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Higher-derivative relations between scalars and gluons

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Abstract: We extend the covariant color-kinematics (CCK) duality introduced by Cheung and Mangan to effective field theories. We focus in particular on the relation between effective theories of gluons only and of gluons coupled to bi-adjoint scalars. We show how to map the equations of motion of those theories, as well as their tree-level scattering amplitudes. As an example of new relations, we find that the amplitudes of pure gluon theory deformed by an F^3 operator, as well as those of the same theory further extended by the BCJ-compatible F^4 operator, can be generated at all multiplicity from amplitudes of the renormalizable theory of gluons and bi-adjoint scalars. To prove this, we identified the appropriate treatment of multi-trace factors in the CCK duality. We also obtain closed-form expressions for the BCJ numerators in D dimensions, which we make explicit for the case of F^3. Finally, we find strong indications of the fact that CCK duality extends to the $(DF)^2$+YM($+\phi$) theory, known to generate a full tower of BCJ-compatible operators beyond the aforementioned F^3 and F^4.

Contents

1 Introduction

Scattering amplitudes are of wide interest in high-energy physics, for their close connection to observables and for their remarkable mathematical properties. The latter aspect has motivated a remarkable revival of activity devoted to scattering amplitudes in recent years, leading to new fascinating discoveries, some of which are not at all obvious from the perspective of Lagrangians or Feynman rules. One prominent example of such unexpected structure is the relation between Yang-Mills (YM) and gravity theories, originally identified by Kawai, Lewellen and Tye (KLT) as a relation between open and closed string amplitudes [1]. In the low-energy limit, this relation enables the calculation of graviton amplitudes at tree level from a product of two, arguably simpler, gluon amplitudes, convoluted with a matrix of kinematic functions dubbed KLT kernel. The basis independence of those relations relies on the low-energy limit of string monodromy relations.

It was later found by Bern, Carrasco and Johansson (BCJ) that the KLT relations are related to a color-kinematics (CK) duality [2]. In a nutshell, Yang-Mills amplitudes can be organized as sums over trivalent graphs, dressed by color and kinematic (or BCJ) numerators that share common structural properties. The color numerators are built from group theory tensors (such as structure constants) and therefore satisfy algebraic linear relations (such as Jacobi identities). The kinematic numerators are functions of the momenta and polarization vectors, which can be chosen to fulfill exactly the same linear relations [2]. This implies in particular the aforementioned low-energy limit of the string monodromy relations, known as BCJ relations. Finding BCJ numerators is typically nontrivial, but methods exist to derive them from known amplitudes [3–6], or to construct them directly [7–18]. The CK duality then permits to recover the KLT relations in an alternative way, namely through the replacement of color numerators by kinematic ones [19, 20], which defines the double copy approach to gravitational amplitudes. The linear relations verified by all numerators now promote gauge invariance to diffeomorphism invariance.

After the discovery of CK duality in YM amplitudes, it was shown to exist for several other theories, including the non-linear sigma model (NLSM) [21–23], theories with matter particles [24–31], and the cubic theory of a bi-adjoint scalar (BAS) [5] (see also [32–34]), whose BCJ numerators are built only out of group theory structure constants, and whose amplitudes generate the aforementioned KLT kernel. Multiplying the numerators of two theories of this set generates a whole web of double-copy theories, some of which are non-gravitational. (See [35] for a recent review.)

The NLSM is a non-renormalizable theory, showing that the double copy applies to effective field theories (EFTs). This is further confirmed by the terms of higher mass dimensions, i.e. higher α' powers, in the low-energy expansion of the original KLT relations [1], which implement the double copy between the gauge and gravitational EFTs obtained from the low-energy limits of open and closed string theories. Remarkably, in the process one also finds that higher α' corrections should be added to the KLT kernel, corresponding to the addition of EFT operators to the cubic bi-adjoint theory [36]. This motivates the study of the double copy in EFTs. The KLT formulation of the double copy was explored in this context and generalized by [37–42]. On the other hand, the CK duality has been studied for higher-derivative corrections to YM theory [43, 44], and bootstrap approaches towards gluon EFT numerators exist [45, 46]. More recently, the notion of CK duality was generalized by considering numerators which contain both kinematic and color information, including rules to build them for scalar particles [47–52]. For instance, these new numerators are needed for a CK-dual approach to a scalar EFT known as Z-theory, which plays a prominent role in double copies of field theories to type I/II superstring theories, in that it encodes all the necessary α' corrections [53–56].

EFTs are defined up to a cutoff scale Λ above which a UV completion kicks in, and calculations are performed up to a fixed order $(E/\Lambda)^n$ for some integer n depending on the required precision and E the characteristic energy of a process. As n increases, new EFT operators are susceptible to contribute and should be included. In a bottom-up approach agnostic about the UV theory, the coefficients of operators at different mass dimensions are taken to be independent. However, assumptions on the UV completions or, for example, on the soft behavior of the amplitudes [57–60] typically correlate the Wilson coefficients. Similarly, the CK duality can be used to constrain the operator content of a theory: for tree-level double-copy consistency in YM theory, the inclusion of the single dimension-six F^3 operator demands the presence of an infinite tower of higher-dimensional operators [43, 61]. An elegant way to capture this tower is through the $(DF)^2$+YM theory [62], which has been shown to supplement Z-theory in the double copies of field theories to bosonic and heterotic string theories [63]. For the NLSM and other theories, the interplay with soft behavior has been studied in [60, 64, 65].

A different but closely related duality between color and kinematics was exposed by Cheung and Mangan [66] at the level of the classical equations of motion (EOMs) instead of that of amplitudes. (See [67–70] for related works.) Writing the NLSM EOM in terms of the chiral current and the YM EOM in terms of the field strength, they uncovered a remarkable analogy with, respectively, the EOM of BAS theory and that of its gauged variant (GBAS). The latter, and its organisation in terms of the covariant derivative, motivated the name covariant color-kinematics (CCK) duality. It was further demonstrated in [66] that CCK duality allows one to relate the color and kinematic algebras of the dual theories, as well as the related conserved currents and, most importantly for the present work, the tree-level scattering amplitudes. Those can be extracted from (functional derivatives of) perturbative solutions to the EOMs with sources, and are therefore natural objects to study under the light of CCK duality. For instance, [66] found new amplitude relations between NLSM/BAS and YM/GBAS theories, as well as new closed-form expressions for BCJ numerators of NLSM and YM tree-level amplitudes at all multiplicity.

In this paper, we study how CCK duality extends to EFTs. More precisely, we work with EFT corrections to YM and GBAS theories. We consider operators of increasing mass dimensions and rely on methods similar to those of [66].[1] The first higher-dimensional operator to be considered in the pure gluon EFT is the only dimension-six correction to YM theory, F^3, to be defined precisely below. We show that the EOM it induces is mapped to that of the gluon EOM in dimension-four GBAS theory. With this new CCK duality at hand, we derive relations between the amplitudes of YM+F^3 and GBAS theories. Interestingly, the relevant GBAS amplitudes also enter the relations with dimension-four YM amplitudes derived in [66], but are treated slightly differently, in terms of the number of scalars and gluons and of the replacement rule for the flavor tensors. We also obtain a closed-form formula for the YM+F^3 BCJ numerators at any multiplicity, which are manifestly gauge invariant in all legs. Those claims, as well as those to follow, have been confirmed by explicit Feynman diagram calculations of the relevant amplitudes.

At dimension eight, we focus on the operators that satisfy the traditional CK duality, and we show that they lead to EOMs CCK-dual to those of a dimension-six extension of GBAS theory, whose dimension-six operators are nothing but those arising from the dimensional reduction of YM+F^3 in $D + n$ to D dimensions. The duality requires a correlated treatment of single- and double-flavor-trace[2] structures in GBAS theory, the

[1] It is relevant to note that Ref. [71] has previously studied the color-kinematics duality in off-shell currents of YM EFT, but we shall not follow this approach.

[2] In this paper, we do not make use of the ordering with respect to the color indices shared by the scalar and the gluon. Therefore, in what follows, "traces" always implicitly refer to flavor traces, namely to the symmetry indices only carried by the scalar.

