Comb Channel Lightcone Bootstrap II: Triple-Twist Anomalous Dimensions

Sebastian Harris[1], **Apratim Kaviraj**[1,2], **Jeremy Mann**[3], **Lorenzo Quintavalle**[4], **Volker Schomerus**[1,5]

[1] *Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany*

[2] *Department of Physics, Indian Institute of Technology - Kanpur, Kanpur 208016, India*

[3] *Department of Mathematics, King's College London, Strand, London, WC2R 2LS, UK*

[4] *Département de Physique, de Génie Physique et d'Optique, Université Laval, Québec, QC G1V0A6, Canada*

[5] *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg*
Zentrum für Mathematische Physik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg

E-mail: sebastian.harris@desy.de, akaviraj@iitk.ac.in, jeremy.mann@kcl.ac.uk,
lorenzo.quintavalle.1@ulaval.ca, volker.schomerus@desy.de

ABSTRACT: We advance the multipoint lightcone bootstrap and compute anomalous dimensions of triple-twist operators at large spin. In contrast to the well-studied double-twist operators, triple-twist primaries are highly degenerate so that their anomalous dimension is encoded in a matrix. At large spin, the degeneracy becomes infinite and the matrix becomes an integral operator. We compute this integral operator by studying a particular non-planar crossing equation for six-point functions of scalar operators in a lightcone limit. The bootstrap analysis is based on new formulas for six-point lightcone blocks in the comb-channel. For a consistency check of our results, we compare them to perturbative computations in the epsilon expansion of phi-cube and phi-four theory. In both cases, we find perfect agreement between perturbative results and bootstrap predictions. As a byproduct of our studies, we extend earlier work of Derkachov and Manashov to compute the anomalous dimension matrices of all triple-twist primaries in scalar phi-cube and phi-four theory to first and second order in epsilon, respectively.

Contents

1 Introduction

1.1 Motivation

The modern conformal bootstrap offers a powerful new window into the dynamics of strongly coupled quantum field theories. Over the last decade, both numerical and analytical techniques have been advanced significantly, mostly in the context of four-point correlation functions. The latter are usually studied for a small set of scalar (or low-spin) field insertions. Despite this restriction, the outcome has been truly impressive. This is particularly true for the flagship applications to the 3d critical Ising model, where the lowest scaling dimensions have been determined with record precision [1–3]. In addition, with the input from numerical studies, the analytic bootstrap has provided accurate predictions for the CFT data of higher spin operators at low twist [4]. While a rigorous method has not yet been explored, the interplay between analytical and numerical methods holds the potential to greatly enhance the efficiency and scope of the bootstrap [4–7]. In spite of all these advances, the precision results for the Ising model remain somewhat singular. While truly impressive numerical bootstrap bounds were obtained e.g. for $O(N)$ Wilson-Fisher fixed points, see [8] and especially [9] (as well as [6] for analytic predictions based on these results), the precision that could be reached so far does not provide a significant improvement over results known in the non-bootstrap literature (e.g. from Monte Carlo simulations). Even for the Ising model, it seems that the existing methods are not able to push precision and scope much further, beyond the levels that were reached ten years ago. At the same time, due to mixing effects and a lack of understanding of multi-twist operators, much of the CFT data at higher twist has eluded both numerical and analytical approaches to solving the bootstrap constraints. It is widely believed that the reason for these limitations lies in the restricted set of observables that have been considered in the past. In this context, the inclusion of multipoint correlations with more than four fields could open the way for a new era of the bootstrap.

All bootstrap studies, be they numerical or analytical, are based on a good knowledge of conformal blocks. In the case of four-point functions, the relevant blocks are indeed well understood and it was this mathematical control that paved the way for the success of the modern bootstrap. Multipoint blocks with more than four external fields are not as well known, but over the last few years several different approaches have been developed. In $d = 1$ dimensions, multipoint blocks are known for any number of legs and any topology [10–13]. For higher dimensions, the results are less exhaustive, but there has been steady progress recently, based on the study of integral formulas, weight-shifting technologies, differential equations and integrability, see e.g. [14–24]. These advances have enabled some first bootstrap studies including a numerical study of (truncated) five-point crossing equations [25, 26], a semi-definite programming analysis of six-point crossing equations [27] and, more relevant to this work, several multipoint extensions of the lightcone bootstrap [22, 24, 28].[1] For these latter applications, the characterization of blocks through a system of differential equations – constructed from limits of Gaudin models in [18–21] – seems particularly suitable. Indeed, while the expressions for some of the higher-order (Casimir and vertex) differential operators are very complex in general, drastic simplifications occur for certain limiting regimes in the space of multipoint cross-ratios. Once sufficiently many distances are made lightlike, it is possible to write down explicit solutions of the differential equations. Much of this was discussed in [24], where we illustrated the simplifications and constructed the resulting lightcone blocks in the context of five-point functions. The methods developed in that work were geared towards extensions to more than five fields, with some first results for six-point (comb-channel) lightcone blocks already present therein.

As outlined in [24], an important motivation for the study of higher-point blocks and crossing equations near lightcone limits is the resolution of multi-twist operators and their dynamics. The simplest class of multi-twist operators, namely double-twist operators, has been extensively studied. The OPE structure of these operators came to light from the four-point lightcone bootstrap in the milestone works of [30, 31]. They showed that if the identity operator in a Lorentzian CFT is separated from the rest of the spectrum by a twist gap, then crossing symmetry implies the existence of an infinite family of large-spin operators whose twists asymptote to the double-twist operators in a generalized free field (GFF) theory. In fact, there exist discrete families of such operators labeled by their twists that organize themselves into Regge trajectories. Such twist accumulations in lightcone bootstrap have been rigorously proven in [32]. In the wake of the original lightcone bootstrap papers, it was eventually shown through several follow-ups [33–39] that the deviation from GFF values in the OPE data can

---

[1]Analytical studies of the crossing equation also include [29], where OPE data for heavy operators is obtained from the Euclidean OPE limit of higher-point correlators in the snowflake channel.

be determined perturbatively in large spin in terms of the low-twist operators in the spectrum. These works hinted towards an important feature of CFTs, namely analyticity in spin, which was made more explicit after the derivation of the Lorentzian inversion formula [40]. Analyticity in spin has been further established by the construction of light-ray operators [41, 42]. The results of lightcone bootstrap have also been studied in AdS/CFT [43–47], where double-twist operators correspond to bound states of two massive objects that have a very small binding energy when rotating around each other at large angular momentum.

In contrast, the CFT data of higher multi-twist operators has remained elusive due to its suppression in four-point functions, as well as the high level of degeneracy in the spectrum – which grows with spin – in perturbative approaches. As the targets of the bootstrap reach higher twists in the spectrum, these operators will play an increasingly important role. While we can only hope to produce asymptotic large-spin expansions of the CFT data from lightcone bootstrap, its impressive accuracy and universal convergence properties in the four-point case suggest that it is worth testing its generalization to higher points. Independently of these considerations, the duality between multi-twist anomalous dimensions and many-body interaction energies in AdS entails an abundance of non-perturbative information that has yet to be studied.

As a starting point, six-point functions and their comb-channel OPE decompositions provide privileged access to triple-twist operators and their anomalous dimensions. In perturbative examples, like scalar ϕ^4 theory in $4 - 2\epsilon$ dimensions, triple-twist fields are composites of three fundamental fields ϕ along with derivatives. It is well known that such triple-twist fields are highly degenerate in the free theory, at least for large spins, i.e. there exist many such operators with identical conformal quantum numbers (weight and spin). As a consequence, the anomalous dimension of triple-twist fields (with given weight and spin) is not just a number but rather an operator that describes how the triple-twist degeneracies are lifted in the interacting theory. In the ϵ-expansion, the one-loop anomalous dimension of triple-twist operators was studied by Kehrein and Wegner [48] in the $O(n)$ model and by Derkachov and Manashov [49, 50] in the $n = 1$ case, where a large degeneracy remains at this order. On the other hand, the later analysis of two-loop anomalous dimensions by Kehrein [51], one-loop anomalous dimensions of ϕ^3 theory in $6 - 2\epsilon$ dimensions by Derkachov-Manashov [52], as well as a combined work [53] on the leading anomalous dimension in the $1/n$ expansion of $O(n)$ models, all observed that the triple-twist degeneracies are fully lifted at these orders. Beyond these results, the diagonalization problem at the lower orders was not fully solved, and higher loop corrections have remained unexplored. The methods developed in [49, 52, 53] were later extended to the large N_c limit of QCD [54–56], where

the integrability of the anomalous dimension operator at weak coupling allowed for a comprehensive description of the triple-twist spectrum at large spin. This approach contributed to the program of solving integrable conformal gauge theories in the planar limit using integrability [57], leading to particularly impressive results on non-perturbative, single-trace anomalous dimensions in $\mathcal{N} = 4$ Super-Yang Mills theory – see e.g. [58] for the state-of-the-art. However, as these last developments are limited to conformal gauge theory and are based on the integrability of their planar limit, it is not clear which properties can generalize to non-planar CFTs with a twist gap.

1.2 Main Results

In this work, we shall advance a new bootstrap approach to triple-twist operators in general and their anomalous dimensions in particular. Following the usual lightcone bootstrap ideology, our analysis is based on the study of a certain crossing symmetry equation for correlation functions of six identical scalar fields ϕ in an appropriate lightcone limit. We shall denote the conformal weight of the external field by $\Delta_\phi = 2h_\phi$. Crossing equations result from the comparison of two different conformal block expansions of correlation functions that are associated with two different sequences in which operator products are performed. For six-point functions, the OPE decompositions belong to one of two possible topologies known as comb and snowflake topology. A lightcone bootstrap study extracting interesting large-spin asymptotics of conformal data from crossings of snowflake channels was performed in [22]. Here, we study a crossing equation in which both expansions use blocks of comb topology, see Figure 1.
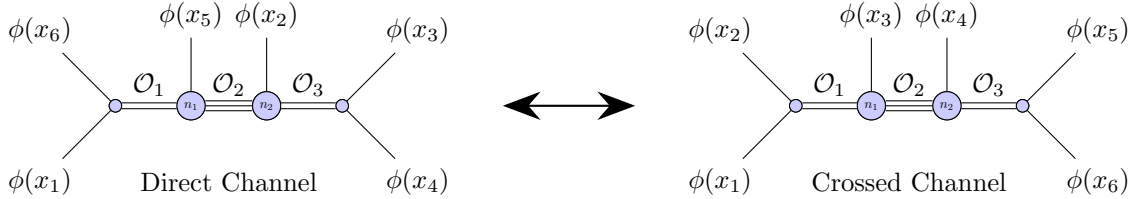
Figure 1. The figure depicts the two sides of the six-point crossing equation we study in this work. With reference to the lightcone limit we are going to introduce below, we shall refer to the left-hand side as the *direct* and to the right-hand side as the *crossed channel*.

Given the topology, different OPE channels are associated with different ways to enumerate the external legs, modulo symmetries of the underlying channel topology. Thus, given one OPE channel, we can generate all others of the same topology by applying some permutation of the external labels. In this sense, the crossing equation visualized in Figure 1 corresponds to the permutation $\varrho = (123)(123456)$. Application of (123456) generates a planar duality. However, the other factor (123) acts only on three

legs and hence it destroys the cyclic order of the external points. We note that the cyclic order in the left diagram of Figure 1 can be restored by cutting the diagram at the middle leg, rotating the left half of the diagram by π, and gluing it back to the right half. This is shown in Figure 2. The relation with such a twisting at the middle leg seems to be the key that unlocks access to triple-twist data.
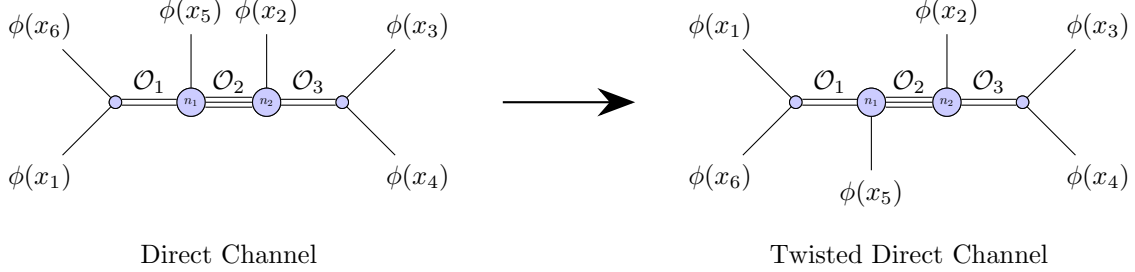
Figure 2. Cutting the direct-channel diagram at the middle leg, rotating the left half by π and gluing the two halves back together restores the cyclic order of the external points.

Let us now discuss the relevant lightcone limits. In this context, we refer to the left-hand side of the crossing equation depicted in Figure 1 as the direct channel and of the right-hand side as the crossed channel. The main goal of the limit we take is to isolate leading-twist contributions in the two outer legs of the direct channel. This is achieved by making the differences x_{16} and x_{34} lightlike. In terms of embedding-space coordinates, this amounts to setting $X_{16} = 0, X_{34} = 0$. We only look at direct-channel contributions with at least one identity exchange, see Figure 3. For these leading contributions, no further limits are necessary in order to get leading twist in the middle leg of the direct channel. However, in order to remove higher-twist contributions in the crossed channel, we must introduce an additional triple-scaling limit where X_{45}, X_{56}, X_{46} tend to zero at the same rate.[2] Hence, in our analysis of the crossing equation we take three limits. A precise definition of the limiting regime in terms of cross-ratios will be given in Section 2.3, see in particular eqs. (2.14–2.17).

By analyzing the non-planar comb channel crossing equation that is depicted in Figure 1 in the lightcone limit (2.14–2.17), we are able to compute the triple-twist anomalous dimension matrix $\gamma_{k\ell}$ in a particular basis of triple-twist operators. The latter is obtained by applying the construction of double-twist primaries twice, first on a pair of fields ϕ to obtain the double-twist primaries $[\phi\phi]_{0,\ell}$ and then to ϕ and $[\phi\phi]_{0,\ell}$ to obtain $\Phi_\ell^{(3)} = [\phi[\phi\phi]_{0,\ell}]$. We refer to this basis of triple-twist operators as "the double-twist basis". The basis vectors are labeled by the spin ℓ of the double-twist constituent. The double-twist basis is the one we use to compute and state the anomalous dimension matrix γ.

---

[2]For Euclidean space, it was shown in [21] that such a triple limit is equivalent to the OPE limit at the middle leg (i.e. projects onto lowest exchanged scaling dimension).

The final result is given in eq. (6.19). It describes the anomalous dimension of triple-twist operators with large STT spin. This is the main result of our paper.

1.3 Outline

Before we conclude this introduction, let us briefly outline the content of the individual sections. Section 2 starts by introducing the relevant notation and describing the general setup, i.e. the crossing equation and the lightcone limits. There, we shall also evaluate the three direct-channel contributions that we want to analyze through the crossed-channel block expansion. Since all three terms involve at least one intermediate identity exchange, the relevant blocks in the direct channel contain at most four external fields and hence are well known.

The study of the relevant crossed-channel blocks is the main subject of Section 3. There, we work out Casimir and vertex operators in the lightcone limits and apply them to direct-channel contributions in order to deduce the parameters of the crossed-channel blocks that contribute. The results are $h_1 = h_3 = 2h_\phi, h_2 = 3h_\phi$ where $h_a = h(\mathcal{O}_a)$ denotes the half-twist of the three intermediate operators. We conclude that the double-twist operators dominate the exchange in the outer legs while triple-twist exchange dominates at the middle leg of the crossed channel, as anticipated. The spin labels J_a of the exchanged operators are all large, but they become large at different rates. While $J_1 \sim J_3 \sim X_{16}^{-1}$, the spin J_2 of the middle leg is sent to infinity as $J_2 \sim X_{16}^{-1} X_{34}^{-1}$. The mixed-symmetry tensor (MST) label κ of the triple-twist field at the middle leg, on the other hand, stays finite. The tensor structure labels n_i turn out to take their maximal allowed value $n_1 = J_1 - \kappa$ and $n_2 = J_3 - \kappa$. The relevant blocks are spelled out in eq. (3.36).

In Sections 4 and 5, we pause our analysis of the crossing equation and discuss some features of multi-twist operators in generalized free field (GFF) theories. More specifically, Section 4 is devoted to the construction of the triple-twist operators. There, we shall discuss the degeneracy of triple-twist primaries and how to parametrize them in terms of certain wave functions. This enables us to introduce the double-twist basis by choosing a basis of wave functions, see eqs. (4.61), (4.62) and (4.40) for explicit formulas. Section 5 is devoted to the calculation of triple-twist OPE coefficients for the operator product between a scalar and a double-twist primary in GFF. The general result is stated in eq. (5.19) and then evaluated more explicitly in the large-spin limit, see Subsection 5.3.3. These formulas are used in our analysis of the crossing equation. In fact, as in the case of the four-point lightcone bootstrap, it will turn out that the dynamical data of scalar six-point functions in a generic CFT approach GFF values when the spins become large.

Section 6 returns to the analysis of the crossing equation in the lightcone limit. There, we shall discuss how the three terms we consider in the direct channel can be reproduced by summing crossed-channel lightcone blocks. The leading term with two identity exchanges confirms the relation between large-spin triple-twist OPE coefficients and their values in GFF. The two next-to-leading order terms with a single identity exchange in the direct channel, finally, furnish our main new result: the leading contribution to the triple-twist anomalous dimension matrix γ. As one might expect, this matrix mixes triple-twist operators in the double-twist basis. The diagonalization of γ is beyond the scope of this work. Nevertheless, we shall discuss the behavior of eigenvalues for some asymptotic regime.

In Section 7, we compare our results with perturbative studies of ϕ^3 theory in $6 - \epsilon$ and ϕ^4 theory in $4 - \epsilon$ dimensions. For ϕ^3 theory, the anomalous dimensions of triple-twist operators with vanishing κ were computed by Derkachov and Manashov. We extend their calculations to non-vanishing κ and verify that the large-spin limits of the resulting anomalous dimensions agree with our bootstrap results. A similar analysis is then performed for ϕ^4 theory. In this case, the anomalous dimensions of triple-twist operators have not appeared in the literature before. We calculate these following the same steps as in Derkachov and Manashov's work and again compare the large-spin limits with our bootstrap results. Once more, we find perfect agreement.

The paper concludes with a summary and a list of interesting open problems in Section 8. Some technical background and further calculations are described in several appendices.

2   Non-planar Six-point Crossing Equation: Overview and Conventions

The six-point crossing equation we aim to study in this work compares two different sequences of operator product expansions that we denote $(16)5(2(34))$ and $(12)3(4(56))$, respectively. The two decompositions are depicted in Figure 1. In order to spell out concrete formulas we need to first introduce the relevant cross-ratios and the labels for the associated conformal blocks. The latter arise from the quantum numbers of the exchanged fields and the choice of tensor structures at the non-trivial vertices. Then we discuss the lightcone limit in which we would like to evaluate the crossing equation. These are chosen to make leading-twist exchanges in the (16) and (34) OPEs dominate, thereby selecting $(16)5(2(34))$ as the direct channel (DC) of our lightcone bootstrap, while projecting on triple-twist exchanges in the middle leg of $(12)3(4(56))$, which we then consider as our crossed channel (CC). The section concludes with concrete formulas for the three direct-channel contributions we would like to match by summing crossed-channel blocks.

2.1 Conformally invariant cross-ratios

To describe the conformal blocks that appear in the DC and CC, we have to make a choice of conformally invariant cross-ratios. The most complex parts of our computations will involve the CC, thus our conventions are mostly tailored to this OPE channel. For a six-point correlator in sufficiently high dimension, i.e. $d > 3$ there are nine independent cross-ratios. For most of our analysis, we choose these to be

$$u_1 := \frac{X_{12}X_{35}}{X_{13}X_{25}}, \quad u_2 := \frac{X_{13}X_{46}}{X_{14}X_{36}}, \quad u_3 := \frac{X_{24}X_{56}}{X_{25}X_{46}},$$

$$v_1 := \frac{X_{14}X_{23}}{X_{13}X_{24}}, \quad v_2 := \frac{X_{25}X_{34}}{X_{24}X_{35}}, \quad v_3 := \frac{X_{36}X_{45}}{X_{35}X_{46}}, \tag{2.1}$$

$$\mathcal{U}_1 := \frac{X_{15}X_{24}}{X_{14}X_{25}}, \quad \mathcal{U}_2 := \frac{X_{26}X_{35}}{X_{25}X_{36}}, \quad \mathcal{U}^6 := \frac{X_{16}X_{24}X_{35}}{X_{14}X_{25}X_{36}}.$$

There is a second set of cross-ratios that we shall use occasionally, namely the *OPE cross-ratios* that were introduced in [21], These are useful to express the asymptotics of blocks in the Lorentzian OPE limit and they are particularly well suited to implement the additional Gram determinant relations that eliminate cross-ratios as we go to dimensions $d < 4$. The nine OPE cross-ratios are denoted by z_1, z_2, z_3, $\bar{z}_1$, $\bar{z}_2$, $\bar{z}_3$, w_1, w_2, and Υ_0. The relation with the cross ratios defined through eqs. (2.1) are somewhat complicated to spell out for generic kinematics. We shall only need this relation in the regime in which all three cross $u_j, \to 0$ go to zero. In terms of the OPE cross-ratios, this regime is mapped to the limit in which the three cross ratios $\bar{z}_j, j = 1, 2, 3$, vanish. In this limit, the relation between our two sets of cross-ratios reads

$$u_2 v_1 v_3 = z_2 \bar{z}_2, \qquad v_2 = 1 - z_2 + \mathcal{O}(\bar{z}_2),$$

$$u_1 v_2 = z_1 \bar{z}_1, \quad v_1 = 1 - z_1 + \mathcal{O}(\bar{z}_1), \quad \mathcal{U}_1 v_1 v_2 = 1 - z_1 - z_2 + w_1 z_1 z_2 + \mathcal{O}(\bar{z}_i),$$

$$u_3 v_2 = z_3 \bar{z}_3, \quad v_3 = 1 - z_3 + \mathcal{O}(\bar{z}_3), \quad \mathcal{U}_2 v_2 v_3 = 1 - z_2 - z_3 + w_2 z_2 z_3 + \mathcal{O}(\bar{z}_j), \tag{2.2}$$

$$\mathcal{U}^6 v_1 v_2 v_3 = 1 - z_1 - z_3 + z_1 z_3 - z_2 \left[1 - w_1 z_1 - w_2 z_3 - z_1 z_3 \left(\Upsilon_0 - w_1 w_2 \right) \right] + \mathcal{O}(\bar{z}_j).$$

Note that the z_j and $\bar{z}_j$ are Dolan-Osborn type of variables [59] and Υ_0 is a rescaled version of the cross-ratio Υ that was introduced in [21],

$$\Upsilon_0 = \Upsilon \sqrt{w_1(1 - w_1)w_2(1 - w_2)}.$$

In terms of OPE cross-ratios, the crossed-channel OPE limit that we mentioned above corresponds to the regime in which

$$\text{OPE}^{(12)3(4(56))} : \bar{z}_1, \bar{z}_2, \bar{z}_3 \ll z_1, z_2, z_3, \Upsilon_0 \ll 1. \tag{2.3}$$

In this limit, the conformal blocks behave as

$$g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}^{(12)3(4(56))} \overset{\text{OPE}^{(12)3(4(56))}}{\sim} \prod_{i=1}^{3} \bar{z}_i^{h_i} z_i^{\bar{h}_i} \Upsilon_0^{\kappa}(1-w_1)^{n_1}(1-w_2)^{n_2}, \tag{2.4}$$

where the n_i labels parameterize the three-point tensor structures at the two innermost vertices, as we will discuss in Subsection 2.2.

Since the relations (2.2) between our cross-ratios (2.1) and the OPE cross-ratios are valid in the OPE limit, they can be used to determine the behavior of u_j, v_j and $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}^6$ in the limiting regime (2.3). As we mentioned above the OPE cross-ratios are also well suited to describe the reduction of cross-ratios that occurs in the $d = 3$ dimensions, due to the vanishing of the Gram determinant $\det(X_{ij})$[21]. Indeed, in terms of OPE cross-ratios, the $d = 3$ constraint simply reads

$$\Upsilon_0^2 = \frac{4z_2\bar{z}_2}{(z_2 - \bar{z}_2)^2} w_1(1-w_1)w_2(1-w_2). \tag{2.5}$$

This may be translated into a much more complicated-looking constraint on $\mathcal{U}_2$. The precise relation is not relevant to our scope. All we should keep in mind is that for $d = 3$ the cross-ratio $\mathcal{U}_2$ may be expressed as some function $\mathcal{U}_2^*(u_i, v_i, \mathcal{U}_1, \mathcal{U}^6)$ of the other eight cross-ratios.

Before we conclude this short discussion of cross-ratios we would like to add one short comment that will become relevant much later in Section 6 when we compute the leading triple-twist anomalous dimensions. It will turn out that some of the direct-channel contributions can be most easily interpreted in terms of a crossed-channel expansion for some CC' that differs from CC by the permutation $\sigma := (13)(46) \in S_6$. From the explicit construction (2.1) it is not difficult to work the following action of σ on our cross-ratios,

$$\sigma : (u_1, v_1, u_2, v_2, u_3, v_3, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}^6) \longrightarrow \left(\mathcal{U}_1 v_1, \frac{u_1}{\mathcal{U}_2}, u_2, \frac{\mathcal{U}^6}{\mathcal{U}_1\mathcal{U}_2}, \mathcal{U}_2 v_3, \frac{u_3}{\mathcal{U}_1}, v_3, \mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_1\mathcal{U}_2 v_2 \right). \tag{2.6}$$

2.2 Tensor structures

In this subsection, we introduce the conventions and notation that we use for two- and three-point correlators of spinning fields, giving special attention to primary fields in mixed-symmetry tensor (MST) representations with two spin labels. Such fields can be exchanged in the middle leg of a six-point OPE channel of comb topology for $d \geq 4$. Since this is the leg in which we also expect to produce triple-twist fields, such representations are quite relevant for us.

Throughout this work, we shall use the embedding space formalism, in which the insertion points x are encoded in lightlike vectors $X \in \mathbb{R}^{2,d}$. Fields in symmetric traceless tensor (STT) representations of the rotation group involve an additional lightlike vector Z that is required to be orthogonal to X,

i.e. $X \cdot Z = 0$. For fields in MST representations with two spin labels, we finally need one more object $W \in \mathbb{C}^{d+2}$ that is required to have a vanishing norm and be perpendicular to both X and Z, i.e. $W \cdot X = 0 = W \cdot Z$. For an MST operator $\mathcal{O}_i(X_i, Z_i, W_i)$ inserted at the embedding space position X_i with polarization vectors Z_i and W_i labeled by some index i, we use the abbreviations

$$X_{ij} := X_i \cdot X_j, \tag{2.7}$$

$$H_{ij} := (X_i \wedge Z_i) \cdot (X_j \otimes Z_j) = (X_i \cdot X_j)(Z_i \cdot Z_j) - (X_i \cdot Z_j)(Z_i \cdot X_j) \tag{2.8}$$

$$J_{i,jk} := (X_i \wedge Z_i) \cdot (X_j \otimes X_k) \tag{2.9}$$

$$U_{i,jk} := (X_i \wedge Z_i \wedge W_i) \cdot (X_j \otimes Z_j \otimes X_k) = (X_i \cdot X_j)(Z_i \cdot Z_j)(W_i \cdot X_k) + \dots \tag{2.10}$$

With this notation, the normalized two-point function of two MST primaries reads

$$\langle \mathcal{O}(X_1, Z_1, W_1)\mathcal{O}(X_2, Z_2, W_2) \rangle := X_{12}^{-(\Delta+J)} H_{12}^{J-\kappa}[(X_1 \wedge Z_1 \wedge W_1) \cdot (X_2 \otimes Z_2 \otimes W_2)]^{\kappa} . \tag{2.11}$$

For STT primaries with $\kappa = 0$ the dependence on the variables W drops out. We can also use the objects defined in eqs. (2.7-2.10) to introduce the three-point tensor structures. For a three-point function of one scalar, one STT, and one MST primary, the three-point function may be expanded as

$$\langle \mathcal{O}_1(X_1, Z_1, W_1)\mathcal{O}_2(X_2, Z_2)\mathcal{O}_3(X_3) \rangle = \tag{2.12}$$

$$= \sum_{n=0}^{J} C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n)} U_{1,23}^{\kappa_1} H_{12}^n J_{2,13}^{J_2-\kappa_1-n} J_{1,23}^{J_1-\kappa_1-n} X_{12}^{-\bar{h}_{12;3}+\kappa_1} X_{13}^{-\bar{h}_{13;2}+\kappa_1+n-J_2} X_{32}^{-\bar{h}_{32;1}+n-J_1} .$$

Here, J_1 and J_2 denote the spins of $\mathcal{O}_1$ and $\mathcal{O}_2$, κ_1 denotes the MST spin of the primary $\mathcal{O}_1$ i.e. the length of the second row of the Young tableaux corresponding to the $SO(d)$ representation of $\mathcal{O}_1$, and $\bar{h}_{ij,k}$ is defined as

$$\bar{h}_{ij,k} := \bar{h}_i + \bar{h}_j - \bar{h}_k \qquad \text{with} \qquad \bar{h} := \frac{1}{2}(\Delta + J + \kappa). \tag{2.13}$$

The label n that enumerates the possible three-point tensor structures, i.e. terms on the right-hand side of equation (2.12), runs over a finite set of non-negative integers with the upper bound J given by the minimum of $J_1 - \kappa_1$ and $J_2 - \kappa_1$.

2.3 Lightcone limits

The lightcone limits we are going to study below are controlled by three different scales that dictate how fast various pairs of insertion points approach the lightcone. The first limits we shall take below are the ones that make leading-twist exchanges in the outer legs of the direct channel dominate. In

terms of our embedding space variables, this means that we shall first take X_{16} and X_{34} to zero. This is then followed by a limit that suppresses subleading-twist exchanges in the crossed channel. This is done by sending X_{45}, X_{56} and X_{46} to zero simultaneously. We shall control these limits by three dimensionless parameters $\epsilon_{16}, \epsilon_{34}$ and ϵ_{456} that multiply the various scalar products X_{ij},

$$X_{16}^{\epsilon} \sim \epsilon_{16} X_{16}\,, \qquad X_{34}^{\epsilon} \sim \epsilon_{34} X_{34}\,, \qquad X_{45}^{\epsilon}, X_{46}^{\epsilon}, X_{56}^{\epsilon} \sim \epsilon_{456} X_{45}, \epsilon_{456} X_{56}, \epsilon_{456} X_{46}\,. \qquad (2.14)$$

such that the lightcone limits may be performed by sending $\epsilon_{16}, \epsilon_{34}$ and ϵ_{456} to zero. There are three different ways to do so that shall be relevant for us. These are distinguished by the relative scaling of the two direct-channel limits that involve ϵ_{16} and ϵ_{34}. The first possibility is to send those two to zero at the same rate before sending ϵ_{456} to zero. This regime is denoted by

$$LCL^{(16,34)} : (\epsilon_{16} = \epsilon_{34}) \ll \epsilon_{456} \ll 1\,. \qquad (2.15)$$

In addition, we shall also consider the regimes in which one of the two direct-channel limits is taken to zero much faster than the other. We denote these regimes by

$$LCL^{(16)} : \epsilon_{16} \ll \epsilon_{34} \ll \epsilon_{456} \ll 1, \qquad (2.16)$$

$$LCL^{(34)} : \epsilon_{34} \ll \epsilon_{16} \ll \epsilon_{456} \ll 1. \qquad (2.17)$$

By definition, the parameters ϵ determine how the scalar products X_{ij} go to zero in the limiting regimes. Since all the differential operators and blocks are later written in terms of cross-ratios, it is most relevant for us to know how the latter behave in the lightcone limits. In terms of the cross-ratios we introduced in eqs. (2.1), the three different regimes are characterized as

$$LCL^{(16)} : \mathcal{U}^6 \ll v_2 \ll u_2 \ll 1\,, \quad LCL^{(34)} : v_2 \ll \mathcal{U}^6 \ll u_2 \ll 1\,. \qquad (2.18)$$

with all the remaining cross-ratios $u_1, u_3, v_1, v_3, \mathcal{U}_1, \mathcal{U}_2$ kept finite. Obviously, in order to reach the lightcone regime $LCL^{(16,34)}$ we should take $\mathcal{U}^6$ and v_2 to zero at the same rate. Under the action of the permutation $\sigma = (13)(46)$ that we introduced and discussed briefly at the end of Subsection 2.1, the two lightcone limits $LCL^{(16)}$ and $LCL^{(34)}$ are exchanged while the symmetric $LCL^{(16,34)}$ is left invariant.

2.4 Leading-twist expansion in the direct channel

With the conventions we have introduced in the previous subsections, the crossing equation of interest takes the form

$$\frac{\Omega_{DC}}{\Omega_{CC}} \sum_{\mathcal{O}_i, n_i} P_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1, n_2)} g_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2}^{DC}(u_i, v_i, \mathcal{U}_i, \mathcal{U}^6) = \sum_{\mathcal{O}_i, n_i} P_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1, n_2)} g_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2}^{CC}(u_i, v_i, \mathcal{U}_i, \mathcal{U}^6). \qquad (2.19)$$

Here, g^{DC} and g^{CC} denote the six-point blocks in the direct and crossed channel, respectively, P are the OPE coefficients that appear in the two expansions, and Ω_{CC} and Ω_{DC} are covariant prefactors. Throughout this paper, we adopt conventions in which the crossed-channel prefactor Ω_{CC} is given by

$$\Omega_{\mathrm{CC}} = (X_{12}X_{34}X_{56})^{-\Delta_\phi}(v_1 u_2 v_3)^{-\frac{\Delta_\phi}{2}}, \tag{2.20}$$

Since we are dealing with six-point functions of identical scalars and both channels possess the same comb topology, the OPE coefficients are the same in both expansions. The conformal blocks and the covariant prefactor, on the other hand, are not the same but we can obtain the direct-channel blocks and covariant prefactors from those of the crossed channel by the relevant (non-planar) permutation of the insertion points. With the help of eq. (2.1) it is not difficult to determine the action of this permutation on our set of crossed-channel cross-ratios. The resulting relation between direct- and crossed-channel blocks reads

$$g^{\mathrm{DC}}_{\mathcal{O};n}(u_1, v_1, u_2, v_2, u_3, v_3, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}^6) = \tag{2.21}$$
$$= g^{\mathrm{CC}}_{\mathcal{O};n}\left(\frac{\mathcal{U}^6}{\mathcal{U}_1}, \frac{u_1 u_2 u_3}{\mathcal{U}_1 \mathcal{U}_2}, \frac{\mathcal{U}_1}{v_3 u_1 u_2}, \frac{1}{\mathcal{U}_2}, \mathcal{U}_2 v_2, v_1 u_2 v_3, \frac{\mathcal{U}_2}{u_1}, \frac{1}{v_3}, \frac{\mathcal{U}_2}{v_3 u_1 u_2}\right).$$

Similarly, the covariant prefactor Ω for the direct channel can be obtained from that of the crossed channel as

$$\Omega_{\mathrm{DC}} = (X_{16}X_{25}X_{34})^{-\Delta_\phi}\left(\frac{u_1 u_2 u_3}{\mathcal{U}_1 \mathcal{U}_2} \cdot \frac{\mathcal{U}_1}{v_3 u_1 u_2} \cdot v_1 u_2 v_3\right)^{-\frac{\Delta_\phi}{2}} = (X_{16}X_{25}X_{34})^{-\Delta_\phi}\left(\frac{v_1 u_2 u_3}{\mathcal{U}_2}\right)^{-\frac{\Delta_\phi}{2}}.$$

In the crossing equation (2.19) only the ratio of the two covariant prefactors appears. This is now easy to evaluate as a function of the nine cross-ratios.

Let us now finally look at a few leading terms in the direct channel as we approach the lightcone regime we specified in the previous subsection. As we send X_{16} and X_{34} to zero, the direct channel is dominated by leading-twist exchanges in the operator products $\phi(X_1) \times \phi(X_6)$ and $\phi(X_4) \times \phi(X_3)$. At leading order, both these OPEs are dominated by the identity exchange. We will analyze the crossing equation to next-to-leading order, i.e. also include contributions for which the identity operator is exchanged in only one of these OPEs while the second OPE involves the first leading operator with a twist above that of the identity. This implies that, in all direct-channel terms we consider, the corresponding conformal blocks reduce to four-point lightcone blocks, see Figure 3.

Hence, at next-to-leading order where we take into account the leading non-trivial primary $\mathcal{O}_*$ that appears in the operator product $\phi(X_1) \times \phi(X_6)$ or $\phi(X_4) \times \phi(X_3)$, the direct-channel conformal block
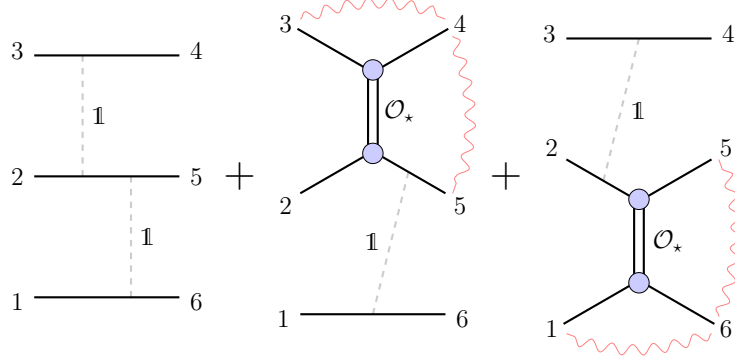
Figure 3. Leading contributions in the DC for the $LCL^{(16,34)}$ limit, where the red wavy lines connect points that are null-separated under this limit. The exact same contributions can be produced from a $(16)2(5(34)$ or a $(16)(25)(34)$ direct channel.

decomposition reads

$$\sum_{\mathcal{O}_i, n_i} P^{(n_1, n_2)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} g^{\mathrm{DC}}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2} \overset{LCL}{\sim} F_{\mathbb{1}, \phi, \mathbb{1}} + F_{\mathcal{O}_\star, \phi, \mathbb{1}} + F_{\mathbb{1}, \phi, \mathcal{O}_\star} + \mathcal{O}(X^{>h_\star}_{34}, X^{>h_\star}_{16}), \tag{2.22}$$

where the three terms on the right-hand side arise from the double identity exchange, the exchange of $\mathcal{O}_*$ in the $\phi(X_1) \times \phi(X_6)$ and in the $\phi(X_4) \times \phi(X_3)$ OPE, respectively. They are given by the following explicit formulas,

$$F_{\mathbb{1}, \phi, \mathbb{1}} = \left(\frac{v_1 u_2 u_3}{\mathcal{U}_2} \right)^{\frac{\Delta_\phi}{2}}, \tag{2.23}$$

$$F_{\mathcal{O}_\star, \phi, \mathbb{1}} = C^2_{\phi \phi \mathcal{O}_\star} \left(\frac{v_1 u_2 u_3}{\mathcal{U}_2} \right)^{\frac{\Delta_\phi}{2}} \left(\frac{\mathcal{U}^6}{\mathcal{U}_1 \mathcal{U}_2} \right)^{h_\star} \left(1 - \frac{u_1 u_2 u_3}{\mathcal{U}_1 \mathcal{U}_2} \right)^{J_\star} {}_2 F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star \\ 2\bar{h}_\star \end{bmatrix} \left(1 - \frac{u_1 u_2 u_3}{\mathcal{U}_1 \mathcal{U}_2} \right), \tag{2.24}$$

$$F_{\mathbb{1}, \phi, \mathcal{O}_\star} = C^2_{\phi \phi \mathcal{O}_\star} \left(\frac{v_1 u_2 u_3}{\mathcal{U}_2} \right)^{\frac{\Delta_\phi}{2}} v_2^{h_\star} (1 - v_1 u_2 v_3)^{J_\star} {}_2 F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star \\ 2\bar{h}_\star \end{bmatrix} (1 - v_1 u_2 v_3), \tag{2.25}$$

In writing these expressions we have not yet applied the lightcone limits that are controlled by ϵ_{456} and that are used to suppress higher-twist exchange in the crossed channel. Once we apply this limit as well, the hypergeometric functions can be approximated through their relevant asymptotic behavior. After multiplying with the covariant factors that appear on the left-hand side of the crossing equation and applying the full lightcone limit, we obtain the following terms in the direct-channel expansion

$$\frac{\Omega_{\mathrm{DC}}}{\Omega_{\mathrm{CC}}} \sum_{\mathcal{O}_i, n_i} P^{(n_1, n_2)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} g^{\mathrm{DC}}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2} \overset{LCL^{(16,34)}}{\sim} \omega^{h_\phi} \left[f_{\mathbb{1}, \phi, \mathbb{1}} + f_{\mathcal{O}_\star, \phi, \mathbb{1}} + f_{\mathbb{1}, \phi, \mathcal{O}_\star} + \mathcal{O}(X^{>h_\star}_{34}, X^{>h_\star}_{16}) \right], \tag{2.26}$$

where

$$\omega = v_1 u_1^2 v_2^2 u_2^3 v_3 u_3^2 \qquad\qquad f_{\mathcal{O}_\star,\phi,\mathbb{1}} = -\frac{C_{\phi\phi\mathcal{O}_\star}^2}{\mathrm{B}_{\bar{h}_\star}} \left(\frac{\mathcal{U}^6}{\mathcal{U}_1 \mathcal{U}_2} \right)^{h_\star} \frac{\log u_2 + \mathcal{O}(u_2^0)}{(\mathcal{U}^6 v_2)^{2h_\phi}}$$

$$f_{\mathbb{1},\phi,\mathbb{1}} = (\mathcal{U}^6 v_2)^{-2h_\phi} \qquad\qquad f_{\mathbb{1},\phi,\mathcal{O}_\star} = -\frac{C_{\phi\phi\mathcal{O}_\star}^2}{\mathrm{B}_{\bar{h}_\star}} v_2^{h_\star} \frac{\log u_2 + \mathcal{O}(u_2^0)}{(\mathcal{U}^6 v_2)^{2h_\phi}} \,. \tag{2.27}$$

These are the three terms of the direct-channel expansion that we would like to express in terms of the crossed-channel lightcone expansion. But before we can do so, we need some extensive preparation. In the next section, we shall first analyze the relevant crossed-channel lightcone blocks. These are much harder to determine than for the direct channel since they involve a non-trivial exchange in all three intermediate channels.

