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A strategy for B-physics observables in the continuum limit

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In a somewhat forgotten paper [1] it was shown how to perform interpolations between relativistic and static computations in order to obtain results for heavy-light observables for masses from, say, $m_{\rm charm}$ to $m_{\rm bottom}$. All quantities are first continuum extrapolated and then interpolated in $1/m_h = 1/m_{\rm heavy}$. Large volume computations are combined with finite volume ones where a relativistic bottom quark is accessible with small $am_{\rm bottom}$. We discuss how this strategy is extended to semi-leptonic form factors and other quantities of phenomenological interest. The essential point is to form quantities where the limit $m_h \to \infty$ is approached with power corrections $\mathrm{O}(1/m_h)$ only. Perturbative corrections $\sim \alpha_s(m_h)^{\gamma+n}$ are cancelled in the construction of the observables. We also point out how such an approach can help to control systematics in semi-leptonic decays with just large volume data. First numerical results with $N_f = 2 + 1$ and lattice spacings down to 0.039 fm are presented in [2].

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1. Introduction

B-physics is a possible portal to physics beyond the Standard Model. This is even more relevant now than in the past since direct searches have not provided evidence for new physics. Small tensions between standard model and experiment exist in B-physics, but others may be hidden by large uncertainties or an inaccurate treatment of the theory. Indeed, precise theory predictions, e.g. for B-meson decays are very challenging, because perturbation theory seems, generically, inaccurate (see below) and non-perturbative computations on the lattice are complicated by the large ratio of scales between the b-quark mass and the non-perturbative scales of QCD including the pion mass.

Because of the large scale ratio, all existing computations for B-physics on the lattice make use of expansions in $1/m_b$, i.e. effective field theories (EFTs), in one way or the other [3]. Also so-called relativistic computations constrain the quark mass dependence of heavy-light observables to forms motivated by an expansion in $1/m_b$ [4–12]. It is important to reduce the number and importance of assumptions made in the EFT treatment to a minimum. A step was taken a while ago [1], combining the relativistic theory,

$$\mathcal{L} = \mathcal{L}_{\rm glue} + \mathcal{L}_{\rm light} + \mathcal{L}_{\rm h}\,, \quad \mathcal{L}_{\rm h} = \bar{\psi}_h(D_\mu\gamma_\mu + m_h)\psi_h\,, \tag{1}$$

with the static effective theory [**?**] ($\bar{\psi}_{\rm stat}\frac{1+\gamma_0}{2} = \bar{\psi}_{\rm stat}\,, \quad \frac{1+\gamma_0}{2}\psi_{\rm stat} = \psi_{\rm stat}$),

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm glue} + \mathcal{L}_{\rm light} + \mathcal{L}_{\rm stat}\,, \quad \mathcal{L}_{\rm stat} = \bar{\psi}_{\rm stat}(D_0 + m_{\rm stat})\psi_{\rm stat}\,. \tag{2}$$

The continuum limit is taken separately in both theories, which in practice, due to accessible a, requires $m_h \lessapprox m_b/2$ in the relativistic case. Results for the physical quark mass, m_b, are obtained by interpolation in $1/m_h$, where the static effective theory yields a point at $1/m_h = 0$, see fig. 1.

The above work [1] built on the introduction of step scaling in the static approximation [13] and in HQET [14] as well as for finite mass heavy quarks [15, 16]. However, the applications to semi-leptonic decays were complicated [17]. Here we discuss that semi-leptonic decays and more can be included by a very simple generalisation. Instead of focusing on the $1/m_h$ expansion of finite volume effects and step scaling as done in the past, the general strategy rests solely on the basic requirement for a combination of static and relativistic results: observables have to admit a simple interpolation in $1/m_h$ as in fig. 1.

2. General strategy

Our general strategy is then to form **suitable quantities** which

i. can be computed in the continuum limit,
ii. can be combined to obtain the desired observables, e.g. decay constants or semi-leptonic form factors,
iii. possess a simple behavior as a function of $1/m_h$, such that they can be interpolated in this variable.

We need low energy quantities such that the $1/m_h$ expansion is applicable. Furthermore, the static limit $m_h \to \infty$ has to exist with no logarithmic corrections but only power corrections $\mathrm{O}(1/m_h)$.