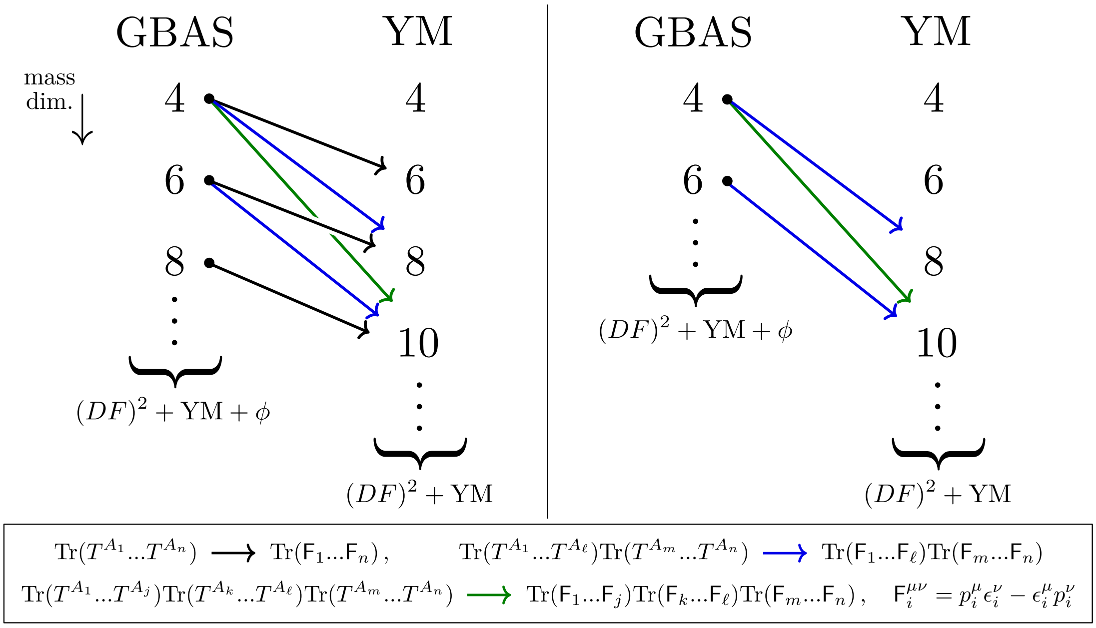


Figure 1. Amplitude relations that were found in this work. The figures on the left and right correspond to Equations (3.8, 4.19, 5.9, 5.10) and Equations (4.29, 5.12), respectively.

latter of which did not contribute at lower mass dimensions. We also observe that the correlation between dimension-six and dimension-eight coefficients demanded by regular CK duality are at the origin of cancellations that make the CCK duality possible. Leveraging this CCK duality, we obtain two different amplitude relations at any multiplicity. One expresses YM dimension-eight amplitudes in terms of dimension-six single-trace and dimension-four double-trace GBAS amplitudes, while the other relation requires only dimension-four double-trace GBAS amplitudes. We are thus led to the remarkable fact that dimension-four renormalizable GBAS theory encodes all tree-level amplitudes of YM up to dimension eight, for a specific choice of operators. This also suggests a straightforward procedure to derive dimension-eight BCJ numerators. Beyond dimension eight, we conjecture that a similar pattern continues for the aforementioned CK-dual $(DF)^2$+YM theory and its GBAS analog, and we perform checks of that conjecture, which includes a proposal for the treatment of an arbitrary number of flavor traces. In particular, we find that the relations that we derived up to dimension eight extend to the full tower of higher-derivative operators defined by the $(DF)^2$+YM theory. We have visualised all amplitude relations in Fig. 1.

The organisation of the paper is as follows. We first review the computation of tree-level scattering amplitudes from EOMs and the CCK duality of Cheung and Mangan [66] in Sec. 2. We then extend this work to the EFT domain: Sec. 3 contains our exploration of YM+F^3, while dimension-eight operators are considered in Sec. 4. We then probe in Sec. 5 the CCK duality involving the $(DF)^2$+YM theory. Finally, we conclude in Sec. 6.

2 Review of the covariant color-kinematics duality

We start by reviewing how to solve equations of motion perturbatively and how to extract tree-level scattering amplitudes from the resulting solutions [72]. After that, we review the covariant color-kinematics duality at the renormalizable level from [66].

2.1 Tree-level scattering amplitudes from equations of motion

For simplicity, let us consider a massless real scalar field φ with a quartic potential. The discussion below readily generalizes to other theories. The corresponding Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{\lambda}{4!}\varphi^4 + J\varphi\,, \tag{2.1}$$

from which the following EOM is obtained,

$$\Box\varphi + \frac{\lambda}{3!}\varphi^3 = J\ . \tag{2.2}$$

The source $J(x)$ is non-dynamical and used to probe the response of the theory to an external perturbation. At a given order $\mathcal{O}(J^n)$ in the source, one can recursively compute the solution $\varphi^{(n)}$ to the EOM in perturbation theory:

$$\begin{aligned}
\varphi^{(1)}(x) &= \left(\frac{1}{\Box}J\right)(x) = -\int \mathrm{d}^4y \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2}J(y)\ , \\
\varphi^{(2)}(x) &= 0\ , \\
\varphi^{(3)}(x) &= -\frac{\lambda}{3!}\left(\frac{1}{\Box}\varphi^{(1)3}\right)(x) = \frac{\lambda}{3!}\int \mathrm{d}^4y\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2}\varphi^{(1)3}(y) \\
&= -\frac{\lambda}{3!}\int\left(\prod_{i=1}^{3}\mathrm{d}^4y_i\frac{\mathrm{d}^4p_i}{(2\pi)^4}\right)\frac{1}{(p_1+p_2+p_3)^2}\left(\prod_{i=1}^{3}\frac{e^{ip_i(x-y_i)}}{p_i^2}J(y_i)\right)\ ,
\end{aligned} \tag{2.3}$$

and so on. In Fourier space, $\varphi(p) \equiv \int \mathrm{d}^4x\, e^{-ip\cdot x}\varphi(x)$ and one finds

$$\begin{aligned}
\varphi^{(1)}(p) &= -\frac{J(p)}{p^2}\ , \\
\varphi^{(2)}(p) &= 0\ , \\
\varphi^{(3)}(p) &= -\frac{\lambda}{3!}\int\left(\prod_{i=1}^{3}\frac{\mathrm{d}^4p_i}{(2\pi)^4}\right)\frac{\delta^{(4)}(p-p_1-p_2-p_3)}{p^2}\frac{J(p_1)}{p_1^2}\frac{J(p_2)}{p_2^2}\frac{J(p_3)}{p_3^2}\ , \\
&\ \ \dots
\end{aligned} \tag{2.4}$$

These perturbative solutions can be represented in terms of Feynman graphs, as shown in Fig. 2. The tree-level scattering amplitudes of the theory are then obtained using the LSZ reduction formula. At n points and for all particles incoming,

$$\mathcal{A}(p_1,...,p_n) = \int\prod_{i=1}^{n}\left(\mathrm{d}^4x_i\frac{ie^{-ip_i\cdot x_i}\,\Box_{x_i}}{(2\pi)^{3/2}}\right)\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle\ . \tag{2.5}$$

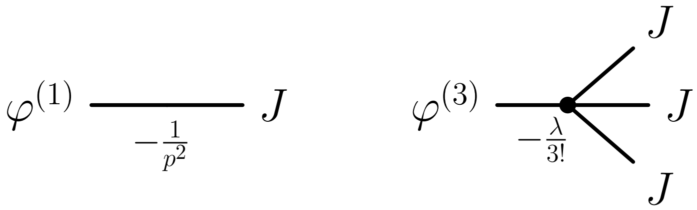


Figure 2. Diagrammatic representation of the perturbative solution to the EOM in the $\lambda\,\varphi^4$ theory.