3 Crossed-channel Blocks in the Lightcone Limit

The goal of this section is to derive an explicit formula for the lightcone conformal blocks relevant for the crossed-channel expansion, see Figure 4. Our main motivation for this is that expanding the leading-twist DC terms discussed in Section 2.4 into these blocks allows us to extract asymptotic conformal data from the crossing equation.

In the first subsection (Sec. 3.1), we spell out explicit formulas for the leading terms of the differential operators in the relevant limit. Then we use these formulas to determine the scaling of eigenvalues in the lightcone limit. In the resulting regime, we shall encounter a very remarkable surprise: it turns out that the spectrum of the two vertex operators can be calculated exactly in terms of the usual tensor structure labels. This will allow us to construct the lightcone limit of conformal blocks in the usual basis for tensor structures, see eq. (2.12). Since the expressions are obtained by solving a system of differential equations, the overall normalization must be determined through continuation from the OPE limit as explained in [24]. This analysis is described in the final subsection. Let us stress, however, that the normalizing prefactors are not needed in the subsequent determination of anomalous dimensions.

3.1 Casimir and vertex operators

In this subsection, we state our conventions regarding Casimir and vertex operators and explicitly spell out the leading terms (in the sense of [24]) of several key differential operators in our lightcone limits.
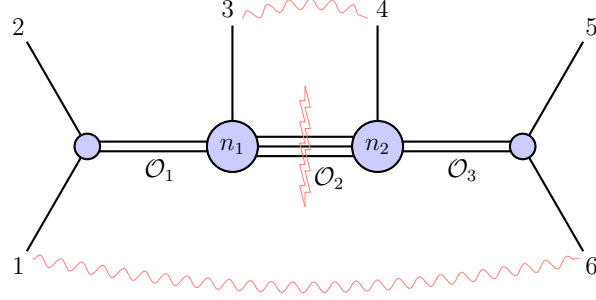
Figure 4. OPE diagram associated with the CC expansion. The $\epsilon_{16}, \epsilon_{34} \to 0$ limits, represented as wavy lines, entail large-spin exchanges at the internal legs that they cover, while the $\epsilon_{456} \to 0$ limit (conformally equivalent to a $\epsilon_{123} \to 0$ limit for the first three fields) is represented as a line that cuts the middle leg, where contributions with lowest twist become more relevant.

We shall denote the usual conformal generators acting on the coordinates of the ith insertion point by $\mathcal{T}_i$. In the embedding space formalism, the conformal generators take the following simple form

$$(\mathcal{T}_i)^{AB} := X_i^A \partial_i^B - X_i^B \partial_i^A. \tag{3.1}$$

In terms of these generators, we can construct seven Casimir operators of the following form

$$\mathcal{D}_{ij}^2 := \frac{1}{4}\text{tr}(\mathcal{T}_i + \mathcal{T}_j)^2 \qquad\qquad \mathcal{D}_{ijk}^2 := \frac{1}{4}\text{tr}(\mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k)^2 \tag{3.2}$$

$$\mathcal{D}_{ij}^4 := \frac{1}{2}\text{tr}(\mathcal{T}_i + \mathcal{T}_j)^4 - 2\left(\mathcal{D}_{ij}^2\right)^2 \qquad \mathcal{D}_{ijk}^4 := \frac{1}{2}\text{tr}(\mathcal{T}_i + \mathcal{T}_j + \mathcal{T}_k)^4 - 2\left(\mathcal{D}_{ijk}^2\right)^2. \tag{3.3}$$

where the indices ij are either $ij = 12$ or $ij = 56$ and $ijk = 123$. These operators can be used to label the representations of the fields exchanged in the internal legs of the diagram. In addition, we can construct two vertex differential operators that label the tensor structures at the two innermost vertices of the comb. We shall work with the following pair[3]

$$\mathcal{V}_1 := \frac{1}{2}\text{tr}\left[(\mathcal{T}_1 + \mathcal{T}_2)^3(\mathcal{T}_3)\right] + \frac{1}{4}\text{tr}(\mathcal{T}_1 + \mathcal{T}_2)^4 + \left(\frac{1}{2}\text{tr}\,\mathcal{T}_3^2 - \mathcal{D}_{123}^2\right)\mathcal{D}_{12}^2 + \frac{d(d-1)}{2}\left[\frac{1}{4}\text{tr}\,\mathcal{T}_3^2 - \mathcal{D}_{12}^2\right],$$
$$\tag{3.4}$$

$$\mathcal{V}_2 := \frac{1}{2}\text{tr}\left[(\mathcal{T}_6 + \mathcal{T}_5)^3(\mathcal{T}_4)\right] + \frac{1}{4}\text{tr}(\mathcal{T}_6 + \mathcal{T}_5)^4 + \left(\frac{1}{2}\text{tr}\,\mathcal{T}_4^2 - \mathcal{D}_{123}^2\right)\mathcal{D}_{56}^2 + \frac{d(d-1)}{2}\left[\frac{1}{4}\text{tr}\,\mathcal{T}_4^2 - \mathcal{D}_{56}^2\right].$$
$$\tag{3.5}$$

After inserting these expressions into our formulas for the nine differential operators and conjugating with the leg factor Ω_{CC} that we defined in eq. (2.20), we obtain a set of reduced differential operators

---

[3]compared to the previous works [19, 20, 24], we are here considering a definition for these vertex operators shifted by quadratic and quartic Casimirs. This form makes the vertex operators behave well under $\mathbb{Z}_2$ transformations that exchange their spinning legs, and display a simpler form under lightcone and large-spin limits.

that act on the nine cross-ratios instead of the six embedding space vectors. In a slight abuse of notation, we will continue to denote these dressed operators by $\mathcal{D}$ and $\mathcal{V}$. Furthermore, we will often conjugate by some additional factor ω in order to simplify the expressions we are dealing with, i.e. we introduce a new set of blocks $\tilde{g}$ by splitting off a simple function of the cross-ratios as follows

$$g^{\text{CC}}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2}(u_a, v_a, \mathcal{U}_i, \mathcal{U}^6) = \omega^{h_\phi} \tilde{g}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2}(u_a, v_a, \mathcal{U}_i, \mathcal{U}^6) , \qquad \omega := v_1 u_1^2 v_2^2 u_2^3 v_3 u_3^2 . \tag{3.6}$$

When acting on the functions $\tilde{g}$, the leading terms of the quadratic Casimir operators read

$$\widetilde{\mathcal{D}}^2_{12} = \epsilon_{16}^{-1} \mathcal{U}_2 \partial_{\mathcal{U}^6} \Big[(1 - \mathcal{U}_1)\left(\vartheta_{u_3} + \vartheta_{\mathcal{U}_1} + \vartheta_{\mathcal{U}^6} - \vartheta_{v_2}\right) - \mathcal{U}_1\left(\vartheta_{u_1} + \vartheta_{\mathcal{U}_2} + 2h_\phi\right)$$

$$- \vartheta_{1 - v_1 - \frac{u_1}{\mathcal{U}_2}} + \mathcal{O}(\epsilon_{456}) \Big] + \mathcal{O}(\epsilon_{16}^0) , \quad (3.7)$$

$$\widetilde{\mathcal{D}}^2_{56} = \epsilon_{16}^{-1} \mathcal{U}_1 \partial_{\mathcal{U}_6} \Big[(1 - \mathcal{U}_2)\left(\vartheta_{u_1} + \vartheta_{\mathcal{U}_2} + \vartheta_{\mathcal{U}_6} - \vartheta_{v_2}\right) - \mathcal{U}_2\left(\vartheta_{u_3} + \vartheta_{\mathcal{U}_1} + 2h_\phi\right)$$

$$- \vartheta_{1 - v_3 - \frac{u_3}{\mathcal{U}_1}} + \mathcal{O}(\epsilon_{456}) \Big] + \mathcal{O}(\epsilon_{16}^0) , \quad (3.8)$$

$$\text{and} \qquad \widetilde{\mathcal{D}}^2_{123} = \epsilon_{16}^{-1} \epsilon_{34}^{-1} \left[\partial_{v_2} \partial_{\mathcal{U}^6} + \mathcal{O}(\epsilon_{456}) \right] + \mathcal{O}\left(\epsilon_{16}^{-1}, \epsilon_{34}^0\right) + \mathcal{O}\left(\epsilon_{16}^0, \epsilon_{34}^{-1}\right) , \tag{3.9}$$

where we used the shorthand notation for Euler operators $\vartheta_x := x \partial_x$. The fourth-order Casimir operator $\widetilde{\mathcal{D}}^4_{123}$ can be conveniently expressed in terms of $\widetilde{\mathcal{D}}^2_{123}$ as

$$\widetilde{\mathcal{D}}^4_{123} = \Big[2(1 - \mathcal{U}_1 - \mathcal{U}_2)\left(3d - 12h_\phi - 4\vartheta_{u_2} - 6\right)\left(2h_\phi + \vartheta_{u_1} + \vartheta_{u_3} + \vartheta_{\mathcal{U}_1} + \vartheta_{\mathcal{U}_2} + \vartheta_{\mathcal{U}_6} - \vartheta_{v_2}\right)$$

$$+ 8\vartheta_{u_2}\left(-d + 6h_\phi + \vartheta_{u_2} + 1\right) + (d - 1)\left(3d - 24h_\phi - 4\right) + 72h_\phi^2 + \mathcal{O}(\epsilon_{456}) \Big] \widetilde{\mathcal{D}}^2_{123} \tag{3.10}$$

$$+ \mathcal{O}\left(\epsilon_{16}^{-1}, \epsilon_{34}^0\right) + \mathcal{O}\left(\epsilon_{16}^0, \epsilon_{34}^{-1}\right) .$$

The remaining quartic Casimirs $\widetilde{\mathcal{D}}^4_{12}$ and $\widetilde{\mathcal{D}}^4_{56}$, the sixth-order Casimir $\widetilde{\mathcal{D}}^6_{123}$, as well as the two vertex operators $\widetilde{\mathcal{V}}_1$ and $\widetilde{\mathcal{V}}_2$ have a more complicated form that we will not display explicitly.

3.2 Large-spin behavior of the eigenvalues

As our notations for the conformal blocks $\tilde{g}$ indicate, the eigenfunctions of our differential operators carry nine quantum numbers. These are the weights h_i and spins J_i of the two outer intermediate operators, the weight h_2 and spins (J_2, κ) of the intermediate operator in the center, as well as the tensor structure labels n_1, n_2 of the two non-trivial vertices. As we shall show in the next subsection, the expansion of the direct-channel terms through crossed-channel blocks requires that all intermediate spins J_i become large such that the spins scale as

$$1 \ll J_1, J_3 \ll J_2 \tag{3.11}$$

while keeping κ finite. For later use, we want to spell out some of the well-known eigenvalues of the various Casimir operators in this regime. Let us begin with the second- and fourth-order Casimir operators $\widetilde{\mathcal{D}}^p_{123}, p = 2, 4$ of the middle leg. In the large-spin regime their eigenvalue equations read

$$\widetilde{\mathcal{D}}^2_{123}\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2} = \left(J_2^2 + \mathcal{O}(J_2)\right)\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2} \tag{3.12}$$

and

$$\widetilde{\mathcal{D}}^4_{123}\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2} = \left(c(h_2,\kappa)J_2^2 + \mathcal{O}(J_2)\right)\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2} \tag{3.13}$$

with

$$c(h,\kappa) = 8h(h + \kappa - d + 1) - 6(d-2)\kappa + 3d^2 - 7d + 4\,. \tag{3.14}$$

For the remaining Casimirs $\widetilde{\mathcal{D}}^p_{12}$ and $\widetilde{\mathcal{D}}^p_{56}$ that are associated with the outer legs, the Casimir eigenvalue equations take the same form, with the obvious replacements $J_2 \to J_1, J_3$ and $c(h_2, \kappa) \to c(h_1, 0), c(h_3, 0)$ respectively.

While the blocks $g_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3;n_1,n_2}$ are eigenfunctions of the Casimir operators that measure the quantum numbers of the three intermediate fields $\mathcal{O}_a$, they need not be eigenfunctions of the two vertex operators in general. Put differently, the eigenvalue equations of the vertex operators $\mathcal{V}$ select a different basis of blocks that is not aligned with the basis labeled by the conventional tensor structure labels n_1, n_2. For a generic choice of spin labels, the eigenvalues and eigenfunctions of the six-point vertex operators are not known. But in the regime in which the spin J_2 is much larger than $J_{1,3}$ there is a very surprising twist to this general description: when applied to the usual basis of blocks that is labeled by the tensor structures $n_1 \leq J_1 - \kappa$ and $n_2 \leq J_3 - \kappa$, the two vertex operators turn out to be upper triangular! Since all matrix elements, and in particular the ones on the diagonal, can be computed explicitly, this implies that the spectrum of the vertex operators is explicitly known in the regime of large J_2. In addition, there is one block, namely the one with the largest possible tensor structure labels $n_1 = J_1 - \kappa$ and $n_2 = J_3 - \kappa$, that is an eigenfunction of the vertex differential operators.

For later use, we want to spell out new closed-form expressions for the eigenvalues and eigenvectors of the vertex differential operators in the regime of large J_2, see Appendix A for details. We parameterize the spectra of $\mathcal{V}_1$ and $\mathcal{V}_2$ as $\mathfrak{t}(h_1, J_1, h_2, J_2, \kappa, h_\phi, d; \nu_1)$ and $\mathfrak{t}(h_3, J_3, h_2, J_2, \kappa, h_\phi, d; \nu_2)$, respectively, where $\nu_{1,2}$ are integers that go from 0 to $J_{1,3} - \kappa$, respectively. As discussed in the appendix, their large J_2 expansion reads

$$\mathfrak{t}(h_i, J_i, h_2, J_2, \kappa, h_\phi, d; \nu) = \frac{J_2^2}{2}\left[[\nu]_{\bar{h}_i}\left([\nu]_{\bar{h}_i} - d\right) - \kappa([\nu]_{\bar{h}_i} - 2)\right] + \mathcal{O}(J_2) \tag{3.15}$$

for the eigenvalues $\mathfrak{t}$ with $i = 1, 3$. In order to render the expression a bit more compact, we have introduced the following shorthand,

$$[\nu]_{\bar{h}_i} = 2(\bar{h}_i - \nu). \tag{3.16}$$

The vertex operator eigenfunction corresponding to the eigenvalue labeled by ν_1, ν_2 is, to leading order in J_2, a simple linear combination of standard basis tensor structures with label n_1, n_2. Both sets of integers range from 0 to $J_i - \kappa$ (see eq. (A.5)). Furthermore, it turns out that the eigenfunction of the vertex differential operators associated with the largest value of the tensor structure labels, i.e. $\nu_1 = J_1 - \kappa$ and $\nu_2 = J_3 - \kappa$ coincides with the blocks with tensor structure labels $n_1 = \nu_1, n_2 = \nu_2$, respectively. This remarkable observation will be crucial for the reasoning below when we construct the lightcone limit of crossed channel blocks.

3.3 Casimir singularity of the direct channel

In order to identify the asymptotics of quantum numbers exchanged in the CC OPE, we shall follow the approach of [4, 33, 34], see also [24] for the extension to multipoint lightcone bootstrap, and act with the CC differential operators on the leading DC contributions. In terms of the rescaled blocks $\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}$ we introduced in eq. (3.6) and using the leading DC contributions identified in eq. (2.26), we can rewrite the crossing equation as

$$f_{\mathbb{1},\phi,\mathbb{1}} + f_{\mathcal{O}_\star,\phi,\mathbb{1}} + f_{\mathbb{1},\phi,\mathcal{O}_\star} + \mathcal{O}(X_{34}^{>h_\star}, X_{16}^{>h_\star}) = \sum_{\mathcal{O},n} P_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n_1n_2)} \tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}. \tag{3.17}$$

Acting on the terms on the left-hand side with the Casimir and vertex operators we described in the first subsection, we find that they all satisfy

$$\frac{\widetilde{\mathcal{D}}_{12}^4 f_{a,\phi,b}}{\widetilde{\mathcal{D}}_{12}^2 f_{a,\phi,b}} \simeq \frac{\widetilde{\mathcal{D}}_{56}^4 f_{a,\phi,b}}{\widetilde{\mathcal{D}}_{56}^2 f_{a,\phi,b}} \simeq c(2h_\phi, 0), \tag{3.18}$$

$$\frac{\widetilde{\mathcal{D}}_{123}^4 f_{a,\phi,b}}{\widetilde{\mathcal{D}}_{123}^2 f_{a,\phi,b}} \simeq c(3h_\phi, (2h_\phi - |h_a + h_b|)(\mathcal{U}_1 + \mathcal{U}_2 - 1)), \tag{3.19}$$

$$\frac{\left(\widetilde{\mathcal{D}}_{123}^4 - 6\widetilde{\mathcal{V}}_s\right) f_{a,\phi,b}}{\widetilde{\mathcal{D}}_{123}^2 f_{a,\phi,b}} \simeq -4(-1 + d - 6h_\phi + 3dh_\phi - 6h_\phi^2), \qquad \text{for} \quad s = 1, 2, \tag{3.20}$$

where $(a, b) \in \{(\mathbb{1}, \mathbb{1}), (\mathbb{1}, \mathcal{O}_\star), (\mathcal{O}_\star, \mathbb{1})\}$ and "$\simeq$" stands for equality at leading order in the ϵ scaling parameters. Thus, if we assume the spins J_i, $i = 1, 2, 3$ of the exchanged operators to be large, the leading DC contributions are reproduced by CC conformal blocks with twists h_i, MST spin κ, and

tensor structures $\mathfrak{t}_i$ fixed to be

$$h_1 = h_3 = 2h_\phi \,, \qquad h_2 = 3h_\phi \,, \qquad \kappa = \mathcal{O}(1) \tag{3.21}$$

$$\mathfrak{t}_1 = \mathfrak{t}_2 = \mathfrak{t}_\star = J_2^2 \left[(h_1 + \kappa)(2h_1 - d) + 2\kappa \right] . \tag{3.22}$$

The large-spin assumption is confirmed by the behavior of the quadratic Casimirs, which give

$$\widetilde{\mathcal{D}}_{123}^2 f_{a,\phi,b} \simeq \frac{2h_\phi \left(2h_\phi - |h_a + h_b| \right)}{v_2 \mathcal{U}^6} f_{a,\phi,b} \,, \tag{3.23}$$

again for $(a,b) \in \{ (\mathbb{1}, \mathbb{1}), (\mathbb{1}, \mathcal{O}_\star), (\mathcal{O}_\star, \mathbb{1}) \}$, as well as

$$\widetilde{\mathcal{D}}_{12}^2 f_{\mathbb{1},\phi,\mathbb{1}} \simeq \widetilde{\mathcal{D}}_{56}^2 f_{\mathbb{1},\phi,\mathbb{1}} \simeq \frac{4h_\phi^2 \mathcal{U}_1 \mathcal{U}_2}{\mathcal{U}^6} f_{\mathbb{1},\phi,\mathbb{1}} \,, \tag{3.24}$$

$$\widetilde{\mathcal{D}}_{12}^2 f_{\mathcal{O}_\star,\phi,\mathbb{1}} \simeq \widetilde{\mathcal{D}}_{56}^2 f_{\mathcal{O}_\star,\phi,\mathbb{1}} \simeq \frac{(2h_\phi - h_\star)^2 \mathcal{U}_1 \mathcal{U}_2}{\mathcal{U}^6} f_{\mathcal{O}_\star,\phi,\mathbb{1}} \,, \tag{3.25}$$

$$\widetilde{\mathcal{D}}_{12}^2 f_{\mathbb{1},\phi,\mathcal{O}_\star} \simeq \left[\widetilde{\mathcal{D}}_{56}^2 + \frac{2h_\phi h_\star (\mathcal{U}_1 - \mathcal{U}_2)}{\mathcal{U}^6} \right] f_{\mathbb{1},\phi,\mathcal{O}_\star} \simeq \frac{2h_\phi (2h_\phi - h_\star) \mathcal{U}_1 \mathcal{U}_2 + 2h_\phi h_\star \mathcal{U}_2}{\mathcal{U}^6} f_{\mathbb{1},\phi,\mathcal{O}_\star} \,. \tag{3.26}$$

In fact, given the scalings $\mathcal{U}^6 \propto \epsilon_{16}$ and $v_2 \propto \epsilon_{34}$, the action of these quadratic Casimirs implies that the most relevant contributions to the CC correspond to exchanged operators whose spins scale as

$$J_1^2 = \mathcal{O}\!\left(\epsilon_{16}^{-1} \right) , \qquad J_2^2 = \mathcal{O}\!\left(\epsilon_{16}^{-1} \epsilon_{34}^{-1} \right) , \qquad J_3^2 = \mathcal{O}\!\left(\epsilon_{16}^{-1} \right) , \tag{3.27}$$

subject to the extra constraint $J_1 = J_3$ for blocks that reproduce $f_{\mathbb{1},\phi,\mathbb{1}}$ and $f_{\mathcal{O}_\star,\phi,\mathbb{1}}$. This is indeed the regime we anticipated in the previous subsection, see eq. (3.11), when we discussed the spectrum of the various differential operators.

In particular, since we are indeed driven into a regime in which J_2 is much larger than all other quantum numbers, we can use the large J_2 spectrum of the vertex differential operators as given in eq. (3.15). Comparing with eq. (3.22), we deduce

$$[\nu_1]_{\bar{h}_1} = 2(2h_\phi + J_1 + \kappa - \nu_1) \overset{!}{=} 4h_\phi + 2\kappa \qquad \Longrightarrow \qquad \nu_1 = J_1 - \kappa \tag{3.28}$$

and likewise $\nu_2 = J_3 - \kappa$. By the observation made at the end of the previous subsection, this implies that the CC lightcone blocks are fixed to have maximal tensor structure labels in the n-basis of three-point tensor structures defined previously, i.e. we have

$$(n_1, n_2) = (J_1 - \kappa, J_3 - \kappa). \tag{3.29}$$

We have thus seen that, for the specific quantum numbers dictated by the crossing equation, the CC lightcone blocks in the usual n-basis of tensor structures satisfy the eigenvalue equations for the vertex

differential operators. Therefore, we will be able to determine the lightcone limit of the relevant blocks from differential equations alone, at least up to an overall normalization.

3.4 Explicit form of six-point triple-twist blocks

In the previous subsection, we understood which quantum numbers must appear in the CC conformal blocks in order to reproduce the DC contributions to the crossing equation (2.27). We now aim to leverage the differential eigenvalue equations for these conformal blocks to determine their functional form. While these equations are hard to solve in full generality, in the lightcone limits we consider and for the twists relevant to the crossing equation $h_1 = h_3 = 2h_\phi$ and $h_2 = 3h_\phi$, the problem greatly simplifies. When defining $\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}$ in eq. (3.6), we extracted out the lightcone OPE asymptotics of the blocks, i.e. a factor $\left(u_1^2 u_2^3 u_3^2\right)^{h_\phi}$ for $u_{\{i\}} \to 0$, Hence, we now look for solutions of the differential operators presented in eqs. 3.1 whose leading term in the regime $u_{\{i\}} \to 0$ does not depend on u_i. In particular, since the limits we are interested in already include $u_2 \to 0$ as a consequence of $X_{45}, X_{56}, X_{46} \ll 1$, we conclude that only the leading term in u_2 of the blocks will contribute, and thus the function $\tilde{g}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}$ has to be independent of u_2. This is perfectly compatible with the fact that all the relevant differential operators for the blocks commute with the Euler operator ϑ_{u_2}, and thus are diagonalized by any power law in u_2, including the case of a constant.

Before we start our diagonalization procedure we shall perform the following simple change of variables

$$
\begin{aligned}
v_1 &\longrightarrow r_1 = 1 - v_1 - \tfrac{u_1}{\mathcal{U}_2}, \\
v_3 &\longrightarrow r_3 = 1 - v_3 - \tfrac{u_3}{\mathcal{U}_1},
\end{aligned}
\tag{3.30}
$$

Once performed, it is easy to see that the differential operators spelled out in eqs. (3.7), (3.8), (3.9), and (3.10) commute with the four Euler operators $\vartheta_{u_1}, \vartheta_{u_3}, \vartheta_{r_1}, \vartheta_{r_3}$. We can therefore work temporarily in an eigenbasis of the latter,

$$
\begin{pmatrix} \vartheta_{u_1} \\ \vartheta_{u_3} \\ \vartheta_{r_1} \\ \vartheta_{r_3} \end{pmatrix} G^{p_1,p_3}_{(J_1,J_2,\kappa,J_3);m_1m_2} = \begin{pmatrix} p_1 \\ p_3 \\ m_1 \\ m_3 \end{pmatrix} G^{p_1,p_3}_{(J_1,J_2,\kappa,J_3);m_1m_2}.
\tag{3.31}
$$

Let us stress that this basis is *not* compatible with the diagonalization of the two quartic Casimirs $\widetilde{\mathcal{D}}^4_{12}, \widetilde{\mathcal{D}}^4_{56}$ of the outer intermediate legs and of the two vertex operators.

Let us now aim to diagonalize the quadratic Casimirs and the quartic Casimir $\widetilde{\mathcal{D}}^4_{123}$ in the basis (3.31). Diagonalizing the Euler operators by simple power laws, and enforcing $\widetilde{\mathcal{D}}^2_{123} \to \frac{J_2^2}{2}$ in the expression (3.10) of $\widetilde{\mathcal{D}}^4_{123}$, we conclude that the eigenvalue equation (3.13) for the quartic Casimir simply reduces to a first-order differential equation that constrains the auxiliary functions G to take

the form

$$G^{p_1,p_3}_{(J_1,J_2,\kappa,J_3);m_1m_2} = v_2^{2h_\phi+\kappa+p_1+p_2}(\mathcal{U}_1+\mathcal{U}_2-1)^\kappa\left(1-v_1-\frac{u_1}{\mathcal{U}_2}\right)^{m_1}\left(1-v_3-\frac{u_3}{\mathcal{U}_1}\right)^{m_2}\widetilde{G}(v_2\mathcal{U}_1,v_2\mathcal{U}_2,v_2\mathcal{U}^6)\,.$$

(3.32)

We can now plug this Ansatz into three eigenvalue equations for the quadratic Casimir. By taking an appropriate linear combination of the form

$$\mathcal{D}^2_{123} + \frac{\mathcal{D}^2_{12}}{v_2\mathcal{U}_2} + \frac{\mathcal{D}^2_{56}}{v_2\mathcal{U}_1}$$

(3.33)

one obtains a relatively simple differential equation that constrains the dependence of the solutions on the variable $v_2\mathcal{U}^6$,

$$\left[\partial_{v_2\mathcal{U}^6}\left(v_2\mathcal{U}^6\partial_{v_2\mathcal{U}^6} + (-2h_\phi - m_1 - m_3 - \kappa)\right) - \left(\frac{J_1^2}{v_2\mathcal{U}_2} + J_2^2 + \frac{J_3^2}{v_2\mathcal{U}_1}\right)\right]\widetilde{G}(v_2\mathcal{U}_1,v_2\mathcal{U}_2,v_2\mathcal{U}^6) = 0\,.$$

(3.34)

We can recognize this as a Bessel-Clifford differential equation [24, Appendix B.1], up to a change of variables. The solutions can be further constrained by imposing the eigenvalue equations for any two of the quadratic Casimir operators. When applied to solutions of the Bessel-Clifford equation (3.34) one obtains two first-order constraints which are easily solved by

$$\widetilde{G}(v_2\mathcal{U}_1,v_2\mathcal{U}_2,v_2\mathcal{U}^6) = \left(\frac{\mathcal{U}^6}{v_2\mathcal{U}_1\mathcal{U}_2}\right)^{2h_\phi+\kappa}\left(\frac{\mathcal{U}^6}{\mathcal{U}_2}\right)^{m_1}\left(\frac{\mathcal{U}^6}{\mathcal{U}_1}\right)^{m_3}(v_2\mathcal{U}_2)^{-p_1}(v_2\mathcal{U}_1)^{-p_3}$$

$$\mathcal{K}_{2h_\phi+m_1+m_3+\kappa}\left(\left[\frac{J_1^2}{\mathcal{U}_2}+J_2^2 v_2+\frac{J_3^2}{\mathcal{U}_1}\right]\mathcal{U}^6\right)\,. \quad (3.35)$$

With this step, we completely determined the auxiliary functions G which we introduced in eq. (3.31). By construction, these functions provide a basis of the function space we are interested in. While generic lightcone conformal blocks correspond to linear combinations of these auxiliary functions, one may check that the constraints (3.18) and (3.20) are satisfied by conformal blocks that are built up of a single function $G^{0,0}_{(J_1,J_2,\kappa,J_3);0,0}$, with the parameters $m_1 = p_1 = m_3 = p_3 = 0$. In conclusion we have shown that the relevant lightcone blocks are given by

$$\tilde{g}_{(J_1,J_2,\kappa,J_3);J_1J_3} \sim \mathcal{N}_{J_1(J_2,\kappa)J_3}(\mathcal{U}_1+\mathcal{U}_2-1)^\kappa\left(\frac{\mathcal{U}^6}{\mathcal{U}_1\mathcal{U}_2}\right)^{2h_\phi+\kappa}\mathcal{K}_{2h_\phi+\kappa}\left(\left[\frac{J_1^2}{\mathcal{U}_2}+J_2^2 v_2+\frac{J_3^2}{\mathcal{U}_1}\right]\mathcal{U}^6\right) \quad (3.36)$$

up to an overall normalization $\mathcal{N}_{J_1(J_2,\kappa)J_3}$. Here we have suppressed the dependence on the quantum numbers h_i since these are determined by the conformal weight h_ϕ of the external scalar through eq. (3.21). Note also that we have dropped the shift by $-\kappa$ from the tensor structure indices since we consider lightcone blocks in the regime where $n_1 = J_1 - \kappa \sim J_1$ and $n_2 \sim J_3$, see eq. (3.27). The explicit

formula (3.36) for lightcone blocks (3.6) is a remarkable result. Let us stress once again that it was obtained from the differential equations alone, without any additional input. The only quantity that requires input from the OPE limit of conformal blocks is the normalization. In the next subsection, we will show that this normalization is given by

$$\mathcal{N}_{J_1(J_2,\kappa)J_3} = \lim_{J_1,J_3 \to \infty} \frac{\Gamma(2\Delta_\phi + 2J_1)}{\Gamma(\Delta_\phi + J_1)} \frac{\Gamma(2\Delta_\phi + 2J_3)}{\Gamma(\Delta_\phi + J_3)} \lim_{J_2 \to \infty} \frac{2\,\Gamma(3\Delta_\phi + 2J_2 + \kappa)}{\Gamma(2\Delta_\phi + J_1 + J_2 + \kappa)\Gamma(2\Delta_\phi + J_2 + J_3 + \kappa)}$$

$$= \pi^{-\frac{1}{2}} 2^{2(J_1+J_2+J_3)+7\Delta_\phi+\kappa-1} \, e^{-(J_1+J_3)} J_1^{J_1+\Delta_\phi} J_2^{\frac{1}{2}-(J_1+J_3+\kappa+\Delta_\phi)} J_3^{J_3+\Delta_\phi}. \tag{3.37}$$

The formulas (3.36) and (3.37) for the triple-twist CC lightcone blocks are the main result of this section and they are key to the lightcone bootstrap analysis that we will describe in the remaining sections of this work.

3.5 Normalization of six-point blocks

The goal of this section is to derive the formula (3.37) for the normalization of CC lightcone blocks for $(h_1, h_2, h_3) = (2, 3, 2) \times h_\phi$ and $(n_1, n_2) = (J_1, J_3)$. Readers solely interested in the derivation of the triple-twist anomalous dimensions can skip this subsection. Indeed, the normalization (3.37) for six-point blocks at GFF twists follows from the results of Section 4, where we determine GFF triple-twist OPE coefficients exactly, and in Section 6, where we determine the large-spin asymptotics of the GFF OPE coefficients in terms of the normalization from the six-point crossing equation. Nonetheless, the method used to derive the normalization in this section also applies to six-point blocks with twists and tensor structures not necessarily equal to the GFF values.

Our derivation of the normalization (3.37) follows the same strategy that was detailed in our previous work for the case of five-point lightcone blocks. The derivation starts from the following observation: a conformal block corresponding to the basis element (n_1, n_2) of monomial tensor structures is the unique solution to the comb-channel Casimir equations with OPE limit boundary condition (2.4), where the OPE limit $\text{OPE}^{(12)3(4(56))}$ following the pattern of coincident points $x_1 \to x_2; x_4 \leftarrow x_5 \leftarrow x_6$, is defined by eq. (2.3) in terms of OPE cross-ratios. Starting from the OPE limit boundary condition (2.4) for the tensor structures $(n_1, n_2) = (J_1, J_3)$, we then use the Casimir equations to interpolate this solution with the lightcone limit $LCL^{(16),(34)}$.

The lack of analytic results on higher-point conformal blocks makes the interpolation procedure difficult. However, the problem is greatly simplified within two limiting regimes.

1. The maximal subset of lightcone limits in $LCL^{(16),(34)}$ that still contains the OPE limit, i.e.

$$\text{LT}^{(12)3(4(56))} : \epsilon_{12}, \epsilon_{456}, \epsilon_{56} \to 0 \iff \bar{z}_1, \bar{z}_2, \bar{z}_3 \to 0. \tag{3.38}$$

This limit restricts the sum over descendants in the conformal block to those with minimal twists $(2h_1, 2h_2, 2h_3)$. However, applying these limits before taking $\epsilon_{16}, \epsilon_{34}$ to zero is opposite to the order prescribed by $LCL^{(16),(34)}$ (which is the order necessary for leading-twist exchange in the direct channel). Therefore, the interpolation is only valid if the limits commute at the level of single blocks. This assumption is supported by the fact that all comb-channel Casimir operators $\mathcal{D}$ satisfy this property at leading order, that is to say

$$\left(\lim_{\epsilon_{12,456,56} \to 0} \lim_{\epsilon_{16,34} \to 0} - \lim_{\epsilon_{16,34} \to 0} \lim_{\epsilon_{12,456,56} \to 0} \right) \vec{\epsilon}^{\,\vec{m}} \mathcal{D}^{\vec{\epsilon}} = 0, \tag{3.39}$$

where $\vec{\epsilon}^{\,-\vec{m}}$ is the leading scaling of the differential operator.

2. Three-dimensional kinematics, where the embedding space vectors X_i (respectively the spacetime vectors x_i) are restricted to a $\mathbb{R}^{2,3}$ subspace (respectively a $\mathbb{R}^{1,2}$ subspace). In the leading-twist limit $\mathrm{LT}^{(12)3(4(56))}$, this amounts to taking $\Upsilon_0 \to 0$ in OPE cross-ratios or equivalently $\mathcal{U}_2 \to 1 - \mathcal{U}_1$ in the cross-ratios of eq. (2.1).

At leading order in both of these limits, the normalized conformal blocks then take the form

$$g_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2} \overset{\bar{z}, \Upsilon_0 \to 0}{\sim} \prod_{i=1}^{3} \bar{z}_i^{h_i} z_i^{\bar{h}_i} \Upsilon_0^{\kappa} (1 - w_1)^{n_1} (1 - w_2)^{n_2} \tilde{F}_{(h_i, J_i, \kappa; n_1, n_2)}(z, w), \tag{3.40}$$

where $\tilde{F}$ is a power series in five variables. The coefficients of this power series are straightforward to derive from the three second-order Casimir equations spelled out in eq. (B.4)–(B.6), with boundary condition $\tilde{F}(0, w) = 1$. We now solve the interpolation problem in the two remaining limits of $LCL^{(16),(34)}$ using the limiting form of the second-order Casimir equations.