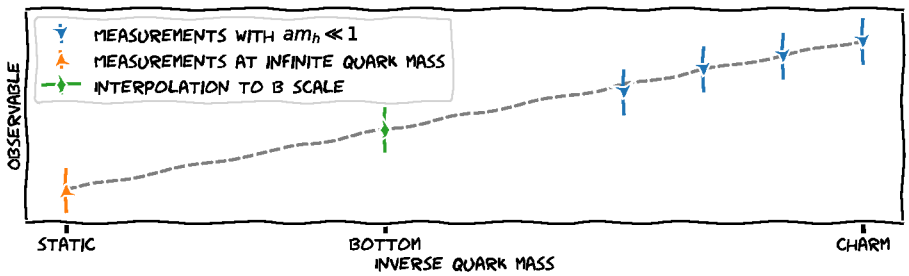


Figure 1: Sketch of interpolation between static and and relativistic data in the continuum.

In our context, logarithmic corrections means $\sim \alpha^{n+\gamma}(m_h)$ while power corrections may (and will) contain logs: $\alpha^n(m_h)/m_h = \mathrm{O}(1/m_h)$.

The step scaling strategy of [1] satisfies these requirements, but it is too restrictive. As a preparation for examples how a general strategy can be applied, we discuss the all-important renormalisation in the static theory.

2.1 Renormalisation and matching in the static theory

While it is not proven, the static effective theory is renormalisable according to all that we know. For a discussion and references we refer to sect. 1.6 of [18]. Beyond renormalisation any effective theory has to also be matched to the fundamental one, such that the observables agree up to power corrections, here $\mathrm{O}(1/m_h)$. These two steps are cleanly separated by using renormalisation group invariants. We briefly recall the facts which we need later, see e.g. [18] for details.

Energies defined and computed with the static Lagrangian eq. (2) are finite after additively renormalising its bare mass,

$$m_{\rm stat} = \text{finite} + \delta m\,, \quad \delta m = \frac{1}{a}[c_0 g_0^2 + \ldots]\,. \tag{3}$$

Since all energies of states with the quantum number of a single b-quark have the property

$$E_n^{\rm stat}(m_{\rm stat}) = E_n^{\rm stat}(0) + m_{\rm stat}\,, \tag{4}$$

energy differences $E_n^{\rm stat} - E_m^{\rm stat}$ are finite and independent of the finite piece in eq. (3). These differences then satisfy our criteria i.-iii. provided the energies correspond to low lying states and momenta are small.

Furthermore there is considerable interest in transition matrix elements of local electroweak operators, e.g. $L_\mu = V_\mu - A_\mu$, where V_μ, A_μ are heavy-light currents. Depending on whether parity changes or not, only A_μ or V_μ contribute. These local fields are renormalised and matched multiplicatively,

$$O_{\rm stat}(x; m_h^{\rm RGI}) = C_O(m_h^{\rm RGI}/\Lambda)\, O_{\rm stat}^{\rm RGI}(x)\,, \quad O_{\rm stat}^{\rm RGI}(x) = Z_O(g_0)\, O_{\rm stat}^{\rm bare}(x; g_0)\,. \tag{5}$$

Note that in the static limit Lorentz symmetry is of course broken and, e.g. $O = V_0$ and $O = V_k$ have different factors $C_{V_0} \neq C_{V_k}$, see [19]. Depending on the lattice regularization one may also have $Z_{V_0} \neq Z_{V_k}$. An important fact is that

$$C_O(m_h^{\rm RGI}/\Lambda) \overset{m_h\to\infty}{\sim} [\log(m_h^{\rm RGI}/\Lambda)]^{\hat\gamma_O} \tag{6}$$

is not finite in the static limit.[1] Therefore matrix elements of $O_{\rm stat}$ by themselves are not "suitable quantities". Instead, ratios of different matrix elements of the same operator and with the same m_h can be used. In the old step scaling method [1, 15, 17], the space-time volume and only it changed between the two matrix elements that form the ratio. This leads to a rather impractical scaling of the momenta for form factors. That restriction is unnecessary.

We turn to examples implementing the general strategy.

