** (2021) 014 [2105.12742].