1. In the $\epsilon_{34} \to 0$ limit with $J_2^2 = \mathcal{O}(\epsilon_{34}^{-1})$, we find

$$\tilde{F}_{(h_i, J_i, \kappa; n_1, n_2)} \left(z_1, 1 - \frac{v_2}{J_2^2}, z_3, \frac{w_1}{J_2^2}, \frac{w_2}{J_2^2} \right) \overset{J_2 \to \infty}{\sim} \mathcal{N}^{4\mathrm{pt}}_{(h_i, J_i, \kappa; n_1, n_2)}$$
$$G_1(z_1, -z_1(1 - w_1)\partial_{v_2}) G_3(z_3, -z_3(1 - w_2)\partial_{v_2}) \mathcal{K}_{\alpha_2}(v_2 J_2^2), \tag{3.41}$$

where $\alpha_2 := h_{1\phi} + h_{3\phi} + n_1 + n_2 + \kappa$ and $G_a(z_a, y_a)$ for $a = 1, 3$ are power series in two variables given by eq. (B.11). The normalization $\mathcal{N}^{4\mathrm{pt}}$ is obtained by comparing the limits $z_1, z_3 \to 0$, $w_1, w_2 \to 1$ before and after $\epsilon_{34} \to 0$. On the one hand, this limit reduces the system to a spinning four-point lightcone block, with $\tilde{F}(0, z_2, 0, 1, 1)$ given by a $_2F_1$ hypergeometric function in z_2. On the other hand, the large-spin limit at $z_1, z_3 = 0$ simplifies to a single modified Bessel-Clifford function multiplied by $\mathcal{N}^{4\mathrm{pt}}$ because $G_a(0, 0) = 1$. Comparing both expressions, we deduce that

$$\frac{2\Gamma(2\bar{h}_2 - \kappa)}{\Gamma(\bar{h}_2 - h_{1\phi} - n_1 - \kappa)\Gamma(\bar{h}_2 - h_{3\phi} - n_2 - \kappa)} \overset{\bar{h}_2 \to \infty}{\sim} \mathcal{N}^{4\mathrm{pt}}_{(h_i, \bar{h}_i - h_i, \kappa; n_1, n_2)}. \tag{3.42}$$

2. In the $\epsilon_{16} \to 0$ limit, we can relate the cross-ratios w_1, w_2 to $\mathcal{U}^6 = \mathcal{O}(\epsilon_{16})$ via the relation

$$w_1 = 1 - \frac{(1 - z_1)v_2}{z_1(1 - v_2)}(1 - \mathcal{U}_1 - \mathcal{U}^6) + \mathcal{O}(\bar{z}), \quad w_2 = 1 - \frac{v_2(1 - z_3)}{(1 - v_2)z_3}\mathcal{U}_1 + \mathcal{O}(\bar{z}). \tag{3.43}$$

We then take the limit $\epsilon_{16} \to 0$ with $J_1^2, \epsilon_{34}J_2^2, J_3^2 = \mathcal{O}(\epsilon_{16}^{-1})$ and $(n_1, n_3) = (J_1 - \kappa - \delta n_1, J_3 - \kappa - \delta n_3)$, where δn_i are positive integers in general, while the blocks in eq. (3.36) correspond to $\delta n_{1,3} = 0$. In this limit, we employ an alternative power series representation (B.15) of $G_a(z_a, y_a)$, as well as the integral representation of the Bessel-Clifford function in [24, eq. (B.4)], to extract the same leading asymptotics of blocks as $G_{\delta n_1 \delta n_2}^{h_1 - 2h_\phi, h_3 - 2h_\phi}$ in eq. (3.32), times additional factors of $\Gamma(2\bar{h}_a)/\Gamma(\bar{h}_a)$, $a = 1, 3$, and $J_2^{-2\alpha_2}$. Combining these factors with the normalization $\mathcal{N}^{\text{4pt}}$ in eq. (3.42) and setting $(h_1, h_2, h_3; \delta n_1, \delta n_3) = (2h_\phi, 3h_\phi, 2h_\phi; 0, 0)$, we obtain the formula for $\mathcal{N}_{J_1(J_2,\kappa)J_3}$ in eq. (3.37).

4 Triple-twist Operators in Generalized Free Field Theory

As in the four-point lightcone bootstrap, the leading terms of the six-point lightcone bootstrap turn out to coincide with generalized free field theory (GFF). For this reason, it is useful to understand the GFF triple-twist operators and their OPE coefficients first, before analyzing the lightcone limit of the crossing equations in more general CFTs. In the current section, we discuss the space of GFF triple-twists with a special focus on the subspace of lowest dimension operators at a fixed spin. The dimension of this subspace, that is to say, the number of linearly independent operators with the same quantum numbers, grows linearly with the spin. We introduce a particular basis of triple-twist operators and then determine the exact form of four-point functions with three scalar and one triple-twist insertion. In reference to [49, 50], we call this four-point function "Derkachov-Manashov wave function of triple-twist operators".

4.1 Minimal-twist descendants in CFT

The first step in our endeavor to study triple-twist operators is related to studying the spaces $\mathcal{H}_\mathcal{O}^\tau$ of minimal-twist descendants of a primary operator $\mathcal{O}$. The latter serve as the building blocks of multi-twist operators. For MST primaries, the minimal-twist descendants fall into irreducible representations of the Lie algebra $\mathfrak{su}(1, 2)$. This will allow us later to fully characterize multi-twist operators as lowest weights in the tensor product of lowest-weight representations of $\mathfrak{su}(1, 2)$. To simplify the exposition, we will focus on the case where all operators are STTs, i.e. $\kappa = 0$, which reduces the relevant representations to those of $\mathfrak{su}(1, 1) \subset \mathfrak{su}(1, 2)$.

4.1.1 Some background from representation theory

As we shall show below, the space of so-called minimal twist descendants of an STT primary, see next subsection for a precise definition, carries an irreducible representation of the Lie algebra $\mathfrak{su}(1,1)$. In order not to clutter the presentation of the CFT constructions too much, we use this section to first collect some representation theoretic facts.

To begin, let us introduce the vector space $\mathcal{H}_{\bar{\tau}}$ that is spanned by polynomials ψ in a single variable α. On this space, we introduce an $\mathfrak{su}(1,1)$ action through the following standard differential operators

$$S_+\psi(\alpha) := -\partial_\alpha\psi(\alpha), \tag{4.1}$$

$$S_0\,\psi(\alpha) := (\alpha\partial_\alpha + \tfrac{1}{2}\bar{\tau})\psi(\alpha), \tag{4.2}$$

$$S_-\psi(\alpha) := (\alpha^2\partial_\alpha + \bar{\tau}\alpha)\psi(\alpha)\ . \tag{4.3}$$

As is well known, these differential operators obey the conditions

$$S_\pm^\dagger = -S_\mp\ ,\quad S_0^\dagger = S_0 \tag{4.4}$$

with respect to the following $\mathfrak{su}(1,1)$-invariant scalar product

$$\langle\psi_1,\psi_2\rangle_{\bar{\tau}} = \frac{\bar{\tau}-1}{\pi}\int_{\mathbb{D}} \mathrm{d}^2\alpha\,(1-\bar{\alpha}\alpha)^{\bar{\tau}-2}\bar{\psi}_1(\bar{\alpha})\psi_2(\alpha),\quad \mathrm{d}^2\alpha := \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)\ . \tag{4.5}$$

Here, $\mathbb{D} := \{\alpha \in \mathbb{C}\,|\,\bar{\alpha}\alpha \le 1\}$ is the unit disk, and the normalization of the measure follows from $\langle 1,1\rangle_{\bar{\tau}} = 1$. The subscript for scalar products and the associated norms denotes an explicit dependence on the weight $\bar{\tau}$ that characterizes the highest-weight representation of $\mathfrak{su}(1,1)$. We will suppress this dependence whenever it is clear from context[4]. For later use, we introduce the orthogonal monomial basis $\psi_M(\alpha) = \alpha^M$ of the space $\mathcal{H}_{\bar{\tau}}$ and note that the norm of these elements is given by

$$\|\psi_M\|_{\bar{\tau}}^2 = \frac{M!}{(\bar{\tau})_M}. \tag{4.6}$$

Let us now discuss a dual representation of the Lie algebra $\mathfrak{su}(1,1)$ on the space of one-variable polynomials. We refer to this representation as $\check{\mathcal{H}}_{\bar{\tau}}$ and to its elements as $\check{\psi}$ to distinguish them from the polynomials ψ in the representation $\mathcal{H}_{\bar{\tau}}$. The dual action of $\mathfrak{su}(1,1)$ is given by the following set of differential operators $\{\check{S}_a\}_{a=0,\pm}$

$$\check{S}_+\check{\psi}(x) := x\check{\psi}(x), \tag{4.7}$$

$$\check{S}_0\,\check{\psi}(x) := (x\partial_x + \tfrac{1}{2}\bar{\tau})\check{\psi}(x), \tag{4.8}$$

$$\check{S}_-\check{\psi}(x) := -(x\partial_x^2 + \bar{\tau}\partial_x)\check{\psi}(x). \tag{4.9}$$

---

[4]We have already done this for the generators S_+, S_0, S_-, which also exhibit an explicit dependence on $\bar{\tau} = 2\bar{h}$.

We can think of the elements $\check{\psi} \in \check{\mathcal{H}}_{\bar{\tau}}$ in the dual vector space as linear functionals on $\mathcal{H}_{\bar{\tau}}$ using the following $\mathfrak{su}(1,1)$-invariant dual pairing

$$(\check{\psi}_1, \psi_2) = \check{\psi}_1(\partial_\alpha)\psi_2(\alpha)|_{\alpha=0} . \tag{4.10}$$

Note that the duality relation is essentially given by a Fourier transform which sends x to a derivative ∂_α with respect to the dual variable. More precisely, there exists an an isomorphism $\mathcal{F}_{\bar{\tau}}$ that maps a polynomial $\check{\psi} \in \check{\mathcal{H}}_{\bar{\tau}}$ to a polynomial $\mathcal{F}_{\bar{\tau}}[\check{\psi}] \in \mathcal{H}_{\bar{\tau}}$ defined by

$$\mathcal{F}_{\bar{\tau}}[\check{\psi}](\alpha) := \check{\psi}(\partial_{\bar{\alpha}})(1 - \bar{\alpha}\alpha)^{-\bar{\tau}}|_{\bar{\alpha}=0}. \tag{4.11}$$

It is easy to see that $\mathcal{F}_{\bar{\tau}}$ transforms monomials to monomials. Concretely,

$$\check{\psi}_M(x) := \frac{x^M}{(\bar{\tau})_M} \qquad \text{is mapped to} \qquad \mathcal{F}_{\bar{\tau}}[\check{\psi}_M](\alpha) = \alpha^M =: \psi_M(\alpha) . \tag{4.12}$$

The transform $\mathcal{F}$ is designed to relate the scalar product (4.5) with pairing (4.10) in the sense that

$$\langle \mathcal{F}_{\bar{\tau}}[\check{\psi}_1], \psi_2 \rangle_{\bar{\tau}} = (\check{\psi}_1, \psi_2) . \tag{4.13}$$

Hence, pulling back the invariant scalar product on $\mathcal{H}_{\bar{\tau}}$ to an invariant product on $\check{\mathcal{H}}_{\bar{\tau}}$ via $\mathcal{F}_{\bar{\tau}}$ gives

$$\langle \check{\psi}_1, \check{\psi}_2 \rangle_{\bar{\tau}} := \langle \mathcal{F}_{\bar{\tau}}[\check{\psi}_1], \mathcal{F}_{\bar{\tau}}[\check{\psi}_2] \rangle_{\bar{\tau}} = (\check{\psi}_1, \mathcal{F}[\check{\psi}_2]) = \check{\psi}_1(\partial_{\bar{\alpha}})\check{\psi}_2(\partial_\alpha)(1 - \bar{\alpha}\alpha)^{-\bar{\tau}}|_{\bar{\alpha},\alpha=0} . \tag{4.14}$$

This concludes our brief discussion of $\mathfrak{su}(1,1)$ and the two highest-weight representations $\check{\mathcal{H}}_{\bar{\tau}}$ and $\mathcal{H}_{\bar{\tau}}$.

4.1.2 Minimal-twist descendants and their wave functions

Consider some primary field $\mathcal{O}$ with conformal weight and MST spin given by (Δ, J, κ). The twist τ of $\mathcal{O}$ is the quantity $\tau \equiv \Delta - J - \kappa$. We shall denote the associated space of descendants by $\mathcal{H}_{\mathcal{O}}$. Note that this space carries an irreducible representation of the conformal algebra. Upon restriction to the subalgebra generated by dilations and rotations, the space $\mathcal{H}_{\mathcal{O}}$ decomposes into an infinity of irreducible subrepresentations which we enumerate by some integer $i = 0, 1, \ldots$. Within each subrepresentation, the descendant fields possess well-defined conformal weight and spin. Hence we can also associate a twist τ_i to each of these subrepresentations. It is not difficult to see that the twists of descendant fields are bounded from below by the twist τ of the primary, i.e. $\tau = \tau_0 \leq \tau_i$. The subspace of states for which the twist τ_i assumes the minimal possible value $\tau_i = \tau$ is denoted by $\mathcal{H}_{\mathcal{O}}^\tau$. We shall refer to elements $\mathcal{H}_{\mathcal{O}}^\tau$ as minimal-twist states and to the associated fields as *minimal-twist fields*. By definition, all primary fields are minimal-twist in this sense. Note that this concept of minimal-twist fields is purely kinematical and it should not be confused with the dynamical concept

of lowest-twist primary. We also stress that the notion of minimal-twist fields makes no reference to general free fields, as opposed to the notions of double- and triple-twist fields we are going to study below. Our goal in this subsection is to analyze the space $\mathcal{H}_{\mathcal{O}}^{\tau}$ of minimal-twist states, at least for primary fields $\mathcal{O}$ with STT spin, i.e. for which $\kappa = 0$. We shall address the more general case in the fourth subsection below.

In order to do so, we shall describe our STT field $\mathcal{O} = \mathcal{O}(X, Z)$ as a homogeneous function in embedding space. Here, X and Z denote two orthogonal, lightlike vectors in embedding space $\mathbb{R}^{2,d}$, i.e. $X^2 = 0 = Z^2$ and $X \cdot Z = 0$. For a field $\mathcal{O}$ of twist τ, we demand the degree of homogeneity under simultaneous rescalings of X and Z to be given by given by $-\tau$,

$$\mathcal{O}(\varpi X, \varpi Z) = \varpi^{-\tau} \mathcal{O}(X, Z) \quad \text{for} \quad \varpi \in \mathbb{R}^* . \tag{4.15}$$

Such a field is primary of conformal weight Δ and spin J if and only if it satisfies the following condition

$$\mathcal{O}(\lambda^{-1}X + \alpha Z, \lambda Z) = \lambda^{\bar{\tau}} \mathcal{O}(X, Z), \tag{4.16}$$

where $\bar{\tau} := \Delta + J$. This property has an interpretation based on the subgroup of conformal transformations that preserve the null space $\text{Span}(P, Z) \subset \mathbb{R}^{2,d}$, namely

$$\mathfrak{su}(1, 1) = \langle \mathcal{T}_- = X\partial_Z, \mathcal{T}_0 = \frac{1}{2}(Z\partial_Z - X\partial_X), \mathcal{T}_+ = Z\partial_X \rangle. \tag{4.17}$$

The infinitesimal version of the eq. (4.16) that characterized the primary field is given by the two conditions $\mathcal{T}_- \mathcal{O} = 0$ and $\mathcal{T}_0 \mathcal{O} = \bar{h}\mathcal{O}$, where the eigenvalue $\bar{h}$ is given by $\bar{h} := \bar{\tau}/2 = \tau + J$. Since the operator $\mathcal{T}_-$ lowers $\bar{h}$ by one unit, the first conditions demand $\mathcal{O}$ to possess lowest weight. The corresponding lowest-weight representation is then spanned by the operators $\mathcal{T}_+^M \mathcal{O}$ with $M = 0, 1, \ldots, \infty$. To make contact with representation theory, we follow the approach of [49] and express states of the minimal-twist Hilbert space as

$$\mathcal{O}_{\check{\psi}}(X, Z) := \check{\psi}(\partial_{\bar{\alpha}})\mathcal{O}(X - \bar{\alpha}Z, Z)|_{\bar{\alpha}=0}, \tag{4.18}$$

where $\check{\psi}(x)$ is a power series in x. With this definition, the action of the generators $\mathcal{T}_a, a = 0, \pm,$ defined in eq. (4.17) can be written as

$$\mathcal{T}_a \mathcal{O}_{\check{\psi}}(X, Z) = \mathcal{O}_{\check{S}_a \check{\psi}}(X, Z) \tag{4.19}$$

where the operators $\check{S}_a, a = 0, \pm,$ were defined in eq. (4.7). So, in more mathematical terms we say that the prescription (4.18) defines as isomorphism between the space $\check{\mathcal{H}}_{\bar{\tau}}$ and the space of minimal-twist fields. This isomorphism intertwines the actions of $\mathfrak{su}(1, 1)$ on the two spaces.

Under the isomorphism between the highest-weight representation $\check{\mathcal{H}}_{\bar{\tau}}$ and the space $\mathcal{H}_{\mathcal{O}}^{\tau}$ of minimal-twist states, the scalar product is sent to the two-point function of the associated operators. More precisely one finds that

$$\langle \check{\psi}_1, \check{\psi}_2 \rangle_{\tau} = \check{\psi}_1(\partial_{\bar{\alpha}}) \check{\psi}_2(\partial_{\alpha}) \langle \mathcal{O}(X_1^{\star} - \bar{\alpha} Z_1^{\star}, Z_1^{\star}) \mathcal{O}(X_2^{\star} + \alpha Z_2^{\star}, Z_2^{\star}) \rangle \qquad (4.20)$$

where the two-point function on the right-hand side is to be computed in the gauge in which the embedding space coordinates of the two fields satisfy the additional conditions $X_1^{\star} \cdot X_2^{\star} = Z_1^{\star} \cdot Z_2^{\star} = 1$ as well as $X_i^{\star} \cdot Z_j^{\star} = 0$.

The transform $\mathcal{F}$ we defined in eq. (4.12) has a simple interpretation in terms of correlation functions in the conformal field theory[5]. In fact, one can check that

$$\psi(\alpha) = \mathcal{F}_{\bar{\tau}}(\check{\psi}) = \frac{\langle \mathcal{O}(X_1, Z_1) \mathcal{O}_{\check{\psi}}(X_2, Z_2) \rangle}{\langle \mathcal{O}(X_1, Z_1) \mathcal{O}(X_2, Z_2) \rangle} \quad \text{with} \quad \alpha := \frac{X_1 \cdot Z_2}{X_1 \cdot X_2}. \qquad (4.21)$$

Thereby we have now assembled all the background that is necessary to work with minimal-twist descendants and in particular to construct multi-twist primaries.

4.2 Triple-twist operators and their wave functions

Our goal now is to construct triple-twist primaries that can be constructed from the external scalar ϕ. We shall first apply the content of the previous subsection to construct double-twist primaries, which will then be useful to construct a "double-twist basis" for triple-twist primaries. In the absence of transverse spin, where the symmetry algebra reduces to $\mathfrak{su}(1,1)$, this basis has already appeared in several areas of the CFT literature, see e.g. [49, 50, 55, 56].

4.2.1 Double-twist primaries

Whereas the notion of minimal-twist fields and the associated structure we discussed in the previous subsubsection was entirely general, we shall focus on generalized free fields (GFF) from now on. Our first goal is to construct double-twist primaries.

Parameterization. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ to be two primaries of spin ℓ_1 and ℓ_2, respectively. Their twists will be denoted by $\tau_a = \Delta_a - \ell_a$. Double-twist operators $\mathcal{O}$ of weight Δ and spin J as well as their descendants can always be expressed as linear combinations of products of derivatives of $\mathcal{O}_1$ and $\mathcal{O}_2$,

$$(Z \partial_X)^M [\mathcal{O}_1 \mathcal{O}_2]_{0,J}(X, Z) = \check{\psi}_{L,M}(\partial_{\bar{\alpha}_1}, \partial_{\bar{\alpha}_2}) : \mathcal{O}_1(X - \bar{\alpha}_1 Z, Z) \mathcal{O}_2(X - \bar{\alpha}_2 Z, Z) : |_{\bar{\alpha}_a = 0}, \qquad (4.22)$$

$$= [\mathcal{O}_1 \mathcal{O}_2]_{\check{\psi}_{L,M}}(X, Z). \qquad (4.23)$$

---

[5]We thank Petr Kravchuk for introducing this parameterization to us.

Here, L is given by $L := J - \ell_1 - \ell_2$ and $\check{\psi}_{L,M}$ is a homogeneous polynomial of degree $L + M$ in its two arguments. The derivative $\partial_{\bar{\alpha}_a}$ acts as the operator $Z_a \partial_{X_a}$ on the field $\mathcal{O}_a$ and it increases both the spin and the conformal dimension by one. Similarly, the action of $(Z\partial_X)^M$ on the primary $[\mathcal{O}_1\mathcal{O}_2]_{0,J}$ does not change the twist, so the corresponding descendants remain at leading twist. Finally, the normal ordered product $:(-):$ that ensures finiteness of eq. (4.22) is equivalent to the identity-subtracted GFF OPE:

$$:\mathcal{O}_1(X_1)\mathcal{O}_2(X_2): := \mathcal{O}_1(X_1)\mathcal{O}_2(X_2) - \langle \mathcal{O}_1(X_1)\mathcal{O}_2(X_2)\rangle \mathbf{1}. \qquad (4.24)$$

In the leading-twist sector, the twist of the operator $\mathcal{O}$ is given by $\tau = \Delta - J = \tau_1 + \tau_2$. Since the action of the generators (4.17) on the operators defined in eq. (4.22) reduces to the action of the generators $\check{S}_{1a}(x_1, \partial_{x_1}) + \check{S}_{2a}(x_2, \partial_{x_2})$ on the polynomials $\check{\psi}(x_1, x_2)$, we deduce that the set of $\check{\psi}_{L,M}$ forms a basis of states in the tensor product of two $\mathfrak{su}(1,1)$ lowest-weight representations, each given by the minimal-twist Hilbert spaces of $\mathcal{O}_{1,2}$ and their descendants. Moreover, the condition that $[\mathcal{O}_1\mathcal{O}_2]_{0,J}$ be a primary translates to $\check{\psi}_{L,0}$ being lowest-weight vectors of weight $\bar{h} = h_1 + h_2 + J$.

In close analogy to eq. (4.21), we can express the dual polynomials $\psi_{L,0}$ in terms of the double-twist three-point function as

$$\psi_{L,0}\left(\frac{X_1 \cdot Z_a}{X_{1a}}, \frac{X_2 \cdot Z_a}{X_{2a}}\right) = \frac{\langle \mathcal{O}_1(X_1, Z_1)\mathcal{O}_2(X_2, Z_2)[\mathcal{O}_1\mathcal{O}_2]_{0,J}(X_a, Z_a)\rangle}{\langle \mathcal{O}_1(X_1, Z_1)\mathcal{O}_1(X_a, Z_a)\rangle \langle \mathcal{O}_2(X_2, Z_2)\mathcal{O}_2(X_a, Z_a)\rangle}, \qquad (4.25)$$

where $\psi_{L,0}$ is again a homogeneous polynomial of degree L in its two arguments. This generic form is obtained by inserting eq. (4.22) into the three-point function above, setting $(X, Z) = (X_a, Z_a)$. The action of the generators (4.17) on the eq. (4.25) reduces to the action of the generators $S_{1a}(\alpha_1, \partial_{\alpha_1}) + S_{2a}(\alpha_2, \partial_{\alpha_2})$ on $\psi(\alpha_1, \alpha_2)$. Now, the lowest-weight condition reads

$$\psi_{L,0}(\lambda\alpha_1 + \alpha_0, \lambda\alpha_2 + \alpha_0) = \lambda^L \psi_{L,0}(\alpha_1, \alpha_2) \qquad (4.26)$$

which can be easily solved through

$$\psi_{L,0}(\alpha_1, \alpha_2) = C_L^{(\bar{\tau}_1, \bar{\tau}_2)}(\alpha_1 - \alpha_2)^L = C_L^{(\bar{\tau}_1, \bar{\tau}_2)}\alpha_{12}^L \qquad (4.27)$$

The corresponding lowest-weight vector $\check{\psi}_{L,0}$ is obtained by applying the inverse of relation (4.12) to each monomial in α_1, α_2,

$$\check{\psi}_{L,0}^{(\bar{\tau}_1, \bar{\tau}_2)}(\partial_{\alpha_1}, \partial_{\alpha_2}) = \frac{C_L^{(\bar{\tau}_1, \bar{\tau}_2)}}{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}} \sum_{k=0}^{L} \frac{(-L)_k}{k!} \frac{\partial_{\alpha_1}^k}{(\bar{\tau}_1)_k} \frac{\partial_{\alpha_2}^{L-k}}{(\bar{\tau}_2)_{L-k}}. \qquad (4.28)$$

Here we have re-introduced the explicit dependence of $\check{\psi}_{L,0}$ on the lowest weights $\bar{\tau}_i$ via the superscript. In case the two fields $\mathcal{O}_1$ and $\mathcal{O}_2$ are identical we can implement permutation symmetry by demanding

additionally that the polynomials $\psi_{L,0}, \check{\psi}_{L,0}$ are symmetric with respect to an exchange of its two arguments. It is easy to check from the above expressions that this permutation of arguments acts as multiplication by $(-1)^L$ on the latter. This means that only double-twist operators with even L define a primary field in generalized free field theory.

Normalization. The constant prefactor $C_L^{(\bar{\tau}_1, \bar{\tau}_2)}$ that appeared in our formulas (4.27) and (4.28) may be determined from the normalization of the double-twist two-point function, i.e.

$$\langle [\mathcal{O}_1 \mathcal{O}_2]_{0,J}(X_1, Z_1)[\mathcal{O}_1 \mathcal{O}_2]_{0,J}(X_2, Z_2)\rangle = \frac{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}(-1)^L}{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}} \frac{H_{12}^J}{X_{12}^{\tau_1 + \tau_2 + 2J}}. \tag{4.29}$$

The second term, proportional to $(-1)^L$, comes from the extra Wick contraction between $\mathcal{O}_1$ and $\mathcal{O}_2$ in the two-point function. Starting with the case $\mathcal{O}_1 \neq \mathcal{O}_2$, we can insert eq. (4.22) to compute the desired two-point function from eq. (4.25). Thereby, we deduce that the normalization condition (4.29) of the two-point functions is equivalent to

$$(\check{\psi}_{L,0}^{(\bar{\tau}_1, \bar{\tau}_2)}, \psi_{L,0}^{(\bar{\tau}_1, \bar{\tau}_2)}) = \|\psi_{L,0}\|_{\bar{\tau}_1, \bar{\tau}_2}^2 = 1$$

In deriving this condition it is convenient to use variables X_i^*, Z_i^* that obey the additional gauge conditions $X_1^* X_2^* = Z_1^* Z_2^* = 1$ and $X_a^* Z_b^* = 0$, as before. We conclude that the coefficients C in eq. (4.28) read

$$C_L^{(\bar{\tau}_1, \bar{\tau}_2)} = \frac{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}(-1)^L}{\sqrt{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}}} \|\alpha_{12}^L\|_{\bar{\tau}_1, \bar{\tau}_2}^{-1} \tag{4.30}$$

with the relevant norm of α_{12}^L given by

$$\|\alpha_{12}^L\|_{\bar{\tau}_1, \bar{\tau}_2}^2 = \sum_{k=0}^{L} \frac{(-L)_k^2}{k!^2} \frac{k!}{(\bar{\tau}_1)_k} \frac{(L-k)!}{(\bar{\tau}_2)_{L-k}} = \frac{L!(\bar{\tau}_1 + \bar{\tau}_2 + L - 1)_L}{(\bar{\tau}_1)_L (\bar{\tau}_2)_L}. \tag{4.31}$$

The corresponding lowest-weight norm has been computed using the orthogonality of the monomial basis $\alpha_1^{k_1} \alpha_2^{k_2}$, along with eq. (4.6) for the norm of monomials. To extend this result to identical fields $\mathcal{O}_1 = \mathcal{O}_2$, we note that the three-point function of $\mathcal{O}_1, \mathcal{O}_2$, and $[\mathcal{O}_1 \mathcal{O}_2]$ in definition (4.25) of $\psi_{L,0}$ will also contain one extra Wick contraction when $\mathcal{O}_1 = \mathcal{O}_2$. Thus, in accordance with the normalization convention (4.29) for the two-point function of double-twist fields, the resulting value of the prefactor C coincides with the value we gave in eq. (4.31) for even L and it vanishes otherwise.

Now that we have completely determined the functions $\psi_{L,0}$ and thereby the GFF three-point function that appears on the right-hand side of eq. (4.25), we can compare with the general form of the spinning

three-point function expressed in an extended monomial basis

$$\langle \mathcal{O}_1(X_1, Z_1)\mathcal{O}_2(X_2, Z_2)\mathcal{O}_3(X_3, Z_3)\rangle = \Omega_{J_1 J_2 J_3}$$

$$\times \sum_{n_{12}, n_{23}, n_{31}} C^{(n_{12}, n_{23}, n_{31})}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} \left(\frac{H_{12}X_{31}X_{23}}{J_{1,23}J_{2,31}}\right)^{n_{12}} \left(\frac{H_{23}X_{12}X_{31}}{J_{2,31}J_{3,12}}\right)^{n_{23}} \left(\frac{H_{31}X_{23}X_{12}}{J_{3,12}J_{1,23}}\right)^{n_{31}},$$

where

$$\Omega_{J_1, J_2, J_3} = \frac{J_{1,23}^{J_1} J_{2,31}^{J_2} J_{3,12}^{J_3}}{X_{12}^{\frac{1}{2}(\bar{\tau}_1 + \bar{\tau}_2 - \tau_3)} X_{23}^{\frac{1}{2}(\bar{\tau}_2 + \bar{\tau}_3 - \tau_1)} X_{31}^{\frac{1}{2}(\bar{\tau}_3 + \bar{\tau}_1 - \tau_2)}} \ . \tag{4.32}$$

The comparison shows that the double-twist three-point function in eq. (4.25) has only a single non-vanishing three-point (as opposed to two-point) tensor structure, namely

$$\frac{J_{3,12}}{X_{13}X_{23}} = \alpha_{12} \ .$$

Consequently, we find that the only non-vanishing OPE coefficient appears when the three tensor structure labels n_{ij} assume the value $(n_{12}, n_{23}, n_{31}) = (0, \ell_2, \ell_1)$. For this special choice, the OPE coefficient is given by[6]

$$c^{(0, \ell_2, \ell_1)}_{\mathcal{O}_1 \mathcal{O}_2 [\mathcal{O}_1 \mathcal{O}_2]_{0,J}} = \frac{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}(-1)^{J - \ell_1 - \ell_2}}{\sqrt{1 + \delta_{\mathcal{O}_1, \mathcal{O}_2}}} \left\| \alpha_{12}^{J - \ell_1 - \ell_2} \right\|_{\bar{\tau}_1, \bar{\tau}_2}^{-1}. \tag{4.33}$$

In writing this result, we have expressed $L = J - \ell_1 - \ell_2$ in terms of the quantum numbers that appear on the left-hand side. The result is valid for identical and non-identical fields and it uses the norm $\left\| \alpha_{12}^L \right\|$ we computed in eq. (4.31).

4.2.2 Double-twist basis of triple-twist primaries

We now come to the main topic of this section, namely the construction of triple-twist operators in GFF. As we explained before, triple-twist operators in GFF are highly degenerate and hence there exists quite some freedom in introducing a basis. Here we shall make some particular choice that we shall refer to as the double-twist basis. This will turn out to be convenient in our discussion of the lightcone bootstrap below. We shall treat the cases of non-identical and identical scalars separately. Let us recall that for now we shall restrict to STT spins.

Non-identical scalars. Consider three non-identical scalars ϕ_1, ϕ_2, ϕ_3. For simplicity, we shall assume that they possess the same conformal weight Δ_ϕ. Their triple-twist primaries can be constructed from the two-fold iteration of eq. (4.22): first for $(\mathcal{O}_1, \mathcal{O}_2) = (\mathcal{O}_\ell, \phi_3)$, where $\mathcal{O}_\ell := [\phi_2 \phi_1]_{0,\ell}$, and then

---

[6]Throughout this work, we denote the OPE coefficients of GFF with lower case letters in order to distinguish them from the OPE coefficients in generic CFTs which are denoted by upper case letters.

for $(\mathcal{O}_1, \mathcal{O}_2) = (\phi_2, \phi_1)$. The result of this iteration is

$$[\phi_3 \mathcal{O}_\ell]_{0,J}(X, Z) = \check{\psi}_{J-\ell,0}(\partial_{\bar{\alpha}_1} + \partial_{\bar{\alpha}_2}, \partial_{\bar{\alpha}_3}) \check{\psi}_{\ell,0}(\partial_{\bar{\alpha}_1}, \partial_{\bar{\alpha}_2}) \phi_3(X - \bar{\alpha}_3 Z) \phi_2(X - \bar{\alpha}_2 Z) \phi_1(X - \bar{\alpha}_1 Z)|_{\bar{\alpha}_i = 0}. \tag{4.34}$$

On the other hand, the most general parameterization of a triple-twist operator at minimal twist is

$$[\phi_3 \phi_2 \phi_1]_{0,J}(X, Z) = \check{\Psi}_J(\partial_{\bar{\alpha}_1}, \partial_{\bar{\alpha}_2}, \partial_{\bar{\alpha}_3}) \phi_3(X - \bar{\alpha}_3 Z) \phi_2(X - \bar{\alpha}_2 Z) \phi_1(X - \bar{\alpha}_1 Z)|_{\bar{\alpha}_i = 0}, \tag{4.35}$$

where $\check{\Psi}$ is a lowest-weight vector of weight $3\Delta_\phi/2 + J$ in the triple tensor product of lowest-weight representations $\mathcal{H}_{\bar{\tau}}$ spanned by minimal-twist descendants of ϕ. Comparing the expressions in the previous two equations, we obtain the lowest-weight vector

$$\check{\Psi}_{\ell,J}^{(12)}(\partial_{\bar{\alpha}_i}) = \check{\psi}_{J-\ell,0}(\partial_{\bar{\alpha}_1} + \partial_{\bar{\alpha}_2}, \partial_{\bar{\alpha}_3}) \check{\psi}_{\ell,0}(\partial_{\bar{\alpha}_1}, \partial_{\bar{\alpha}_2}). \tag{4.36}$$

Here we placed a superscript (12) at the triple-twist polynomials $\check{\Psi}$ to stress they these are associated with a particular choice of basis that comes with the special iterative construction (4.34) of triple-twist operators, in which we form double-twist operators $\mathcal{O}_\ell$ of $\mathcal{O}_1 = \phi_1$ and $\mathcal{O}_2 = \phi_2$ first. We shall refer to the polynomials $\check{\Psi}_{\ell,J}^{(12)}$ as a *double-twist basis* of triple-twist operators. Note that the basis vectors of the double-twist basis are enumerated by the spin ℓ of the intermediate double-twist operator.

As in our discussion of minimal-twist and double-twist operators, we can pass from the polynomials $\check{\Psi}_J$ to a dual set of polynomials Ψ_J with the help of the triple-twist four-point function. The general prescription is

$$\Psi_J \left(\frac{X_1 \cdot Z_b}{X_{1b}}, \frac{X_2 \cdot Z_b}{X_{2b}}, \frac{X_3 \cdot Z_b}{X_{3b}} \right) := \frac{\langle \prod_{i=1}^{3} \phi_i(X_i)[\phi_3 \phi_2 \phi_1]_{0,J}(X_b, Z_b) \rangle}{\langle \prod_{i=1}^{3} \phi_i(X_i)\phi_i(X_b) \rangle}. \tag{4.37}$$

Homogeneity and gauge-invariance of the correlation function then imply that the dual state Ψ_J is a translation-invariant and homogeneous polynomial of degree J, i.e.

$$\Psi_J(\lambda \alpha_i + \alpha_0) = \lambda^J \Psi_J(\alpha_i) \Rightarrow \Psi_J(\alpha_1, \alpha_2, \alpha_3) = \alpha_{31}^J \, \Psi_J \left(0, \frac{\alpha_{12}}{\alpha_{13}}, 1 \right). \tag{4.38}$$

In Appendix C, we explicitly compute the exact form of these polynomials for our double-twist basis, by inserting eq. (4.36) into eq. (4.37). The final result of this calculation is

$$\Psi_{\ell,J}^{(12)}(\alpha_1, \alpha_2, \alpha_3) = \left\| g_{\ell,J}^{(12)} \right\|^{-1} g_{\ell,J}^{(12)}(\alpha_1, \alpha_2, \alpha_3), \tag{4.39}$$

with functions g of the form

$$g_{\ell,J}^{(12)}(\alpha_1, \alpha_2, \alpha_3) = \alpha_{21}^\ell \alpha_{31}^{J-\ell} {}_2F_1 \left[\begin{matrix} \ell - J, \Delta_\phi + \ell \\ 2\Delta_\phi + 2\ell \end{matrix} \right] \left(\frac{\alpha_{12}}{\alpha_{13}} \right). \tag{4.40}$$

In complete analogy with our discussion of double-twist operators, the normalization of the polynomial $\Psi_{\ell,J}^{(12)}$ which we displayed in eq. (4.39) can be read off from the normalization of the two-point functions of triple-twist operators. The relevant norm of $g_{\ell,J}^{(12)}$ can be given explicitly,

$$\left\| g_{\ell,J}^{(12)} \right\|^2 = \left\| \alpha_{12}^{\ell} \right\|_{\Delta_\phi,\Delta_\phi}^2 \left\| \alpha_{a3}^{J-\ell} \right\|_{2\Delta_\phi+2\ell,\Delta_\phi}^2. \tag{4.41}$$

The norm that appears on the right-hand side can be found in eq. (4.31). Before we conclude our discussion of non-identical scalars we note that any permutation $\sigma \in S_3$ of the three scalar fields gives rise to a double-twist basis by the same construction. The double-twist basis that is associated with the triple-twist operators $[\phi_{\sigma(3)}[\phi_{\sigma(2)}\phi_{\sigma(1)}]_\ell]$ is described by the polynomials

$$\Psi_{\ell,J}^{(\sigma(1)\sigma(2))}(\alpha_i) := \Psi_{\ell,J}^{(12)}(\alpha_{\sigma(i)}) \ .$$

Identical scalars. In the case of three identical scalars $\phi_{1,2,3} = \phi$, the permutation of scalar fields yields an equivalent basis. This is reflected by the extra Wick contractions in the four-point function (4.37) that define the wave functions Ψ_J, and that act as a symmetrizer in the case of the double-twist basis, such that

$$\Psi_{\ell,J}(\alpha_i) = \frac{1}{\sqrt{6}} \sum_{\sigma \in A_3} \Psi_{\ell,J}^{(\sigma(1)\sigma(2))}(\alpha_1,\alpha_2,\alpha_3). \tag{4.42}$$

Similarly, the extra Wick contractions in the two-point function of $[\phi\,[\phi\phi]_{0,\ell}]_{0,J}$ lead to the expression

$$\langle [\phi\,[\phi\phi]_{0,k}]_{0,J}(X_1^\star, Z_1^\star)[\phi\,[\phi\phi]_{0,\ell}]_{0,J}(X_2^\star, Z_2^\star)\rangle = \frac{1}{6} \sum_{\sigma \in S_3} \langle \Psi_{k,J}^{(12)}, \Psi_{\ell,J}^{(\sigma(1)\sigma(2))}\rangle. \tag{4.43}$$

Finally, imposing permutation symmetry of the triple-twist field and a normalization consistent with the two-point function, the dual polynomial in the double-twist basis may be seen to take the form

$$\check{\Psi}_{\ell,J}(\partial_{\alpha_i}) = \frac{1}{6\sqrt{6}} \sum_{\sigma \in S_3} \check{\psi}_{J-\ell,0}(\partial_{\bar{\alpha}_{\sigma(1)}} + \partial_{\bar{\alpha}_{\sigma(2)}}, \partial_{\bar{\alpha}_{\sigma(3)}})\check{\psi}_{\ell,0}(\partial_{\bar{\alpha}_{\sigma(1)}}, \partial_{\bar{\alpha}_{\sigma(2)}}) \ . \tag{4.44}$$

In summary, considering three identical scalars imposes a complete symmetrization on wave functions. In addition, the two-point function of triple-twist operators contains a projector onto the permutation-symmetric subspace, very much in the same way as in our discussion of double-twist operators for identical scalars, see in particular eq. (4.29). This projection also reduces the dimension of the space of triple-twist primaries as follows

$$\dim [\phi_3\phi_2\phi_1]_{0,J} = \begin{cases} J, & \phi_1 \neq \phi_2 \neq \phi_3, \\ J/2, & \phi_1 = \phi_2, \\ \left\lfloor \frac{J+2}{2} \right\rfloor - \left\lfloor \frac{J+2}{3} \right\rfloor, & \phi_1 = \phi_2 = \phi_3 \ . \end{cases} \tag{4.45}$$

The first line gives the number of independent triple-twist operators of spin J for three non-identical scalars. In the second line, we have displayed the result for an intermediate case in which just two of the three scalars are identical. The third line, finally, applies to three identical scalars, We note that for large values of the spin J the triple-twist degeneracy $\dim[\phi_3\phi_2\phi_1]_{0,J}$ becomes infinite. For three identical scalars, the degeneracy behaves as $J/6 + \mathcal{O}(1)$.