The n-point correlator computed without source is obtained from the one-point function with a source, $\langle 0|\varphi(x_n)|0\rangle_J$, by taking functional derivatives:

$$\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_n)|0\rangle = (-i)^{n-1}\left(\frac{\delta^{n-1}}{\delta J(x_1)...\delta J(x_{n-1})}\langle 0|\varphi(x_n)|0\rangle_J\right)\Bigg|_{J=0} , \tag{2.6}$$

where, at tree-level, $\langle 0|\varphi(x_n)|0\rangle_J$ is simply the solution to the equation of motion with the source, evaluated at the point x_n. Following the terminology of [66], we will refer to $\varphi(x_n)$ as the *root* leg of the corresponding diagrams, and to $\varphi(x_{1,...,n-1})$ as the *leaf* legs. For illustration,

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_4)|0\rangle &= (-i)^3\left(\frac{\delta^3}{\delta J(x_1)...\delta J(x_3)}\langle 0|\varphi(x_4)|0\rangle_J\right)\Bigg|_{J=0} = (-i)^3\frac{\delta^3\varphi^{(3)}(x_4)}{\delta J(x_1)...\delta J(x_3)}\\ &= -i\lambda\int\prod_{i=1}^{4}\left(\frac{\mathrm{d}^4p_i e^{-ip_i\cdot x_i}}{(2\pi)^4 p_i^2}\right)(2\pi)^4\delta^{(4)}\!\left(\sum_i p_i\right) ,\end{aligned} \tag{2.7}$$

and

$$\mathcal{A}(p_1,...,p_4) = \int\prod_{i=1}^{4}\left(\mathrm{d}^4x_i\frac{-ie^{ip_i\cdot x_i}\,\Box_{x_i}}{(2\pi)^{3/2}}\right)\langle 0|\mathrm{T}\varphi(x_1)...\varphi(x_4)|0\rangle = -i\frac{\lambda}{(2\pi)^2}\,\delta^{(4)}\!\left(\sum_i p_i\right) , \tag{2.8}$$

consistently with the Feynman rules of the Lagrangian in Eq. (2.1). In the rest of this paper, we write the amplitudes without momentum-conserving delta function and powers of 2π or i.

Before closing this section, let us emphasize a point used later on: non-linear terms depending on the source in the equations of motion are irrelevant on shell. For concreteness, let us add the term $J\varphi$ to the right-hand side (r.h.s.) of the EOM in Eq. (2.2). This has the effect of turning on $\varphi^{(2)}$,

$$\varphi^{(2)}(x) = \frac{1}{\Box}\left(J\varphi^{(1)}\right)(x) = \int \mathrm{d}^4y\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2}J(y)\int \mathrm{d}^4z\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{e^{iq\cdot(y-z)}}{q^2}J(z) . \tag{2.9}$$

Differentiating with respect to $J(x_2)$ and $J(x_3)$ and applying the LSZ formula, one finds

$$-i\,\mathcal{A}(p_1,p_2,p_3) = p_2^2 + p_3^2 = 0 , \tag{2.10}$$

i.e. the new term in the EOM has no effect on the on-shell scattering amplitudes. More generally, terms of the form $J\varphi^n$ in the EOM would generate subdiagrams like that of Fig. 3, leading to amplitudes proportional to the (vanishing) square of the momentum flowing through the source.

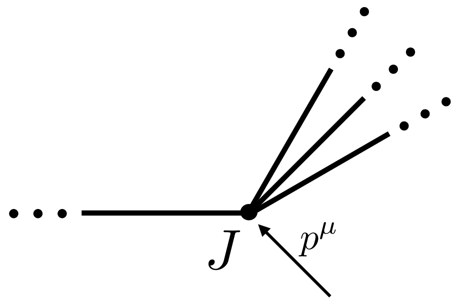


Figure 3. Portion of diagram arising from a non-linear term $J\varphi^3$ involving the source J in the equation of motion of φ. It leads to amplitudes proportional to p^2, where p^μ is the momentum flowing through the source.

2.2 Covariant color-kinematics duality between GBAS and YM

We now turn to a short review of the covariant color-kinematics (CCK) duality introduced by Cheung and Mangan [66]. It establishes maps between equations of motion of different theories and, therefore, between their tree-level scattering amplitudes.

Let us consider the Yang-Mills theory example, which will be most useful for our purposes. Starting from the Yang-Mills (YM) Lagrangian with a source $J^a_\mu(x)$,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + A^{a\mu} J^a_\mu \ , \tag{2.11}$$

where the field-strength tensor is $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$, one derives the usual YM EOM,

$$D^\mu F^a_{\mu\nu} = -J^a_\nu \ , \tag{2.12}$$

where $D_\mu F^a_{\nu\rho} \equiv \partial_\mu F^a_{\nu\rho} + g\, f^{abc} A^b_\mu F^c_{\nu\rho}$ and f^{abc} are group structure constants which verify the Jacobi identity. Upon differentiating the EOM above and using the Bianchi identity, [66] showed that the following equation can be derived:

$$D^2 F^a_{\mu\nu} + g\, f^{abc} F^b_{\rho[\mu} F^{c\rho}_{\nu]} = -D_{[\mu} J^a_{\nu]} \ , \tag{2.13}$$

where we defined $X_{[\mu\nu]} \equiv X_{\mu\nu} - X_{\nu\mu}$, and where DJ could be replaced by ∂J without affecting the on-shell scattering amplitudes, as explained above. This equation has the crucial property that the spacetime indices of the gluon field strength are not contracted with those of covariant derivatives. Since $D^2 = \Box$+non-linear interaction terms dependent on A_μ and $F_{\mu\nu}$, given a solution A_μ and $F_{\mu\nu}$ at a given order in the source, one can solve for $F_{\mu\nu}$ at the next order by simply inverting $\Box$, without making the relation between $F_{\mu\nu}$ and A_μ explicit. Consequently, one can reinterpret Eq. (2.13) as describing the propagation of six flavors of colored scalars

$$\lambda\, \phi^{aA} \leftrightarrow F^a_{\mu\nu} \, , \tag{2.14}$$

with a cubic interaction. (We have included a factor of λ in accordance with dimensional analysis.) Moreover, that cubic interaction can be expressed in terms of a structure constant f^{ABC}, to be constructed below, which verifies the Jacobi identity. Therefore, the scalars form a bi-adjoint multiplet whose first symmetry group has been gauged. Below, we will refer to those two groups as color and flavor, respectively. This theory is known as the gauged bi-adjoint scalar theory (GBAS). Its Lagrangian reads

$$\mathcal{L}_{\text{GBAS}} = \mathcal{L}_{\text{YM}} + \frac{1}{2}D^\mu\phi^{aA}D_\mu\phi^{aA} - \frac{g\,\lambda}{3}f^{abc}f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC} + J^{aA}\phi^{aA} \tag{2.15}$$

where λ has mass dimension one and leads to the following EOM

$$D^2\phi^{aA} + g\,\lambda\,f^{abc}f^{ABC}\phi^{bB}\phi^{cC} = J^{aA}\ , \tag{2.16}$$

from which we can read off the map of the scalar source into the gluon one,

$$\lambda\,J^{aA} \leftrightarrow -D_{[\mu}J^a_{\nu]}\ , \tag{2.17}$$

as well as the map for the flavor structure constant in terms of the space-time metric,

$$f^{A_1A_2A_3} \leftrightarrow -\frac{1}{4}\eta^{\nu_3][\mu_1}\eta^{\nu_1][\mu_2}\eta^{\nu_2][\mu_3}\ . \tag{2.18}$$

Having connected the EOMs of the two theories, we can also connect their one-point functions with sources, and therefore their scattering amplitudes. This is however nontrivial given *i*) that the bi-adjoint scalar still interacts with gluons, and *ii*) that the sources for both fields are correlated according to Eq. (2.17).