2.2 Large volume

The form factors for $B \to \pi\ell\nu$ decays are very relevant for the determination of $V_{\rm ub}$, the test of CKM unitarity as well as the search for deviations from the Standard Model. The QCD matrix elements of the vector current determine two independent form factors, denoted by $h_\parallel$ and $h_\perp$ in the HQET basis. Labelling the hadronic states by the spatial momenta of the hadrons and using non-relativistic normalisation[2] for the B-meson state, the form factors are defined by ($E_\pi^2 = \vec{p}_\pi^{\,2} + m_\pi^2$)

$$\langle \pi(\vec{p}_\pi)|V^k(0)|B(\vec{0})\rangle = \sqrt{2} p_\pi^k \, h_\perp(E_\pi), \quad \langle \pi(\vec{p}_\pi)|V^0(0)|B(\vec{0})\rangle = \sqrt{2}\, h_\parallel(E_\pi)\,. \tag{10}$$

Normalizing by the form factor at a reference energy $E_\pi^{\rm ref}$, we define

$$\tau_x = \log\left(h_x(E_\pi)/h_x(E_\pi^{\rm ref})\right), \quad x \in \{\perp, \parallel\}, \tag{11}$$

where in the static theory matching and renormalization factors cancel,

$$\lim_{m_h\to\infty} \tau_x = \tau_x^{\rm stat} = \log\left(h_x^{\rm stat}(E_\pi)/h_x^{\rm stat}(E_\pi^{\rm ref})\right) = \log\left(h_x^{\rm stat,bare}(E_\pi)/h_x^{\rm stat,bare}(E_\pi^{\rm ref})\right). \tag{12}$$

More precisely, the static limit is approached with power corrections only,

$$\tau_x = \tau_x^{\rm stat} + {\rm O}(E_\pi/m_h, \Lambda/m_h), \tag{13}$$

and i) as well as iii) are satisfied. The strategy now consists in the following steps.

a) Compute the lhs of eq. (13) in the relativistic theory for $am_h \ll 1$, such that a standard Symanzik analysis applies and take the continuum limit. In practice one is limited to $m_h \lessapprox m_b/2$ or similar.
b) A separate continuum limit in the static theory yields $\tau_x^{\rm stat}$.
c) The b-quark mass scale can be reached by an interpolating fit, e.g.

$$\tau_x^{\rm fit} = \sum_{n=0}^{N} t_{x,n} \left(\frac{E_\pi}{m_H}\right)^n \tag{14}$$

[1] Also the renormalisation factors

$$Z_O(g_0) = 1 + Z_1^O g_0^2 + Z_2^O g_0^4 + \ldots \tag{7}$$

have a logarithmic dependence on a since $g_0^2 \sim -1/\log(a\Lambda)$, but it is sub-leading.

[2] The normalization conditions in a finite spatial $L \times L \times L$ volume are

$$\langle B(\vec{p})|B(\vec{p}\,')\rangle = 2L^3\delta_{\vec{p},\vec{p}\,'}, \quad \langle B^*(\vec{p},\lambda)|B^*(\vec{p}\,',\lambda')\rangle = 2L^3\delta_{\vec{p},\vec{p}\,'}\,\delta_{\lambda,\lambda'}, \tag{8}$$

$$\langle \pi(\vec{p})|\pi(\vec{p}\,')\rangle = 2E_{\pi,\vec{p}}L^3\delta_{\vec{p},\vec{p}\,'}, \tag{9}$$

with $k = 1, 2, 3$ the polarisation of the vector meson state in the rest frame.

to all continuum results, including $\tau_x^{\rm stat}$. Here the heavy-light pseudo-scalar mass, m_H, is used as a proxy for the heavy quark mass and the fit function is finally evaluated at $m_H = m_B$ where m_h becomes $m_{\rm b}$.

Eq. (14) is a model since logarithmic modifications of the power corrections ($n \geq 1$) are neglected and one has to truncate at some order N. However, since we do an interpolation and not an extrapolation the model dependence is expected to be small, but this will have to be assessed case-by-case. The leading $n = 0$ term is free of logs by construction.

We end this part by a modification which we expect to be useful in practice. It is always good to have normalising factors defined in such a way that they are numerically very precise. Here this suggests $\vec{p}^{\rm ref} = \vec{0}$ or equivalently $E_\pi^{\rm ref} = m_\pi$ We recommend this for $\tau_\parallel$, but the form factor $h_\perp$ is not defined for vanishing momentum. Hence, we propose to switch to a different matrix element of the spatial vector current. The vector meson decay constant,

$$\hat{f}_V \epsilon_k^\lambda = \langle B^*(\vec{0},\lambda)|V^k(0)|0\rangle\,, \tag{15}$$

can be determined precisely from zero momentum two-point functions of the vector current. We thus propose to use

$$\tau_\parallel = \log\left(h_\parallel(E_\pi)/h_\parallel(m_\pi)\right)\,, \quad \tau_\perp = \log\left(E_\pi h_\perp(E_\pi)/(L_{\rm ref}\hat{f}_V)\right) \tag{16}$$

in practice. The factors E_π and $L_{\rm ref}$ get the mass dimensions straight and a particular choice $L_{\rm ref} = 1/f_\pi$ is motivated by lowest order HMChPT where $E_\pi f_\pi h_\perp(E_\pi)/\hat{f}_V = g_{BB^*\pi}/\sqrt{2}$ [20].