4.3 Generalization to arbitrary MST spin

The generalization of previous results to multi-twist operators with non-zero MST spin is tantamount to an extension of $\mathfrak{su}(1,1)$ to $\mathfrak{su}(1,2)$. This extension is straightforward, but leads to lengthier formulas due to the addition of two more raising operators that do not commute amongst themselves. To avoid these technicalities, we will restrict ourselves to a brief summary of results and refer to Appendix C.2 for the full derivations.

Minimal-twist descendants. In embedding space, MST primaries $\mathcal{O}(X,Z,W)$ are described in terms of three pairwise-orthogonal null vectors X,Z,W. While the first two vectors $X,Z \in \mathbb{R}^{2,d}$ are assumed to be real, components of the third vector $W \in \mathbb{C}^{d+2}$ can be complex. At fixed twist τ, the condition to be a primary of MST spin κ is now

$$\mathcal{O}(\lambda^{-1}X + \alpha Z + \beta W, \lambda\zeta^{-1}Z - \gamma W, \zeta W) = \lambda^{\bar{\tau}-\kappa}\zeta^{\kappa-J}\mathcal{O}(X,Z,W). \tag{4.46}$$

This is a lowest-weight condition for a representation of $\mathfrak{su}(1,2)$. The latter contains five new generators, in addition to the three generators of the subalgebra $\mathfrak{su}(1,1)$ we displayed in eq. (4.17). The additional generators take the form

$$X\partial_W, Z\partial_W, W\partial_W - Z\partial_Z, W\partial_Z, W\partial_X . \tag{4.47}$$

Since the Lie-algebra $\mathfrak{su}(1,2)$ includes three raising operators, its lowest-weight representations can be realized on a space of polynomials in three variables which we shall denote as α, β, γ. In complete analogy with eq. (4.18) we can parameterize minimal-twists descendants in terms of polynomials $\check{\psi}$,

$$\mathcal{O}_{\check{\psi}}(X,Z,W) = \check{\psi}(\partial_{\bar{\alpha}}, \partial_{\bar{\beta}}, \partial_{\bar{\gamma}})\mathcal{O}(X - \bar{\alpha}Z - \bar{\beta}W, Z + \bar{\gamma}W, W)|_{\bar{\alpha},\bar{\beta},\bar{\gamma}=0}. \tag{4.48}$$

Once again, we can pass to the dual wave function ψ by computing the two-point function of a primary with the descendent field $\mathcal{O}_{\check{\psi}}$, see eq. (4.18),

$$\psi\left(\frac{X_1 \cdot Z_2}{X_{12}}, \frac{X_1 \cdot W_2}{X_{12}}, \frac{X_1 \otimes Z_1 \cdot X_2 \wedge W_2}{H_{12}}\right) = \frac{\langle \mathcal{O}(X_1, Z_1, W_1)\mathcal{O}_{\check{\psi}}(X_2, Z_2, W_2)\rangle}{\langle \mathcal{O}(X_1, Z_1, W_1)\mathcal{O}(X_2, Z_2, W_2)\rangle} . \tag{4.49}$$

The duality relation between $\check{\psi}$ and ψ is given more explicitly by the following simple prescription that generalizes eq. (4.10),

$$\psi(\alpha, \beta, \gamma) = \check{\psi}(\partial_{\bar{\alpha}}, \partial_{\bar{\beta}}, \partial_{\bar{\gamma}})(1 - \bar{\alpha}\alpha - \bar{\beta}\beta)^{\kappa - \bar{\tau}}(1 + \bar{\gamma}\gamma - (\bar{\gamma}\bar{\alpha} - \bar{\beta})(\gamma\alpha - \beta))^{J - \kappa}|_{\bar{\alpha}, \bar{\beta}, \bar{\gamma} = 0}. \tag{4.50}$$

There exists an obvious pairing $(\cdot, \cdot)$ between a wave function $\check{\psi}$ and its dual ψ of the form

$$(\check{\psi}_1, \psi_2) = \check{\psi}_1(\partial_\alpha, \partial_\beta, \partial_\gamma)\psi_2(\alpha, \beta, \gamma)|_{\alpha, \beta, \gamma = 0} . \tag{4.51}$$

As in our previous discussion of STT representations, this pairing determines an $\mathfrak{su}(1, 2)$ invariant scalar product on lowest-weight modules and their duals through

$$\langle \check{\psi}_1, \check{\psi}_2 \rangle_{\bar{\tau} - \kappa, J - \kappa} = (\check{\psi}_1, \psi_2) = \langle \psi_1, \psi_2 \rangle_{\bar{\tau} - \kappa, J - \kappa} \tag{4.52}$$

The norms of monomials that are relevant for the applications of this paper are computed in Appendix C.2, see in particular eqs. (C.14) and (C.16).

Spinning double-twist primaries. For applications to triple-twist composites of scalar fields, we consider double-twist operators of the form $[\phi\mathcal{O}_\ell]_{0,J,\kappa}$, where $\mathcal{O}_\ell$ is an STT primary of STT spin ℓ. The double-twist operator can be expressed in terms of a tensor product of representations of $\mathfrak{su}(1, 2)$ as follows:

$$[\phi\mathcal{O}_\ell]_{0,J,\kappa}(X, Z, W) = \tag{4.53}$$

$$= \check{\psi}_{J-\ell,\kappa,0}(\partial_{\bar{\alpha}_a}, \partial_{\bar{\beta}_a}, \partial_{\bar{\gamma}_a}, \partial_{\bar{\alpha}_3}, \partial_{\bar{\beta}_3}) :\phi(X - \bar{\alpha}_3 Z - \bar{\beta}_3 W)\mathcal{O}_\ell(X - \bar{\alpha}_a Z - \bar{\beta}_a W, Z + \bar{\gamma}_a W):|_{\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i = 0}$$

The dual wave functions $\psi_{J-\ell,\kappa,0}(\alpha_a, \beta_a, \gamma_a, \alpha_3, \beta_3)$ can be computed from the double-twist three-point functions as, see also eq. (4.25),

$$\psi_{J-\ell,\kappa,0}\left(\frac{X_a \cdot Z_b}{X_{ab}}, \frac{X_a \cdot W_b}{X_{ab}}, \frac{X_a \otimes Z_a \cdot X_b \wedge W_b}{H_{ab}}, \frac{X_3 \cdot Z_b}{X_{3b}}, \frac{X_3 \cdot W_b}{X_{3b}}\right) = \frac{\langle \mathcal{O}(X_a, Z_a)\phi(X_3)[\phi\mathcal{O}_\ell]_{0,J,\kappa}(X_b, Z_b, W_b)\rangle}{\langle \mathcal{O}_\ell(X_a, Z_a)\mathcal{O}_\ell(X_b, Z_b)\rangle\langle\phi(X_1)\phi(X_b)\rangle}.$$

The lowest-weight condition (which follows from homogeneity and gauge-invariance of the three-point function) is now

$$\psi_{L,\kappa,0}(\lambda\alpha_1 + \alpha_0, \zeta\beta_1 + \beta_0 + \gamma_0\alpha_1, \lambda\alpha_2 + \alpha_0, \zeta\beta_2 + \beta_0 + \gamma_0\alpha_2, \lambda^{-1}\zeta\gamma_2 + \gamma_0) = \lambda^L \zeta^\kappa \psi_{L,\kappa,0}(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_2). \tag{4.54}$$

This condition fixes solutions up to a multiplicative constant, see eq. (4.27) for the related result in the case of STT representations,

$$\psi_{L,\kappa,0}(\alpha_a, \beta_a, \gamma_a, \alpha_3, \beta_3) = \psi_{L,\kappa,0}(0, 0, 0, 1, 1) \, \alpha_{a3}^{J-\ell}(\beta_{a3} - \gamma_a\alpha_{a3})^\kappa. \tag{4.55}$$

The multiplicative constant is determined by requiring that $\psi_{L,\kappa,0}$ be of unit norm, just as in our previous discussion. Thereby we find

$$\psi_{L,\kappa,0}(0,0,0,1,1) = \left\| \alpha_{12}^{J-\ell}(\beta_{12} - \gamma_2\alpha_{12})^\kappa \right\|^{-1}.$$

The square of the norm on the right-hand side can be expressed as a finite sum of the quantities we defined in eq. (4.31). The explicit expression is given by eq. (C.22) for $(L, \bar{\tau}_a, \bar{\tau}_3) = (J-\ell, 2\Delta_\phi+2\ell, \Delta_\phi)$. In the case $\mathcal{O}_\ell = [\phi\phi]_{0,\ell}$ and in the large-spin limit $J > \ell \gg 1$, it is easy to check that the terms labeled by $q > 0$ in eq. (C.22) are subleading and of relative order ℓ^{-q}, such that

$$\psi_{J-\ell,\kappa,0}(0,0,0,1,1) = \kappa!^{-1} \left\| \alpha_{a3}^{J-\ell} \right\|_{2\Delta_\phi+2\ell,\Delta_\phi+\kappa}^{-1} \left(1 + \mathcal{O}(\ell^{-1})\right). \tag{4.56}$$

Hence, in this limiting regime we can compute the prefactor of the wave function (4.55) by computing the norm (4.31) for $\alpha_{a3}^{J-\ell}$ instead of α_{12}^L.

Double-twist basis of triple-twist operators. Let us now finally construct the double-twist basis for triple-twist operators with arbitrary MST spin, the most general we can obtain from three external scalars. Following our reasoning in the discussion of STT spins, we can define MST triple-twist wave functions in terms of the corresponding triple-twist four-point function as

$$\Psi_{J,\kappa}\left(\frac{X_i \cdot Z_b}{X_{ib}}, \frac{X_i \cdot W_b}{X_{ib}}\right) = \frac{\langle \prod_{i=1}^3 \phi(X_i)[\phi\phi\phi]_{0,J,\kappa}(X_b, Z_b, W_b)\rangle}{\prod_{i=1}^3 \langle \phi(X_i)\phi(X_b)\rangle}. \tag{4.57}$$

As it stands, the formula holds for any triple-twist operator. If we want the wave function $\Psi_{J,\kappa}(\alpha_i, \beta_i)$ to describe a primary, we have to impose that it is a highest-weight vector in the tensor product of three $\kappa = 0$ representations of $\mathfrak{su}(1,2)$. This condition is equivalent to the homogeneity and gauge-invariance of the four-point function in eq. (4.57), just as in our previous discussion, see eq. (4.38),

$$\Psi_{J,\kappa}(\lambda\alpha_i + \alpha_0; \zeta\beta_i + \beta_0 + \gamma_0\alpha_i) = \lambda^J \zeta^\kappa \Psi_{J,\kappa}(\alpha_i; \beta_i). \tag{4.58}$$

The dependence in β_i is easy to solve and takes the form

$$\Psi_{J,\kappa}(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) = \alpha_{21}^{J-\kappa} \left(\alpha_{21}\beta_{31} - \alpha_{31}\beta_{21}\right)^\kappa \Psi_{J,\kappa}\left(0, 0, 1, 0, \frac{\alpha_{31}}{\alpha_{21}}, 1\right). \tag{4.59}$$

Using the properties (4.58) of the wave function Ψ, we can absorb the factor α_{21}^J into Ψ to obtain

$$\Psi_{J,\kappa}(\alpha_i; \beta_i) = \omega^\kappa \, \Psi_{J,\kappa}(\alpha_i; \beta_i^\star) \qquad with \quad \omega(\alpha_i; \beta_i) := \frac{(\alpha_{13}\beta_{12} - \alpha_{12}\beta_{13})^\kappa}{\alpha_{12}^\kappa} \tag{4.60}$$

and $(\beta_i^\star) = (\beta_1^\star, \beta_2^\star, \beta_3^\star) := (0, 0, 1)$. For the double-twist basis $[\phi\mathcal{O}_\ell]_{0,J,\kappa}$ with $\mathcal{O}_\ell := [\phi\phi]_{0,\ell}$ of triple-twist operators, the wave functions $\Psi_{\ell,J,\kappa}(\alpha_i; \beta_i)$ are obtained by inserting eqs. (4.53) and (4.59) into

eq. (4.57). The calculation is performed in Appendix C.2, with the result

$$\Psi_{\ell,J,\kappa}(\alpha_i;\beta_i) = \left\|g_{\ell,J,\kappa}^{(12)}\right\|^{-1} \sum_{\sigma \in S_3} g_{\ell,J,\kappa}^{(12)}(\alpha_i;\beta_i), \tag{4.61}$$

where

$$g_{\ell,J,\kappa}^{(12)}(\alpha_i;\beta_i) = \frac{(\alpha_{12}\beta_{13} - \alpha_{13}\beta_{12})^{\kappa}}{\alpha_{12}^{\kappa}} g_{\ell,J}^{(12)}(\alpha_i), \tag{4.62}$$

This formula expresses the wave function for the double-twist basis of MST triple-twist primaries for three identical scalars through the function $g_{\ell,J}^{(12)}$ we constructed in order to obtain STT triple-twist operators, see eq. (4.40). The norm that appears as a prefactor in eq. (4.61) is now given by

$$\left\|g_{\ell,J,\kappa}^{(12)}\right\|^2 = \left\|\alpha_{12}^\ell\right\|_{\Delta_\phi,\Delta_\phi}^2 \left\|\alpha_{a3}^{J-\ell}(\beta_{a3} - \gamma_a\alpha_{a3})\right\|_{2\Delta_\phi+2\ell,\ell,\Delta_\phi}^2. \tag{4.63}$$

Explicit formulas for the first and second factor on the right-hand side can be found in eqs. (4.31) and (C.22), respectively. Note that the non-normalized wave function at $\kappa > 0$ reduces to its STT counterpart, i.e.

$$g_{\ell,J,\kappa}^{(12)}(\alpha_1, 0, \alpha_2, 0, \alpha_3, 1) \equiv g_{\ell,J}^{(12)}(\alpha_i;\beta_i^\star) = g_{\ell,J}^{(12)}(\alpha_i) \tag{4.64}$$

However, the norm of the MST wave function does not reduce to the norm of the STT wave function in general. Nonetheless, at leading order in the large-spin limit $J > \ell \gg 1$, the identity (4.56) implies the simple shift relation

$$\left\|g_{\ell,J,\kappa}^{(12)}\right\|^2 = \left\|g_{\ell-\kappa,J-\kappa}^{(12)}\right\|^2\Big|_{\Delta_\phi \to \Delta_\phi + \kappa} \left(1 + \mathcal{O}(\ell^{-1})\right). \tag{4.65}$$

Since we are only going to analyze a few leading terms of the crossing symmetry equation in the lightcone limit, this behavior of the normalization will be sufficient for what comes below.

5 Triple-twist OPE Coefficients in Generalized Free Field Theory

In the previous subsection, we computed the four-point functions of the GFF triple-twist operators $[\phi[\phi\phi]_{0,\ell}]_{0,J,\kappa}$ in the double-twist basis, see eq. (4.39) for $\kappa = 0$ and eq. (4.61) for $\kappa > 0$. In this section, our goal is to expand the result in four-point blocks and to determine the associated OPE coefficients of the GFF. The latter are given in eq. (5.19). In the first subsection, we shall review some relevant background on triple-twist conformal blocks in the lightcone limit. We shall see that these are equivalent to the functions $g^{(12)}$ used before when we constructed the triple-twist wave functions. This observation will then allow us in the second subsection to compute the OPE coefficients directly from eq. (4.61). In the final subsection, we evaluate the triple-twist OPE coefficients explicitly in the large-spin limit.

5.1 Conformal blocks at GFF twists

The four-point functions with three scalar fields ϕ and one triple-twist insertion of the form $[\phi\phi\phi]_{0,J,\kappa}$ can be directly expanded in lightcone blocks. Indeed, since in GFF all scalar fields must be Wick-contracted with one of the constituents of the triple-twist operator, the correlation function does not exhibit any dependence in the distances X_{12}, X_{23}, X_{13}. The absence of these singularities implies that, among all the operators that appear in the OPE of any two of the scalar fields, only the leading double-twist families contribute to the triple-twist four-point function. It is exactly these terms that are captured by the lightcone blocks.

Four-point blocks for correlation functions of three scalar fields and one field in an MST representation depend on three cross-ratios which we shall denote by $z, \bar{z}$ and w. The cross-ratios $z, \bar{z}$ are constructed from the insertion points X_i of the four fields as usual. The third cross-ratio w involves the embedding space coordinate Z that is associated with the spin of the fourth field. Since the lightcone limit is obtained by sending $\bar{z}$ to zero, the desired lightcone blocks have a non-trivial dependence on z and w only. Before taking the lightcone limit, the blocks depend on three quantum numbers: the weight Δ_1 and spin J_1 of the intermediate field in the OPE of the scalar ϕ with itself, and the tensor structure label n at the non-trivial vertex. In the lightcone limit, $\Delta_1 = 2\Delta_\phi + J_1$ and it therefore suffices to label lightcone blocks by two quantum numbers J_1 and n. Before we state the result, we recall that the triple-twist fields $[\phi\phi\phi]_{0,J,\kappa}$ we insert at the fourth point are labeled by J, κ. We will only need the lightcone blocks for the tensor structure $n = J_1 - \kappa$. These blocks are given by

$$g_{J_1, n=J_1-\kappa}(z, w) = \left(\frac{z}{1-z}\right)^{\Delta_\phi} (1-w)^{-\kappa} \left(\frac{z(1-w)}{1-z}\right)^{J_1} {}_2F_1\left[\begin{matrix}\Delta_\phi + J_1, J_1 - J \\ 2\Delta_\phi + 2J_1\end{matrix}\right]\left(\frac{z(w-1)}{1-z}\right) . \quad (5.1)$$

This formula for the four-point lightcone blocks is derived in Appendix D.1. Our analysis there starts from the conformal block decomposition of the scalar six-point function in OPE cross-ratios,

$$\langle \phi(X_1)\ldots\phi(X_6)\rangle = \sum_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3;n_1,n_2} P^{(n_1 n_2)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \frac{g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1 n_2}(\bar{z}_1, z_1, \bar{z}_2, z_2, \Upsilon_0, \bar{z}_3, z_3, w_1, w_2)}{(X_{12}X_{34}X_{56})^{\Delta_\phi}(\bar{z}_2 z_2)^{\Delta_\phi/2}}. \quad (5.2)$$

The blocks of the scalar six-point functions that appear on the right-hand side depend on nine cross-ratios. First, by taking the OPE limit in the fields that are inserted at X_5 and X_6, we remove the two cross-ratios z_3 and $\bar{z}_3$. Next, the second OPE limit with the scalar field inserted at x_4 further removes the three cross-ratios $z_2, \bar{z}_2, \Upsilon_0$ and factorizes the monomial tensor structure $(1 - w_2)^{n_2}$. It is this second OPE that produces the triple-twist field we are after. Finally, we take the lightcone limit $\bar{z}_1 \to 0$, whose only effect is a factor of $\bar{z}_1^{\Delta_\phi}$ to account for double-twist exchange in the first leg. Hence, after this sequence of limits, the blocks depend only on two cross-ratios z_1 and w_1, while

the dependence on all other cross-ratios is reduced to power laws. The original six-point blocks are labeled by nine quantum numbers. We fix these to the values they take in our triple-twist four-point function:

$$(h_1, \bar{h}_1) = (\Delta_\phi, \Delta_\phi + J_1), \qquad (h_2, \bar{h}_2) = \left(\frac{3}{2}\Delta_\phi, \frac{3}{2}\Delta_\phi + J_2 + \kappa\right) \tag{5.3}$$

$$(h_3, \bar{h}_3) = (\Delta_\phi, \Delta_\phi + J_3), \qquad (n_1, n_2) = (J_1 - \kappa, J_3 - \kappa). \tag{5.4}$$

For this choice of quantum numbers of intermediate fields and tensor structures, one finds in the above limit

$$g_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2} \overset{\bar{z}_i, z_{2,3}, \Upsilon_0 \to 0}{\sim} (\bar{z}_2 z_2)^{\frac{3}{2}\Delta_\phi} z_2^{J_2 + \kappa} \Upsilon_0^\kappa (1 - w_2)^{J_3 - \kappa} (\bar{z}_3 z_3)^{h_3} z_3^{J_3} \bar{z}_1^{\Delta_\phi} g_{J_1, n_1 = J_1 - \kappa}(z_1, w_1), \tag{5.5}$$

where $g_{J_1, J_1 - \kappa}$ on the right-hand side is the function we introduced in eq. (5.1). We have thereby shown that eq. (5.1) gives the lightcone limit of four-point blocks of three scalars and one MST primary. Remarkably, our result (5.1) for the triple-twist lightcone block resembles the expressions for the triple-twist wave functions $g_{J_1, J_2, \kappa}^{(12)}(\alpha_i, \beta_i)$ derived in the previous subsection, see eqs. (4.62) and (4.40). Indeed, the two formulas agree provided one makes the identification

$$z_2 \equiv \alpha_1 - \alpha_3, \quad \frac{z_1(w_1 - 1)}{1 - z_1} \equiv \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}. \tag{5.6}$$

The result (5.5) and the identification (5.6) are derived in Appendix D.2. Let us stress, in particular, that the normalization of the $g^{(12)}$-functions matches the normalization of blocks. Indeed, in the (12) OPE limit where z_1 goes to zero, the lightcone blocks behave as

$$g_{J_1; J_1 - \kappa}(z_1, w_1) \overset{z_1 \to 0}{\sim} z_1^{\Delta_\phi} z_1^{J_1} (1 - w_1)^{J_1 - \kappa} + \dots. \tag{5.7}$$

On the other hand, according to the identification (5.6), the OPE limit corresponds to $\alpha_1 \to \alpha_2$. In this limit, the function $g^{(12)}$ behaves as

$$g_{J_1, J_2, \kappa}^{(12)}(\alpha_1, 0, \alpha_2, 0, \alpha_3, 1) \overset{\alpha_1 \to \alpha_2}{\sim} \alpha_{12}^{J_1} + \dots. \tag{5.8}$$

Inserting the identification rules (5.6) then confirms that the relation between blocks and $g^{(12)}$ is compatible with the canonical normalization (2.4) of conformal blocks.

In conclusion, we have shown that the conformal block decomposition for the GFF four-point function with one insertion of the triple-twist field $[\phi[\phi\phi]_{0,J_3}]_{0,J_2,\kappa}$ takes the form

$$\Psi_{J_3, J_2, \kappa}(\alpha_i, \beta_i) = \sum_{J_1=0}^{J_2} p_{J_1}^{J_3}(J_2, \kappa) \, g_{J_1, J_2, \kappa}^{(12)}(\alpha_i, \beta_i). \tag{5.9}$$

where

$$p_k^\ell(J, \kappa) := p_{[\phi\phi]_{0,k}}^{(\ell-\kappa)}(\ell, J, \kappa) := c_{\phi\phi[\phi\phi]_{0,k}} c_{[\phi\phi]_{0,k}\phi[\phi[\phi\phi]_{0,\ell}]_{0,J,\kappa}}^{(\ell-\kappa)} \tag{5.10}$$

is the product of GFF OPE coefficients in the triple-twist four-point function, and we have expressed the lightcone conformal blocks through the function g using the identification (5.6).

5.2 OPE coefficients in terms of Racah coefficients

In the previous subsection, we obtained the conformal block decomposition (5.9) of the triple-twist wave functions $\Psi_{J_1, J_2, \kappa}$. The blocks were expressed in terms of the same function $g_{J_1, J_2, \kappa}^{(12)}(\alpha_i)$ that we used to construct the triple-twist function in Subsections 4.2 and 4.3, see eq. (4.62). But our two formulas for the triple-twist wave function are not quite the same. On the one hand, equation (4.61) is the sum of a single g-polynomial with permuted arguments:

$$\Psi_{J_3, J_2, \kappa} = \frac{1}{\sqrt{6}} \left\| g_{J_3, J_2, \kappa}^{(12)} \right\|^{-1} \sum_{\sigma \in S_3} g_{J_3, J_2, \kappa}^{(\sigma(1)\sigma(2))} \quad \text{with} \quad g^{(\sigma(1)\sigma(2))}(\alpha_i, \beta_i) := g^{(12)}(\alpha_{\sigma(i)}, \beta_{\sigma(i)}) \tag{5.11}$$

The right-hand side, including the normalizing prefactor, was fully determined in the previous subsection. The block expansion (5.9), on the other hand, does not involve a summation over permutations and it contains some OPE coefficients that we have not computed yet. We will now fill that gap by bringing eq. (5.11) into the form of the expansion (5.9). This is achieved by expanding the polynomials $g_{J_3, J_2, \kappa}^{(\sigma(1)\sigma(2))}$ into a linear combination of polynomials $g_{J_1, J_2, \kappa}^{(12)}$, where J_1 denotes the spin of the double-twist field exchanged in the s-channel of the four-point function.

It will be convenient to express these coefficients in terms of the basis-change matrix between the orthonormal bases for $(\sigma(1)\sigma(2))$ and (12), which we define as

$$\Psi_{\ell, J, \kappa}^{(\sigma(1)\sigma(2))} = \sum_{k=0}^{J_2} W_{k\ell}^\sigma \Psi_{k, J, \kappa}^{(12)} \quad \text{for} \quad \sigma \in S_3. \tag{5.12}$$

The function $\Psi_{\ell, J, \kappa}^{(ij)}$ denotes the normalized g-function which we introduced in eq. (4.61) with a normalization that is given in eq. (4.63) and that is independent of i, j. We can also compute the coefficients W that appear on the right-hand side through the scalar product of wave functions as

$$W_{k\ell}^\sigma(J, \kappa) := \langle \Psi_{k, J, \kappa}^{(12)}, \Psi_{\ell, J, \kappa}^{(\sigma(1)\sigma(2))} \rangle \quad \text{for} \quad \sigma \in S_3. \tag{5.13}$$

Since they are defined as transformations between different choices of an orthonormal basis in the same space, the matrix elements $W_{k\ell}^\sigma$ of eq. (5.12) constitute an orthogonal representation of the permutation group S_3 on $\mathbb{R}^{J_2}$. In particular, the composition of permutations is mapped to a product of W matrices,

$$W_{km}^{\sigma\sigma'} = \sum_{\ell=0}^{J_2} W_{k\ell}^\sigma W_{\ell m}^{\sigma'}, \quad W_{k\ell}^{\sigma^{-1}} = W_{\ell k}^\sigma. \tag{5.14}$$

Using these relations, all permutation matrices can be expressed as products of the 2-cycle matrices $W^{(12)}$ and $W^{(13)}$, i.e.

$$W^{(23)} = W^{(12)}W^{(13)}W^{(12)}, \quad W^{(123)} = W^{(12)}W^{(13)}, \quad W^{(132)} = W^{(13)}W^{(12)}.$$

Moreover, the matrix $W^{(12)}$ is diagonal in our double-twist basis:

$$\sum_{k=0}^{J_2} W_{k\ell}^{(12)} \Psi_{k,J,\kappa}^{(12)} = \Psi_{\ell,J,\kappa}^{(21)} = (-1)^\ell \Psi_{\ell,J,\kappa}^{(12)} \implies W_{k\ell}^{(12)} = (-1)^\ell \delta_{k\ell}. \tag{5.15}$$

Hence, we can compute all the matrix elements W^σ for any choice of the permutation from the matrix elements of $W^{(13)}$. We shall often omit the superscript (13) when we are dealing with this special generating W-matrix, i.e. we shall write $W_{k\ell}^{(13)} \equiv W_{k\ell}$. The matrix elements $W_{k\ell}(J_2, \kappa)$ are often called *Racah coefficients* for $\mathfrak{su}(1,1)$ when $\kappa = 0$ and for $\mathfrak{su}(1,2)$ in case $\kappa > 0$. The Racah coefficients of $\mathfrak{su}(1,1)$ are known exactly and can be expressed in terms of $_4F_3(1)$ hypergeometric functions, see e.g. [56, eq. (36)]. In the large-spin limit $J_2 > J_1, J_3 \gg 1$, the formula $g_{J_1,J_2,\kappa}^{(12)}(\alpha_1, 0, \alpha_2, 0, \alpha_3, 1) = g_{J_1,J_2}^{(12)}(\alpha_1, \alpha_2, \alpha_3)$ combined with the shift relation (4.65) implies

$$W_{J_3 J_1}(J_2, \kappa) \sim W_{J_3 J_1}(J_2, 0)|_{\Delta_\phi \to \Delta_\phi + \kappa} . \tag{5.16}$$

In the next section, we will derive a formula for the large-spin asymptotics of the Racah coefficients from the functional form of six-point comb-channel lightcone blocks in that regime.

Coming back to the expansion (5.11) of the wave functions, we can now insert eq. (5.12) to obtain

$$\Psi_{J_3,J_2,\kappa} = \frac{1}{\sqrt{6}} \sum_{J_1=0}^{J_2} \left\{ \left\| g_{J_1}^{(12)} \right\|^{-1} \sum_{\sigma \in S_3} W_{J_1 J_3}^\sigma \right\} g_{J_1,J_2,\kappa}^{(12)}. \tag{5.17}$$

Note that here the summation over the permutation group only applies to the matrices W^σ. The sum is therefore proportional to the S_3 projection operator

$$\mathbb{S}_{k\ell} := \frac{1}{3!} \sum_{\sigma \in S_3} W_{k\ell}^\sigma(J, \kappa) = \frac{1}{3} \left(\Pi_{k\ell} + 2(\Pi W \Pi)_{k\ell} \right), \quad \text{with} \quad \Pi_{k\ell} = \frac{1 + (-1)^\ell}{2} \delta_{k\ell}. \tag{5.18}$$

In rewriting the sum over permutations we have used the relations of eq. (5.14) to express all six terms through $W^{(12)}$ and $W^{(13)} = W$. Note that Π is simply the projector for the S_2 subgroup generated by $W^{(12)}$. Now, the only $g^{(ij)}$-function that appears in eq. (5.17) is $g^{(12)}$, just as in the case of the conformal block expansion (5.9). By comparing the two expansions, we therefore deduce that the GFF OPE coefficients of eq. (5.10) take the form

$$p_{J_1}^{J_3}(J_2, \kappa) = \sqrt{6} \left\| g_{J_1,J_2,\kappa}^{(12)} \right\|^{-1} \mathbb{S}_{J_1 J_3}(J_2, \kappa). \tag{5.19}$$

This concludes the derivation of the main result of this subsection.

5.3 Large-spin crossing kernel and OPE coefficients

In order to prepare for our analysis of the crossing equation, we finally want to determine the triple-twist OPE coefficients (5.19) of the six-point function in the limit of large spins. This requires, in particular, to determine the large-spin limit of the Racah coefficients $W_{k\ell}^{(13)}(J, \kappa)$. We shall first do so by combining some existing results in the literature. For convenience, we shall then sketch a second derivation that is based on the study of six-point blocks with GFF scaling dimensions. The results on the large-spin limit of the triple-twist OPE coefficients are collected at the end.

5.3.1 Hankel transform from large-spin Racah coefficients

The goal here is to evaluate the Racah matrix elements $W_{k\ell}^{(13)}(J, \kappa) \equiv W_{k\ell}(J, \kappa)$ which were defined by eq. (5.12), in the following large-spin limit

$$J, k, \ell \to \infty, \quad \frac{k\ell}{J} = \text{finite}. \tag{5.20}$$

To describe the limiting form of the Racah matrix elements, we first use the fact that to leading order in the limit (5.20), the dependence of W on the MST spin variable κ can simply be absorbed in a shift of the conformal weight,

$$W_{k\ell}^{(\Delta_\phi)}(J, \kappa) = W_{k\ell}^{(\Delta_\phi+\kappa)}(J, 0) \tag{5.21}$$

This property of the Racah matrix coefficients is explained below eq. (5.19). This reduces the study of the large-spin limit for the Racah coefficients of $\mathfrak{su}(1,2)$ to the well-studied case of $\mathfrak{su}(1,1)$. After defining $\nu := \Delta - 1 := \Delta_\phi + \kappa - 1$, we can now use the result in [53, eq. (5.1.9)] (see also [56, eq. (39)]) on the asymptotics of Racah coefficients in the intermediate regime

$$\ell = \tau J, \quad J \to \infty, \quad \tau \in [0, 1), \quad k \in \mathbb{Z}_+. \tag{5.22}$$

In [53, App. B], this regime is studied by direct computation of the scalar product in eq. (5.12). Equivalently, in [56, Sec. 3.5] (also in [55, App. A] for $\Delta = 2$), the asymptotics are derived from a scaling limit of the second-order recurrence relation satisfied by the Racah coefficients. This recurrence relation follows from the representation of the (13) Casimir operator of $\mathfrak{su}(1,1)$ as a tridiagonal matrix in the orthonormal eigenbasis $\Psi_{k,J}^{(12)}$ of the corresponding (12) Casimir operator, see e.g. [56, eq. (38)]. The result is a second-order differential equation in $\tau = k/J$ with polynomial solution

$$W_{k,\tau J}^{(13)} \sim (-1)^J \sqrt{\frac{2\tau}{J}} \left(\tau^2(1 - \tau^2)\right)^{\frac{\nu}{2}} \left(\frac{\Gamma_{k+2\nu+1}(2k + \nu + 1)}{k! \, \Gamma_{\nu+k+1}^2}\right)^{\frac{1}{2}} P_k^{(\nu,\nu)}(1 - 2\tau^2), \tag{5.23}$$

where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials. The multiplicative prefactor ensures that the functions $W_{k(\tau J)}^{(13)}$, when seen as polynomials in τ, are unit-normalized with respect to the Jacobi polynomials'

scalar product, which is the limiting form of the $\mathfrak{su}(1,1)$-invariant scalar product in the orthonormal basis $\psi_{\tau J}^{(23)}$. We can now retrieve the large-spin limit (5.20) we are interested in by setting $\tau = \mathcal{O}(k^{-1})$ and taking $k \to \infty$. In this limit, the leading asymptotics of the Jacobi polynomials are given by a Mehler-Heine type formula [60, Thm. 8.1.1]:

$$P_k^{(\alpha,\beta)}\left(1 - \frac{z^2}{2k^2}\right) \overset{k\to\infty}{\sim} \left(\frac{2k}{z}\right)^\alpha J_\alpha(z),$$ (5.24)

where $J_\nu(z)$ is the Bessel function of the first kind. Setting $\alpha = \nu = \Delta - 1$ and $z = 2k\tau = 2k\ell/J$, we obtain

$$W_{k\ell}^{(\Delta)}(J,\kappa) = (-1)^J \sqrt{\frac{2}{J}} \sqrt{\frac{2k\ell}{J}} J_{\Delta+\kappa-1}\left(2\frac{k\ell}{J}\right),$$ (5.25)

Note that $W_{k\ell}$ in eq. (5.25) is the kernel of an integral transform named after Hankel that has been well explored in the theory of special functions, see e.g. [61, § 10.22(v)]:

$$\mathcal{H}_\nu[f](y) := \int_0^\infty \mathrm{d}x \, \sqrt{xy} J_\nu(xy) f(x),$$ (5.26)

It is well known that the Hankel transform $\mathcal{H}_\nu$ squares to the identity, i.e. it satisfies $\mathcal{H}_\nu \circ \mathcal{H}_\nu = 1$. This ensures that the limiting form of the Racah coefficients in eq. (5.25) continues to define an orthogonal representation of the S_2 generated by the permutation (13).

5.3.2 Six-point crossing kernel

To define the action of the integral transform, we first rewrite the product of blocks and GFF OPE coefficients at leading order as

$$P_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(J_1,J_3)} g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3; J_1 J_3}(u_i,v_i,\mathcal{U}_i,\mathcal{U}^6) = \omega(u_i,v_i)^{h_\phi} \frac{J_2^{2\Delta_\phi-1}}{\kappa!} \frac{32}{\Gamma_{\Delta_\phi}^3} \delta(J_1 - J_3) f_{J_1(J_2,\kappa)J_3}(\mathcal{U}_0,v_2;\mathcal{U}_1,\mathcal{U}_2),$$ (5.27)

where the prefactor is given by $\omega(u_i,v_i) = u_1^2 u_2^3 u_3^2 v_1 v_2^2 v_3$. In addition, we have introduced a new name for the cross-ratio $\mathcal{U}_0 := \mathcal{U}^6/(\mathcal{U}_1\mathcal{U}_2)$ that appears here and will do so frequently in the next section. The conformal blocks in GFF normalization, i.e. with the GFF OPE coefficients are finally given by

$$f_{J_1(J_2,\kappa)J_3}(\mathcal{U}_0,v_2;\mathcal{U}_1,\mathcal{U}_2) := (J_1 J_3)^{\frac{\Delta_\phi+\kappa-1}{2}} \mathcal{U}_0^{\Delta_\phi+\kappa} \mathcal{K}_{\Delta_\phi+\kappa}\left(J_1^2 \mathcal{U}_0\mathcal{U}_2 + J_2^2 v_2 \mathcal{U}_0\mathcal{U}_1\mathcal{U}_2 + J_3^2 \mathcal{U}_0\mathcal{U}_1\right).$$ (5.28)

The action of the permutation $\sigma = (13)(64)$ on cross-ratios follows from its action on the two-point invariants, $\sigma : X_{ij} \mapsto X_{\sigma(i)\sigma(j)}$. The result was displayed in eq. (5.29). From there, it is easy to show that σ leaves the prefactor $\omega(u_i,v_i)$ invariant and transforms the remaining cross-ratios in the argument of $f_{J_1(J_2,\kappa)J_3}$ as

$$\sigma : (\mathcal{U}_0, v_2, \mathcal{U}_1, \mathcal{U}_2) \mapsto (v_2, \mathcal{U}_0, \mathcal{U}_2, \mathcal{U}_1).$$ (5.29)

At the level of the GFF-normalized conformal blocks $f_{J_1(J_2,\kappa)J_3}(\mathcal{U}_0, v_2; \mathcal{U}_1, \mathcal{U}_2)$, this permutation of cross-ratios is realized by the following integral identity:

$$\int_0^\infty \mathrm{d}J_1' \, W_{J_1 J_1'}^{(\Delta_\phi)}(J_2, \kappa) \int_0^\infty \mathrm{d}J_3' \, W_{J_3 J_3'}^{(\Delta_\phi)}(J_2, \kappa) f_{J_1'(J_2,\kappa)J_3'}(\mathcal{U}_0, v_2; \mathcal{U}_1, \mathcal{U}_2) = f_{J_1(J_2,\kappa)J_3}(v_2, \mathcal{U}_0; \mathcal{U}_2, \mathcal{U}_1),$$
(5.30)

where $W_{k\ell}^{(\Delta)}(J, \kappa)$ are given by eq. (5.25). This integral identity is derived in appendix E. We conclude that the product of Racah coefficients coincides with the six-point crossing kernel for the permutation $\sigma = (13)(64)$. In particular, since $\mathcal{H}_\nu \circ \mathcal{H}_\nu = 1$, our expression for the crossing kernel provides an orthogonal representation of the permutation σ, which is a product of disjoint two-cycles that squares to the identity. More specifically:

$$\int_0^\infty \mathrm{d}\ell \, W_{k\ell}^{(\Delta_\phi)}(J_2, \kappa) W_{\ell m}^{(\Delta_\phi)}(J_2, \kappa) = \delta(k - m).$$
(5.31)

Let us note that the integral identity (5.30) only determines $W_{kl}^{(\Delta_\phi)}(J_2, \kappa)$ up to a sign. As a result, the non-analytic prefactor $(-1)^{J_2}$ in our formula (5.25) for the large spin limit of the Racah coefficients is invisible to the six-point crossing property we studied here.

5.3.3 Lightcone limits and large-spin OPE coefficients

To find the limiting form of the GFF OPE coefficients in eq. (5.19) relevant to the lightcone bootstrap, it is useful to review how it arises from the lightcone limit of the GFF six-point function.