The complication *i*) arises since we artificially separated the gluon field and its field strength. In order to compute scattering amplitudes as sketched in Sec. 2.1, we could use $\langle 0|A^a_\mu|0\rangle_J$ or $\langle 0|F^a_{\mu\nu}|0\rangle_J$. Both fields interpolate single-gluon states and can be related after gauge fixing. So using either of them will simply change the differential operators that act on the nth field in the LSZ reduction formula. For instance, in an axial gauge where $q^\mu A^a_\mu = 0$ for an arbitrary reference vector q,

$$|g^a(p,h)\rangle = \epsilon^\nu_h A^a_\nu(p)|0\rangle = \frac{iq^{[\mu}\epsilon^{\nu]}_h F^a_{\mu\nu}(p)}{2\,q\cdot p}|0\rangle\ , \tag{2.19}$$

for a gluon of momentum p, helicity h and color a. Reference [66] proposes to use the field strength, related to $\langle 0|\phi^{aA}|0\rangle_J$ in the dual theory through the CCK replacement rule,

$$\lambda\Big[\langle 0|\phi^{aA}|0\rangle_J\Big]_{\text{GBAS}} \xrightarrow[\text{CCK}]{} \Big[\langle 0|F^a_{\mu\nu}|0\rangle_J\Big]_{\text{YM}}, \tag{2.20}$$

I included a λ above given our above normalizationswhich we make explicit below. Differentiating with respect to sources, this implies a duality between GBAS scattering amplitudes involving at least one scalar and YM amplitudes. However, one should note that the computation of $\langle 0|\phi^{aA}|0\rangle_J$ is affected by the fact *ii*): in the perturbative solution for ϕ^{aA}, the same source generates both gluons and scalars. Therefore, n-point scattering amplitudes of gluons in the YM theory are mapped to combinations of amplitudes with different numbers of scalars in the GBAS theory; specifically $2 \le m \le n$ scalars and $n-m$ gluons (where we used the fact that tree-level GBAS amplitudes with a single scalar are zero).

Although the GBAS scalar EOM of Eq. (2.16) is in one-to-one correspondence with the YM field strength EOM of Eq. (2.13), the gluon EOMs in the two theories are different. The YM gluon propagates according to Eq. (2.12), whereas the GBAS gluon EOM in principle includes a scalar current of the form $\phi D\phi$. However, this term can be ignored when restricting to single-trace GBAS amplitudes calculated from $\langle 0|\phi^{aA}|0\rangle_J$, in which case the two gluon EOMs become identical. The CCK duality can thus be phrased as a map from GBAS amplitudes with only a single trace of flavor group generators to YM amplitudes.

What happens in practice is best described through an example, so let us focus on the three-point gluon amplitude $\mathcal{A}_{\text{YM}}(1,2,3)$. It can be computed from the three-point correlator $\langle 0|TA^a_\mu(x)A^b_\nu(y)F^c_{\rho\sigma}(z)|0\rangle$, using the usual LSZ reduction formula with the exception that the third polarization should be replaced by $iq^{[\rho}\epsilon_3^{\sigma]}/(2\,q\cdot p_3)$. That correlator can itself be derived from $\langle 0|F^c_{\rho\sigma}(z)|0\rangle_J$, upon differentiation with respect to $J^a_\mu(x)$ and $J^b_\nu(y)$, before fixing all sources to zero. By the CCK duality of the EOMs, this is equivalent to acting on $\langle 0|\phi^{cC}|0\rangle_J$. Now, which amplitudes of the regular gauged bi-adjoint theory are generated by $\langle 0|\phi^{cC}|0\rangle_J$? We have that

$$(-i)\frac{\delta}{\delta J^{b\nu}(y)}\langle 0|\phi^{cC}(z)|0\rangle_J = \int \mathrm{d}^4y' \left[\frac{\delta J^{b'\nu'}(y')}{\delta J^{b\nu}(y)}\langle 0|A^{b'}_{\nu'}(y')\phi^{cC}(z)|0\rangle_J + \frac{\delta J^{b'B}(y')}{\delta J^{b\nu}(y)}\langle 0|\phi^{b'B}(y')\phi^{cC}(z)|0\rangle_J\right] \tag{2.21}$$

where, ignoring non-linear terms involving the source,

$$\frac{\delta J^{b'\nu'}(y')}{\delta J^{b\nu}(y)} = \delta^{b'}_b\,\delta^{\nu'}_\nu\,\delta^{(4)}(y-y')\ , \quad \frac{\delta J^{b'B}(y')}{\delta J^{b\nu}(y)} = -\delta^{b'}_b\,\delta^{B,\rho\sigma}\partial_{[\rho}\eta_{\sigma]\nu}\delta^{(4)}(y-y')\,. \tag{2.22}$$

The second equation here arises from the relation between the sources in Eq. (2.17) and results in an external polarization of the scalars given by $-ip_{[\mu}\epsilon_{\nu]}$. Differentiating once more, using the LSZ formula and matching to GBAS amplitudes, one finds

$$\begin{aligned}\mathcal{A}_{\text{YM}}(g_1,g_2,g_3) =& \frac{i\delta^{\alpha\beta}_{A_3}q_{[\alpha}\epsilon_{3\beta]}}{2\,q\cdot p_3}\Bigg[\delta^{\mu\nu}_{A_1}\delta^{\rho\sigma}_{A_2}\left(-ip_{1[\mu}\epsilon_{\nu]1}\right)\left(-ip_{2[\rho}\epsilon_{\sigma]2}\right)\mathcal{A}_{\text{GBAS}}(\phi_1^{A_1},\phi_2^{A_2},\phi_3^{A_3}) \\ &+\left\{\delta^{\rho\sigma}_{A_2}\left(-ip_{2[\rho}\epsilon_{\sigma]2}\right)\mathcal{A}_{\text{GBAS}}(g_1,\phi_2^{A_2},\phi_3^{A_3}) + (1\leftrightarrow 2)\right\} \\ &+\mathcal{A}_{\text{GBAS}}(g_1,g_2,\phi_3^{A_3})\Bigg]\ ,\end{aligned} \tag{2.23}$$

where the last amplitude on the r.h.s. actually vanishes.

In general, an explicit restriction to single traces has to be performed on the GBAS side. However, the amplitudes on the r.h.s. of Eq. (2.23) only involve a single trace of flavor generators and can therefore be kept. Actually, one obtains simpler formulae by making those flavor factors explicit, i.e. using flavor-ordered GBAS amplitudes. Let us look at the first line of Eq. (2.23) above: $\mathcal{A}(\phi_1^{A_1},\phi_2^{A_2},\phi_3^{A_3})$ comes with a factor of $\lambda\,f^{A_1A_2A_3}$ defined in Eq. (2.18). Contracting with the momentum and polarization factors, one finds

$-4i\,\mathrm{Tr}(\mathsf{F}_1\mathsf{F}_2\tilde{\mathsf{F}}_3)$ where

$$\mathsf{F}_i^{\mu\nu} \equiv p_i^{[\mu}\epsilon_i^{\nu]}\ , \quad \tilde{\mathsf{F}}_i^{\mu\nu} \equiv -\frac{q^{[\mu}\epsilon_i^{\nu]}}{2\,p_i\cdot q}\ , \quad \mathrm{Tr}(\mathcal{O}_{\mu\nu}) \equiv \frac{1}{2}\eta^{\mu\nu}\mathcal{O}_{\mu\nu}\ . \tag{2.24}$$

Using the antisymmetry of $\overset{(\sim)}{\mathsf{F}}$, one can rewrite $-4i\mathrm{Tr}(\mathsf{F}_1\mathsf{F}_2\tilde{\mathsf{F}}_3) = -2i\,\mathrm{Tr}(\mathsf{F}_1\mathsf{F}_2\tilde{\mathsf{F}}_3 - \mathsf{F}_2\mathsf{F}_1\tilde{\mathsf{F}}_3)$ and compare with the usual relation, $f^{A_1A_2A_3} = -2i\mathrm{Tr}(T^{A_1}T^{A_2}T^{A_3} - T^{A_2}T^{A_1}T^{A_3})$. The flavor factor of the term on the second line of Eq. (2.23) is $\delta^{A_2A_3} = -\frac{1}{4}\eta^{\nu_3][\mu_2}\eta^{\nu_2][\mu_3}$, so that we find $2\,\mathrm{Tr}(\mathsf{F}_2\tilde{\mathsf{F}}_3)$, to be compared with $\delta^{A_2A_3} = 2\mathrm{Tr}(T^{A_2}T^{A_3})$. We can therefore rewrite Eq. (2.23) as

$$\mathcal{A}_{\text{GBAS}}(\phi_1^{A_1},\phi_2^{A_2},\phi_3^{A_3}) + \Big\{\mathcal{A}_{\text{GBAS}}(g_1,\phi_2^{A_2},\phi_3^{A_3}) + (1\leftrightarrow 2)\Big\} \xrightarrow[\text{CCK}]{} \mathcal{A}_{\text{YM}}(g_1,g_2,g_3)\ , \tag{2.25}$$

where the CCK replacement rule on the flavor factors is

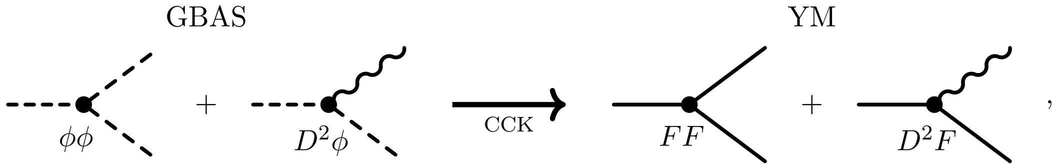
$$\lambda^{n-2}\,\mathrm{Tr}(T^{A_1}T^{A_2}...T^{A_n}) \xrightarrow[\text{CCK}]{} \mathrm{Tr}(\mathsf{F}_1\mathsf{F}_2...\tilde{\mathsf{F}}_n)\,. \tag{2.26}$$