2.3 Step scaling

With the above, the E_π-dependence (q^2 in a general frame) of the form factors can be computed and compared to experiments. However, the absolute normalisation is lost. A determination of $|V_{ub}|$ does also need the absolute normalisation. We thus also want a strategy for the computation of $\hat{f}_V$ and $h_\parallel(m_\pi)$. It is provided by the step scaling approach of [1]. Compared to the strategy above, the energy variable is replaced by the extent of a finite volume and a crucial point is that one can compute quantities directly in the relativistic theory when the size of the volume is of the order of $L_1 \approx 0.5$fm, where lattice spacings are accessible such that $am_b \ll 1$. The matching factors $C_x(m_b^{\rm RGI}/\Lambda)$ are then replaced by a direct relativistic computation and step scaling functions independent of C_x transport this information to large volume. A slight complication is that in finite volume we need a non-perturbative proxy for the quark masses. The natural choice is a finite volume heavy-light "mass" $m_H(L)$ with the property $\lim_{L\to\infty} m_H(L) = m_H$. The first step therefore is the determination of the finite-size dependence of $m_H(L)$.

We introduce one length-scale $L_{\rm ref}$ to form dimensionless observables and write (in principle more steps σ can be inserted)

$$L_{\rm ref} m_{\rm B} = L_{\rm ref} m_H(L_1) + \sigma_m(u_1, y_2) + \rho_m(u_2, y_B)\,, \tag{17}$$

$$\sigma_m(u_1, y_2) = L_{\rm ref}[m_H(L_2) - m_H(L_1)]\,, \quad \rho_m(u_2, y_B) = L_{\rm ref}[m_{\rm B} - m_H(L_2)]\,. \tag{18}$$

Here all quantities refer to the same heavy quark mass and the light quark masses are set to zero in finite volume. Since the functions σ_m and ρ_m are differences of energies, they are finite in the static

theory; δm drops out. Steps a)-c) can therefore be carried out for ρ, σ as written down for τ. The variables u_i are proxys for the sizes L_i in the form of values of running couplings, $u_i = \bar{g}^2_{\rm GF}(L_i)$ [21] and the variables $y_i = L_{\rm ref}\, m_H(L_i)$ are proxys for the b-quark mass. They are obtained recursively going from large volume to small,

$$y_{\rm B} \equiv L_{\rm ref}\, m_{\rm B}\,, \quad y_2 = y_{\rm B} - \rho_m(u_2, y_B)\,, \quad y_1 = y_2 - \sigma_m(u_1, y_2)\,. \tag{19}$$

Following this chain imposes that the quark mass is set to the physical b-quark mass. Of course, starting from a different input, e.g. $m_B \to m_D$, one uses a different heavy quark mass in all subsequent steps.

2.3.1 Quark mass

When small lattice spacings are available and a non-perturbative renormalisation of the quark mass is carried out [22], the function

$$\pi_m(u_1, y) = \frac{m_H(L_1)}{m_h^{\rm RGI}} = \frac{y}{L_{\rm ref} m_h^{\rm RGI}} \tag{20}$$

can be computed in the continuum limit. Combined with y_1 it yields the renormalisation group invariant b-quark mass

$$m_{\rm b}^{\rm RGI} = \frac{1}{L_{\rm ref}} \frac{y_1}{\pi_m(u_1, y_1)}\,. \tag{21}$$

2.3.2 Decay constants and other multiplicatively renormalised matrix elements

As discussed above, the vector meson decay constant is likely to play an important role in a precise determination of $b \to u$ semi-leptonic decays and the pseudo-scalar one yields a relevant crosscheck through leptonic B decays.