Subleading contributions to the lightcone limit of the correlator correspond *either* to individual subleading contributions to the crossed-channel large-spin OPE coefficients, *or* to families of individually leading terms whose overall asymptotics are dampened by relative phases upon summation. This fact is well known from the four-point bootstrap (see e.g. [62, App. B]), where the sign factor $(-1)^J$ relates the contributions of t-channel identity exchange and u-channel identity exchange in the s-channel OPE coefficients. While the subtlety is avoided in four-point functions of identical operators by the constraint that J has to be an even integer, there is no simple analog of this constraint in the case of six-point functions[7].

To see the damping effect of relative phases in action, let us now discuss the lightcone limit of the GFF six-point function of the defining scalar field in more detail. The correlator consists of a sum over permutations $\sigma \in S_6$ of Wick contractions $\left(X_{\sigma(1)\sigma(2)} X_{\sigma(3)\sigma(4)} X_{\sigma(5)\sigma(6)}\right)^{-\Delta_\phi}$. By dividing out the prefactor Ω_{CC} in eq. (2.20) and applying the lightcone limit $\bar{z}_2 \to 0$, we observe at leading order that triple-twist operators are only exchanged in the crossed-channel decompositions of the six Wick

---

[7]The analogous constraint could only be expressed in an orthonormal eigenbasis of the S_3 projection operator in eq. (5.18), of which we do not know any examples that are analytically tractable.

contractions $\left(X_{\sigma(1)6}X_{\sigma(2)5}X_{\sigma(3)4}\right)^{-\Delta_\phi}$ with $\sigma \in S_3$. To obtain the OPE coefficients that reproduce these terms in the crossed channel, recall that the triple-twist two-point function is normalized by

$$\langle[[\phi\phi]_{0,k}\phi]_{0,J_2,\kappa}(P_1^\star,Z_1^\star,W_1^\star)[\phi[\phi\phi]_{0,\ell}]_{0,J_2,\kappa}(P_2^\star,Z_2^\star,W_2^\star)\rangle = \mathbb{S}_{k\ell}, \tag{5.32}$$

where $\mathbb{S}$ is the S_3 projection operator defined in eq. (5.18). This implies that the product of OPE coefficients in the crossed-channel decomposition of the correlator corresponding to $\tau_2 = 3\Delta_\phi + J_2 + \kappa$ exchanges is given by

$$p_{J_1(J_2,\kappa)J_3} = \sum_{k,\ell} p_{J_1}^k \mathbb{S}_{k\ell} p_{J_3}^\ell = 6\left\|g_{J_1,J_2,\kappa}^{(12)}\right\|^{-1}\left\|g_{J_3,J_2,\kappa}^{(12)}\right\|^{-1}\mathbb{S}_{J_1 J_3}. \tag{5.33}$$

By tracking the contribution of separate Wick contractions (or by comparison with the case of three non-identical scalars), we can decompose the six-point OPE coefficients into a sum of terms

$$p_{J_1(J_2,\kappa)J_3} = \sum_{\sigma \in S_3} p_{J_1(J_2,\kappa)J_3}^\sigma, \quad p_{J_1(J_2,\kappa)J_3}^\sigma := \left\|g_{J_1,J_2,\kappa}^{(12)}\right\|^{-1}\left\|g_{J_3,J_2,\kappa}^{(12)}\right\|^{-1}W_{J_1 J_3}^\sigma, \tag{5.34}$$

such that

$$\lim_{\bar{z}_2 \to 0} \sum_{J_1,J_2,\kappa,J_3} p_{J_1(J_2,\kappa)J_3}^\sigma \frac{g_{(h_i,J_i,\kappa;J_1-\kappa,J_3-\kappa)}^{\mathrm{CC}}(\bar{z}_1,z_1,\bar{z}_2,z_2,\Upsilon_0,\bar{z}_3,z_3,w_1,w_2)}{(X_{12}X_{34}X_{56})^{\Delta_\phi}(\bar{z}_2 z_2)^{\Delta_\phi/2}} = \left(X_{\sigma(1)6}X_{\sigma(2)5}X_{\sigma(3)4}\right)^{-\Delta_\phi}.$$

With this equation, the stage is set to make the damping effect that we would like to expose manifest. To do so, let us now take the full lightcone limit $LCL^{(16,34)}$. The only leading Wick contraction in this limit is $(X_{16}X_{25}X_{34})^{-\Delta_\phi}$. Thus, only the contribution $p_{J_1(J_2,\kappa)J_3}^{(1)}$ (proportional to $W_{J_1 J_3}^{(1)} = \delta_{J_1 J_3}$) survives in the decomposition (5.34). But note that the other contributions $p_{J_1(J_2,\kappa)J_3}^\sigma$ only differ from $p_{J_1(J_2,\kappa)J_3}^{(1)}$ by $\mathcal{O}(1)$ factors, namely components of the orthogonal matrices W^σ! Hence, at the level of individual summands, they should be just as leading as $p_{J_1(J_2,\kappa)J_3}^{(1)}$. The only explanation for their absence in the limit is therefore the cancellation of these individual contributions between themselves, which is the effect that we set out to highlight at the beginning of this section.

With the qualitative goal of the section, namely the exposition of cancellations between OPE coefficients in the large-spin limit, achieved, let us now move forward to its quantitative goal and determine the concrete asymptotic form of the contribution that survives the cancellation. Following the analysis of Section 3.3, the crossed-channel sum localizes according to the large-spin limit (3.27). The asymptotics of the coefficients $p_{J_1(J_2,\kappa)J_3}^{(1)}$ in this limit follow from the large-spin asymptotics of the norm:

$$\left\|g_{J_1,J_2,\kappa}^{(12)}\right\|^2 \overset{J_1,J_2\to\infty}{\sim} \Gamma(\Delta_\phi)^3 \, \kappa! \, \pi^{-\frac{1}{2}} 2^{2J_2+4J_1+\kappa+7\Delta_\phi-4} \, e^{-2J_1} J_2^{1-2J_1-\kappa-3\Delta_\phi} J_1^{1+2J_1-2\kappa}. \tag{5.35}$$

Note that, from the six-point OPE coefficients $p^{(1)}_{J_1(J_2,\kappa)J_3}$, we can also deduce the limiting form of the four-point OPE coefficients $p^{\ell}_{J_1}(J_2,\kappa)$ implied by the lightcone limit $LCL^{(16,34)}$. Since relation (5.33) reduces to $\sum_{\ell} p^{\ell}_{J_1} p^{\ell}_{J_3} = p_{J_1(J_2,\kappa)J_3}$, we deduce that $p^{\ell}_{J_i}$ must be proportional to the matrix elements of a square-root of the identity matrix. Moreover, the crossing kernel $W^{(13)}_{J_1 J'_1}(J_2,\kappa) W^{(13)}_{J_3 J'_3}(J_2,\kappa)$ must map the CC OPE coefficients $p^{\ell}_{J_i}$ to the CC$'$ OPE coefficients $(p')^{\ell'}_{J'_i}$ while preserving the six-point OPE coefficients $p_{J_1(J_2,\kappa)J_3}$. As a result, the triple-twist OPE coefficients of the crossed channel CC and the dual crossed channel CC$'$ reduce to

$$p^{\ell}_{J_i}(J_2,\kappa) \sim \left\| g^{(12)}_{J_i,J_2,\kappa} \right\|^{-1} \delta(J_i - \ell) , \qquad (p')^{\ell'}_{J'_i}(J_2,\kappa) \sim \left\| g^{(12)}_{J'_i,J_2,\kappa} \right\|^{-1} W_{J_i \ell}(J_2,\kappa), \qquad (5.36)$$

where the large-spin limit of the norm is given by eq. (5.35) and the limiting behavior of the Racah coefficient can be found in eq. (5.25).

6 Triple-twist CFT Data from Six-Point Crossing Equation

We are finally prepared to study triple-twist operators in general CFTs, following our exposition in Section 2. Concretely, we begin in Section 6.1 by analyzing the leading direct-channel contribution that stems from the exchange of two identity fields. We will be able to reconstruct this term through crossed-channel contributions. As in the four-point lightcone bootstrap, this analysis puts strong constraints on the large-spin limit of the OPE coefficients, essentially implying that the large-spin limit of triple-twist OPE coefficients approaches their GFF counterparts. After completing and discussing the analysis of the leading term, we turn our attention to the two subleading terms in which one identity field gets exchanged in the direct channel. These subleading terms contain logarithms of two cross-ratios which we shall interpret in terms of triple-twist anomalous dimensions. Since triple-twist operators are highly degenerate (infinitely degenerate in the large-spin limit), their anomalous dimensions are encoded in a matrix (resp. operator in the large-spin limit).

We determine the relevant operator in Section 6.2, see eq. (6.19). This formula is the main result of our work. Some special eigenfunctions and eigenvalues are discussed at the end of this section, leaving a more systematic study of the eigenvalue problem for a future publication.

The direct-channel contributions in the crossing equation we are about to study were spelled out in Section 2.4, see eq. (2.26). Our goal is to match these by summing up crossed-channel contributions,

$$\frac{1 + \frac{C^2_{\phi\phi\mathcal{O}_\star}}{B_{\bar{h}_\star}}(\mathcal{U}^{h_\star}_0 + v^{h_\star}_2) \log u_2^{-1} + \mathcal{O}(\epsilon^{h_\star}_{16} \epsilon^{h_\star}_{34})}{(v_2 \mathcal{U}_0 \mathcal{U}_1 \mathcal{U}_2)^{\Delta_\phi}} = \sum_{\substack{h_i, J_i, n_i \\ \kappa, \ell}} P^{(n_1)}_{\mathcal{O}_1 \mathcal{O}^{\ell}_2} P^{(n_2)}_{\mathcal{O}_3 \mathcal{O}^{\ell}_2} \omega^{-h_\phi} g^{(12)3(4(56))}_{(h_1,J_1)(h_2,J_2,\kappa)(h_3,J_3);n_1 n_2}. \qquad (6.1)$$

In comparison to the earlier expression of the direct-channel terms, we have now divided by the factor $\omega^{h_\phi} := (u_1^2 u_2^3 u_3^2 v_1 v_2^2 v_3)^{h_\phi}$ and expressed the ratio of the leg factors Ω through cross-ratios using eq. (2.1). Furthermore, we also split the OPE coefficients as

$$P^{(n_1,n_2)}_{\mathcal{O}_1 \mathcal{O}_2^\ell \mathcal{O}_3} = \sum_{\ell=0}^{J_2} P^{(n_1)}_{\mathcal{O}_1 \mathcal{O}_2^\ell} P^{(n_2)}_{\mathcal{O}_3 \mathcal{O}_2^\ell}, \qquad P^{(n)}_{\mathcal{O}_i \mathcal{O}_2^\ell} := C_{\phi\phi\mathcal{O}_i} C^{(n)}_{\mathcal{O}_i \phi \mathcal{O}_2^\ell}, \tag{6.2}$$

where ℓ denotes an extra label for the degeneracy of operators $\mathcal{O}_2^\ell$ with the same twist and spin (h_2, J_2, κ).

In Section 3.3, we established that, to the order we consider here and in all relevant lightcone limits $LCL^{(16,34)}$, $LCL^{(16)}$, or $LCL^{(34)}$, the action of the Casimir and vertex differential operators on the direct-channel contributions implies that the crossed-channel conformal block decomposition indeed localizes to exchanges of the double-twist operators $(\mathcal{O}_1, \mathcal{O}_3) = ([\phi\phi]_{0,J_1}, [\phi\phi]_{0,J_3})$ and some triple-twist operators $\mathcal{O}_2 = [\phi\phi\phi]^\Psi_{0,J_2,\kappa}$. At the same time, the tensor structures are fixed to the values $n_1 = J_1 - \kappa$, $n_2 = J_3 - \kappa$ and large spins.

Note that at this stage, the degeneracy label Ψ of the triple-twist operator is not yet determined, as it cannot be measured by any of the conformally invariant operators we considered. However, we saw in Section 4 that the degenerate triple-twist operators can be expanded in the basis of twofold double twists $[\phi[\phi\phi]_{0,\ell}]_{0,J_2,\kappa}$. We now use this basis to parameterize the operator exchanges of the middle leg in the crossed channel and determine the anomalous dimension matrix that lifts the degeneracy.

6.1 Leading order solution to crossing: triple-twist OPE coefficients

Focusing on the most leading contribution $f_{\mathbb{1},\phi,\mathbb{1}}$ which involves two identity exchanges, we see that the constraint $J_1 = J_3$ imposed by (3.24) implies that the product (6.2) of OPE coefficients at leading order has to be of the form

$$P^{(n_1,n_2)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} \sim \delta(J_1 - J_3) \, P^{(J_1-\kappa,J_1-\kappa)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_1}. \tag{6.3}$$

At leading order, the crossing equation therefore reduces to

$$\frac{1}{(v_2 \mathcal{U}^6)^{\Delta_\phi}} = \sum_{\kappa=0}^{\infty} \int dJ_1 dJ_2 dJ_3 \, \delta(J_1 - J_3) \, P^{(J_1-\kappa,J_1-\kappa)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_1} \mathcal{N}_{J_1(J_2,\kappa)J_1} \times \tag{6.4}$$

$$(\mathcal{U}_1 + \mathcal{U}_2 - 1)^\kappa \left(\frac{\mathcal{U}^6}{\mathcal{U}_1 \mathcal{U}_2} \right)^{\Delta_\phi + \kappa} \mathcal{K}_{\Delta_\phi + \kappa} \left(\left[J_1^2(\mathcal{U}_1^{-1} + \mathcal{U}_2^{-1}) + J_2^2 v_2 \right] \mathcal{U}^6 \right).$$

Here, we inserted our formula (3.36) for the CC lightcone blocks. The homogeneity of the direct-channel contribution on the left-hand side imposes that the OPE coefficients at large spin take the form

$$P^{(J_1-\kappa,J_1-\kappa)}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_1} \mathcal{N}_{J_1(J_2,\kappa)J_1} = J_1^{2(\Delta_\phi + \kappa)-1} J_2^{2\Delta_\phi - 1} b_\kappa, \tag{6.5}$$

for some sequence $(b_\kappa)_{\kappa=0}^\infty$ that still needs to be determined. We can then perform the double-integral over (J_1, J_2) with the help of the following integral identity,

$$\int_{\mathbb{R}_+^2} \frac{\mathrm{d}x\mathrm{d}y}{xy} x^{\Delta_\phi+\kappa} y^{\Delta_\phi} \mathcal{K}_{\Delta_\phi+\kappa}(x+y) = \frac{1}{2}\Gamma(\Delta_\phi)^2\,\Gamma(\Delta_\phi+\kappa). \tag{6.6}$$

After multiplying both sides of the crossing equation (6.4) by $(v_2\mathcal{U}^6)^{\Delta_\phi}$, we obtain the following condition on the unknown coefficients b_k:

$$1 = (\mathcal{U}_1+\mathcal{U}_2)^{-\Delta_\phi} \sum_{\kappa=0}^\infty \left(1 - \frac{1}{\mathcal{U}_1+\mathcal{U}_2}\right)^\kappa \Gamma(\Delta_\phi)^2\,\Gamma(\Delta_\phi+\kappa)\frac{b_\kappa}{8}. \tag{6.7}$$

Assuming $|\mathcal{U}_1+\mathcal{U}_2| > 1$, this equation can be solved by the binomial series, provided we choose b_κ as

$$b_\kappa = 8\,\frac{(\Delta_\phi)_\kappa}{\kappa!\,\Gamma(\Delta_\phi)^2\,\Gamma(\Delta_\phi+\kappa)}\;. \tag{6.8}$$

For these values of the coefficients b_κ, the OPE coefficients (6.5) must then take the form

$$P_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(J_1-\kappa,J_3-\kappa)} \sim \frac{J_1^{2(\Delta_\phi+\kappa)-1} J_2^{2\Delta_\phi-1}}{\kappa!} \frac{8}{\Gamma(\Delta_\phi)^3} \frac{\delta(J_1-J_3)}{\mathcal{N}_{J_1(J_2,\kappa)J_1}}. \tag{6.9}$$

This formula coincides with the asymptotics of the GFF OPE coefficients $p_{J_1(J_2,\kappa)J_3}^{\sigma=(1)}$ defined in eq. (5.34) if and only if

$$\lim_{J_1\to\infty} \lim_{J_2\to\infty} \frac{J_1^{2(\Delta_\phi+\kappa)-1} J_2^{2\Delta_\phi-1}}{\kappa!} \frac{8}{\Gamma_{\Delta_\phi}^3} \frac{\left\|g_{J_1,J_2,\kappa}^{(12)}\right\|^2}{\mathcal{N}_{J_1(J_2,\kappa)J_1}} = 1. \tag{6.10}$$

Using the explicit formulas for the denominator $\mathcal{N}$ in eq. (3.37) and the numerator $\|g\|^2$ in eq. (5.35), one can check that this identity is indeed satisfied. Repeating the derivation of Section 5.3.3, we can then deduce the asymptotics of the four-point OPE coefficients. In summary, at leading order in the lightcone limit $LCL^{(16),(34)}$, the solution to crossing coincides with the GFF triple-twist OPE coefficients (5.36), i.e.

$$P_{\mathcal{O}_1\mathcal{O}_2^\ell}^{(n)} = C_{\phi\phi[\phi\phi]_{0,J_1}} C_{[\phi\phi]_{0,J_1}\phi[\phi[\phi\phi]_{0,\ell}]_{0,J_2}}^{(n)} \;\sim\; \delta_{n,J_1-\kappa}\, p_{J_1}^\ell(J_2,\kappa). \tag{6.11}$$

To retrieve the full GFF OPE coefficients (5.36) from the above asymptotics induced by the lightcone limit, we must symmetrize by the S_3 projector $\mathbb{S}_{k\ell}$ of eq. (5.18). More generally, we conjecture that a decomposition of the form $P = \sum_\sigma P^\sigma$ with $P^\sigma = P^{(1)} W^\sigma$ continues to hold for general CFTs at finite spin. This would be a natural generalization of double-twist OPE coefficients for the four-point function, where the decomposition $P_\mathcal{O} = P_\mathcal{O}^t + (-1)^{J_\mathcal{O}} P_\mathcal{O}^u$ of s-channel OPE coefficients can be defined non-perturbatively in terms of double-discontinuities of the four-point function [40].

6.2 Crossing at NLO: triple-twist anomalous dimensions

We now analyze the subleading contributions of the direct channel, see eq. (6.4). Note that the two terms that arise from the exchange of a single identity field in the direct channel both contain a factor of $\log u_2$. The same type of logarithms also appears in the subleading terms of the four-point lightcone bootstrap where they were traced back to anomalous contributions to the conformal weights of the intermediate fields. The same is true for the logarithmic contributions in the direct channel of the six-point lightcone bootstrap. We shall first discuss this for the term of the form $\mathcal{U}_0^{h_\star} \log u_2^{-1}$ in the first part of the analysis before we turn to the second logarithmic term in the second part.

6.2.1 Diagonal contributions to the anomalous dimension matrix

In the crossed channel CC, the direct-channel term proportional to $\mathcal{U}_0^{h_\star} \log u_2^{-1}$ arises from the anomalous dimensions of the double-twist operators $\mathcal{O}_\ell = [\phi\phi]_{0,\ell}$ that constitute the triple-twist operators $\mathcal{O}_2^\ell = [\phi\mathcal{O}_\ell]_{0,J_2,\kappa}$. Indeed, recall that the large-spin expansion of double-twist anomalous dimensions from the four-point bootstrap takes the form

$$h_{\mathcal{O}_\ell} = \Delta_\phi + \frac{\gamma_0}{2}\ell^{-2h_\star} + \mathcal{O}(\ell^{-2h_\star-1}) + \mathcal{O}(\ell^{-2h_>}),\tag{6.12}$$

where the constant γ_0 is given by

$$\gamma_0 := -2\frac{C_{\phi\phi\mathcal{O}_\star}^2}{\mathrm{B}_{\bar{h}_\star}}\frac{\Gamma_{\Delta_\phi}^2}{\Gamma_{\Delta_\phi-h_\star}^2},\tag{6.13}$$

and $2h_>$ is the twist of the next leading-twist operator in the OPE of ϕ with itself. If we then insert this expansion of $h_{\mathcal{O}_l}$ into the middle-leg twist $h_2 = h_\phi + h_{\mathcal{O}_\ell} + \dots$ of the crossed channel CC, we reproduce the direct channel by virtue of the relation

$$\frac{C_{\phi\phi\mathcal{O}_\star}^2}{\mathrm{B}_{\bar{h}_\star}}\frac{U^{h_\star}}{(UVX_1X_2)^{\Delta_\phi}} = \sum_{\kappa=0}^{\infty}\int \mathrm{d}^3J \frac{8}{\Gamma_{\Delta_\phi}^3}J_2^{2\Delta_\phi-1}\left(\frac{\gamma_0}{2J_1^{2h_\star}}\right)\delta(J_1-J_3)f_{J_1J_2J_3}(U,V;X_1,X_2),\tag{6.14}$$

where we used the shorthand $(U,V,X_1,X_2) = (\mathcal{U}_0,v_2,\mathcal{U}_1,\mathcal{U}_2)$. With the help of eq. (5.27), we can translate the integrand on the right-hand side of eq. (6.14) into a matrix product

$$I = \int \mathrm{d}k\,\mathrm{d}\ell\, p_{J_1}^k\,\gamma_{k\ell}\,p_{J_3}^\ell\ .\tag{6.15}$$

of the anomalous dimension operator γ with the GFF OPE coefficients $p_\ell^{J_i}$ of eq. (5.36). We thereby deduce the following contributions to the triple-twist anomalous dimension operator:

$$\mathrm{DC}^{(1)}: \mathcal{U}_0^{h_\star}\log u_2^{-1} \quad \leftrightarrow \quad \mathrm{CC}: \gamma_{k\ell}^{(1)} = \gamma_0\frac{\delta(k-\ell)}{\ell^{2h_\star}}\tag{6.16}$$

We have placed the superscript (1) in order to show that this is only part of the anomalous dimension operator, namely the part that is needed in order to reproduce the direct-channel term $\mathcal{U}_0^{h_\star}\log u_2^{-1}$. It still remains to analyze the second term.

6.2.2 Non-diagonal contributions to the anomalous dimension matrix

Let us now address the term that involves the product $v_2^{h_\star} \log u_2^{-1}$. It is not difficult to understand how to recover this term from the crossed channel, based on the following crucial observation: the two logarithmic terms in the direct channel are actually mapped to each other by a simple exchange of the external insertion points via the permutation $\sigma = (13)(46)$. Indeed, we have already computed the action of σ on our cross-ratios in eq. (5.29). From these formulas, it is evident that σ maps $\mathcal{U}_0$ to v_2 while leaving the product $v_2 \mathcal{U}_0 \mathcal{U}_1 \mathcal{U}_2$ invariant. Hence, the second logarithmic term in the direct channel possesses a similar interpretation in terms of the crossed-channel expansion as the first one, as long as we expand in terms of the crossed channel $CC' = CC \circ \sigma$. Hence, if we apply the identity (6.14) with the choice $(U, V, X_1, X_2) = (v_2, \mathcal{U}_0, \mathcal{U}_2, \mathcal{U}_1)$ we can rewrite the second term in terms of lightcone blocks for the crossed channel CC'. Before we can read off the contribution to the anomalous dimension matrix γ, we have to transform back from the lightcone blocks in the CC' channel to those in the crossed channel CC. But this is exactly what we discussed in Section 5.3.2, see in particular the integral relation (5.30). In conclusion, we deduce that

$$\frac{C_{\phi\phi\mathcal{O}_\star}^2}{B_{\bar{h}_\star}} \frac{v_2^{h_\star}}{(\mathcal{U}_0 v_2 \mathcal{U}_1 \mathcal{U}_2)^{\Delta_\phi}} = \tag{6.17}$$

$$= \sum_{\kappa=0}^{\infty} \int d^3 J \frac{8}{\Gamma_{\Delta_\phi}^3} J_2^{2\Delta_\phi - 1} \left\{ \int dJ_1' W_{J_1' J_1}^{(\Delta_\phi)} \frac{\gamma_0}{2 J_1'^{2h_\star}} W_{J_1' J_3}^{(\Delta_\phi)} \right\} \delta(J_1 - J_3) f_{J_1 J_2 J_3}(\mathcal{U}_0, v_2; \mathcal{U}_1, \mathcal{U}_2).$$

After translating the integrand into the matrix product of the form (6.15) by means of eq. (5.27), we obtain the following expression for the second contribution to the anomalous dimension matrix

$$\text{DC}: v_2^{h_\star} \log u_2^{-1} \quad \leftrightarrow \quad \text{CC}: \gamma_{k\ell}^{(2)} = \int dk' d\ell' \, W_{kk'}^{(\Delta_\phi)} \gamma_{k'\ell'}^{(1)} W_{\ell'\ell}^{(\Delta_\phi)} \tag{6.18}$$

The full anomalous dimension matrix. In summary, we have found that crossing symmetry entails the following anomalous dimension matrix for triple-twist operators $[\phi[\phi\phi]_{0,\ell}]_{0,J,\kappa}$:

$$\frac{\gamma_{k\ell}(J, \kappa)}{\gamma_0} = \frac{\delta(k - \ell)}{\ell^{2h_\star}} + \int_0^\infty dm \frac{W_{km}^{(\Delta_\phi)}(J, \kappa) W_{m\ell}^{(\Delta_\phi)}(J, \kappa)}{m^{2h_\star}} + \mathcal{O}(J^{-2h_\star}). \tag{6.19}$$

The second term can be evaluated explicitly using [61, eq. (10.22.58)]:

$$\int_0^\infty dm \frac{W_{km}^{(\Delta_\phi)}(J, \kappa) W_{m\ell}^{(\Delta_\phi)}(J, \kappa)}{m^{2h_\star}} = \tag{6.20}$$

$$= \frac{2}{\sqrt{k\ell}} \frac{\Gamma_{\Delta_\phi + \kappa - h_\star}}{\Gamma_{\Delta_\phi + \kappa} \Gamma_{h_\star}} \left(\frac{k\ell}{k^2 + \ell^2} \right)^{\Delta_\phi + \kappa} \left(\frac{k^2 + \ell^2}{J^2} \right)^{h_\star} {}_2F_1 \left[\begin{array}{c} \frac{\Delta_\phi + \kappa - h_\star}{2}, \frac{\Delta_\phi + \kappa + 1 - h_\star}{2} \\ \Delta_\phi + \kappa \end{array} \right] \left(\frac{4k^2\ell^2}{(k^2 + \ell^2)^2} \right).$$

Equations (6.19) and (6.20) constitute the main result of this work. They provide the leading correction dimension of triple-twist operators in the double-twist basis.

6.3 Double-twist limit and first correction

We do not intend to give a complete analysis of the eigenvalue equation $\gamma \cdot \Psi_\lambda = \lambda \, \Psi_\lambda$ for the anomalous dimensions matrix γ, but would like to study the eigenvalues near a limiting regime that recovers the behavior of triple-twist operators expected from the four-point bootstrap. To this end, we reintroduce the independent scaling of the degeneracy labels k, ℓ (which behave like J_1, J_3) and the triple-twist spin J (which behaves like J_2) that followed from the six-point lightcone bootstrap, i.e. we shall assume that $k^2, \ell^2 = \mathcal{O}(\epsilon_{16}^{-1})$, while $J^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{34}^{-1})$. For this scaling behavior we observe that the two terms $\gamma^{(1)}$ and $\gamma^{(2)}$ that contribute to γ scale differently. Considering the term $\gamma^{(1)}$ first, it is obvious from its definition (6.16) that

$$\gamma^{(1)}_{k\ell}(J, \kappa) := \frac{\delta(k - \ell)}{k^{2h_\star}} = \mathcal{O}(\ell^{-2h_\star - 1}). \tag{6.21}$$

The non-diagonal term $\gamma^{(2)}$, on the other hand, can be seen from eq. (6.20) above to scale as

$$\gamma^{(2)}_{k\ell}(J, \kappa) := \int_0^\infty \mathrm{d}m \frac{W^{(\Delta_\phi)}_{km}(J, \kappa) W^{(\Delta_\phi)}_{m\ell}(J, \kappa)}{m^{2h_\star}} = \mathcal{O}(\ell^{2h_\star - 1} J^{-2h_\star}). \tag{6.22}$$

Thus, the non-diagonal term is suppressed relative to the diagonal term in the regime $\ell^2 \ll J$, i.e. for $\epsilon_{16} \gg \epsilon_{34}$. In this case, the anomalous dimension matrix is diagonalized by wave functions $\Psi^{(12)}_{\ell_\lambda, J, \kappa}$ with eigenvalue $\gamma_0 \, \ell_\lambda^{-2h_\star}$ at leading order. The wave functions correspond to the triple-twist operators $[\phi[\phi\phi]_{0,\ell_\lambda}]_{0,J,\kappa}$, while the eigenvalues coincide with the leading anomalous dimensions of the constituent double-twist operators $[\phi\phi]_{0,\ell_\lambda}$. We therefore call this regime the "double-twist" limit. Let us stress that the limiting behavior that arises from the leading diagonal term $\gamma^{(1)}$ is expected from a bootstrap analysis of the four-point function $\langle \mathcal{O}_\ell(X_a, Z_a)\phi(X_3)\phi(X_4)\mathcal{O}_\ell(X_c, Z_c)\rangle$, where $\mathcal{O}_\ell = [\phi\phi]_{0,\ell}$.

To compute the first correction to the triple-twist anomalous dimensions away from the double-twist limit, we consider the perturbation theory of the free Hamiltonian $\gamma^{(1)}_{k\ell}$ by the small perturbation $\gamma^{(2)}_{k\ell}$. In this case, the first correction to the eigenvalue is given by

$$\gamma([\phi[\phi\phi]_{0,\ell_\lambda}]_{0,J,\kappa}) \overset{\ell_\lambda^2 \ll J}{\sim} \gamma_0 \left(\ell_\lambda^{-2h_\star} + \ell_\lambda \, \gamma^{(2)}_{\ell_\lambda \ell_\lambda}(J, \kappa) + \dots \right)$$

$$= \frac{\gamma_0}{\ell_\lambda^{2h_\star}} + 2\gamma_0 \left(\frac{\ell_\lambda}{J} \right)^{2h_\star} \frac{\Gamma(\Delta_\phi + \kappa - h_\star)}{\Gamma(\Delta_\phi + \kappa + h_\star - 1)} \frac{\Gamma(2h_\star - 1)}{\Gamma(h_\star)^2} + \mathcal{O}\left(\frac{\ell_\lambda^{2h_\star}}{J^{h_\star}} \right). \tag{6.23}$$

In the second line, we evaluated the matrix elements $\gamma^{(2)}_{\ell\ell}(J, \kappa)$ directly using [61, eq. (10.22.57)].

The dual double-twist limit. We can also study a dual limit $\ell^2 \gg J$ (i.e. $\epsilon_{16} \ll \epsilon_{34}$), where $\gamma^{(2)}_{k\ell}$ is the leading contribution and $\gamma^{(1)}_{k\ell}$ is the small perturbation. To diagonalize the operator in this limit, recall that the two matrices $\gamma^{(1)}_{k\ell}$ and $\gamma^{(2)}_{k\ell}$ are conjugate to one another via the Racah matrix

$W_{k\ell}^{(\Delta_\phi)}$, which realizes the change of basis $\Psi_{\ell,J,\kappa}^{(12)} \mapsto \Psi_{\ell,J,\kappa}^{(32)}$. As a result, the anomalous dimension operator is diagonalized by $\Psi_{\ell'_\lambda,J,\kappa}^{(32)}$ at leading order, with eigenvalues $\gamma_0\, \ell_\lambda'^{-2h_\star}$ and first correction of order $(J/\ell'_\lambda)^{-2h_\star}$. This means that the dual limit again localizes to the double-twist basis, and one can move from one regime to the other via the identification $\ell'_\lambda = J/\ell_\lambda$.

7 Comparison to ϵ-Expansion in Scalar Field Theories

In this section, we show that our results for the anomalous dimension (6.19) of triple-twist operators in the large-spin limit of eq. (5.20) agrees with the anomalous dimension operator of certain fixed points that can be studied perturbatively through the ϵ-expansion. More specifically, we will compute the triple-twist anomalous dimensions of ϕ^3-theory in $d = 6 - 2\epsilon$ dimensions at one loop and of ϕ^4-theory in $d = 4 - 2\epsilon$ dimensions at two loops and finite spin following the approach of Derkachov and Manashov, which first appeared in [49] and was later reviewed in [50]. The first subsection is devoted to the fixed point of scalar ϕ^3-theory. We shall first review the results of Derkachov and Manashov on the anomalous dimensions of triple-twist operators with vanishing MST spin $\kappa = 0$. These are stated in eqs. (7.3) and (7.4) and then evaluated in the large-spin limit, see eq. (7.11) where we find perfect agreement with our formula (6.19). We also extend the one-loop analysis to triple-twist operators with non-zero MST spin κ. Once again the large-spin limit is shown to agree with the results of the lightcone bootstrap. In the second subsection, we then turn our attention to the Wilson-Fisher fixed points of scalar ϕ^4-theory. We shall show that the bootstrap result (6.19) makes a prediction for the two-loop anomalous dimensions in this case. The latter are then computed perturbatively through the evaluation of the relevant Feynman diagrams. The calculation mimics the approach Manashov and Derkachov used for ϕ^3-theory. The resulting formulas for the two-loop anomalous dimensions of triple-twist operators at the Wilson-Fisher fixed point are new. In the large-spin limit, they are shown to reproduce eq. (6.19). Finally. we shall conclude with a few comments on the relation of our bootstrap analysis with a recent study of one-loop twist-four anomalous dimensions in [63].

7.1 ϕ^3 theory in $6 - \epsilon$ dimensions

In this section, we compare the bootstrap results we obtained in Section 6 with known results for ϕ^3 theory in $d = 6 - 2\epsilon$ dimensions. The fixed point of a scalar field with interaction $g\phi^3$ was studied to leading order in the epsilon expansion in [50, 53]. Following [50, eq. (4.11)], the beta function of the ϕ^3 model and the anomalous dimension of its fundamental scalar ϕ are given by

$$\beta(g) = -g\left(\epsilon + \frac{3}{256\pi^3}g^2 + \mathcal{O}(g^4)\right), \quad \Delta_\phi(g) = 2 - \epsilon + \frac{g^2}{768\pi^3} + \mathcal{O}(g^4). \tag{7.1}$$
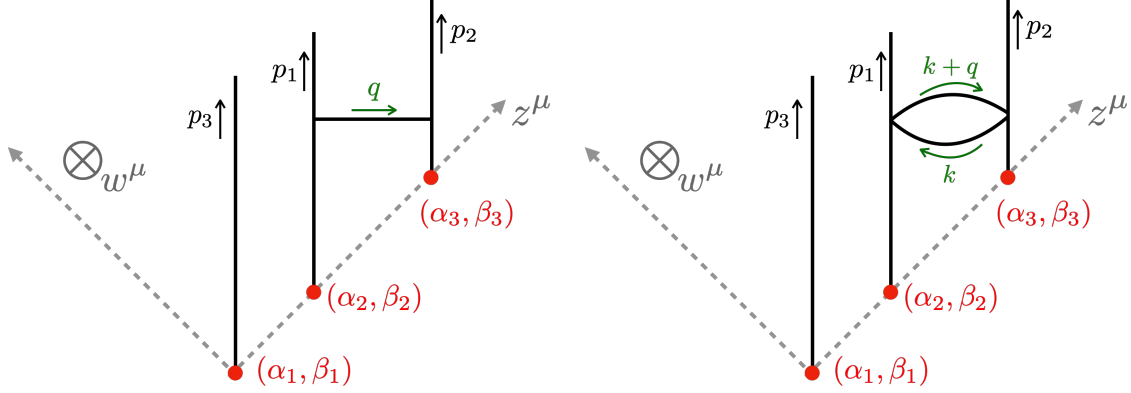
Figure 5. First Feynman diagrams in the ϵ-expansion of ϕ^3 theory (left) and ϕ^4 theory (right) that contribute to the anomalous dimensions of triple-twist operators in the large-spin limit $1 \ll k^2, \ell^2 = \mathcal{O}(J)$. Starting from a disconnected contribution to the six-point function, we Fourier transform three fields to $\tilde{\phi}(p_i)$ and take the normal-ordered product $\mathbb{O}(\alpha_i; \beta_i)$ of the three others, inserted along the null directions spanned by the orthogonal polarization vectors z, w. The poles of these diagrams in the ϵ-expansion are then renormalized by the anomalous dimensions of the triple-twist operators that appear in the lightcone OPE of the normal-ordered product $\mathbb{O}(\alpha_i; \beta_i)$.

Therefore, at this order in the epsilon expansion, the coupling and scaling dimension at the fixed point are given by [64, 65]

$$g_\star^2 = -\frac{256\pi^3}{3}\epsilon + \mathcal{O}(\epsilon^2), \quad \Delta_\phi(g_\star) = 2 - \frac{10}{9}\epsilon + \mathcal{O}(\epsilon^2). \tag{7.2}$$

These values determine the substractions we need to perform in passing from the triple-twist conformal dimensions to the anomalous contributions,

$$\gamma_{[\phi\phi\phi]_{0,J}} = \Delta_{[\phi\phi\phi]_{0,J}} - 3\Delta_\phi(g_\star) - J = -\frac{g_\star^2}{(4\pi)^3}\mathbb{H} + \mathcal{O}(g_\star^4) = \frac{4\epsilon}{3}\mathbb{H} + \mathcal{O}(\epsilon^2), \tag{7.3}$$

where $\mathbb{H} = \mathcal{O}(\epsilon^0)$ denotes some operator on the space of triple-twist wave functions. In general, these wave functions depend on six variables $\alpha_i, \beta_i, i = 1, 2, 3$. Derkachov and Manashov computed the leading contributions to the triple-twist anomalous dimensions at the fixed point only for STT triple-twist operators, i.e. for operators with $\kappa = 0$. Their approach is based on the study of the correlation function $\langle \phi(x_1)\phi(x_2)\phi(x_3)\mathbb{O}(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)\rangle$, where $\mathbb{O} =:\!\phi\phi\phi\!:$ is the normal-ordered product of three fundamental fields at lightlike separation (see Fig. 5), which can then be expanded into a linear combination of triple-twist operators following the methods of Section 4. In perturbation theory, the six-point Feynman diagrams that contribute to this correlator exhibit collinear singularities when the three points are lightlike separated. The corresponding $1/\epsilon$ pole is then renormalized by the anomalous dimensions of the triple-twist operators that span $\mathbb{O}$, and these anomalous dimensions are expressed in terms of an integral operator acting on the space of triple-twist wave functions $\Psi_{J,\kappa}$. By acting on

the wave functions $\Psi^{(12)}_{\ell,J,\kappa}$, we can then compare the resulting anomalous dimension in the double-twist basis with our prediction from the lightcone bootstrap at large spin.

7.1.1 Anomalous dimensions for vanishing MST spin

In the case $\kappa = 0$ the wave functions only depend on the three variables α_i. So, in order to state the results of Derkachov and Manashov for γ we shall drop the variables β_i. Recall from eq. (5.9) that the triple-twist wave functions $\Psi(\alpha_1, \alpha_2, \alpha_3)$ can be expanded in the basis of double-twist wave functions (4.39). On these wave functions, the Hamiltonian $\mathbb{H}$ takes the following nice form [50, eqs. (5.13),(5.17a)]

$$\mathbb{H} = \sum_{1 \leq i < k \leq 3} \mathbb{H}_{ik}, \qquad \mathbb{H}_{ik} = \left(\frac{1}{\mathbb{C}^2_{ik}} - \frac{1}{24} \delta_{\mathbb{C}^2_{ik},2} \right). \tag{7.4}$$

Here, the operators $\mathbb{C}^2_{ik}$ denote (quadratic) Casimir operators of $\mathfrak{su}(1,1)$ with generators given by sums $S_i + S_k$ of operators that act on α_i and α_k, respectively, see eqs. (4.1-4.3) for concrete expressions. The action of the resulting Casimir operators on the triple-twist wave functions $\Psi^{(ik)}_{\ell,J}$ is given by

$$\mathbb{C}^2_{ik} \Psi^{(ik)}_{\ell,J}(\alpha_1, \alpha_2, \alpha_3) := (\Delta_\phi + \ell)(\Delta_\phi + \ell - 1)\Psi^{(ik)}_{\ell,J}(\alpha_1, \alpha_2, \alpha_3), \tag{7.5}$$

Below we shall explain that the same formulas (7.4) and (7.3) also give the anomalous dimensions of triple-twist primaries in MST representations, i.e. with $\kappa \neq 0$, provided one replaces the Casimir operators $\mathbb{C}^2$ of $\mathfrak{su}(1,1)$ by those of the Lie algebra $\mathfrak{su}(1,2)$. But before we do so, let us first compare the original result of Derkachov and Manashov for fields with $\kappa = 0$ with the outcome of our bootstrap analysis.