In terms of flavor-ordered amplitudes, this means

$$\mathcal{A}_3^{\text{YM}} = A_{\phi\phi\phi}[123]\,\mathsf{F}[12\tilde{3}] + A_{\phi\phi\phi}[213]\,\mathsf{F}[21\tilde{3}] + A_{\phi g\phi}[13]\,\mathsf{F}[1\tilde{3}] + A_{g\phi\phi}[23]\,\mathsf{F}[2\tilde{3}]\,, \tag{2.27}$$

where we used the shorthand notation $\mathsf{F}[\sigma\tilde{n}] \equiv \mathrm{Tr}(\mathsf{F}_{\sigma_1}...\mathsf{F}_{\sigma_{|\sigma|}}\tilde{\mathsf{F}}_n)$ with σ being the permutation of the ϕ-scalar subset of the $(1,...,n-1)$ particles. The flavor-ordered GBAS amplitudes on the r.h.s. have $[\sigma n]$ arguments specifying the flavor traces that have been isolated (along with powers of λ), and have subscript making explicit which of the particles are scalars and gluons.

Diagrammatically, the CCK map for this three-point amplitude (or one-point function expanded to $\mathcal{O}(J^2)$) is thus the following:



(2.28)

where the vertices are schematically labeled by the terms which generate them in the YM-field-strength and GBAS EOMs of Eq. (2.13) and Eq. (2.16). Dashed lines represent ϕ scalars, solid ones represent the YM field strength F, while wavy ones are gluons. Note that on the r.h.s. only pure-gluon amplitudes are generated (external gluons are interpolated by *both* A_μ and $F_{\mu\nu}$), while on the left-hand side (l.h.s.) we start from pure-scalar and mixed scalar-gluon amplitudes. The EOM evolution of the fields is pictured in these diagrams from left to right, from the initial root leg to the final leaf legs (or sources). Starting from a scalar root leg, the restriction to the single-trace sector of the GBAS theory is achieved by allowing scalar legs to branch into scalars and gluons, while forbidding gluons to branch back into scalars.

Extending the above to n-point scatterings, [66] found

$$A_{\text{YM},n} = \sum_{\Phi \in \mathbb{P}^{+}(1 \ldots n-1)} \sum_{\sigma \in S(\Phi)} A_{\text{GBAS}}[\sigma n] \, \mathsf{F}[\sigma \tilde{n}] \,, \tag{2.29}$$

where the first sum runs over all different choices of $m-1$ scalars (with $2 \leq m \leq n$), captured by the non-empty power set $\mathbb{P}^{+}(1 \ldots n-1)$, which is the set of all non-empty subsets of $(1, \ldots, n-1)$, while the $S(\Phi)$ set captures all permutations of Φ. We stress once more that factors of the dimensionful coupling λ are taken out of the above formula through the definition of the flavor-ordered amplitude, as required for a correct matching of amplitude dimensions. The same will apply to all formulae of that sort in what follows.

2.3 Derivation of Yang-Mills numerators

The CCK duality as presented in Eq. (2.29) derives YM amplitudes from GBAS ones with fewer gluons and more scalars, but not quite from pure-scalar amplitudes yet. Conversely, it is also known how to relate amplitudes in the opposite direction: namely to obtain GBAS amplitudes with fewer gluons and more scalars, or to get GBAS amplitudes from YM ones, through the so-called *transmutation* operators of [73]. Combining both techniques, Cheung and Mangan [66] derived a closed-form expression for the BCJ numerators of YM at any multiplicity in the trace basis. These allow for an explicit decomposition of gluon amplitudes in terms of single-trace pure-scalar GBAS (i.e. BAS) amplitudes,

$$\mathcal{A}_{\text{YM},n} = \sum_{\sigma \in S(1 \ldots n-1)} A^{\text{BAS}}_{\phi^n}[\sigma n] \, K^{(4)}[\sigma n] \,. \tag{2.30}$$

The numerator superscript (4) distinguishes it from analogous objects derived below at higher EFT order. From these trace-basis numerators $K^{(4)}[\sigma n]$ with any ordering σ, one can straightforwardly obtain BCJ numerators for the YM theory (in the adjoint basis).[3] They are therefore directly relevant for the regular CK duality and the BCJ approach to the double copy.

Because we will follow the same procedure to derive numerators in the EFT below, we now review this at three points. Let us consider the expression of the three-point YM amplitude in terms of the polarization vector ϵ_i of the gluon i. It has been shown in [73] that acting with the operator $\partial_{\epsilon_1 \cdot \epsilon_3}$ on that amplitude generates a GBAS amplitude according to the transmutation relation $-2\, \partial_{\epsilon_1 \cdot \epsilon_3} \mathcal{A}_{\text{YM},3} = A^{\text{GBAS}}_{\phi g \phi}[13]$. Acting now with this operator on both sides of Eq. (2.27), we can solve for the mixed scalar-gluon amplitude in terms of a pure scalar amplitude,

$$A^{\text{GBAS}}_{\phi g \phi}[13] = A^{\text{BAS}}_{\phi^3}[123] \, \mathsf{G}[1,2,3] \,, \tag{2.31}$$

[3] The procedure is strictly identical to the one through which one generates adjoint color structures from traces of color generators [6, 40, 74]. In this analogy, $K^{(4)}$ plays the role of a trace and the resulting BCJ numerators have the required adjoint-like properties. Note that the trace-basis numerators $K^{(4)}$ are more redundant than regular BCJ ones, since certain trace-like structures give rise to vanishing adjoint-like objects.

where

$$\mathsf{G}[\sigma,\tau,\rho] \equiv -\frac{(p_\sigma)_\mu (\mathsf{F}_\tau)^{\mu\nu}\, q_\nu}{(p_{\sigma\rho})^\alpha\, q_\alpha}\,, \tag{2.32}$$

with q an arbitrary reference momentum, $p_\sigma = p_{\sigma_1} + ... + p_{\sigma_{|\sigma|}}$ and

$$(\mathsf{F}_\sigma)^{\mu\nu} = (\mathsf{F}_{\sigma_1})^{\mu}{}_{\mu_1}(\mathsf{F}_{\sigma_2})^{\mu_1}{}_{\mu_2}...(\mathsf{F}_{\sigma_{|\sigma|}})^{\mu_{|\sigma|-1}\nu}\,. \tag{2.33}$$

Inserting this back into Eq. (2.27), we obtain the YM amplitude in terms of pure scalar GBAS amplitudes and, hence, the three-point numerator:

$$K^{(4)}[123] = \mathsf{F}[12\tilde{3}] + \mathsf{G}[1,2,3]\,\mathsf{F}[1\tilde{3}]\,. \tag{2.34}$$

The other numerator, $K^{(4)}[213]$, can be derived in a similar way, or simply obtained as a permutation of the particle labels in the above numerator.

3 Effective-field-theory extension to dimension six

The derivation above relies on the precise form of the equations of motion, i.e. of the interactions. It is therefore natural to ask whether these can be modified while maintaining the CCK duality. One possible modification is to deform the action by the addition of higher-dimensional operators, while keeping the spectrum untouched. It is known that a regular CK duality exists at least for some of those deformations, including the lowest-order dimension-six correction to the Yang-Mills theory consisting of a trace of three field-strength tensors [43].