Our strategy for their determination parallels the one for the quark mass and uses the already known quark mass proxys y_i. Decay constants are matrix elements of the currents V_μ and A_μ which are multiplicatively renormalised and matched in the static theory. We obtain a basic equation which is form-identical to (17) by defining the finite and infinite volume observables to be the logarithms of the dimensionless matrix elements. We write it generically for any multiplicatively renormalised matrix element, made dimensionless by our reference scale $L_{\rm ref}$, [3]

$$\log(\mathcal{M}_\infty) = \log(\mathcal{M}(L_1)) + \sigma_{\mathcal{M}}(u_1, y_2) + \rho_{\mathcal{M}}(u_2, y_B), \tag{24}$$

$$\sigma_{\mathcal{M}}(u_1, y_2) = \log(\mathcal{M}(L_2)\,/\,\mathcal{M}(L_1))\,,\ \rho_{\mathcal{M}}(u_2, y_B) = \log(\mathcal{M}_\infty\,/\,\mathcal{M}(L_2))\,. \tag{25}$$

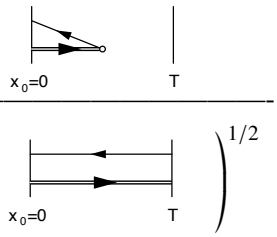
[3] We here focus on matrix elements of a single operator. There are also cases where a single operator in QCD is mapped to more operators in the static theory which mix under renormalisation and under matching. Furthermore, mixing can already be present in the relativistic theory. Then eq. (5) holds with a N_O-component vector O and $N_O \times N_O$ matrices C, Z. We then need to consider N_O different matrix elements of the operators; $\mathcal{M}$ is $N_O \times N_O$ with the first index corresponding to the different operators and the second one to the different matrix elements. The master equations (24-25) are changed to

$$\mathcal{M}_\infty = \rho_{\mathcal{M}}(u_2, y_B)\, \sigma_{\mathcal{M}}(u_1, y_2)\, \mathcal{M}(L_1), \tag{22}$$

$$\rho_{\mathcal{M}}(u_2, y_B) = \mathcal{M}_\infty\, \mathcal{M}(L_2)^{-1}\,, \quad \sigma_{\mathcal{M}}(u_1, y_2) = \mathcal{M}(L_2)\, \mathcal{M}(L_1)^{-1}\,. \tag{23}$$

In particular, for the **vector decay constant**, we choose in our **practical implementation** [2]

$$\mathcal{M}(L) \propto \hat{f}_V(L) = \langle B^*(\vec{0},k);L|V^k(0)|\Omega;L\rangle \quad = \quad \frac{\text{x}_0\text{=0} \qquad \text{T}}{\left(\qquad\qquad\qquad\qquad\right)^{1/2}} \,. \tag{26}$$



Here the states are finite volume states, $|\Omega;L\rangle$ has vacuum quantum numbers, $|B^*(\vec{0},k);L\rangle$ is a zero momentum B^*-meson state with polarisation k. The matrix element is constructed from correlation functions with Schrödinger functional boundary conditions [23, 24] as indicated in the equation. A precise definition is $\log(L^{3/2}\hat{f}_V(L)/\sqrt{2}) = \Phi_{19}$ with Φ_{19} of [25], but with the projection to zero topological charge as for the coupling [21].

3. Lines of Constant Physics and continuum limits

Lines of Constant Physics (LCP) specify how bare parameters are scaled as the resolutions a/L_i of the lattice theory change. They can differ from quantity to quantity. Indeed, it is useful to define the LCP adapted to the physics involved.

The overall scale $L_{\rm ref}$ could be taken e.g. as the gradient flow (GF) scales $\sqrt{t_0}$ [26] or w_0 [27]. Since we work with finite volumes, it is more natural to define the scale through the GF running coupling in one of the used volumes. In our numerical example [2] practical considerations lead to the choice [28]

$$L_{\rm ref} = 4L_0\,, \quad \bar{g}^2(L_0) = u_0 = 3.949\,, \quad [L_1 = 2L_0,\ L_2 = 4L_0 = L_{\rm ref}]\,, \tag{27}$$

in terms of the GF coupling defined exactly as in [21]. The arguments $u_i = \bar{g}^2(L_i)$ of our scaling functions are then given by the precisely known [21] coupling step scaling functions

$$u_1 = \sigma(u_0), \quad u_2 = \sigma(u_1)\,. \tag{28}$$

Taking the knowledge of $\sigma(u_i)$ for granted, we need three LCP's for the above strategy. In principle, they only differ by the values of the renormalised coupling u and the proxy for the heavy quark mass y. For the latter one chooses a set of values compatible with $am_h \ll 1$. The LCPs are the set of conditions

$$\mathcal{S}_0 = \left\{\bar{g}^2(g_0,L/a) = u, \quad \frac{L}{a}M_H(g_0,am_h,L/a) = y\,, \quad m_l^{\rm PCAC} = 0,\ l = 1\ldots N_{\rm l}\right\}, \quad M_H = am_H\,, \tag{29}$$

where the dimensionless arguments are the bare coupling, g_0, bare heavy quark mass, $am_h = 1/(2\kappa_h) - 4$, and resolution L/a. The bare light quark masses are fixed by the last $N_{\rm l}$ conditions.