To compare formulas (7.4) and (7.3) with our result (6.19) for the triple-twist anomalous dimensions in the large-spin limit $J \gg 1$, $\ell^2 = \mathcal{O}(J)$, we study the matrix elements of (7.3) in the $\psi^{(12)}_\ell$ basis, that is

$$\Gamma_{k\ell}(J, \kappa = 0) = \frac{4}{3} \epsilon \, \langle \psi^{(12)}_k, \mathbb{H}\psi^{(12)}_\ell \rangle. \tag{7.6}$$

We want to evaluate the asymptotics of these matrix elements in the limit (5.20) with $k^2 \sim \ell^2 \sim J$ and show that these coincide with eq. (6.19). To begin with, let us look at the two terms in (7.4) that involve the Casimir operators $(\mathbb{C}^2_{12})^{-1}$ and $(\mathbb{C}^2_{23})^{-1}$. The first term is diagonal in the (12) double-twist basis, see eq. (7.5), and the asymptotic behavior of its matrix elements is obtained from

$$\langle \psi^{(12)}_{k,J}, \mathbb{C}^2_{12}\psi^{(12)}_{\ell,J} \rangle = (\ell^2 + \mathcal{O}(\ell))\delta_{k\ell} \; . \tag{7.7}$$

The second term that involves the Casimir $\mathbb{C}^2_{23}$ may be expressed in terms of the $\mathfrak{su}(1,1)$ Racah coefficients $W^{(\Delta_\phi)}_{k\ell}(J, 0)$ that provide the transformation from the (12) to the (23) double-twist basis,

see eq. (5.12). The large-spin asymptotics of the Racah coefficients was given in eq. (5.25). From there we see that the matrix elements of the Casimir elements $\mathbb{C}^2_{23}$ behave as

$$\langle \psi^{(12)}_{k,J}, \mathbb{C}^2_{23}\psi^{(12)}_{\ell,J}\rangle = \mathcal{O}(J^2/\ell^2). \tag{7.8}$$

Consequently, in the limiting regime in which we want to compare the results of Derkachov and Manashov with our expression for anomalous dimensions, the terms $(\mathbb{C}^2_{12})^{-1}$ and $(\mathbb{C}^2_{23})^{-1}$ contribute at order $1/J$. There is one more Casimir operator to look at, namely the operator $\mathbb{C}^2_{13}$. Since the latter operator is quadratic in the generators S_i, it can be rewritten in terms of the Casimir operator $\mathbb{C}^2_{123}$ that is built from the sum $S_1 + S_2 + S_3$ as follows:

$$\mathbb{C}^2_{13} = \mathbb{C}^2_{123} - \mathbb{C}^2_{12} - \mathbb{C}^2_{23} - 3h_\phi(h_\phi - 1) = J^2 + \mathcal{O}(\ell^2, J^2/\ell^2). \tag{7.9}$$

The statement on the asymptotic behavior of $\mathbb{C}^2_{13}$ then follows from the fact that the Casimir operator $\mathbb{C}^2_{123}$ is constant and takes the value $(3h_\phi + J)(3h_\phi + J - 1)$. As a result, the contribution $(\mathbb{C}^2_{13})^{-1}$ to the operator $\mathbb{H}$ is subleading in the limit $J = \mathcal{O}(\ell^2) \gg 1$. Finally, for none of the pairs $(i,k) \in \{(1,2),(2,3),(1,3)\}$ does the Casimir operator $\mathbb{C}^2_{ik}$ possess eigenvectors with eigenvalue $\lambda = 2$ in this regime. Indeed, the spectrum of the Casimir operator of $\mathfrak{su}(1,1)$ is given by $\lambda = (\Delta_\phi + \ell)(\Delta_\phi + \ell - 1)$ and hence, given that $\Delta_\phi = 2 + \mathcal{O}(\epsilon)$, the eigenvalue $\lambda = 2$ is only assumed for $\ell = 0$. Since the constant term in $\mathbb{H}$ is proportional to $\delta_{\mathbb{C}^2_{ik},2}$, this term can never contribute. Putting all this together we have now shown that the large-spin limit of the matrix elements $\Gamma_{k\ell}$ defined in eq. (7.6) reduces to

$$\frac{3}{4\epsilon}\Gamma_{k\ell}(J,0) = \frac{\delta_{k\ell}}{(\Delta_\phi + \ell)(\Delta_\phi + \ell - 1)} + \langle W^{(2)}(J,0)\psi^{(23)}_{k,J}, \frac{1}{\mathbb{C}^2_{23}}W^{(2)}(J,0)\psi^{(23)}_{\ell,J}\rangle + \mathcal{O}(J^{-2}). \tag{7.10}$$

Here we have used that $\Delta_\phi = 2 + \mathcal{O}(\epsilon)$. In the large-spin limit, we can then approximate the sum over double-twist basis elements in the second term by an integral, and replace the Racah coefficients $W^{(\Delta_\phi=2)}_{k\ell}(J,\kappa=0)$ by their large-spin limit (5.25). In conclusion, the large-spin limit of Derkachov and Manashov's one-loop anomalous dimension operator is given by

$$\Gamma_{k\ell}(J,0) = \frac{4}{3}\epsilon\left(\frac{\delta(k-\ell)}{\ell^2} + \int_0^\infty \mathrm{d}m \frac{W^{(\Delta_\phi)}_{km}W^{(\Delta_\phi)}_{m\ell}}{m^2}\right) + \mathcal{O}(J^{-2}). \tag{7.11}$$

In order to compare the large-spin limit of the one-loop anomalous dimensions with our formula (6.19), we still need to determine the two parameters $h_\star$ and γ_0 that appear in eq. (6.19). In scalar ϕ^3 theory, the leading-twist operator in the OPE of ϕ with itself is $\mathcal{O}_\star = \phi$ and hence $h_\star = \Delta_\phi/2 = 1 + \mathcal{O}(\epsilon)$. Moreover, for the coefficient γ_0 that was defined in (6.13) we obtain the following value,

$$\gamma_0 = -2\frac{\Gamma^3_{\Delta_\phi}}{\Gamma^4_{\Delta_\phi/2}}C^2_{\phi\phi\phi} = -2C^2_{\phi\phi\phi} = 4\epsilon/3 + \mathcal{O}(\epsilon^2) \tag{7.12}$$

In the last step we have inserted the well-known value of the OPE coefficient $C^2_{\phi\phi\phi} = -2\epsilon/3 + \mathcal{O}(\epsilon^2)$ in scalar ϕ^3 theory, see e.g. [66, 67]. In conclusion, we have established that the large-spin limit (7.11) of the one-loop anomalous dimension matrix Γ found by Derkachov and Manashov indeed coincides with our general result (6.19) for γ once we plug in the parameters of scalar ϕ^3 theory and set the MST spin κ to $\kappa = 0$.

7.1.2 Generalization to non-zero MST spin

In the previous subsection, we have evaluated our general formula (6.19) for the large-spin behavior of the anomalous dimension matrix for scalar ϕ^3 theory and compared it with the existing one-loop calculation of the same quantity. Derkachov and Manashov performed their calculation for triple-twist operators in STT representations only. In this subsection, we want to briefly describe how their formula (7.3,7.4) can be extended to triple-twist primaries with $\kappa \neq 0$ and we shall sketch how the resulting formula for $\mathbb{H}$ compares with our bootstrap analysis.

It is actually not difficult to generalize the original derivation of one-loop, M-twist anomalous dimensions in [50, Sec. 4] to non-zero MST spin. This only requires to add an additional orthogonal null vector $w \in \mathbb{C}^d$ such that $z^2 = w^2 = z \cdot w = 0$. The normal-ordered product of M operators then takes the form[8]

$$\mathbb{O}^{(2)}(\alpha_i; \beta_i) := :\phi(x + \alpha_1 z + \beta_1 w) \ldots \phi(x + \alpha_M z + \beta_M w): . \tag{7.13}$$

Among the two diagrams in [50, Fig. 2] that contribute to $\langle \phi\phi\phi\mathbb{O} \rangle$ at one loop, the rightmost diagram can only produce anomalous dimensions for $[\phi\,\phi^2]_{0,J}$, which therefore do not exist for non-zero MST spin $\kappa > 0$. The former property follows from the fact that the diagram connects two external legs to the same vertex, as explained at the beginning of section 7.2.2. For the remaining diagram, with the labeling of external legs and arguments of $\mathbb{O}$ given on the left-hand-side of Fig. 5, one can check that the $1/\epsilon$ divergence is renormalized by the anomalous dimension operator $(4\epsilon/3)\mathbb{H}_{12}$, where

$$\mathbb{H}_{12}\Psi_{J,\kappa}(\alpha_i; \beta_i) = \prod_{i=1}^{3} \int_0^1 dt_i\, \delta(t_1 + t_2 + t_3 - 1)\Psi_{J,\kappa}(\alpha_{12}^{t_1}, \alpha_{21}^{t_2}, \alpha_3; \beta_{12}^{t_1}, \beta_{21}^{t_2}, \beta_3), \tag{7.14}$$

and $\alpha_{12}^t := (1 - t)\alpha_1 + t\alpha_2$. Using the highest-weight condition (4.58), we can then express the β_i-dependence of the MST wave function as in eq. (4.60). In addition, one can easily check that the factor ω defined in eq. (4.60) obeys

$$\omega(\alpha_{12}^{t_1}, \alpha_{21}^{t_2}, \alpha_3; \beta_{12}^{t_1}, \beta_{21}^{t_2}, \beta_3) = \omega(\alpha_i; \beta_i) . \tag{7.15}$$

---

[8]Note that our notation differs from [50] in that $u^{\mathrm{DM}} = z^{\mathrm{ours}}$ and $z_i^{\mathrm{DM}} = \alpha_i^{\mathrm{ours}}$.

Hence, we have

$$\mathbb{H}_{12}\,\omega(\alpha_i;\beta_i)^\kappa\,\Psi_{J,\kappa}(\alpha_i,\beta_i^\star) = \omega(\alpha_i;\beta_i)^\kappa\,\mathbb{H}_{12}\Psi_{J,\kappa}(\alpha_i,\beta_i^\star). \tag{7.16}$$

Using the property (4.64) of the (non-normalized) double-twist basis $g_{\ell,J,\kappa}^{(12)}(\alpha_i,\beta_i)$ we can finally determine the action of the operator $\mathbb{H}_{12}$ on the function by taking the limit $\alpha_1 \to \alpha_2$. At leading order, we find

$$\mathbb{H}_{12}g_{\ell,J}^{(12)}(\alpha_i) \overset{\alpha_1\to\alpha_2}{\sim} \alpha_{13}^{J-\ell}\mathbb{H}_{12}\alpha_{12}^\ell = \frac{\alpha_{12}^\ell\alpha_{13}^{J-\ell}}{(2+\ell)(1+\ell)}. \tag{7.17}$$

Since $(2+\ell)(1+\ell)$ is the eigenvalue of the second-order Casimir operator $\mathbb{C}_{12}^2$ for the $\mathfrak{su}(1,2)$ generators $S_1 + S_2$ with corresponding eigenvector $\Psi_{\ell,J,\kappa}^{(12)}(\alpha_i;\beta_i)$, we deduce that $\mathbb{H}_{12} = (\mathbb{C}_{12}^2)^{-1}$ even for $\kappa > 0$. After summing over permutations of the external legs on the leftmost diagram of Fig. 5 that produced $\mathbb{H}_{12}$, we then find that the one-loop anomalous dimension operator at $\kappa > 0$ is given by

$$\mathbb{H} = \sum_{1\leq i<k\leq 3} \frac{1}{\mathbb{C}_{ik}^2}. \tag{7.18}$$

While formally identical to the expression we reviewed before in eq. (7.4), there is an important difference: in eq. (7.4), the symbol $\mathbb{C}^2$ denotes the quadratic Casimir elements of the Lie algebra $\mathfrak{su}(1,1)$. Instead, the extension to operators with $\kappa \neq 0$ we derived here involves the quadratic Casimir element of the Lie algebra $\mathfrak{su}(1,2)$.

As in the previous subsection for triple-twist primaries with $\kappa = 0$ we can now calculate the matrix elements of the operator (7.18) for $\kappa \neq 0$ in the large-spin limit and then compare with with our result (6.19) from the lightcone bootstrap. At large spin and finite κ, where $J \sim J - \kappa$, we thus retrieve the same shift relation $\Delta_\phi \mapsto \Delta_\phi + \kappa$ for $\Delta_\phi = 2$ as seen from the property (5.25) of Racah coefficients. Note that the expression (7.18) does not have the Kronecker delta contributions like (7.4). Those terms are important only for $\ell = 0$ of the basis vectors ψ_ℓ^{12}, and hence can be neglected in the comparison which is done in the limiting regime where $\ell^2 \sim J \gg 1$. The conclusion generalized that of the previous subsection: when applied to triple-twist operators in scalar ϕ^3 theory, our bootstrap result (6.19) coincides with the large-spin limit of the one-loop anomalous dimension matrix (7.18) even for operators with non-vanishing MST spin κ.

Before we conclude our discussion of ϕ^3 theory, we would like to note one additional outcome of our new formula for the one-loop anomalous dimension of operators with $\kappa \neq 0$. It concerns the special case in which $\kappa = J$. For this maximal value of κ, the MST triple-twist families are non-degenerate, with a unique wave function

$$\Psi_{\kappa,\kappa}(\alpha_i;\beta_i) = (\alpha_{21}\beta_{31} - \alpha_{31}\beta_{21})^\kappa$$

up to normalization. We can insert this simple expression for the wave function directly into our integral formula (7.14) to obtain

$$\gamma_{[\phi\phi\phi]_{0,J=\kappa,\kappa}} = \frac{4}{3}\epsilon \prod_{i=1}^{3} \int_0^1 dt_i\, \delta(t_1+t_2+t_3-1)\,(t_3^\kappa+t_1^\kappa+t_2^\kappa) = \frac{4\epsilon}{(\kappa+1)(\kappa+2)}. \tag{7.19}$$

This is a new result for the anomalous dimension of triple-twist operators with maximal $\kappa = J$ in ϕ^3 theory. Note that it is a corollary of our perturbative analysis, so its validity does not require any large-spin limit.

7.2 ϕ^4 theory in $4-\epsilon$ dimensions

In this subsection, we will consider the Wilson-Fisher CFT, i.e. the fixed point of ϕ^4 theory in ϵ-expansion. While the two-loop, lowest triple-twist anomalous dimension operator in this model was obtained by Kehrein in [51], it is given as an operator on the whole conformal multiplets rather than the primaries, which makes it difficult to compare with a CFT derivation (it is also not obvious how these expressions behave in the large-spin limit). Instead, we will derive some of Kehrein's results using the parameterization of Derkachov and Manashov, as explained in e.g. [50], to check the results predicted by the six-point lightcone bootstrap.

7.2.1 Prediction from six-point lightcone bootstrap

For Wilson-Fisher theory in $d = 4-2\epsilon$ dimensions, the large-spin triple-twist anomalous dimensions predicted by the lightcone bootstrap first appear at second order in the ϵ expansion. To understand why this is the case, recall the scaling dimensions of the two lowest-lying scalars ϕ and ϕ^2 [68].

$$\Delta_\phi = 1 - \epsilon + \frac{\epsilon^2}{27} + \mathcal{O}(\epsilon^3), \quad \Delta_{\phi^2} = 2 - \frac{4}{3}\epsilon + \mathcal{O}(\epsilon^2) = 2\Delta_\phi + \frac{2}{3}\epsilon + \mathcal{O}(\epsilon^2). \tag{7.20}$$

We will use the general formula (6.19) to compute the triple-twist anomalous dimension operator in the limit $J \sim \mathcal{O}(\ell^2) \gg 1$. In the limit $\epsilon \to 0$ the leading-twist operators in $\phi \times \phi$ OPE are ϕ^2 and higher-spin currents $[\phi\phi]_{0,\ell \geq 2}$, all appearing with twist two. Now if we set $\mathcal{O}_\star \equiv \phi^2$ as the leading-twist operator for nonzero epsilon in our formula, then the triple-twist anomalous dimension is proportional to

$$\gamma_0\Big|_{\mathcal{O}_\star=\phi^2} = -\frac{4+\mathcal{O}(\epsilon)}{\Gamma\left(\Delta_\phi - \Delta_{\phi^2}/2\right)^2} = -\frac{4}{9}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{7.21}$$

Moreover, while the stress tensor and broken higher-spin currents $[\phi\phi]_{0,\ell \geq 2}$ have lower twist than ϕ^2, their anomalous dimensions first appear at second order with [68, 69]

$$\tau_{[\phi\phi]_{0,\ell}} = 2 - 2\epsilon + \frac{2}{27}\left(1 - \frac{6}{\ell(\ell+1)}\right)\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{7.22}$$

This absence of anomalous dimensions at leading order implies a further suppression of γ_0:

$$\gamma_0\Big|_{\mathcal{O}_\star=[\phi\phi]_{0,\ell}} = -\frac{B_{1+\ell}^{-1}C_{\phi\phi[\phi\phi]_{0,\ell}}^2 + \mathcal{O}(\epsilon)}{\Gamma\left(\Delta_\phi - \tau_{[\phi\phi]_{0,\ell}}/2\right)^2} = \frac{4}{729}\left(-\frac{6}{\ell(\ell+1)}\right)^2 B_{1+\ell}^{-1}C_{\phi\phi[\phi\phi]_{0,\ell}}^2\epsilon^4 + \mathcal{O}(\epsilon^5). \qquad (7.23)$$

Accordingly, if we first expand both sides of the crossing equation up to order ϵ^2, then only ϕ^2 appears at leading twist in the lightcone OPEs of the direct channel. We conclude that our formula (6.19) for the universal large-spin anomalous dimension operator with $\mathcal{O}_\star = \phi^2$ and $\gamma_0 = -4\epsilon^2/9$ is a prediction for the two-loop anomalous dimension of triple-twist operators in Wilson-Fisher theory.

Let us comment on why we are allowed to consider an infinite family of minimal twist operators $\mathcal{O}_\star$ with a vanishing twist gap. What we are really doing is isolating the deviation of the Wilson-Fisher theory from the free field theory at a given order of ϵ. At $O(\epsilon)$ this only isolates the contribution of ϕ^2. Now even though ϕ^2 has a higher twist than the family of currents with $\ell \geq 2$, this is not the case in $\epsilon < 0$ (i.e. $d > 4$) where the interacting CFT still exists (although not a stable IR fixed point). The OPE data could also be analytically continued to any ϵ. Such an order-by-order analysis in ϵ can be justified by the observation that the sum over ℓ in eq. (7.23) converges. Let us also mention that a similar analysis of anomalous dimensions for double-trace operators in [70] was shown to give correct results up to order $\mathcal{O}(\epsilon^5)$.

7.2.2 Explicit check with Feynman diagrams

In order to verify our bootstrap results in Wilson-Fisher theory we now need to compute the anomalous dimension matrix of triple-twist operators to order ϵ^2 in the epsilon expansion. This can be done with the methods of [50], i.e. by computing the UV divergences of the correlation function $\langle\mathbb{O}(\alpha_1,\alpha_2,\alpha_3;\beta_1,\beta_2,\beta_3)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\rangle$, where $\tilde{\phi}$ is the Fourier transform of ϕ and $\mathbb{O}$ denotes the normal-ordered product of fundamental fields at lightlike separation

$$\mathbb{O}(\alpha_i;\beta_i) =:\prod_i \phi(y_i):, \quad y_i := \alpha_i z + \beta_i w. \qquad (7.24)$$

There are five diagrams that can contribute up to two loops. One of them is shown on the right of Figure (5) while the remaining ones are displayed in Figure (6).

Out of the five diagrams that contribute to the connected part[9] of this correlation function, only the diagram on the right-hand side of Fig. 5 is responsible for the anomalous dimension of triple-twist

---

[9] There is a sixth diagram at one loop, labeled by "6" in [51] and "(b)" in [49], which contributes to the disconnected parts $\langle\phi\phi\rangle\langle\phi\mathbb{O}\rangle$. From a bootstrap perspective, this clearly cannot contribute to the triple-twist anomalous dimension, as it involves identity exchange in one of the OPEs. Moreover, Derkachov and Manashov showed that the anomalous dimension operator generated by this diagram exchanges three ϕ's for one single-trace operator $\partial^2\phi$. This gives the anomalous dimension operator the structure of a block triangular matrix, whose eigenvalue problem reduces to that of the triple-twist-triple-twist block.
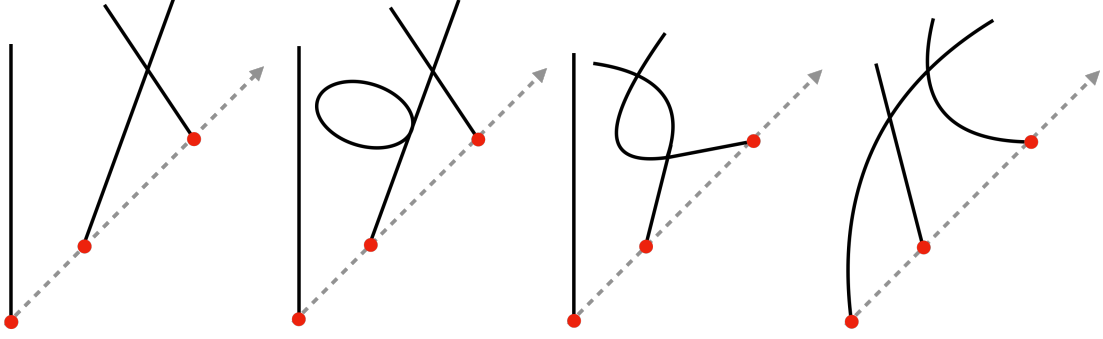
Figure 6. Four of the five Feynman diagrams that contribute to the $\langle \phi(x_1)\phi(x_2)\phi(x_3)\mathbb{O}(\alpha_i; \beta_i)\rangle$ four-point function up to two loops. The fifth diagram is depicted on the right of Fig. 5. Contrary to the latter, these four diagrams shown here share the property that two of the external legs are connected to the same vertex. It follows that all their $1/\epsilon$ poles are renormalized by the anomalous dimension of $[\phi\,\phi^2]_{0,J}$.

operators in the large-spin limit (5.20), while the four remaining diagrams in Fig. 6 can only produce anomalous dimensions for $[\phi\,\phi^2]_{0,J}$. To see why this is the case, note that all four diagrams in Fig. 6 share the property that two external legs are connected to a single vertex. Without loss of generality, we can label each of these external legs by a momentum p_1 or p_2. In momentum space, this property implies that the Feynman integral only depends on the sum $p_1 + p_2$ of the two momenta. After Fourier transforming back to position space, the corresponding diagram must then be localized to coincident points $x_1 = x_2$. Consequently, the $1/\epsilon$ divergences of such a diagram can only be renormalized by anomalous dimension operators that localize the normal-ordered product $\mathbb{O}(\alpha_i, \beta_i)$ to a subset with two coincident points $(\alpha_{\sigma(1)}, \beta_{\sigma(1)}) = (\alpha_{\sigma(2)}, \beta_{\sigma(2)})$, for some permutation $\sigma \in S_3$. Indeed, it is only in these cases that the condition $x_1 = x_2$ can be reproduced by Wick contraction. Finally, in expanding the normal ordered product $\mathbb{O}(\alpha_i; \beta_i)$ into a linear combination of primaries, the restriction to coincident points imposes that only the double-twist operators $[\phi\,\phi^2]_{0,J}$ (i.e. the $\ell = 0$ elements in the double-twist basis) can appear. Note that the same mechanism occurs for triple-twist anomalous at one loop, where only the leftmost diagram of Fig. 6 contributes to the correlator, reproducing the result of Derkachov-Manashov and Kehrein-Wegner that $[\phi\,\phi^2]_{0,J}$ are the only triple-twist operators to acquire an anomalous dimension at one loop in Wilson-Fisher theory.

Having established that only the right diagram of Fig. 5 contributes to the anomalous dimension of triple-twist operators outside of $[\phi\,\phi^2]_{0,J}$, we can isolate it and express the two-loop correlator as

$$\langle \mathbb{O}(\alpha_i; \beta_i)\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\rangle = \cdots + \frac{g^2}{2} \sum_{1 \leq i \neq j \neq k \leq 3} I(y_i, y_j; p_i, p_j)\langle \phi(y_k)\tilde{\phi}(p_k)\rangle + \mathcal{O}(g^3),$$

where the ellipsis denotes the four diagrams of Fig. 6, while

$$I(y_1, y_2; p_1, p_2) = e^{i(p_1 y_1 + p_2 y_2)} \int \frac{d^d q}{(2\pi)^d} \frac{e^{iq(y_1 - y_2)}}{(q + p_1)^2 (q - p_2)^2} I_{11}(q), \quad I_{11}(q) := \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k+q)^2}.$$

corresponds to the connected part of the diagram on the right-hand side of Fig. 5. We can now proceed to compute $I(y_1, y_2; p_1, p_2)$ using

$$I_{11}(q) = \frac{\Gamma\left(2 - \frac{d}{2}\right) \Gamma\left(\frac{d-2}{2}\right)^2}{\Gamma(d-2)} (q^2)^{\frac{d-4}{2}}. \tag{7.25}$$

After this, we introduce Feynman parameters to help carry out the remaining loop momentum integral, i.e.

$$\frac{1}{q^{2\epsilon}(q+p_1)^2 (q+p_2)^2} = \frac{\Gamma(2+\epsilon)}{\Gamma(\epsilon)} \prod_{i=1}^{3} \int_0^1 dt_i \, \delta\left(\sum_{i=1}^3 t_i - 1\right) t_3^{\epsilon-1} (Q^2 + M^2)^{-2-\epsilon}, \tag{7.26}$$

where $Q := q + t_1 p_1 - t_2 p_2$ and $M^2 := t_1 t_2 s_{12} := t_1 t_2 (p_1 + p_2)^2$. Apart from the $(Q^2 + M^2)^{-2-\epsilon}$ in eq. (7.26), the remaining dependence in the loop momentum comes from the phase $e^{iQ \cdot y_{12}}$. However, after expanding the latter into a power series in y_{12}^μ, it follows from Lorentz invariance that any higher order term in y_{12} will be proportional to y_{12}^2. Since $y_{12} \in \mathrm{Span}(z, w)$, where z, w are mutually orthogonal null vectors, these terms must however vanish[10]. We can therefore drop this phase and explicitly integrate over Q to obtain

$$G(y_1, y_2; p_1, p_2) = \frac{\Gamma(1-\epsilon)^2 \Gamma(2\epsilon)}{(4\pi)^{2-\epsilon} \Gamma(2-2\epsilon) s_{12}^{2\epsilon}} \prod_{i=1}^3 \int_0^1 dt_i \frac{t_3^{\epsilon-1}}{(t_1 t_2)^{2\epsilon}} \delta\left(\sum_{i=1}^3 t_i - 1\right) e^{i\left(p_1 y_{12}^{t_1} + p_2 y_{21}^{t_2}\right)}, \tag{7.27}$$

where $y_{12}^t := (1-t)y_1 + t y_2$. The $\Gamma(2\epsilon)$ produces a simple pole, which is eliminated by adding a two-loop anomalous dimension to the composite operator $\mathbb{O}$. Following the conventions of [50, Sec. 4], it takes the form

$$\gamma = -\frac{4\epsilon^2}{9} \left(\mathbb{H}_{12} + \mathbb{H}_{23} + \mathbb{H}_{13}\right), \tag{7.28}$$

where

$$\mathbb{H}_{12} \Psi^{(2)}(\alpha_i; \beta_i) = \prod_{i=1}^3 \int_0^1 dt_i \, \delta(t_1 + t_2 + t_3 - 1) \, t_3^{-1} \Psi^{(2)}(\alpha_{12}^{t_1}, \alpha_{21}^{t_2}, \alpha_3; \beta_{12}^{t_1}, \beta_{21}^{t_2}, \beta_3). \tag{7.29}$$

Using once again eq. (4.60), we can repeat the derivation of the one-loop anomalous dimension of ϕ^3 theory from Subsection 7.1 and 7.1.2 with $[\mathbb{H}_{12}, \omega^\kappa] = 0$ and

$$\mathbb{H}_{12} g_{\ell, J}^{(12)}(\alpha_i) \overset{\alpha_1 \to \alpha_2}{\sim} \alpha_{13}^{J-\ell} \mathbb{H}_{12} \alpha_{12}^\ell = \frac{\alpha_{12}^\ell \alpha_{13}^{J-\ell}}{(1+\ell)\ell}. \tag{7.30}$$

---

[10]The same argument can found in [50], below eq. (4.19), for the one-loop diagram of ϕ^3 theory on the left-hand side of Fig. 5.

Given $\Delta_\phi = 1$ in $d = 4$, we then equate $\mathbb{H}_{12}$ with the inverse $\mathfrak{su}(1,2)$ Casimir operator and write $\mathbb{H} = \sum_{i,k} 1/\mathbb{C}_{ik}^2$. At large spin, we thus retrieve the same result as the lightcone bootstrap with $\mathcal{O}_\star = \phi^2$ and $\gamma_0 = -4\epsilon^2/9$. In conclusion, we have not only computed the anomalous dimension (7.28) of triple-twist operators in Wilson-Fisher theory to order ϵ^2 in the epsilon expansion but also shown that the large-spin limit of this two-loop result agrees beautifully with the prediction of our bootstrap analysis, i.e. with formula (6.19).

As in our discussion of the perturbative anomalous dimensions of triple-twist operators with non-vanishing MST spin κ we want to add one small comment concerning the primaries with maximal value $\kappa = J$ of the MST spin. We have pointed out before that this primary is not degenerate. Therefore its anomalous dimension is a number that we can easily compute from eqs. (7.28) and (7.29). The result is

$$\gamma_{[\phi\phi\phi]_{0,J=\kappa,\kappa}} = -\frac{4\epsilon^2}{9} \prod_{i=1}^{3} \int_0^1 \mathrm{d}t_i \, \delta(t_1 + t_2 + t_3 - 1) \left(t_3^{\kappa-1} + t_1^{\kappa-1} + t_2^{\kappa-1} \right) = -\frac{4\epsilon^2}{3\kappa(\kappa+1)}. \qquad (7.31)$$

Of course, this formula receives corrections at higher orders in ϵ but it does not require any large spin limits. Our formulas (7.28) and (7.29) equally apply to all triple-twist operators with $\kappa \leq J$. However, due to the triple-twist degeneracies, extracting anomalous dimensions requires solving the mixing problem, i.e. diagonalizing the matrix we computed. This is beyond the scope of our work.

7.2.3 Comment about one-loop, twist-four anomalous dimensions

There is another test of our bootstrap analysis we can perform in the context of Wilson-Fisher theories. Recently, the authors of [63] studied the one-loop anomalous dimension of the twist-four primaries $[\phi \, [\phi\phi^2]_{0,\ell}]_{0,J,0}$. While the analysis we have presented was restricted to identical scalars and hence does not quite cover these twist-four operators, it is not too difficult to extend our framework accordingly. More specifically, after a minimal adjustment of our setup to the six-point function $\langle \phi(X_1)\phi(X_2)\phi^2(X_3)\phi^2(X_3)\phi(X_5)\phi(X_6)\rangle$ of non-identical scalars, we expect to recover a formula like eq. (6.23) that describes the anomalous dimension of the primaries $[\phi \, [\phi\phi^2]_{0,\ell}]_{0,J,\kappa}$ near the double-twist limit $\ell^2 \ll J$. However, while the $(J/\ell)^{-2h_\star}$ term remains proportional to $\gamma_0 \propto C_{\phi\phi\mathcal{O}_\star}^2$, the prefactor of the $\ell^{-2h_\star}$ term will be modified to $\gamma_0 \propto C_{\phi^2\phi^2\mathcal{O}_\star} C_{\phi\phi\mathcal{O}_\star}$. Given the leading-twist operators $\mathcal{O}_\star$ of the $\phi \times \phi$ OPE listed in Section 7.2.1, the latter contribution will therefore be subleading in the ϵ-expansion due to the suppression of the OPE coefficient $C_{\phi^2\phi^2\mathcal{O}_\star}$. Focusing on the former contribution, we can then use the same reasoning as in Section (7.2.1) to deduce that $\mathcal{O}_\star = \phi^2$ is the dominant exchange in the ϵ-expansion. Now, while $\gamma_0|_{\mathcal{O}_\star=\phi^2} = -\frac{4}{9}\epsilon^2$ is of second order in ϵ, the pole $\Gamma(\Delta_\phi - h_\star) = \mathcal{O}(\epsilon^{-1})$ in the $(J/\ell)^{-2h_\star}$ term of eq. (6.23) implies that this anomalous dimension does indeed start at one loop.

Finally, after plugging in the scaling dimensions Δ_ϕ and $h_\star = \Delta_{\phi^2}/2$ of eq. (7.20), our analysis predicts that the one-loop anomalous dimension of $[\phi\,[\phi\phi^2]_{0,\ell}]_{0,J,0}$ in Wilson-Fisher theory in the double-twist limit is given by

$$\gamma\left([\phi\,[\phi\phi^2]_{0,\ell}]_{0,J,0}\right) \overset{1\ll \ell^2 \ll J}{\sim} \frac{8}{3}\,\epsilon\left(\frac{\ell}{J}\right)^2 + \dots \tag{7.32}$$

Quite remarkably, this does indeed coincide with the anomalous dimension correction at the bottom of [63, Tab. 1], for even ℓ and $k = -m = \ell + 1 \gg 1$.

8 Conclusions and Outlook

In this paper, we employed a higher-point lightcone bootstrap approach to extract large-spin CFT data of triple-twist operators. Just like in the four-point lightcone bootstrap, our analysis of the six-point crossing equation in lightcone limits demonstrates that triple-twist CFT data asymptotes to generalized free field theory at large spin. The corresponding scalar/double-twist/triple-twist OPE coefficients $p_\ell^{J_i}$ are presented in eq. (5.19), with an asymptotic form given by eq. (5.36). Beyond this expected behavior, we successfully derived the triple-twist anomalous dimension matrix $\gamma_{k\ell}(J,\kappa)$ displayed in (6.19), and valid in the large-spin limit $1 \ll k,\ell,J$ with $k^2,\ell^2 = \mathcal{O}(J)$. Each of these results can describe triple-twist operators in a mixed-symmetry representation with two spin labels, J and κ.

To achieve our goal, we considered a six-point comb-to-comb crossing under a triple lightcone limit, tailored to project onto lowest-twist contributions for the direct channel and triple-twist exchanges in the crossed channel. To reproduce the finite number of terms of the direct channel from the crossed-channel expansion, we followed the approach of [24] and made crucial use of the Casimir and vertex differential operators that can be used to characterize conformal blocks [18–21]. This allowed us to determine both the relevant scaling of crossed-channel eigenvalues, cf. Section 2, and the crossed-channel lightcone blocks corresponding to those eigenvalues in Section 3. The expressions of six-point lightcone conformal blocks (3.36) and their normalization (3.37) are new, and their derivation constitutes a significant step forward in the higher-point bootstrap program.

To correctly interpret and express our results, it was necessary to discuss the parameterization of triple-twist operators and their degeneracy in generalized free field theory. Our discussion in Section 4 addressed this in the language of [49, 50], and generalized their framework to account for mixed-symmetry representations within the basis of triple-twist operators $[\phi[\phi\phi]_{0,\ell}]_{0,J,\kappa}$. This was central to the derivation of the Racah coefficients $W_{k\ell}^{(\Delta)}(J,\kappa)$ in the large-spin limit $1 \ll k = \mathcal{O}(J/\ell)$ presented

in eq. (5.25). These Racah coefficients are an important ingredient in the anomalous dimension matrix (6.19) of triple-twist operators.

Let us stress again that our results only apply to triple-twist operators supported on a subset of the entire range $0 \leq \ell \leq J$ spanned by the degeneracy parameter ℓ.[11] As we pointed out above, our formula (6.19) captures the regime $1 \ll \ell \ll J$. Near the 'edges' of this region of validity, i.e. when either $1 \ll \ell \ll \sqrt{J}$ or $\sqrt{J} \ll \ell \ll J$, the anomalous dimension matrix becomes diagonal in the double-twist basis and hence the triple-twist wave functions align with the elements of the double-twist basis. At the same time, the anomalous dimensions are simply given by the anomalous dimension of the corresponding double-twist constituent. But as ℓ moves further toward the center of the region of validity, i.e. $\ell = \mathcal{O}(\sqrt{J})$, mixing occurs within the double-twist basis and we obtain a strong deviation from the double-twist behavior and therefore a genuine feature of triple-twist operators that goes beyond the reach of the four-point lightcone bootstrap. In particular, the anomalous dimension matrix now scales as $J^{-h_\star}$.

It would be particularly interesting to explore the $\ell = \mathcal{O}(J)$ region, which is furthest away from the double-twist approximation and goes well beyond the domain of validity of formula (6.19). From explicit examples like ϕ^3 theory at one loop [50, 53], we expect this regime to describe the triple-twist operators with the lowest absolute anomalous dimension at large spin, scaling like $J^{-2h_\star}$ instead of $J^{-h_\star}$. However, to reach such operators from the six-point crossing equation of this paper, it appears necessary to relax the $\epsilon_{34} \to 0$ limit so that all three intermediate spins $J_a \propto \epsilon_{16}^{-1/2}$ scale in the same way. In the absence of the lightcone limit on X_{34}, we would then need to include all higher-twist exchange in the (34) OPE of the direct channel. As a result, any comprehensive analysis requires at least significant further work on lightcone blocks in both channels. On the other hand, the holographic description of triple-twist operators in AdS provides a useful and complementary approach to resolving their dynamics in this regime. This exploration of triple-twist dynamics based on effective field theory in AdS is the subject of upcoming work [71].