In this section, let us thus consider the $\mathcal{O}(1/\Lambda^2)$ amplitudes of such a YM+F^3 theory:[4]

$$\mathcal{L}^{(6)}_{\rm YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{g}{3\,\Lambda^2}f^{abc}F^{a}_{\mu}{}^{\nu}F^{b}_{\nu}{}^{\rho}F^{c}_{\rho}{}^{\mu} + A^a_\mu J^{a\,\mu}_A\,. \tag{3.1}$$

where Λ is an energy scale. We will find that a CCK duality is still present, which will be expressed in terms of scattering amplitudes at the end of Sec. 3.1. In terms of one-point functions in the presence of sources, it reads

$$\Big[\langle 0|A^a_\mu|0\rangle^{(4,1)}_J\Big]_{\rm GBAS} \xrightarrow[\text{CCK}]{} \Big[\langle 0|A^a_\mu|0\rangle^{(6)}_J\Big]_{\rm YM}\,, \tag{3.2}$$

where $\big[\langle 0|\Phi|0\rangle^{(m[,n])}_J\big]_{\rm Th.}$ denotes the one-point function of the field Φ computed in the theory Th. at mass dimension m, in the n-trace sector (only for the GBAS theory). This duality therefore relates the *renormalizable* GBAS to the dimension-six YM+F^3 effective field theory. The subscript J indicates that the one-point function is computed in the presence of sources and, in the GBAS theory, the two sources are correlated as in Eq. (2.17). Finally, the CCK map is extended to a new treatment of flavor traces, different from that of Eq. (2.27), which is presented below.

[4]To avoid confusion with the terminology used there, we stress that the F^3 operator is not related to the F^3 replacement rule of [66], where higher-derivative interactions are not considered.

3.1 Covariant color-kinematics duality between GBAS and YM+F^3

To establish this CCK duality, we inspect the EOM of the YM+F^3 theory,

$$D^\mu F^a_{\mu\nu} + \frac{g}{\Lambda^2}\, f^{abc}\, F^b_{\mu\rho} D_\nu F^{c,\mu\rho} = -J^a_\nu \ , \tag{3.3}$$

derived using the Bianchi identity, dropping non-linear terms where sources multiply other fields, and truncating to $\mathcal{O}(1/\Lambda^2)$ by using the renormalizable YM EOM of Eq. (2.12) in terms that are already suppressed by $1/\Lambda^2$.

As field-strength and covariant-derivative indices are not contracted together in the second term of the l.h.s., this EOM can be mapped to the gluon EOM in the GBAS theory,

$$D^\mu F^a_{\mu\nu} + g\, f^{abc}\, \phi^{bA} D_\nu \phi^{cA} = -J^a_\nu \,, \tag{3.4}$$

through the same replacement as in the previous section, $F^a_{\mu\nu} \leftrightarrow \lambda\, \phi^{aA}$, but with the notable difference that the field strength in the first term, $D^\mu F^a_{\mu\nu}$, does not get mapped. Instead, the variable of interest in this term remains the gluon field A_μ and not $F_{\mu\nu}$. The EFT power counting makes this partial map consistent, when solving the EOM perturbatively in J *and* in $1/\Lambda^2$ as follows. Denoting $F^{(d)}$ the solution at order $\mathcal{O}\big(1/\Lambda^{d-4}\big)$, the EOM of Eq. (3.3) can be rewritten as

$$\begin{aligned} &D^\mu F^{(6)a}_{\mu\nu} + \frac{g}{\Lambda^2}\, f^{abc}\, F^{(4)b}_{\alpha\beta} D_\nu F^{(4)c,\alpha\beta} = 0 \,, \\ &D^\mu F^{(4)a}_{\mu\nu} = -J^a_\nu \,. \end{aligned} \tag{3.5}$$

We do not consider $F^{(d>6)}$ since we have dropped terms of order $\mathcal{O}\big(1/\Lambda^4\big)$ when deriving Eq. (3.3). Following the steps of the previous section, we can therefore interpret $F^{(4)}$ as a scalar propagating in a gluon background, while considering $F^{(6)}$ as the field strength tensor of that gluon.[5] The EOM of $F^{(4)}$ can then be rewritten as

$$D^2 F^{(4)a}_{\mu\nu} + g\, f^{abc} F^{(4)b}_{\rho[\mu} F^{(4)c\rho}_{\nu]} = -D_{[\mu} J^a_{\nu]} \ , \tag{3.6}$$

just as in the renormalizable case discussed in Sec. 2.2.

Thanks to this duality between the gluon EOM in YM+F^3 and the gluon EOM in GBAS (Eq. (3.3) and Eq. (3.4)), YM+F^3 amplitudes are therefore encoded in GBAS ones. To be precise and as anticipated in Eq. (3.2), the EFT power counting implies that the relevant GBAS amplitudes are those obtained from the single-trace part of $\langle 0|A^a_\mu|0\rangle_J$ with at least two scalars. This means that n-point gluon amplitudes in YM+F^3 are mapped to combinations of amplitudes with $2 \le m \le n-1$ scalars and $n-m \ge 1$ gluon(s). We stress here the difference with Sec. 2.2, where the relevant GBAS object is the single-trace

[5] One may wonder why we do not also try to interpret $F^{(6)}$ as a scalar. It turns out that manipulating the dimension-six gluon EOM of Eq. (3.3) as done to obtain Eq. (2.13) leads to the following EOM for $F^{(6)}$,

$$D^2 F^a_{\mu\nu} + f^{abc} F^b_{\rho[\mu} F^{c\rho}_{\nu]} - \frac{g}{\Lambda^2} f^{abc} \left(f^{bde} F^d_{\mu\nu} F^e_{\rho\sigma} F^{c\rho\sigma} + D_{[\mu} F^{c\rho\sigma} D_{\nu]} F^b_{\rho\sigma} \right) = -D_{[\mu} J^a_{\nu]} \ ,$$

which cannot easily be recast as a scalar EOM because of the presence of covariant derivatives with uncontracted indices.

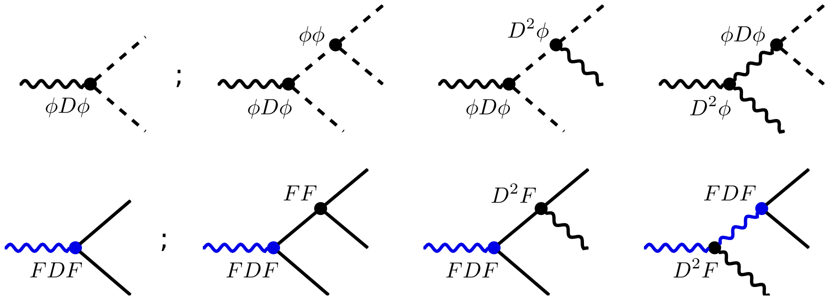


Figure 4. Diagrammatic representation of the perturbative solution to the gluon EOM in the YM+F^3 theory at three and four points, using the field-strength tensor as an independent function. Blue lines correspond to EOM solutions at dimension six and end at blue vertices, indicating dimension-six interactions.

part of the scalar one-point function $\langle 0|\phi^{aA}|0\rangle_J$, and where the relevant amplitudes have $2 \leq m \leq n$ scalars and $n-m \geq 0$ gluons. [unification here too?]

To relate the GBAS amplitudes to pure gluon ones, the external polarizations are again determined by Eq. (2.17) in the same way as in the previous section. Similarly to Eq. (2.26), the flavor traces are replaced by combinations of momenta and polarization vectors,

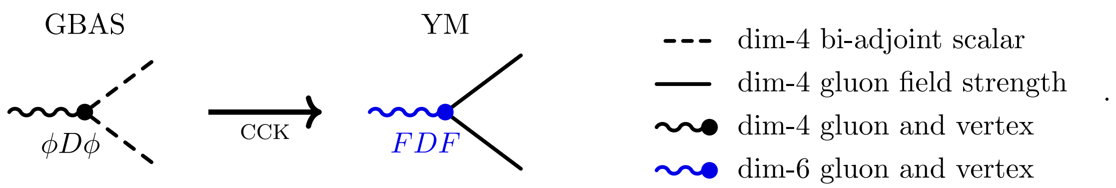
$$\lambda^{n-2}\,\mathrm{Tr}(T^{A_1}T^{A_2}...T^{A_n}) \to \frac{1}{\Lambda^2}\,\mathrm{Tr}(\mathsf{F}_1\mathsf{F}_2...\mathsf{F}_n)\,, \tag{3.7}$$

but $\tilde{\mathsf{F}}$ no longer appears since the generating correlator $\langle 0|A^a_\mu|0\rangle_J$ now features the gluon field. The explicit factor of $1/\Lambda^2$ clearly shows that this CCK duality generates higher-derivative interactions. [unification, still?]