The lattice approximants for the step scaling functions (we use the special case $L_{\rm ref} = 2L_1$) are

$$\Sigma_m(u,y,a/L) \;=\; 2L/a\ [M_H(g_0,am_h,2L/a) - M_H(g_0,am_h,L/a)]_{\mathcal{S}_0}\,, \tag{30}$$

$$\Sigma_{\mathcal{M}}(u,y,a/L) \;=\; \log\left(\mathcal{M}(g_0,am_h,2L/a)\,\mathcal{M}(g_0,am_h,L/a)^{-1}\right)_{\mathcal{S}_0}\,, \tag{31}$$

where we are interested in $u = u_1$ and a range of y.

Analogously, the small volume function is given by

$$\pi_m(u, y) = \lim_{a\to 0} \Pi_m(u, y, a/L), \quad \Pi_m(u, y, a/L) = \left.\frac{M_H(g_0, am_h, L/a)}{M_h^{\rm RGI}(g_0, am_h, L/a)}\right|_{\mathcal{S}_0}, \quad M_h^{\rm RGI} = am_h^{\rm RGI}. \tag{32}$$

In large volume, massless light quarks are not desired. We therefore introduce a massive LCP, $\mathcal{S}_{\rm m}$. It may correspond to the physical point of isoQCD [3] or the symmetric point of CLS ($m_\pi = m_{\rm K} \approx 410\,{\rm MeV}$) [29]). It is implied that the volume is large enough such that finite size effects are negligible. $\mathcal{S}_{\rm m}$ applies straight forwardly to the pure large volume quantities $\mathcal{M} = \lim_{a/L_{\rm ref}\to 0} \mathcal{M}(g_0, am_h, am_i)$, while for the functions connecting large volume and finite L one has, e.g.

$$\rho_{\mathcal{M}}(u, y) = \lim_{a\to 0} R_{\mathcal{M}}(u, y, a/L), \quad R_{\mathcal{M}}(u, y, a/L) = \log\left(\frac{\mathcal{M}(\tilde{g}_0, a\tilde{m}_h, am_i)_{\mathcal{S}_{\rm m}}}{\mathcal{M}(\tilde{g}_0, a\tilde{m}_h, L/a)_{\mathcal{S}_0}}\right), \tag{33}$$

where $\mathcal{S}_0$ defines the parameters in the denominator and $\mathcal{S}_{\rm m}$ in the numerator. The improved bare parameters $\tilde{g}_0, a\tilde{m}_i$ are defined in [30, 31]. Since the light quark masses vanish for $\mathcal{S}_0$, we have $\mathcal{M}(\tilde{g}_0, a\tilde{m}_h, L/a)_{\mathcal{S}_0} = \mathcal{M}(g_0, am_h, L/a)_{\mathcal{S}_0}$.

4. Conclusion and Outlook

We have pointed out that the core idea of [1] is applicable beyond the step scaling in volume. Extrapolations of the relativistic theory to the b-quark mass can be turned into interpolations in any situation where the static theory result is free from logarithms in the heavy quark mass m_h. Generically such logs are present. They originate from loop-corrections in the matching of the effective theory to QCD. Because of the simple structure of the static theory it is easy to find functions where all m_h-dependence cancels. For matrix elements of local operators one takes ratios of matrix elements between different states. This means that the simple interpolation of fig. 1 can determine the energy dependence of the form factors, but not the normalisation. The latter becomes accessible by adding step scaling in the volume. A single step with just one σ appears to be enough and first results are encouraging [2].

The somewhat involved step scaling part has to be done only once but including a continuum limit. The result can then be used with any action and one can concentrate on the particular challenges appearing in large volume, e.g. controlling excited state effects [32] and form-factor parameterisations [33, 34].

We note that the strategy is applicable beyond the explicit cases discussed. E.g. footnote 3 applies to the 2×2 mixing problem encountered in $B\bar{B}$ mixing in the standard model with twisted mass fermions or with exact lattice chiral symmetry [35].

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