In addition to studying higher corrections to our analysis of six-point functions, it would certainly be relevant to go to correlation functions with more than six operator insertions. Some parts of our analysis seem relatively straightforward to extend. Indeed, it should be possible to bootstrap twist-M operators of the form $[\phi^{\times M}]_{0,J,\{\kappa\}}$ by studying a comb-channel crossing equation for $2M$

---

[11]As we discussed above, a triple-twist operator with anomalous dimension λ in any CFT is characterized by a wave function Ψ_λ, which admits an expansion in the double-twist basis $\Psi_{\ell,J,\kappa}$. The values of ℓ that 'contribute significantly' to this expansion may be referred to as the *support* of the triple-twist operator. The following statements in the text apply to triple-twist operators with support in certain subregions of the degeneracy parameter ℓ.

insertions of ϕ. A natural generalization of the six-point crossing equation we used above to $M > 3$ is depicted in Figure 7. By taking the lightcone limits $X_{1(2M)}, X_{3(2M-2)}, \ldots, X_{M(M+1)} \to 0$ and
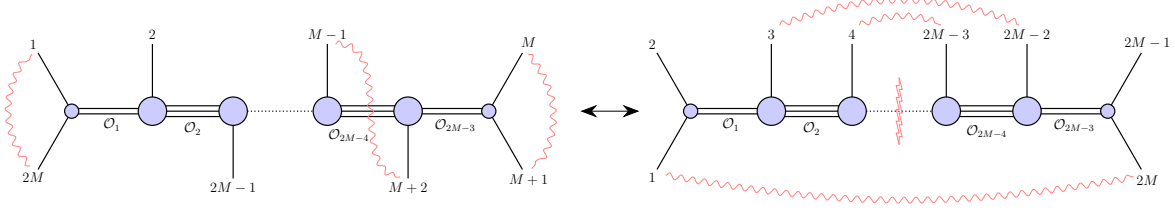

Figure 7. Proposed comb-to-comb duality to entail exchanges of twist-M operators in the middle leg of the crossed channel (right). The red wavy lines connect points whose distances are taken to be null and are meant to project on lowest-twist exchanges on all the internal legs of the direct channel (left). The CC middle-leg leading-twist limit $u_{M-1} \to 0$ is only represented on the right as a red line that cuts the diagram in half.

projecting to leading twist in the crossed-channel middle leg via $u_{M-1} \to 0$, we expect the direct-channel contribution with a maximal number of identity exchanges to be reproduced by a sequence of multi-twist exchanges $\mathcal{O}_a = [\phi^{\times(a+1)}]_{0,J_a,\{\kappa\}}$ for $a = 1, \ldots, M-1$, with increasingly large spin $J_a \ll J_{a+1}$. At next-to-leading order, the $M-1$ exchanges of one single leading-twist operator $\mathcal{O}_\star$ in the direct channel should then entail a large-spin anomalous dimension matrix of the form

$$\gamma = \gamma^{(1)} + \sum_{a=3}^{M} W^{(1a)} \gamma^{(1)} \left(W^{(1a)}\right)^{-1}, \tag{8.1}$$

where $\gamma^{(1)}$ has eigenvalues $\gamma_0 J_1^{-2h_\star}$ and $W^{(1a)}$ realizes the permutation $(1a) \in S_M$ in the space of twist-M operators. In fact, the $M-1$ terms we displayed are the natural extension of the two terms in eq. (6.19). To write down the corresponding twist-M OPE coefficients and anomalous dimensions explicitly, we can generalize the basis arising from the iterated construction of double-twist operators. In fact, in accordance with the connection to conformal blocks observed in this paper, the Derkachov-Manashov wave functions for these basis elements should correspond to eigenfunctions of the Gaudin integrable model on the M-punctured sphere associated with $\mathfrak{su}(1, r)$, where $r - 1$ is the number of non-vanishing transverse spins, in the same comb-channel limit used for conformal blocks, see [18, 19]. The only minor difference is that instead of imposing M-point conformal invariance as in the case of conformal blocks, the diagonal action of $\mathfrak{su}(1, r)$ on twist-M wave functions is now determined by the non-trivial quantum numbers of the twist-M primary that we exchange in the middle leg. In the absence of transverse spins, i.e. for $r = 1$, the method of separation of variables determines the dual wave functions $\tilde{\Psi}$ explicitly in terms of a product of $M-1$ Jacobi polynomials. These results should allow us to study the anomalous dimension matrix explicitly and diagonalize it in certain limits.

The analysis of higher multi-twist operators is particularly interesting in connection with recent work on large-charge effective descriptions of 3d CFTs[12] [72] (see also the follow-ups [73, 74]). In this context, it would be helpful to obtain a bootstrap understanding of the emergence of the giant vortex phase in the regime where the squared charge Q^2 approaches the spin J. One could possibly probe this regime by picking special configurations of Q pairs of charge ± 1 operator insertions at near-to-light-like separation, letting the $Q \to \infty$ limit and the lightcone limit be controlled by the same parameter[13]. As noted in [72], the asymptotics of such a regime are determined by a delicate balance between the combinatorics of different contributions to the OPEs in the correlator and the suppression of higher-twist terms by the lightcone limit. Concretely, [72] sketches a qualitative description of the situation to argue for a match between the boundaries of the Regge phase and giant vortex phase. However, this qualitative description is based on the assumption that the twist gap is exactly one. In contrast, a bootstrap analysis could provide important insight into the general dependence on the twist gap and the behavior that arises for generic values of it. Moreover, the intrinsically model-agnostic approach of the bootstrap could point towards direct generalizations to space-time dimensions other than three. Finally, a putative bootstrap description of the conjectured Regge-Giant-Vortex phase transition may help elucidate the qualitative similarities between the multi-twist Hilbert space of Regge theory (studied in this work) on the one hand, and the Hilbert space of fluctuations around the giant vortex on the other hand.

---

[12]We thank Alessio Miscioscia for enjoyable discussions on that topic.

[13]Note that this would differ from the analysis of lower-point correlators of highly charged operators in [75] that successfully provided a bootstrap perspective on the superfluid phase.

A Large-Spin Vertex Operators

This appendix provides a simple expression for the basis of conformal tensor structures for STT-MST-scalar correlators

$$\langle \mathcal{O}_1(X_1, Z_1)\mathcal{O}_2(X_2, Z_2, W_2)\mathcal{O}_3(X_3)\rangle \tag{A.1}$$

that diagonalizes the vertex operator

$$\mathcal{V} = \frac{1}{2}\operatorname{tr}\left(\mathcal{T}_1^3\mathcal{T}_3\right) - \frac{1}{16}\operatorname{tr}\left(\mathcal{T}_1^2\right)\operatorname{tr}\left(\mathcal{T}_2^2\right) + \frac{1}{8}\operatorname{tr}\left(\mathcal{T}_1^2\right)\operatorname{tr}\left(\mathcal{T}_3^2\right) + \frac{d(d-1)}{8}\left[\operatorname{tr}\left(\mathcal{T}_3^2\right) - \operatorname{tr}\left(\mathcal{T}_2^2\right)\right] + \frac{1}{4}\operatorname{tr}\left(\mathcal{T}_1^4\right), \tag{A.2}$$

in the regime

$$d, J_1, h_1, h_2, \kappa, \Delta_3 \ll J_2. \tag{A.3}$$

Concretely, in terms of the standard basis

$$\{\Omega_3 \mathcal{X}^n\}_{n=0}^{J_1-\kappa}, \quad \text{where} \quad \mathcal{X} = \frac{H_{21}X_{23}X_{31}}{J_{1,23}J_{2,13}} \quad \text{and} \quad \Omega_3 = \frac{U_{2,13}^\kappa J_{1,23}^{J_1-\kappa_2} J_{2,13}^{J_2-\kappa}}{X_{21}^{\bar{h}_{21;3}-\kappa} X_{23}^{\bar{h}_{23;1}-\kappa+J_1} X_{31}^{\bar{h}_{31;2}+J_2}}, \tag{A.4}$$

we find that a basis that diagonalizes $\mathcal{V}$ is given by

$$\{\Omega_3 g_\nu(\mathcal{X})\}_{\nu=0}^{J_1-\kappa} \quad \text{where} \quad g_\nu(\mathcal{X}) = (1-\mathcal{X})^{J_1-\kappa-\nu}\mathcal{X}^\nu + \mathcal{O}(1/J_2). \tag{A.5}$$

The corresponding eigenvalues are

$$\mathcal{V}g_\nu(\mathcal{X}) = J_2^2\left[(h_1 + J_1 - \nu)\left(2(h_1 + J_1 - \nu) - d\right) - 2\kappa_2\left(h_1 + J_1 - \nu - 1\right)\right]g_\nu(\mathcal{X}) + \mathcal{O}(J_2), \tag{A.6}$$

which can be verified to correspond to the parametrization of eigenvalues presented in [24, eq. (4.15)] with $N = J_1 - \nu$, once one sets $\kappa = 0$ and factors in the different shifts by Casimir operators in the definition of the vertex operator. Note that the basis $\{g_\nu\}_{\nu=0}^{J_1-\kappa}$ can be expressed as a simple polynomial basis if one redefines the prefactor and cross-ratios in use. Indeed, with

$$\tilde{\Omega}_3 := \Omega_3(1-\mathcal{X})^{J_1-\kappa} \qquad\qquad \tilde{\mathcal{X}} := \frac{\mathcal{X}}{1-\mathcal{X}}, \tag{A.7}$$

we find

$$\Omega_3 g_\nu(\mathcal{X}) = \tilde{\Omega}_3 \tilde{\mathcal{X}}^\nu. \tag{A.8}$$

Moreover, we would like to point out that the g_ν–basis is preferable over the standard basis for a description of vertex operators at finite J_2 in the sense that $\mathcal{V}$ is simply a tridiagonal matrix in the g_ν–basis, whereas in the standard basis the second diagonal above the main diagonal is non-vanishing.

B Normalization of Six-point Lightcone Blocks

In Subsection 3.5, we outlined the main steps in the derivation of the normalization $\mathcal{N}_{J_1(J_2,\kappa)J_3}$ of eq. (3.37) that enters formula (3.36) for large spin, crossed-channel lightcone blocks. In this appendix, we will provide explicit formulas for the second-order Casimir equations and their solutions that appear in the derivation.

B.1 Casimir equations for lightcone blocks.

Given the parameterization (3.40) of normalized lightcone blocks, the second-order Casimir equations with eigenvalues $\lambda_1(h_1,\bar{h}_1), \lambda_2(h_2,\bar{h}_2,\kappa), \lambda_3(h_3,\bar{h}_3)$ can be expressed as

$$(\mathcal{D}_{12}^2 - \lambda_1) g_{\mathcal{O}_i;n_s} \overset{\Upsilon_0,\bar{z}\to 0}{\sim} \prod_{i=1}^3 \bar{z}_i^{h_i} z_i^{\bar{h}_i} \prod_{s=1}^2 (1-w_s)^{n_s} \Upsilon_0^\kappa \; \mathcal{D}_1 \cdot \tilde{F}_{(h_i,\bar{h}_i,\kappa;n_s)}(z_1,z_2,z_3,w_1,w_2), \tag{B.1}$$

$$(\mathcal{D}_{456}^2 - \lambda_2) g_{\mathcal{O}_i;n_s} \overset{\Upsilon_0,\bar{z}\to 0}{\sim} \prod_{i=1}^3 \bar{z}_i^{h_i} z_i^{\bar{h}_i} \prod_{s=1}^2 (1-w_s)^{n_s} \Upsilon_0^\kappa \; \mathcal{D}_2 \cdot \tilde{F}_{(h_i,\bar{h}_i,\kappa;n_s)}(z_1,z_2,z_3,w_1,w_2), \tag{B.2}$$

$$(\mathcal{D}_{56}^2 - \lambda_3) g_{\mathcal{O}_i;n_s} \overset{\Upsilon_0,\bar{z}\to 0}{\sim} \prod_{i=1}^3 \bar{z}_i^{h_i} z_i^{\bar{h}_i} \prod_{s=1}^2 (1-w_s)^{n_s} \Upsilon_0^\kappa \; \mathcal{D}_3 \cdot \tilde{F}_{(h_i,\bar{h}_i,\kappa;n_s)}(z_1,z_2,z_3,w_1,w_2), \tag{B.3}$$

where, using the notation $j_i := J_i - \kappa$, the differential operators on the right-hand side are given by

$$\mathcal{D}_1 = \vartheta_{z_1}(\vartheta_{z_1} + 2\bar{h}_1 - 1) - z_1 \vartheta_{z_1}(\bar{h}_{12;\phi} - \kappa + \vartheta_{z_1} + \vartheta_{z_2} - \vartheta_{w_1} + w_1(n_1 - j_2 - \vartheta_{z_2} + \vartheta_{w_1})), \tag{B.4}$$

$$\mathcal{D}_3 = \vartheta_{z_3}(\vartheta_{z_3} + 2\bar{h}_3 - 1) - z_3 \vartheta_{z_3}(\bar{h}_{23;\phi} - \kappa + \vartheta_{z_2} + \vartheta_{z_3} - \vartheta_{w_2} + w_2(n_2 - j_2 - \vartheta_{z_2} + \vartheta_{w_2})), \tag{B.5}$$

and

$$\mathcal{D}_2 = \vartheta_{z_2}(\vartheta_{z_2} + 2\bar{h}_2 - \kappa - 1) - z_2(\bar{h}_{12;\phi} - \kappa + \vartheta_{z_1} + \vartheta_{z_2} - \vartheta_{w_1} + w_1(n_1 - j_1 - \vartheta_{z_1} + \vartheta_{w_1})) \tag{B.6}$$
$$(\bar{h}_{23;\phi} - \kappa + \vartheta_{z_2} + \vartheta_{z_3} - \vartheta_{w_2} + w_2(n_2 - j_3 - \vartheta_{z_3} + \vartheta_{w_2})),$$

This system of differential equations $\mathcal{D}_a \tilde{F} = 0$ with initial condition $\tilde{F}(0,w_s) = 1$ can be solved in terms of a hypergeometric power series in z_i, w_s, or expressed as an integral representation.

B.2 The limit $\epsilon_{34} \to 0$.

Now, to interpolate with the lightcone limit of crossed-channel blocks, we will directly solve the system of equations in the limit $\epsilon_{34} \to 0$. To do this, we will make the change of variables to $v_2 = 1 - z_2 = \mathcal{O}(\epsilon_{34})$ and $\mathcal{X}_s = 1 - w_s = \mathcal{O}(\epsilon_{34})$. Then in the limit $\epsilon_{34} \to 0$ with $J_2^2 = \mathcal{O}(\epsilon_{34}^{-1})$, we find at leading order

$$\mathcal{D}_{1,3} = \mathcal{D}_{\bar{h}_{1,3},h_{2\phi}+n_{1,2}}(z_{1,3}, \vartheta_{z_{1,3}}; \mathcal{X}_{1,2}, \vartheta_{\mathcal{X}_{1,2}}; -\partial_{v_2}) + \mathcal{O}(\epsilon_{14}), \tag{B.7}$$

$$\mathcal{D}_2 = \partial_{v_2}(\vartheta_{v_2} + h_{1\phi} + h_{3\phi} + \kappa + \vartheta_{\mathcal{X}_1} + \vartheta_{\mathcal{X}_2}) - J_2^2 + \mathcal{O}(\epsilon_{34}^0), \tag{B.8}$$

where the limiting form of the outer-leg Casimirs in eq. (B.7) are given by

$$\mathcal{D}_{\bar{h},\alpha}(z,\vartheta_z;x,\vartheta_x;\tau) := \vartheta_z(2\bar{h}+\vartheta_z-1) - z(\bar{h}+\vartheta_z)(\bar{h}+\alpha+\vartheta_x+\vartheta_z) - zx\tau(\bar{h}+\vartheta_z). \tag{B.9}$$

We assume that the solution takes the form

$$\tilde{F}_{(h_i,\bar{h}_i,\kappa;n_s)}(z_1,1-v_2,z_3,1-\mathcal{X}_1,1-\mathcal{X}_2) \overset{v_2,\mathcal{X}_s\to 0}{\sim}$$

$$\mathcal{N}^{4\mathrm{pt}}_{(h_i,\bar{h}_i,\kappa,n_s)} G_1(z_1,-z_1\mathcal{X}_1\partial_{v_2})G_3(z_3,-z_3\mathcal{X}_2\partial_{v_2})\mathcal{K}_{\alpha_2}(J_2^2 v_2), \tag{B.10}$$

where $\alpha_2 := h_{1\phi} + h_{3\phi} + n_1 + n_2 + \kappa$ and for $a=1,3$:

$$G_a(z_a,y_a) = \sum_{m,\nu=0}^{\infty} \frac{(\bar{h}_a+\alpha_a+\nu)_m}{m!}\frac{1}{\nu!}\frac{(\bar{h}_a)_{m+\nu}}{(2\bar{h}_a)_{m+\nu}} z_a^m y_a^\nu, \quad (\alpha_1,\alpha_3) := (h_{2\phi}+n_1+\kappa, h_{2\phi}+n_2+\kappa). \tag{B.11}$$

To prove this ansatz and determine $\mathcal{N}^{4\mathrm{pt}}$, we compare the solution of the quadratic Casimir equation $\mathcal{D}_2\tilde{F}_{\mathcal{O}_i;n_s}(z_1=0,z_2,z_3=0,w_s=1)=0$, which is $_2F_1\big(\bar{h}_2+h_{1\phi}+n_1+\kappa, \bar{h}_2+h_{3\phi}+n_2+\kappa; 2\bar{h}_2-\kappa; z_2\big)$ when normalized by the OPE limit, to the expression (B.10) evaluated at $z_1,z_3,\mathcal{X}_1,\mathcal{X}_2=0$. Using the identity

$$\lim_{\bar{h}\to\infty} \frac{\Gamma(\bar{h}+c-a)\Gamma(\bar{h}+c-b)}{2\,\Gamma(2\bar{h}+c)} \, _2F_1\begin{bmatrix} \bar{h}+a,\bar{h}+b \\ 2\bar{h}+c \end{bmatrix}(1-\bar{h}^{-2}x) = \mathcal{K}_{a+b-c}(x), \tag{B.12}$$

we deduce that $\mathcal{N}^{4\mathrm{pt}}$ is given by eq. (3.42).

B.3 The limit $\epsilon_{16}\to 0$

To reach the kinematical limit of crossed-channel lightcone blocks, we change variables to

$$\mathcal{X}_1 = \frac{(1-z_1)v_2}{z_1}x_1, \quad \mathcal{X}_2 = \frac{v_2(1-z_3)}{z_3}x_2, \quad (x_1,x_2) := (1-\mathcal{U}_1-\mathcal{U}^6, \mathcal{U}_1). \tag{B.13}$$

Next, we rewrite the functions G_a as

$$G_a(y_a,z_a) = \frac{1}{(1-z_a)^{\alpha_a}} \sum_{\nu_a=0}^{\infty} \frac{1}{\nu!}\frac{(\bar{h}_a)_\nu}{(2\bar{h}_a)_\nu}\left(\frac{y_a}{1-z_a}\right)^\nu \, _2F_1\begin{bmatrix} \bar{h}_a,\bar{h}_a-\alpha_a \\ 2\bar{h}_a-\nu \end{bmatrix}(z_a) \tag{B.14}$$

$$= \frac{1}{(1-z_a)^{\alpha_a}} \sum_{m_a=0}^{\infty} \frac{z_a^{m_a}}{m_a!}\frac{(\bar{h}_a)_{m_a}(\bar{h}_a-\alpha_a)_{m_a}}{(2\bar{h}_a)_{m_a}} \, _1F_1\begin{bmatrix} \bar{h}_a \\ 2\bar{h}_a+m_a \end{bmatrix}\left(\frac{y_a}{1-z_a}\right). \tag{B.15}$$

After applying the identity $y^{-\alpha}\mathcal{K}_\alpha(y) = \mathcal{K}_{-\alpha}(y)$ and inserting the integral representation of $\mathcal{K}_{-\alpha}$, we then obtain the following expression for $F(v_i,x_s) \equiv v_1^{\alpha_1}v_2^{\alpha_2}v_3^{\alpha_3}\tilde{F}(1-v_i,1-\mathcal{X}_s(v_s,v_{s+1},x_s))$:

$$F_{(h_i,\bar{h}_i,\kappa:n_s)}(v_i,x_s) \overset{\epsilon_{34}\to 0}{\sim} \frac{\mathcal{N}^{4\mathrm{pt}}_{(h_i,\bar{h}_i,\kappa;n_s)}}{2J_2^{2\alpha_2}} \int_0^\infty \frac{\mathrm{d}t}{t^{1-\alpha_2}} e^{-\left(t+\frac{v_2 J_2^2}{t}\right)} \tag{B.16}$$

$$\sum_{m_a=0}^{\infty} \frac{z_a^{m_a}}{m_a!}\frac{(\bar{h}_a)_{m_a}(h_{a2;\phi}-\delta n_a)_{m_a}}{(2\bar{h}_a)_{m_a}} \, _1F_1\begin{bmatrix} \bar{h}_a \\ 2\bar{h}_a+m_a \end{bmatrix}(tx_{[a]}),$$

where $(x_{[1]}, x_{[3]}) := (x_1, x_2)$ and $(\delta n_1, \delta n_3) := (J_1 - n_1, J_3 - n_2)$. We now define the lightcone limit $X_{16} = \mathcal{O}(\epsilon_{16})$, $\epsilon_{16} \to 0$ where the cross-ratios and quantum numbers scale with ϵ_{16} as follows:

$$\left(v_a, \mathcal{U}_1, \mathcal{U}^6; J_2, J_a, \delta n_a\right) \propto (1, 1, \epsilon_{12}; \epsilon_{12}^{-1/2}, \epsilon_{12}^{-1/2}, 1). \tag{B.17}$$

To retrieve the limiting form (3.36) of blocks in this regime, we must first recall the prefactor that relates $F = \prod_a v_a^{\alpha_a} \tilde{F}$, defined by eq. (3.40) to $\tilde{g}$, defined by eq. (3.6):

$$\tilde{g} = x_0^{\kappa} \prod_{a=1,3} z_a^{\delta n_a} x_{[a]}^{j_a - \delta n_a} F, \quad x_0 := \frac{z_1 z_2 z_3 \Upsilon_0}{v_1 v_2 v_3}. \tag{B.18}$$

We then write the expansion (B.16) as

$$x_0^{-\kappa} \tilde{g} = \mathcal{N}_{(h_i, \bar{h}_i, \kappa; n_s)}^{\text{4pt}} \prod_{a=1,3} x_{[a]}^{-(h_a + \delta n_a + \kappa)} \sum_{m_a=0}^{\infty} \frac{z_a^{m_a}}{m_a!} \frac{(\bar{h}_a)_{m_a} (h_{a2;\phi} - \delta n_a)_{m_a}}{(2\bar{h}_a)_{m_a}} I_{\bar{h}_1 \bar{h}_3}^{m_1 m_3}, \tag{B.19}$$

where

$$I_{\bar{h}_1 \bar{h}_3}^{m_1 m_3} := \frac{1}{2} J_2^{-2\alpha_0} \int_0^{\infty} \frac{\mathrm{d}k}{k^{1+\alpha_0}} e^{-\left(kJ_2^2 + \frac{v_2}{k}\right)} \prod_{a=1,3} (kJ_2^2 x_{[a]})^{\bar{h}_a} \, {}_1F_1 \begin{bmatrix} \bar{h}_a \\ 2\bar{h}_a + m_a \end{bmatrix} (kJ_2^2 x_a), \tag{B.20}$$

and $\alpha_0 := 2h_\phi + \kappa + \delta n_1 + \delta n_2$. In the limit $J_2^2 = \mathcal{O}(\epsilon_{16}^{-1})$, we can expand the confluent hypergeometric functions at large argument using [61, eq. (13.7.1)], which yields

$$(kJ_2^2 x_{[a]})^{\bar{h}_a} \, {}_1F_1 \begin{bmatrix} \bar{h}_a \\ 2\bar{h}_a \end{bmatrix} (kJ_2^2 x_{[a]}) = \frac{\Gamma(2\bar{h}_a)}{\Gamma(\bar{h}_a)} (kJ_2^2 x_{[a]})^{-m_a} \exp\left(kJ_2^2 x_{[a]} - \frac{\bar{h}_a^2}{kJ_2^2 x_{[a]}} \right) \left(1 + \mathcal{O}(\epsilon_{12}^{1/2})\right).$$

Given $x_1 + x_2 = 1 - \mathcal{U}^6$ and $x_2 = \mathcal{U}_1$, we then obtain

$$I_{\bar{h}_1 \bar{h}_3}^{m_1 m_3} \overset{\epsilon_{16} \to 0}{\sim} J_2^{-2(m_1 + m_3)} \left(\mathcal{U}^6\right)^{\alpha_0 + m_1 + m_2} \mathcal{K}_{\alpha_0 + m_1 + m_2} \left(\left[\frac{\bar{h}_1^2}{1 - \mathcal{U}_1} + J_2^2 v_2 + \frac{\bar{h}_3^2}{\mathcal{U}_1} \right] \mathcal{U}^6 \right). \tag{B.21}$$

Inserting this leading form back into eq. (B.19), one finds that $I^{m_1 m_3}$ is suppressed by a factor $J_2^{-2(m_1 + m_3)}$ relative to I^{00}. We then take the leading term $m_a = 0$ to retrieve the asymptotics of crossed-channel lightcone blocks:

$$\tilde{g}_{(h_i, h_i + J_i, \kappa; J_{[s]} - \delta n_{[s]})} \overset{\epsilon_{16} \to 0}{\sim} \frac{\mathcal{N}_{(h_i, h_i + J_i, \kappa; J_{[s]} - \delta n_{[s]})}^{\text{4pt}}}{J_2^{2(h_1 + h_2 + J_1 + J_3 - 2h_\phi - \kappa - \delta n_1 - \delta n_2)}} \times$$

$$\times \prod_{a=1,3} \frac{\Gamma(2\bar{h}_a)}{\Gamma(\bar{h}_a)} G_{(J_1, J_2, \kappa, J_3); \delta n_1 \delta n_3}^{h_1 - 2h_\phi, h_3 - 2h_\phi}(v_2, \mathcal{U}_1, 1 - \mathcal{U}_1, \mathcal{U}^6). \tag{B.22}$$

In the case $(h_1, h_2, h_3; \delta n_1, \delta n_3) = (2h_\phi, 3h_\phi, 2h_\phi; 0, 0)$, one deduces formula (3.37) for the normalization of six-point blocks with GFF twists exchanged.

C   Leading-Twist GFF Correlators from Representation Theory

In this appendix, we address the derivation of the GFF correlator of a triple-twist operator and three scalars displayed in (4.40). After introducing the necessary background on the highest-weight representation theory of $\mathfrak{su}(1,2)$, we then derive the generalization of the latter, displayed in (4.61), to triple-twist operators in MST representations.

C.1   STT sector: derivation of eq. (4.40)

Performing the Wick contractions in the GFF correlator that defines $\Psi_{\ell,J}$ leads to

$$\Psi_{\ell,J} = \check{\Psi}_{\ell,J}(\partial_{\bar{\alpha}_i}) \sum_{\sigma \in A_3} \prod_{i=1}^{3} X_{ib}^{\Delta_\phi} (X_{\sigma(i)b} + \bar{\alpha}_i Z X_{\sigma(i)})^{-\Delta_\phi}. \tag{C.1}$$

Inserting (4.31) for $\check{\psi}$ in eq. (4.36) furthermore yields

$$\check{\Psi}_{L,0}(\partial_{\bar{\alpha}_1}, \partial_{\bar{\alpha}_2}) = c_{J-\ell} c_\ell \sum_{n=0}^{J-\ell} \sum_{k=0}^{n} \sum_{j=0}^{\ell} \frac{(\ell - J)_n}{n!} \binom{n}{k} \frac{(-\ell)_j}{j!} \frac{\partial_{\bar{\alpha}_1}^{k+j} \partial_{\bar{\alpha}_2}^{\ell+n-k-j} \partial_{\bar{\alpha}_3}^{J-\ell-n}}{(2\Delta_\phi + 2\ell)_n (\Delta_\phi)_{J-\ell-n} (\Delta_\phi)_j (\Delta_\phi)_{\ell-j}}. \tag{C.2}$$

Therefore,

$$\Psi_{\ell,J}(\alpha_1, \alpha_2, \alpha_3) = \left\| \alpha_{12}^\ell \right\|_{\Delta_\phi,\Delta_\phi}^{-1} \left\| \alpha_{12}^{J-\ell} \right\|_{\Delta_\phi,2\Delta_\phi+2\ell}^{-1} \sum_{\sigma \in A_3} g_{\ell,J}^{(12)}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}), \tag{C.3}$$

with

$$g_{\ell,J}^{(12)} = \sum_{n=0}^{J-\ell} \sum_{k=0}^{n} \sum_{j=0}^{\ell} \frac{(\ell - J)_n}{n!} \binom{n}{k} \frac{(-\ell)_j}{j!} \frac{\alpha_1^{k+j} \alpha_2^{\ell+n-k-j} \alpha_3^{J-\ell-n} (\Delta_\phi)_{k+j} (\Delta_\phi)_{\ell+n-k-j}}{(2\Delta_\phi + 2\ell)_n (\Delta_\phi)_j (\Delta_\phi)_{\ell-j} (-1)^J}. \tag{C.4}$$

On the one hand, by translation symmetry, we obtain the same function by evaluating $\alpha_1 = 0$ while setting $\alpha_2 = -\alpha_{12}$ and $\alpha_3 = -\alpha_{13}$. On the other hand, the condition $\alpha_1 = 0$ directly trivialises two of the sums in eq. C.4 and we conclude that

$$g_{\ell,J}^{(12)} = \sum_{n=0}^{J-\ell} \frac{(\ell - J)_n}{n!} \frac{\alpha_2^{\ell+n} \alpha_3^{J-\ell-n} (\Delta_\phi)_{\ell+n}}{(2\Delta_\phi + 2\ell)_n (\Delta_\phi)_\ell (-1)^J} = \alpha_{12}^\ell \alpha_{13}^{J-\ell} \sum_{n=0}^{J-\ell} \frac{(\ell - J)_n (\Delta_\phi + \ell)_n}{(2\Delta_\phi + 2\ell)_n n!} \left(\frac{\alpha_{12}}{\alpha_{13}} \right)^n. \tag{C.5}$$

C.2   MST sector and representation theory of $\mathfrak{su}(1,2)$

C.2.1   Generators and action on polynomials

The generators of $\mathfrak{sl}(3)$ in a lowest-weight representation are given in [76, Sec. 3.1] as first order differential operators $\mathrm{T}_{ij}(x,y,z,\partial_x,\partial_y,\partial_z; m, n)$ and $\mathrm{H}_i(x,y,z,\partial_x,\partial_y,\partial_z; m, n)$ acting on power series in three variables x, y, z. In this realization, the unit-normalized highest-weight vector is 1. The relation to generators in our notation is then given by

$$S_{ij}(\alpha, \beta, \gamma, \partial_\alpha, \partial_\beta, \partial_\gamma) := \mathrm{T}_{ij}(\alpha, \beta, \gamma, -\partial_\alpha, -\partial_\beta, \partial_\gamma; J - \kappa, \kappa - \bar{\tau}), \quad 1 \leq i \neq j \leq 3, \tag{C.6}$$

$$S_{0i}(\alpha, \beta, \gamma, \partial_\alpha, \partial_\beta, \partial_\gamma) := \frac{1}{2} \mathrm{H}_i(\alpha, \beta, \gamma, \partial_\alpha, \partial_\beta, \partial_\gamma; J - \kappa, \kappa - \bar{\tau}), \quad i = 1, 2 \tag{C.7}$$

The lowest-weight vector in this realization is $S_{ki} \cdot 1 = 0$, $k > i$, with weights $(S_{01}, S_{02}) \cdot 1 = (\bar{\tau} - \kappa, \kappa - J)$. The Hilbert space is then spanned by vectors $\psi_{r,s,t}(\alpha, \beta, \gamma) = S_{12}^r S_{23}^s S_{13}^t \cdot 1$. When $J - \kappa > 0$ is an integer, the corresponding basis is restricted to $s \leq J - \kappa$.

C.2.2 Duality, scalar product and norms

Define $(\delta, j) := (\bar{\tau} - \kappa, J - \kappa)$. The action of the triangular subalgebra spanned by (S_{12}, S_{23}, S_{13}) is then given by

$$\left(e^{\bar{\alpha} S_{12}} \cdot \psi\right)(\alpha, \beta, \gamma) = (1 - \bar{\alpha}\alpha)^{-\delta} \psi\left(\frac{\alpha}{1 - \bar{\alpha}\alpha}, \frac{\beta}{1 - \bar{\alpha}\alpha}, \gamma - \bar{\alpha}(\alpha\gamma - \beta)\right), \tag{C.8}$$

$$\left(e^{\bar{\gamma} S_{23}} \cdot \psi\right)(\alpha, \beta, \gamma) = (1 + \bar{\gamma}\gamma)^j \psi\left(\alpha - \bar{\gamma}\beta, \beta, \frac{\gamma}{1 + \bar{\gamma}\gamma}\right), \tag{C.9}$$

$$\left(e^{\bar{\beta} S_{13}} \cdot \psi\right)(\alpha, \beta, \gamma) = \frac{(1 + \bar{\beta}(\alpha\gamma - \beta))^j}{(1 - \bar{\beta}\beta)^\delta} \psi\left(\frac{\alpha}{1 - \bar{\beta}\beta}, \frac{\beta}{1 - \bar{\beta}\beta}, \frac{\gamma}{1 + \bar{\beta}(\gamma\alpha - \beta)}\right). \tag{C.10}$$

From these properties, it is easy to show that the duality $\check{\psi} \mapsto \psi$ in eq. (4.50) satisfies

$$\psi(\alpha, \beta, \gamma) = \check{\psi}(\partial_{\bar{\alpha}}, \partial_{\bar{\beta}}, \partial_{\bar{\gamma}}) \frac{\left[1 + \bar{\gamma}\gamma - (\bar{\alpha}\bar{\gamma} - \bar{\beta})(\alpha\gamma - \beta)\right]^j}{(1 - \bar{\alpha}\alpha - \bar{\beta}\beta)^\delta}\bigg|_0 \tag{C.11}$$

$$= \check{\psi}(\partial_{\bar{\alpha}}, \partial_{\bar{\beta}}, \partial_{\bar{\gamma}}) e^{\bar{\alpha} S_{12}} e^{\bar{\gamma} S_{23}} e^{\bar{\beta} S_{13}} \cdot 1 \tag{C.12}$$

$$= \mathrm{L}\left\{\check{\psi}(S_{12}, S_{23}, S_{13})\right\} \cdot 1, \tag{C.13}$$

where $\mathrm{L}\{\}$ now denotes left-ordering of S_{12} with respect to S_{23} (both of which commute with S_{13}). The inverse map $\psi \mapsto \check{\psi}$ is complicated by the fact that $S_{12}^r S_{23}^s S_{13}^t \cdot 1$ is no longer a monomial in α, β, γ. This, in turn, reflects the fact that the monomials β and $\alpha\gamma$ have the same homogeneity degree with respect to the Cartan subalgebra of $\mathfrak{sl}(3)$, such that the overlaps of states $(r, s, t) \neq (r', s', t')$ need not vanish for $(r - s, s + t) = (r' - s', s' + t')$. Fortunately, we need only compute norms that come from the two following subsets of states:

1. $\psi \in \mathrm{Span}(S_{12}^r S_{13}^t \cdot 1)$. Scalar products between these states are obtained by applying eq. (4.50) with $\bar{\gamma} = 0 = \gamma$, and for monomials we obtain

$$\partial_\alpha^r \partial_\beta^t \longleftrightarrow (\delta)_r (\delta - j + r)_t \, \alpha^r \beta^t, \quad \langle \alpha^r \beta^t, \alpha^{r'} \beta^{t'} \rangle_{\delta, j} = \frac{\delta_{rr'} \delta_{tt'} r! t!}{(\delta)_r (\delta - j + r)_t}. \tag{C.14}$$

In the specific case $j = 0$, it will be useful to instead rewrite the monomial norms in terms of shifted norms for the $\mathfrak{su}(1, 1)$ subgroup, $\|\alpha^r\|_{\bar{\tau}}^2 = r!/(\bar{\tau})_r$, such that

$$\left\|\psi(\alpha)\beta^t\right\|_{\delta, 0}^2 = t! \, \|\psi(\alpha)\|_{\bar{\tau}+t}^2. \tag{C.15}$$

2. $\psi \in \mathrm{Span}(S_{12}^r S_{23}^s \cdot 1)$. Scalar products between these states are obtained by applying eq. (4.50) with $\bar{\beta} = 0 = \beta$, and for monomials we obtain

$$\partial_\alpha^r \partial_\gamma^s \longleftrightarrow (\delta - s)_r (-j)_s (-1)^s \alpha^r \gamma^s, \quad \langle \alpha^r \gamma^s, \alpha^{r'} \gamma^{s'} \rangle_{\delta,j} = \frac{\delta_{rr'} \delta_{ss'} r! s!}{(\bar{\tau} - \kappa - s)_r (\kappa - J)_s (-1)^s}. \quad \text{(C.16)}$$

It is again useful to rewrite these norms in terms of the $\mathfrak{su}(1,1)$ norms $\|\alpha^r\|_{\bar{\tau}}^2 = r!/(\bar{\tau})_r$, such that

$$\|\psi(\alpha)\gamma^s\|_{\delta,j}^2 = \binom{j}{s}^{-1} \|\psi(\alpha)\|_{\delta-s}^2. \quad \text{(C.17)}$$

C.2.3 Norm and dual of the double-twist wave function

Up to normalization, the unique lowest-weight vector of weight $(\bar{\tau}_1 + \bar{\tau}_2 + J + \kappa, J - \kappa)$ in the tensor product of two $\mathfrak{su}(1,2)$ representations of weights $(\bar{\tau}_1, 0)$ and $(\bar{\tau}_2, \ell)$ is given by

$$\frac{\psi_{J-\ell,\kappa,0}(\alpha_a, \beta_a, \gamma_a, \alpha_3, \beta_3)}{\psi_{J-\ell,\kappa,0}(0,0,0,1,1)} = \alpha_{a3}^{J-\ell} (\beta_{a3} - \gamma_a \alpha_{a3})^\kappa \quad \text{(C.18)}$$

$$= \alpha_3^{J-\ell} \beta_3^\kappa (1 - \alpha_3^{-1}\alpha_a)^{J-\ell} \left[1 - \beta_3^{-1}\alpha_3\gamma_a + \beta_3^{-1}(\gamma_a\alpha_a - \beta_a)\right]^\kappa. \quad \text{(C.19)}$$

In the last line, we expressed the dependence on $\alpha_a, \beta_a, \gamma_a$ in the same way as eq. (C.11) with $\bar{\alpha}_a = \alpha_3^{-1}$, $\bar{\gamma}_a = -\beta_3^{-1}\alpha_3$, and $\bar{\beta}_a = 0$. This implies that $\psi_{J-\ell,\kappa,0} \in \mathrm{Span}(S_{12}^r S_{13}^t \cdot 1) \otimes \mathrm{Span}(S_{12}^r S_{23}^s \cdot 1)$, and that the norm can be computed in terms of scalar products of vectors in this subspace. In particular, we can set $\beta_2 = 0$ and expand the norm in terms of the norms of eq. (C.16),(C.17) as follows:

$$\left\|\alpha_{a3}^L (\beta_{a3} - \gamma_a \alpha_{a3})^\kappa\right\|_{\bar{\tau}_a, \ell, \bar{\tau}_3}^2 = \left\|\alpha_{a3}^L (\gamma_a \alpha_{a3} + \beta_3)^\kappa\right\|_{\bar{\tau}_a, \ell, \bar{\tau}_3}^2 \quad \text{(C.20)}$$

$$= \sum_{q=0}^{\kappa} \binom{\kappa}{q}^2 \left\|\beta_1^{\kappa-q} \gamma_2^q \alpha_{12}^{L+q}\right\|_{\bar{\tau}_a, \ell, \bar{\tau}_3}^2 \quad \text{(C.21)}$$

$$= \frac{\kappa!}{\ell!} \sum_{q=0}^{\kappa} \binom{\kappa}{q} \frac{(\ell-q)!}{(\bar{\tau}_3)_{\kappa-q}} \left\|\alpha_{a3}^{L+q}\right\|_{\bar{\tau}_a - q, \bar{\tau}_3 + \kappa - q}^2. \quad \text{(C.22)}$$

It also follows from this analysis that the dual wave function for $\partial_{\bar{\beta}_a} = 0$ is given by

$$\check{\psi}(x_a, 0, z_a, x_3, y_3) = \frac{\psi_{J-\ell,\kappa,0}(0,0,0,1,1)}{\psi_{J-\ell,0}(0,1)} \sum_{q=0}^{\kappa} \binom{\kappa}{q} \frac{z_a^q}{(-1)^q(-\ell)_q} \frac{y_3^{\kappa-q}}{(\bar{\tau}_3)_{\kappa-q}} \check{\psi}_{L+q,0}^{(\bar{\tau}_a - q, \bar{\tau}_3 + \kappa - q)}(x_a, x_3). \quad \text{(C.23)}$$