It thus follows that the n-point dimension-six YM+F^3 amplitude is encoded in single-trace GBAS amplitudes through

$$\mathcal{A}^{(6)}_{\mathrm{YM},n} = \frac{1}{\Lambda^2}\sum_{\phi\in\mathbb{P}^{++}(1...n-1)}\;\sum_{\sigma\in S(\phi)/Z_{|\phi|}} A_{\mathrm{GBAS}}[\sigma]\;\mathsf{F}[\sigma]\,. \tag{3.8}$$

where $\mathsf{F}[\sigma] \equiv \mathrm{Tr}(\mathsf{F}_{\sigma_1}...\mathsf{F}_{\sigma_{|\sigma|}})$. This equation is similar to Eq. (2.29), with important differences arising from the fact that the nth particle is now a gluon. Since at least two scalars are required in the GBAS amplitudes, there appears the set of all subsets of $(1...n-1)$ containing at least two elements, denoted $\mathbb{P}^{++}(1,...,n-1)$. In addition, the set $S(\phi)/Z_{|\phi|}$ contains all permutations that result in inequivalent traces (using cyclicity). The three-point CCK map at dimension-six is for instance the following:



(3.9)

We emphasize the remarkable fact that the higher-derivative amplitudes of YM+F^3 are captured by the GBAS amplitudes *without* higher-derivative interactions. For example, at 3 and 4 points, Eq. (3.8) is written as

$$\mathcal{A}^{(6)}_{\text{YM},3} = \frac{1}{\Lambda^2} A^{\text{GBAS}}_{\phi\phi g}[12]\,\mathsf{F}[12]\,, \tag{3.10}$$

and

$$\begin{aligned}\mathcal{A}^{(6)}_{\text{YM},4} = &\frac{1}{\Lambda^2}\Big(A^{\text{GBAS}}_{\phi\phi\phi g}[123]\,\mathsf{F}[123] + A^{\text{GBAS}}_{\phi\phi\phi g}[132]\,\mathsf{F}[132] \\ &+ A^{\text{GBAS}}_{\phi\phi gg}[12]\,\mathsf{F}[12] + A^{\text{GBAS}}_{\phi g\phi g}[13]\,\mathsf{F}[13] + A^{\text{GBAS}}_{g\phi\phi g}[23]\,\mathsf{F}[23]\Big)\,.\end{aligned} \tag{3.11}$$

The diagrams that enter the GBAS calculation at these orders and their YM analogues are illustrated in Fig. 4.

3.2 Derivation of YM+F^3 numerators

Equation (3.8) derives YM+F^3 amplitudes from a sum of GBAS amplitudes. Exactly as in Sec. 2.3, the transmutation operation can be used to reduce the latter to BAS amplitudes (i.e. pure-scalar single-trace tree-level GBAS amplitudes) and hence isolate the BCJ numerators in the trace basis.

For example, at three points, we use Eq. (3.10) and a symmetrized version of Eq. (2.31), namely

$$A^{\text{GBAS}}_{\phi\phi g}[12] = \frac{1}{2}A^{\text{GBAS}}_{\phi^3}[231]\;\mathsf{G}[2,3,1] + \frac{1}{2}A^{\text{GBAS}}_{\phi^3}[321]\;\mathsf{G}[3,2,1]\,, \tag{3.12}$$

to conclude that

$$K^{(6)}[123] = \frac{1}{2\Lambda^2}\mathsf{F}[21]G[2,3,1]\,. \tag{3.13}$$

The derivation of BCJ numerators at any multiplicity also follows that of [66], with an extra symmetrization that relates to the fact that the root leg is a gluon rather than a scalar in the CCK duality at dimension six. The resulting closed-form expression is

$$K^{(6)}[12...n] \;=\; \frac{1}{\Lambda^2}\sum_{\ell=1}^{n-2}\sum_{\tau}\frac{1}{|\tau_1|+1}\,\mathsf{F}[\tau_1\ell]\prod_{i=2}^{|\tau|}\mathsf{G}\big[(\tau_1...\tau_{i-1})_{<\tau_i},\tau_i,(\tau_1...\tau_{i-1})_{>\tau_i}\ell\big]\,, \tag{3.14}$$

and permutations thereof, with the second sum running over $\tau\in\text{part}(\ell+1,...,n,1,...,\ell-1)$. This expression relies on the notation of [66] with small modifications that we discuss now. The function part(σ) is defined as the set of all ordered partitions of the set σ into subsets whose elements follow the ordering of σ. For example, 1 should appear on the right of n if both appear in the same subset of a partition. We also require that the first subset of every partition (i.e. τ_1) contains the first element of σ but *never* n. Finally, the greater-than symbol $>$ and less-than symbol $<$ also refer to the ordering $(\ell+1,...,n,1,...,\ell-1)$. Namely, $(\tau_1...\tau_{i-1})_{<\tau_i}$ are the elements in $\tau_1\cup...\cup\tau_{i-1}$ on the left of the first element of τ_i in $(\ell+1,...,n,1,...,\ell-1)$, and $(\tau_1...\tau_{i-1})_{>\tau_i}$ are the elements in $\tau_1\cup...\cup\tau_{i-1}$ on the right of the first element of τ_i.

At lowest orders, the part function is

$$
\begin{aligned}
\text{part}(23) &= \{2,3\} \\
\text{part}(234) &= \{\{23,4\},\{2,34\},\{2,3,4\},\{2,4,3\}\} \\
\text{part}(341) &= \{\{31,4\},\{3,41\},\{3,4,1\},\{3,1,4\}\}\,,
\end{aligned} \tag{3.15}
$$

such that Eq. (3.14) for $n=4$ yields

$$
\begin{aligned}
\Lambda^2\, K^{(6)}[1234] = \;& \frac{1}{3}\mathsf{F}[231]\mathsf{G}[23,4,1] + \frac{1}{3}\mathsf{F}[312]\mathsf{G}[3,4,12] \\
& + \frac{1}{2}\mathsf{F}[21]\left(\mathsf{G}[2,34,1] + \mathsf{G}[2,3,1]\mathsf{G}[23,4,1] + \mathsf{G}[2,4,1]\mathsf{G}[2,3,41]\right) \\
& + \frac{1}{2}\mathsf{F}[32]\left(\mathsf{G}[3,41,2] + \mathsf{G}[3,4,2]\mathsf{G}[34,1,2] + \mathsf{G}[3,1,2]\mathsf{G}[3,4,12]\right)\,,
\end{aligned} \tag{3.16}
$$

where we remind the reader that $\mathsf{F}[\sigma] \equiv \mathrm{Tr}(\mathsf{F}_{\sigma_1}\ldots\mathsf{F}_{\sigma_{|\sigma|}})$. We have cross-checked Eq. (3.8) and Eq. (3.14) against explicit Feynman diagram calculations in amplitudes with up to seven external particles.