C.2.4 Derivation of eq. (4.61)

The wave function $\Psi_{\ell,J,\kappa}(\alpha_i, \beta_i)$ corresponding to $[\phi \mathcal{O}_\ell]_{0,J,\kappa}$, $\mathcal{O}_\ell := [\phi\phi]_{0,\ell}$ is given by

$$\Psi_{\ell,J,\kappa}(\alpha_i, \beta_i) = \sum_{\sigma \in A_3} \Psi_{\ell,J,\kappa}^{(12)}(\alpha_{\sigma(i)}, \beta_{\sigma(i)}), \quad \text{(C.24)}$$

where each permutation amounts to a Wick contraction in the six-point function of ϕ. The (non-symmetric) wave function $\Psi^{(12)}$ corresponding to a single permutation takes the form

$$\Psi^{(12)}_{\ell,J,\kappa}(\alpha_i,\beta_i) = \check{\psi}_{J-\ell,\kappa,0}(\partial_{\bar{\alpha}_a},\partial_{\bar{\beta}_a},\partial_{\bar{\gamma}_a},\partial_{\bar{\alpha}_3},\partial_{\bar{\beta}_3})\check{\psi}_{\ell,0}(\partial_{\bar{\alpha}_1},\partial_{\bar{\alpha}_2})\prod_{i=1}^{3}(1-\xi_i\alpha_i-\eta_i\beta_i)^{-\Delta_\phi}, \qquad \text{(C.25)}$$

where

$$(\xi_1,\xi_2,\xi_3) := (\bar{\alpha}_1+\bar{\alpha}_a,\bar{\alpha}_2+\bar{\alpha}_a,\bar{\alpha}_3) \qquad \text{(C.26)}$$

$$(\eta_1,\eta_2,\eta_3) := (\bar{\beta}_a-\bar{\gamma}_a\bar{\alpha}_1,\bar{\beta}_a-\bar{\gamma}_a\bar{\alpha}_2,\bar{\beta}_3). \qquad \text{(C.27)}$$

The lowest-weight condition uniquely defines $\Psi^{(12)}_{\ell,J,\kappa}(\alpha_i,\beta_i)$ in terms of $\Psi^{(12)}_{\ell,J,\kappa}(\alpha_1,0,\alpha_2,0,\alpha_3,1)$, which reduces to

$$\Psi^{(12)}_{\ell,J,\kappa}(\alpha_1,0,\alpha_2,0,\alpha_3,1) = \check{\psi}_{J-\ell,\kappa,0}|_{\partial_{\bar{\beta}_a},\partial_{\bar{\gamma}_a}=0}\check{\psi}_{\ell,0}(1-\bar{\alpha}_1\alpha_1)^{-\Delta_\phi}(1-\bar{\alpha}_2\alpha_2)^{-\Delta_\phi}(1-\bar{\alpha}_3\alpha_3-\bar{\beta}_3\beta_3)^{-\Delta_\phi}|_0.$$

The dual polynomial $\check{\psi}_{J-\ell,\kappa,0}(x_a,0,0,x_3,y_3)$ is a special case of eq. (C.23), which after insertion leads to

$$\Psi^{(12)}_{\ell,J,\kappa}(\alpha_1,0,\alpha_2,0,\alpha_3,1) = C\,\check{\psi}^{(\bar{\tau}_a,\Delta_\phi+\kappa)}_{J-\ell,0}(\partial_{\bar{\alpha}_1}+\partial_{\bar{\alpha}_2},\partial_{\bar{\alpha}_3})\check{\psi}^{(\Delta_\phi,\Delta_\phi)}_{\ell,0}(\partial_{\bar{\alpha}_1}+\partial_{\bar{\alpha}_2},\partial_{\bar{\alpha}_3})\frac{\partial^\kappa_{\bar{\beta}_3}}{(\Delta_\phi)_\kappa}$$
$$\times(1-\bar{\alpha}_1\alpha_1)^{-\Delta_\phi}(1-\bar{\alpha}_2\alpha_2)^{-\Delta_\phi}(1-\bar{\alpha}_3\alpha_3-\bar{\beta}_3\beta_3)^{-\Delta_\phi}|_{\bar{\alpha}_i,\bar{\beta}_3=0},$$

where $\bar{\tau}_a := 2\Delta_\phi+2\ell$ and C is the ratio of $\psi_{J-\ell,\kappa,0}(0,0,0,1,1)$ by $\psi_{J-\ell,0}(0,1)$. The action of $(\Delta_\phi)^{-1}_\kappa\partial^\kappa_{\bar{\beta}_3}|_{\bar{\beta}_3=0}$ has the effect of shifting the exponent of $(1-\bar{\alpha}_3\alpha_3)$ by $-\kappa$. The remaining expression is equivalent to the double-twist basis wave function $g^{(12)}_{\ell,J}$ at $\kappa=0$, but with shifted arguments $(J,\ell,\Delta_\phi)\to(J-\kappa,\ell-\kappa,\Delta_\phi+\kappa)$. Since the $g^{(12)}_{\ell,J}$ as defined in eq. (4.40) is invariant under this shift, we retrieve the formula

$$\Psi^{(12)}_{\ell,J,\kappa}(\alpha_1,0,\alpha_2,0,\alpha_3,1) = C\,\psi^{(12)}_{\ell,J}(\alpha_1,\alpha_2,\alpha_3). \qquad \text{(C.28)}$$

To retrieve the full result $\Psi_{\ell,J,,\kappa}$, we first reintroduce the full dependence on the β_i by virtue of the lowest-weight condition:

$$\Psi^{(12)}_{\ell,J,\kappa}(\alpha_i,\beta_i) = \frac{(\alpha_{12}\beta_{13}-\alpha_{13}\beta_{12})^\kappa}{\alpha^\kappa_{12}}\Psi^{(12)}_{\ell,J,\kappa}(\alpha_1,0,\alpha_2,0,\alpha_3,1). \qquad \text{(C.29)}$$

The final result then reduces to

$$\Psi_{\ell,J,\kappa}(\alpha_i,\beta_i) = C(\alpha_{12}\beta_{13}-\alpha_{13}\beta_{12})^\kappa\sum_{\sigma\in S_3}\alpha^{-\kappa}_{\sigma(1)\sigma(2)}\psi^{(12)}_{\ell,J}(\alpha_{\sigma(i)}). \qquad \text{(C.30)}$$

D OPE Limits to Lower-Point Functions

The scope of this appendix is to use OPE limits from six-point conformal blocks to four-point ones to work out some of the results relevant for Section 5. In Appendix D.1, we provide a derivation of the GFF four-point blocks presented in (5.1). In the second part, Appendix D.2, we both derive the relation between cross-ratios and the coordinates α_i presented in (5.6), as well as deriving the multiplicative prefactor used in (5.9) to express the GFF four-point correlator $\langle \phi(X_1)\phi(X_2)\phi(X_3)[\phi\phi\phi]_{0,J,\kappa}(X_b, Z_b)\rangle$ in terms of linear combinations of four-point lightcone blocks $g_{\mathcal{O}_1;n_1}(z_1, w_1)$.

D.1 GFF four-point blocks from six-point OPE limit

The spinning four-point lightcone blocks can be obtained from an OPE limit of six-point lightcone blocks. For this reason, we consider again the conformal block decomposition of the six-point function in OPE cross-ratios:

$$\langle \phi(X_1)\dots\phi(X_6)\rangle = \sum_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3;n_1,n_2} P^{(n_1 n_2)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3} \frac{g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1 n_2}(\bar{z}_1, z_1, \bar{z}_2, z_2, \Upsilon_0, \bar{z}_3, z_3, w_1, w_2)}{(X_{12}X_{34}X_{56})^{\Delta_\phi}(\bar{z}_2 z_2)^{\Delta_\phi/2}}. \tag{5.2}$$

Taking the OPE limit in the second and third leg along with the lightcone limit $\bar{z}_1 \to 0$, we obtain the lightcone blocks for the four-point function in eq. (4.37) (and eq. (4.57) for $\kappa > 0$):

$$g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1 n_2} \sim \bar{z}_1^{h_1} g_{\mathcal{O}_1;n_1}(z_1, w_1)(\bar{z}_2 z_2)^{h_2} z_2^{J_2+\kappa} \Upsilon_0^\kappa (1-w_2)^{n_2}(\bar{z}_3 z_3)^{h_3} z_3^{J_3}. \tag{D.1}$$

The multiplicative prefactor relating linear combinations of $\psi_{\mathcal{O}_1;n_1}$ to the four-point correlator itself is derived in Appendix D.2. After setting

$$g_{\mathcal{O}_1;n_1}(z_1, w_1) = z_1^{\bar{h}_1}(1-w_1)^{n_1} f_{\mathcal{O}_1;n_1}(z_1, w_1) \tag{D.2}$$

, the second-order Casimir equation for the spinning lightcone block reduces to $\mathcal{D}_1 f_{\mathcal{O}_1;n_1} = 0$, where

$$\mathcal{D}_1 = \vartheta_{z_1}(\vartheta_{z_1} + 2\bar{h}_1 - 1) - z_1 \vartheta_{z_1}(\bar{h}_{12;\phi} - \kappa + \vartheta_{z_1} + \vartheta_{z_2} - \vartheta_{w_1} + w_1(n_1 - j_2 - \vartheta_{z_2} + \vartheta_{w_1})). \tag{D.3}$$

The corresponding solution satisfying the OPE-limit normalization $f_{\mathcal{O}_1;n_1}(0, w_1) = 1$ is given by

$$f_{\mathcal{O}_1;n_1}(z_1, w_1) = F_1(\bar{h}_1; \bar{h}_{12;\phi} - \kappa, n_1 + \kappa - J; 2\bar{h}_1; z_1, w_1 z_1). \tag{D.4}$$

In comparing the generic conformal block decomposition of a spinning four-point function with the GFF correlator of eq. (4.37) ((4.57) for $\kappa > 0$), one notices an apparent discrepancy in the number of degrees of freedom: two cross-ratios z_1, w_1 in the conformal block decomposition, but one cross-ratio $\frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}$ for the homogeneous, gauge-invariant polynomial $\Psi_{J_2,\kappa}(\alpha_i, \beta_i)$. This discrepancy is resolved by

a special property of GFF OPE coefficients derived above eq. (4.33): all OPE coefficients with tensor structure $n_1 + \kappa < J_1(< J_2)$ vanish. For the single non-vanishing tensor structure $n_1 = J_1 - \kappa$ and the GFF quantum numbers

$$(h_1, \bar{h}_1) = (\Delta_\phi, \Delta_\phi + J_1), \quad (h_2, \bar{h}_2) = \left(\frac{3}{2}\Delta_\phi, \frac{3}{2}\Delta_\phi + J_2 + \kappa\right), \tag{D.5}$$

the four-point lightcone block further reduces to a one-variable hypergeometric function that is polynomial of order $J_2 - J_1$:

$$F_1\left(\Delta_\phi + J_1; 2\Delta_\phi + J_2 + J_1, J_1 - J_2; 2\Delta_\phi + 2J_1; z_1, z_1 w_1\right) = \frac{{}_2F_1\left[\begin{matrix}\Delta_\phi + J_1, J_1 - J_2 \\ 2\Delta_\phi + 2J_1\end{matrix}\right]\left(\frac{z_1(w_1-1)}{1-z_1}\right)}{(1-z_1)^{\Delta_\phi + J_1}}. \tag{D.6}$$

Once this last expression is plugged in (D.2), we recover the expression of GFF four-point lightcone blocks displayed in eq. (5.1).

D.2   From triple-twist wave functions to conformal blocks

The scope of this appendix is, on the one hand, to provide a derivation of equation (5.6) relating the cross-ratios z_1, z_2, and w_1 to the coordinates α_i and, on the other hand, to derive the multiplicative prefactor used to express the GFF four-point correlator $\langle\phi(X_1)\phi(X_2)\phi(X_3)[\phi\phi\phi]_{0,J,\kappa}(X_b, Z_b)\rangle$ in terms of linear combinations of four-point lightcone blocks $g_{\mathcal{O}_1;n_1}(z_1, w_1)$.

We will use the results of [21, Section 3.2], which allow the computation of OPE limits directly in embedding space, and provide a direct construction of the lower-point degrees of freedom in terms of the higher-point ones. This can be directly applied to two consecutive OPE limits for the six-point conformal blocks $g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1 n_2}(z_i, \bar{z}_i, w_1, w_2, \Upsilon_0)$ to recover the four-point degrees of freedom associated with three external scalars and one external MST.

Since we are ultimately interested in four-point lightcone blocks, and since the Lorentzian OPE limits can be taken by making points approach each other along a null direction, we can start by considering the lightcone limits

$$X_{12}, X_{56}, X_{46}, X_{45} \to 0 \qquad \text{with} \quad \frac{X_{45}}{X_{46}} = \text{const} \tag{D.7}$$

under which three cross-ratios are constrained to $\bar{z}_1, \bar{z}_2, \bar{z}_3 = 0$ while the remaining ones are determined in terms of embedding space positions via (2.2) and (2.1). With these limits taken, we can then perform the first OPE limit between the fields at position X_5 and X_6 by first enforcing

$$X_6 = X_c \qquad\qquad\qquad X_5 = X_c + \epsilon_c Z_c \tag{D.8}$$

with Z_c being a null embedding space vector due to $X_{56} = 0$, to then take $\epsilon \to 0$. Due to the projective nature of embedding space vectors, Z_c remains a finite and non-trivial degree of freedom after the limit, and its normalization can be chosen such that it becomes a standard polarization vector associated with the field $\mathcal{O}_c(X_c, Z_c)$ produced in this first OPE limit, where $X_c = X_5 = X_6$. Using (2.2) and (2.1), one can readily verify that this limit constrains

$$z_3 = \epsilon_c \frac{X_3 \cdot Z_c}{X_3 \cdot X_c} + \dots \tag{D.9}$$

and leaves us with the finite cross-ratios $z_1, z_2, w_1, w_2, \Upsilon_0$ that remain the degrees of freedom of a spinning five-point correlator under two lightcone limits.

The second OPE limit can now be taken in two steps that are similar to what we just performed. We first require

$$X_4 = X_c + \epsilon_b Z_b \tag{D.10}$$

$\epsilon_b \to 0$ such that we produce a first polarization vector, and then we take

$$Z_c = Z_b - \tilde{\epsilon}_b W_b \tag{D.11}$$

with $\tilde{\epsilon}_b \to 0$, producing a second polarization vector associated with the MST spin. Naming $X_b = X_4 = X_c$ after the limit, we have this way constructed the degrees of freedom of an MST field $\mathcal{O}_b(X_b, Z_b, W_b)$ produced in the $\phi(X_4) \times \mathcal{O}_c(X_c, Z_c)$ OPE.

Performing these limits in (2.2) once the cross-ratios are expressed in terms of embedding space vectors via (2.1), we see that this finally constrains

$$z_3(1 - w_2) = \epsilon_c \left(\frac{1}{\epsilon_b} + \frac{X_c \cdot Z_c}{X_3 \cdot X_c} \right) + \dots, \qquad \frac{\Upsilon_0}{(1 - w_1)(1 - w_2)} = \left(\frac{\beta_{21}}{\alpha_{21}} - \frac{\beta_{23}}{\alpha_2 3} \right) \tilde{\epsilon}_b + \dots, \tag{D.12}$$

and leaves us with the two finite cross-ratios

$$z_1 = 1 - \frac{X_{1b} X_{23}}{X_{13} X_{2b}}, \qquad w_1 = 1 + \frac{J_{b,12}}{J_{b,23}} \frac{X_{23} X_{3b}}{(X_1 \otimes X_2) \cdot (X_b \wedge X_3)}. \tag{D.13}$$

To understand how this reflects in the conformal block expansion of $\Psi(\alpha_1, \alpha_2, \alpha_3)$, we can take one final OPE limit $X_1 \equiv X_a$, $X_2 = X_a + \epsilon_a Z_a$ with $\epsilon_a \to 0$ to relate the z_1, w_1 cross-ratios in the full OPE limit to the α_i's. In this case, we find

$$z_1 = 1 - v_1 = \frac{(X_1 \otimes X_2)(X_3 \wedge X_b)}{X_{13} X_{2b}} = \frac{J_{a,3b}}{X_{a3} X_{ab}} \epsilon_a + \dots, \qquad 1 - w_1 = \mathcal{X} = \frac{H_{ab} X_{a3} X_{b3}}{J_{a,b3} J_{b,3a}} + \dots \tag{D.14}$$

and

$$z_2 = 1 - v_2 = 1 - \frac{X_{2c} X_{34}}{X_{24} X_{3c}} = \epsilon_b \left(\frac{X_2 Z_b}{X_{2c}} - \frac{X_3 Z_b}{X_{3c}} \right) + \dots = \epsilon_b(\alpha_2 - \alpha_3) + \dots. \tag{D.15}$$

On the other hand,

$$\alpha_2 - \alpha_1 = \frac{Z_b X_2}{X_b X_2} - \frac{Z_b X_1}{X_b X_1} = \frac{(X_a X_b)(Z_a Z_b) - (X_a Z_b)(X_b Z_a)}{X_{ab}^2}\epsilon_a + \cdots = \frac{H_{ab}}{X_{ab}^2}\epsilon_a + \mathcal{O}(\epsilon^2) \qquad \text{(D.16)}$$

$$\alpha_i - \alpha_3 = \frac{(X_3 \otimes X_i)(X_b \wedge Z_b)}{X_{ib} X_{3b}} = \frac{J_{b,3a}}{X_{ab} X_{3b}} + \ldots, \quad \forall i = 1, 2, \qquad \text{(D.17)}$$

which implies

$$\frac{z_1(w_1 - 1)}{1 - z_1} = -\epsilon_a \frac{J_{a,3b}}{X_{a3} X_{ab}} \frac{H_{ab} X_{a3} X_{b3}}{J_{a,b3} J_{b,3a}} = \frac{H_{ab}}{X_{ab}^2}\epsilon_a \frac{X_{ab} X_{3b}}{J_{b,3a}} = \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}. \qquad \text{(D.18)}$$

Since the comb is symmetric under the exchange of X_1 and X_2, eqs. (D.15) and (D.18) provide the relation between OPE cross-ratios and αs sought for in eq. (5.6).

With these identities, the relation between four-point and six-point wave functions stated in eq. (5.9) follows immediately. In order to recall which lightcone blocks are relevant, let us reproduce eq. (5.2) from the main text. It reads

$$\langle \phi(X_1) \ldots \phi(X_6) \rangle = \sum P_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)} G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)}, \qquad \text{(D.19)}$$

$$\text{with} \qquad G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)} = \frac{\psi_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3; n_1 n_2}(\bar{z}_1, z_1, \bar{z}_2, z_2, \Upsilon_0, \bar{z}_3, z_3, w_1, w_2)}{(X_{12} X_{34} X_{56})^{\Delta_\phi}(\bar{z}_2 z_2)^{\Delta_\phi/2}}. \qquad \text{(D.20)}$$

In Appendix D.1, we computed the lightcone limit of $G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)}$ that is given in Section 5.1 as

$$G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)} = \frac{\Upsilon_0^\kappa (z_1 \bar{z}_1 z_2 \bar{z}_2 z_3 \bar{z}_3)^{\Delta_\phi} {}_2F_1\begin{bmatrix} \Delta_\phi + J_1, J_1 - J_2 \\ 2\Delta_\phi + 2J_1 \end{bmatrix}\left(\frac{z_1(w_1-1)}{1-z_1}\right)}{(X_{12} X_{34} X_{56})^{\Delta_\phi}} \frac{z_1^{J_1} z_2^{J_2+\kappa} z_3^{J_3}(1-z_1)^{\Delta_\phi + J_1}}{(1-w_1)^{\kappa - J_1}(1-w_2)^{\kappa - J_3}}. \qquad \text{(D.21)}$$

If we use that

$$\frac{(z_1 \bar{z}_1 z_2 \bar{z}_2 z_3 \bar{z}_3)^{\Delta_\phi}}{(X_{12} X_{34} X_{56})^{\Delta_\phi}} = \frac{1}{X_{3b}^{\Delta_\phi} X_{ab}^{2\Delta_\phi}} + \ldots \qquad \text{(D.22)}$$

along with equation (D.18), our expression (D.21) simplifies to

$$G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)} = \frac{\Upsilon_0^\kappa {}_2F_1\begin{bmatrix} \Delta_\phi + J_1, J_1 - J_2 \\ 2\Delta_\phi + 2J_1 \end{bmatrix}\left(\frac{\alpha_{21}}{\alpha_{23}}\right)}{(X_{3b} X_{ab}^2)^{\Delta_\phi}}(1-w_1)^{-\kappa}(1-w_2)^{J_3 - \kappa}\left(\frac{\alpha_{12}}{\alpha_{23}}\right)^{J_1} z_2^{J_2+\kappa} z_3^{J_3}. \qquad \text{(D.23)}$$

With the help of equations (D.12) and (D.15) we deduce the following equivalent formula

$$G_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}^{(n_1 n_2)} = \frac{(\alpha_{23}\epsilon_b)^{J_2+\kappa}\left(\frac{\alpha_{12}}{\alpha_{23}}\right)^{J_1} {}_2F_1\begin{bmatrix} \Delta_\phi + J_1, J_1 - J_2 \\ 2\Delta_\phi + 2J_1 \end{bmatrix}\left(\frac{\alpha_{21}}{\alpha_{23}}\right)}{(X_{3b} X_{ab}^2)^{\Delta_\phi}}\left(\left(\frac{\beta_{21}}{\alpha_{21}} - \frac{\beta_{23}}{\alpha_{23}}\right)\tilde{\epsilon}_b\right)^\kappa (\epsilon_c/\epsilon_b)^{J_3}. \qquad \text{(D.24)}$$

Removing the ϵ parameters that we used for bookkeeping and using the $1 \leftrightarrow 2$ symmetry of the comb channel, we conclude

$$G_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}^{(n_1n_2)} = \prod_{i=1}^{3} \langle \phi(X_i)\phi(X_b) \rangle \alpha_{13}^{J_2} \alpha_{21}^{J_1}{}_2 F_1 \left[\begin{matrix} \Delta_\phi + J_1, J_1 - J_2 \\ 2\Delta_\phi + 2J_1 \end{matrix} \right] \left(\frac{\alpha_{12}}{\alpha_{13}} \right) \frac{(\alpha_{13}\beta_{12} - \alpha_{12}\beta_{13})^\kappa}{\alpha_{12}^\kappa} \qquad \text{(D.25)}$$

This indeed confirms the claim on the relation between four-point lightcone blocks and the triple-twist wave functions in GFF that we stated at the end of Section 5.1.

E   Six-point crossing kernel

To derive eq. (5.30), we begin by expressing $f_{J_1'(J_2,\kappa)J_3'}$ in terms of the integral representation of the modified Bessel function:

$$f_{J_1'(J_2,\kappa)J_3'}(v_2, \mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2) = \frac{1}{2} \int_0^\infty \frac{dt}{t^{1+\Delta}} e^{-t\mathcal{U}_0 - \frac{J_2^2}{t}v_2\mathcal{U}_1\mathcal{U}_2} \left(J_1'^{\Delta - \frac{1}{2}} e^{-\frac{\mathcal{U}_1}{t}J_1'^2} \right) \left(J_3'^{\Delta - \frac{1}{2}} e^{-\frac{\mathcal{U}_2}{t}J_3'^2} \right). \qquad \text{(E.1)}$$

Next, we use the following identity from [61, eq. (10.22.51)]:

$$\int_0^\infty da\, a^{\nu+1} J_\nu(ab) e^{-p^2 a^2} = \frac{b^\nu}{(2p^2)^{\nu+1}} e^{-\frac{b^2}{4p^2}}, \qquad \text{(E.2)}$$

which translates to an identity for the Hankel transform of a Gaussian:

$$\int_0^\infty dJ_1' W_{J_1 J_1'} J_1'^{\Delta - \frac{1}{2}} e^{-\frac{\mathcal{U}_1}{t}J_1'^2} = k^{\Delta - \frac{1}{2}} \left(\frac{t}{J_2\mathcal{U}_1} \right)^\Delta e^{-\frac{J_1^2 t}{J_2^2\mathcal{U}_1}}. \qquad \text{(E.3)}$$

The same applies for the integral over J_1, so the LHS of eq. (5.30) can be expressed as

$$\text{LHS} = \frac{(J_1 J_3)^{\Delta - \frac{1}{2}}}{(J_2^2\mathcal{U}_1\mathcal{U}_2)^\Delta} \int_0^\infty \frac{1}{2} \frac{dt}{t} t^\Delta e^{-\frac{J_2^2\mathcal{U}_1\mathcal{U}_2}{t}v_2 - t\mathcal{U}_0 - \frac{t}{J_2^2\mathcal{U}_1}J_1^2 - \frac{t}{J_2^2\mathcal{U}_2}J_3^2}. \qquad \text{(E.4)}$$

After the change of variables

$$\tilde{t} := \frac{J_2^2\mathcal{U}_1\mathcal{U}_2}{t}, \qquad \text{(E.5)}$$

we obtain explicitly the integral transform of the modified Bessel function with permuted arguments, i.e.

$$\text{LHS} = (J_1 J_3)^{\Delta - \frac{1}{2}} v_2^\Delta \mathcal{K}_\Delta \left(J_1^2 v_2\mathcal{U}_2 + J_2^2\mathcal{U}_0 v_2\mathcal{U}_1\mathcal{U}_2 + J_3^2 v_2\mathcal{U}_1 \right). \qquad \text{(E.6)}$$

References

[1] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *"Solving the 3D Ising Model with the Conformal Bootstrap"*, Phys. Rev. D 86, 025022 (2012), arxiv:1203.6064.

[2] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, *"Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents"*, J. Stat. Phys. 157, 869 (2014), arxiv:1403.4545.

[3] F. Kos, D. Poland and D. Simmons-Duffin, *"Bootstrapping Mixed Correlators in the 3D Ising Model"*, JHEP 1411, 109 (2014), arxiv:1406.4858.

[4] D. Simmons-Duffin, *"The Lightcone Bootstrap and the Spectrum of the 3d Ising CFT"*, JHEP 1703, 086 (2017), arxiv:1612.08471, https://link.springer.com/article/10.1007/JHEP03(2017)086.

[5] S. Caron-Huot, D. Mazac, L. Rastelli and D. Simmons-Duffin, *"Dispersive CFT Sum Rules"*, JHEP 2105, 243 (2021), arxiv:2008.04931.

[6] J. Liu, D. Meltzer, D. Poland and D. Simmons-Duffin, *"The Lorentzian inversion formula and the spectrum of the 3d* $O(2)$ *CFT"*, JHEP 2009, 115 (2020), arxiv:2007.07914.

[7] N. Su, *"The Hybrid Bootstrap"*, arxiv:2202.07607.

[8] F. Kos, D. Poland and D. Simmons-Duffin, *"Bootstrapping the* $O(N)$ *vector models"*, JHEP 1406, 091 (2014), arxiv:1307.6856.

[9] F. Kos, D. Poland, D. Simmons-Duffin and A. Vichi, *"Bootstrapping the O(N) Archipelago"*, JHEP 1511, 106 (2015), arxiv:1504.07997.

[10] V. Rosenhaus, *"Multipoint conformal blocks in the comb channel"*, Journal of High Energy Physics 2019, 39 (2019), http://dx.doi.org/10.1007/JHEP02(2019)142.

[11] J.-F. Fortin, W.-J. Ma and W. Skiba, *"All Global One- and Two-Dimensional Higher-Point Conformal Blocks"*, arxiv:2009.07674.

[12] K. Alkalaev, A. Kanoda and V. Khiteev, *"Wilson networks in AdS and global conformal blocks"*, Nucl. Phys. B 998, 116413 (2024), arxiv:2307.08395.

[13] J.-F. Fortin, W.-J. Ma, S. Parikh, L. Quintavalle and W. Skiba, *"One- and two-dimensional higher-point conformal blocks as free-particle wavefunctions in* $AdS_3^{\otimes m}$ *"*, JHEP 2401, 031 (2024), arxiv:2310.08632.

[14] V. Gonçalves, R. Pereira and X. Zhou, *"20′ Five-Point Function from* $AdS_5 \times S^5$ *Supergravity"*, JHEP 1910, 247 (2019), arxiv:1906.05305.

[15] J.-F. Fortin, W.-J. Ma and W. Skiba, *"Seven-point conformal blocks in the extended snowflake channel and beyond"*, Phys. Rev. D 102, 125007 (2020), arxiv:2006.13964.

[16] S. Hoback and S. Parikh, *"Dimensional reduction of higher-point conformal blocks"*, JHEP 2103, 187 (2021), arxiv:2009.12904.

[17] D. Poland and V. Prilepina, *"Recursion relations for 5-point conformal blocks"*, JHEP 2110, 160 (2021), arxiv:2103.12092.

[18] I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *"From Gaudin Integrable Models to d-dimensional Multipoint Conformal Blocks"*, Phys. Rev. Lett. 126, 021602 (2021), arxiv:2009.11882.

[19] I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *"Gaudin models and multipoint conformal blocks: general theory"*, JHEP 2110, 139 (2021), arxiv:2105.00021.

[20] I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *"Gaudin models and multipoint conformal blocks. Part II. Comb channel vertices in 3D and 4D"*, JHEP 2111, 182 (2021), arxiv:2108.00023.

[21] I. Buric, S. Lacroix, J. A. Mann, L. Quintavalle and V. Schomerus, *"Gaudin models and multipoint conformal blocks III: comb channel coordinates and OPE factorisation"*, JHEP 2206, 144 (2022), arxiv:2112.10827.

[22] A. Antunes, M. S. Costa, V. Goncalves and J. V. Boas, *"Lightcone bootstrap at higher points"*, JHEP 2203, 139 (2022), arxiv:2111.05453.

[23] J.-F. Fortin, S. Hoback, W.-J. Ma, S. Parikh and W. Skiba, *"Feynman rules for scalar conformal blocks"*, JHEP 2210, 097 (2022), arxiv:2204.08909.

[24] A. Kaviraj, J. A. Mann, L. Quintavalle and V. Schomerus, *"Multipoint lightcone bootstrap from differential equations"*, JHEP 2308, 011 (2023), arxiv:2212.10578.

[25] D. Poland, V. Prilepina and P. Tadić, *"Improving the five-point bootstrap"*, arxiv:2312.13344.

[26] D. Poland, V. Prilepina and P. Tadić, *"The five-point bootstrap"*, JHEP 2310, 153 (2023), arxiv:2305.08914.

[27] A. Antunes, S. Harris, A. Kaviraj and V. Schomerus, *"Lining up a Positive Semi-Definite Six-Point Bootstrap"*, arxiv:2312.11660.

[28] C. Bercini, V. Gonçalves and P. Vieira, *"Light-Cone Bootstrap of Higher Point Functions and Wilson Loop Duality"*, Phys. Rev. Lett. 126, 121603 (2021), arxiv:2008.10407, https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.126.121603.

[29] T. Anous, A. Belin, J. de Boer and D. Liska, *"OPE statistics from higher-point crossing"*, JHEP 2206, 102 (2022), arxiv:2112.09143.

[30] A. L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, *"The Analytic Bootstrap and AdS Superhorizon Locality"*, JHEP 1312, 004 (2013), arxiv:1212.3616.

[31] Z. Komargodski and A. Zhiboedov, *"Convexity and Liberation at Large Spin"*, JHEP 1311, 140 (2013), arxiv:1212.4103.

[32] S. Pal, J. Qiao and S. Rychkov, *"Twist Accumulation in Conformal Field Theory: A Rigorous Approach to the Lightcone Bootstrap"*, Commun. Math. Phys. 402, 2169 (2023), arxiv:2212.04893.

[33] L. F. Alday and A. Zhiboedov, *"An Algebraic Approach to the Analytic Bootstrap"*, JHEP 1704, 157 (2017), arxiv:1510.08091.

[34] L. F. Alday, *"Large Spin Perturbation Theory for Conformal Field Theories"*, Phys. Rev. Lett. 119, 111601 (2017), arxiv:1611.01500.

[35] A. Kaviraj, K. Sen and A. Sinha, *"Analytic bootstrap at large spin"*, JHEP 1511, 083 (2015), arxiv:1502.01437.

[36] A. Kaviraj, K. Sen and A. Sinha, *"Universal anomalous dimensions at large spin and large twist"*, JHEP 1507, 026 (2015), arxiv:1504.00772.

[37] D. M. Hofman, D. Li, D. Meltzer, D. Poland and F. Rejon-Barrera, *"A Proof of the Conformal Collider Bounds"*, JHEP 1606, 111 (2016), arxiv:1603.03771.

[38] D. Li, D. Meltzer and D. Poland, *"Conformal Collider Physics from the Lightcone Bootstrap"*, JHEP 1602, 143 (2016), arxiv:1511.08025.

[39] D. Li, D. Meltzer and D. Poland, *"Non-Abelian Binding Energies from the Lightcone Bootstrap"*, JHEP 1602, 149 (2016), arxiv:1510.07044.

[40] S. Caron-Huot, *"Analyticity in Spin in Conformal Theories"*, JHEP 1709, 078 (2017), arxiv:1703.00278.

[41] M. Kologlu, P. Kravchuk, D. Simmons-Duffin and A. Zhiboedov, *"The light-ray OPE and conformal colliders"*, JHEP 2101, 128 (2021), arxiv:1905.01311.

[42] P. Kravchuk and D. Simmons-Duffin, *"Light-ray operators in conformal field theory"*, JHEP 1811, 102 (2018), arxiv:1805.00098.

[43] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, *"Eikonal Approximation in AdS/CFT: From Shock Waves to Four-Point Functions"*, JHEP 0708, 019 (2007), hep-th/0611122.

[44] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, *"Eikonal Approximation in AdS/CFT: Conformal Partial Waves and Finite N Four-Point Functions"*, Nucl. Phys. B 767, 327 (2007), hep-th/0611123.

[45] L. Cornalba, M. S. Costa and J. Penedones, *"Eikonal approximation in AdS/CFT: Resumming the gravitational loop expansion"*, JHEP 0709, 037 (2007), arxiv:0707.0120.

[46] A. L. Fitzpatrick, J. Kaplan and M. T. Walters, *"Universality of Long-Distance AdS Physics from the CFT Bootstrap"*, JHEP 1408, 145 (2014), arxiv:1403.6829.

[47] A. L. Fitzpatrick, J. Kaplan, M. T. Walters and J. Wang, *"Eikonalization of Conformal Blocks"*, JHEP 1509, 019 (2015), arxiv:1504.01737.

[48] S. K. Kehrein and F. Wegner, *"The Structure of the spectrum of anomalous dimensions in the N vector model in (4-epsilon)-dimensions"*, Nucl. Phys. B 424, 521 (1994), hep-th/9405123.

[49] S. E. Derkachov and A. N. Manashov, *"The Spectrum of the anomalous dimensions of the composite operators in epsilon expansion in the scalar phi**4 field theory"*, Nucl. Phys. B 455, 685 (1995), hep-th/9505110.

[50] S. E. Derkachov and A. N. Manashov, *"Anomalous dimensions of composite operators in scalar field theories"*, J. Math. Sci. 168, 837 (2010).

[51] S. K. Kehrein, *"The Spectrum of critical exponents in phi**2 in two-dimensions theory in D = (4-epsilon)-dimensions: Resolution of degeneracies and hierarchical structures"*, Nucl. Phys. B 453, 777 (1995), hep-th/9507044.

[52] S. E. Derkachov, S. K. Kehrein and A. N. Manashov, *"High-gradient operators in the N-vector model"*, Nucl. Phys. B 493, 660 (1997), cond-mat/9610106.

[53] S. E. Derkachov and A. N. Manashov, *"Spectrum of anomalous dimensions in scalar field theory"*.

[54] V. M. Braun, S. E. Derkachov and A. N. Manashov, *"Integrability of three particle evolution equations in QCD"*, Phys. Rev. Lett. 81, 2020 (1998), hep-ph/9805225.

[55] V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, *"Baryon distribution amplitudes in QCD"*, Nucl. Phys. B 553, 355 (1999), hep-ph/9902375.

[56] A. V. Belitsky, *"Integrability and WKB solution of twist - three evolution equations"*, Nucl. Phys. B 558, 259 (1999), hep-ph/9903512.

[57] G. P. Korchemsky, *"Review of AdS/CFT Integrability, Chapter IV.4: Integrability in QCD and N<4 SYM"*, Lett. Math. Phys. 99, 425 (2012), arxiv:1012.4000.

[58] N. Gromov, A. Hegedus, J. Julius and N. Sokolova, *"Fast QSC Solver: tool for systematic study of N=4 Super-Yang-Mills spectrum"*, arxiv:2306.12379.

[59] F. A. Dolan and H. Osborn, *"Conformal partial waves and the operator product expansion"*, Nucl. Phys. B 678, 491 (2004), hep-th/0309180.

[60] G. Szego, *"Orthogonal polynomials"*, *"volume 23"*, American Mathematical Soc. (1939).

[61] *"NIST Digital Library of Mathematical Functions"*, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds., http://dlmf.nist.gov/.

[62] L. F. Alday, A. Bissi and T. Lukowski, *"Large spin systematics in CFT"*, JHEP 1511, 101 (2015), arxiv:1502.07707.

[63] J. Henriksson, P. Kravchuk and B. Oertel, *"Missing local operators, zeros, and twist-4 trajectories"*, arxiv:2312.09283.

[64] J. A. Gracey, *"Four loop renormalization of ϕ^3 theory in six dimensions"*, Phys. Rev. D 92, 025012 (2015), https://link.aps.org/doi/10.1103/PhysRevD.92.025012.

[65] O. F. de Alcantara Bonfim, J. E. Kirkham and A. J. McKane, *"Critical Exponents to Order ϵ^3 for ϕ^3 Models of Critical Phenomena in Six ϵ-dimensions"*, J. Phys. A 13, L247 (1980), [Erratum: J.Phys.A 13, 3785 (1980)].

[66] C. Hasegawa and Y. Nakayama, *"ϵ-expansion in critical ϕ^3-theory on real projective space from conformal field theory"*, Modern Physics Letters A 32, 1750045 (2017).

[67] R. Gopakumar, A. Kaviraj, K. Sen and A. Sinha, *"A Mellin space approach to the conformal bootstrap"*, JHEP 1705, 027 (2017), arxiv:1611.08407.

[68] K. G. Wilson and J. Kogut, *"The renormalization group and the ϵ expansion"*, Physics Reports 12, 75 (1974), https://www.sciencedirect.com/science/article/pii/0370157374900234.

[69] S. É. Derkachov, J. A. Gracey and A. N. Manashov, *"Four loop anomalous dimensions of gradient operators in ϕ^4 theory"*, The European Physical Journal C - Particles and Fields 2, 569 (1998), https://doi.org/10.1007/s100529800706.

[70] P. Dey and A. Kaviraj, *"Towards a Bootstrap approach to higher orders of epsilon expansion"*, JHEP 1802, 153 (2018), arxiv:1711.01173.

[71] P. Kravchuk and J. Mann, *"In preparation"*.

[72] G. Cuomo and Z. Komargodski, *"Giant Vortices and the Regge Limit"*, JHEP 2301, 006 (2023), arxiv:2210.15694.

[73] G. Cuomo, Z. Komargodski and S. Zhong, *"Chiral Modes of Giant Superfluid Vortices"*, arxiv:2312.06095.

[74] G. Cuomo, J. M. V. P. Lopes, J. Matos, J. Oliveira and J. Penedones, *"Numerical tests of the large charge expansion"*, arxiv:2305.00499.

[75] D. Jafferis, B. Mukhametzhanov and A. Zhiboedov, *"Conformal Bootstrap At Large Charge"*, JHEP 1805, 043 (2018), arxiv:1710.11161.

[76] S. E. Derkachov, *"Factorization of the R-matrix. I."*, math/0503396.