4 Effective-field-theory extension to dimension eight

The YM+F^3 theory of Eq. (3.1) does not satisfy the usual CK duality at dimension eight, i.e. $\mathcal{O}(1/\Lambda^4)$. However, the duality can be restored at that order by including a specific dimension-eight interaction [43], resulting in

$$
\mathcal{L}^{(8)}_{\text{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{g}{3\Lambda^2}f^{abc}\,F^{a}_{\mu}{}^{\nu}F^{b}_{\nu}{}^{\rho}F^{c}_{\rho}{}^{\mu} - \frac{g^2}{4\,\Lambda^4}f^{abe}f^{ecd}F^a_{\mu\nu}F^b_{\rho\sigma}F^{c\mu\nu}F^{d\rho\sigma} + A^a_\mu J^{a\,\mu}_A\,. \tag{4.1}
$$

In this section, we derive a CCK duality up to $\mathcal{O}(1/\Lambda^4)$ between this theory and the following GBAS theory:

$$
\begin{aligned}
\mathcal{L}^{(6)}_{\text{GBAS}} =& \mathcal{L}^{(6)}_{\text{YM}} + \frac{1}{2}D^\mu\phi^{aA}D_\mu\phi^{aA} - \frac{g\,\lambda}{3}f^{abc}f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC} + J^{aA}\phi^{aA} \\
& - \frac{g^2}{4}f^{abe}f^{ecd}\phi^{aA}\phi^{bB}\phi^{cA}\phi^{dB} - \frac{g}{2\,\Lambda^2}f^{abe}f^{ecd}F^a_{\mu\nu}F^{c\,\mu\nu}\phi^{bA}\phi^{dA}\,,
\end{aligned} \tag{4.2}
$$

which (except for the ϕ^3 interaction) results from the dimensional reduction of $\mathcal{L}^{(6)}_{\text{YM}}$ after projection on the massless modes, where the flavors of bi-adjoint scalars correspond to the Lorentz indices of the gauge field along the compact manifold. This theory therefore satisfies the BCJ relations for all flavor structures (i.e. beyond single-trace) [75] (see also [76, 77]). However, we have dropped all double-trace operators appearing at dimension six in the Lagrangian of Eq. (4.2), consistently with the EFT power counting of the CCK replacement rule in Eq. (3.7). We will show that this rule generalizes to dimension eight, so that the CCK duality combines double-trace dimension-four and single-trace dimension-six GBAS amplitudes to generate purely gluonic dimension-eight amplitudes. The resulting CCK relation, expressed in terms of one-point functions and using the notation introduced in Eq. (3.2), reads

$$\Big[\langle 0|A_\mu^a|0\rangle_J^{(6,1)} + \langle 0|A_\mu^a|0\rangle_J^{(4,2)}\Big]_{\text{GBAS}} \xrightarrow[\text{CCK}]{} \Big[\langle 0|A_\mu^a|0\rangle_J^{(8)}\Big]_{\text{YM}} . \tag{4.3}$$

The corresponding relation in terms of scattering amplitudes and the explicit treatment of double traces will be given in Sec. 4.2.

4.1 Covariant color-kinematics duality between GBAS and YM+F^3+F^4

At $\mathcal{O}(1/\Lambda^2)$, it was found in the previous section that the gluon EOM of the YM EFT can be mapped onto the gluon EOM of the GBAS theory. To extend this duality one order higher, we compare the following EOM in the pure-gluon theory at $\mathcal{O}(1/\Lambda^4)$,

$$D^\mu F^a_{\mu\nu} + \frac{g}{\Lambda^2} f^{abc} F^b_{\mu\rho} D_\nu F^{c\mu\rho} + 4\,\frac{g^2}{\Lambda^4} f^{abe} f^{ecd} F^c_{\mu\nu} D^\mu F^b_{\rho\sigma} F^{d\rho\sigma} = -J^a_\nu \, , \tag{4.4}$$

with the EOMs in the GBAS theory up to $\mathcal{O}(1/\Lambda^2)$,

$$\begin{aligned} D^\mu F^a_{\mu\nu} + g f^{abc} \phi^{bA} D_\nu \phi^{cA} &+ \frac{g}{\Lambda^2} f^{abc} F^b{}_{\mu\rho} D_\nu F^{c\mu\rho} \\ &+ 4 \frac{g^2}{\Lambda^2} f^{abe} f^{ecd} F^c_{\mu\nu} D^\mu \phi^{bA} \phi^{dA} = -J^a_\nu \, , \end{aligned} \tag{4.5}$$

$$\begin{aligned} D^2 \phi^{aA} + \lambda f^{abc} f^{ABC} \phi^{bB} \phi^{cC} &- g^2 f^{abe} f^{ecd} \phi^{bB} \phi^{cB} \phi^{dA} \\ &- \frac{g^2}{\Lambda^2} f^{abe} f^{ecd} F^b_{\mu\nu} F^{c\mu\nu} \phi^{dA} = J^{aA} \, . \end{aligned} \tag{4.6}$$

To derive the EOMs in this form, which is suggestive of the covariant color-kinematics duality, we used the lower-order EOMs iteratively in combination with the Jacobi identity. In particular, the dimension-eight term in Eq. (4.4) receives contributions from iterations at dimension six, indicating an intricate interplay between different mass dimensions. We comment further on this point in Sec. 4.5.

At the order we are considering, we can decompose the field strength of the pure-gluon theory as $F = F^{(4)} + F^{(6)} + F^{(8)}$, where, as previously, $F^{(d)}$ refers to the field strength solving the gluon EOM at $\mathcal{O}(1/\Lambda^{d-4})$. As in Sec. 3.1, we expand $F^{(6)}$ and $F^{(8)}$ in terms of gluons, whereas only $F^{(4)}$ is interpreted as a scalar and taken to evolve through the field-strength EOM of Eq. (2.13), which we repeat here:

$$D^2 F^{(4)a}_{\mu\nu} + g\, f^{abc} F^{(4)b}_{\rho[\mu} F^{(4)c\rho}_{\nu]} = -D_{[\mu} J^a_{\nu]} \, . \tag{4.7}$$

We start by inspecting the dimension-eight term in the pure-gluon EOM in Eq. (4.4),

$$D^\mu F^a_{\mu\nu} = -4\,\frac{g^2}{\Lambda^4} f^{abe} f^{ecd} F^c_{\mu\nu} D^\mu F^b_{\rho\sigma} F^{d\rho\sigma} + \dots \, . \tag{4.8}$$

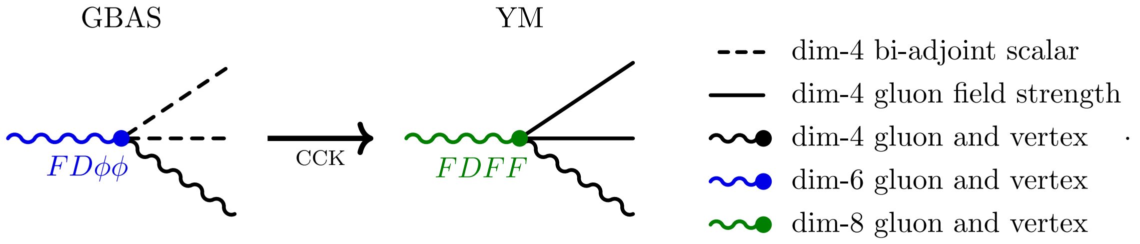
For solutions up to $\mathcal{O}(1/\Lambda^4)$, the field strengths on the r.h.s. need only satisfy the renormalizable YM equation of motion. So, at this order, we can actually solve

$$D^\mu F^a_{\mu\nu} = -4 \frac{g^2}{\Lambda^4} f^{abe} f^{ecd}\, F^c_{\mu\nu}\, D^\mu F^{(4)b}_{\rho\sigma}\ F^{(4)d\rho\sigma} + \dots \, , \tag{4.9}$$

where $F^{(4)}$ satisfies the renormalizable YM EOM with source J given in Eq. (2.13). We could have added a superscript (4) to the remaining field strength on the r.h.s. of Eq. (4.9) as well, but as it stands the above has a clear correspondence with the last term ($FD\phi\phi$) of the GBAS EOM in Eq. (4.5). Indeed, when interpreting the flavor structures in terms of Lorentz indices as in $F^a_{\mu\nu} \leftrightarrow \lambda\, \phi^{aA}$, we know that $F^{(4)}$ maps to $\phi^{(4)}$ which solves the GBAS EOM at dimension four with source DJ or, importantly, any EOM like Eq. (4.6) which reduces to it at $\mathcal{O}(1/\Lambda^0)$ and in the single-trace sector. Therefore, the solution to the pure-gluon theory also solves the following EOM,

$$D^\mu F^a_{\mu\nu} = -4\frac{g^2\lambda^2}{\Lambda^4} f^{abe} f^{ecd}\, F^c_{\mu\nu}\, D^\mu \phi^{bA}\ \ \phi^{dA} + \ldots \tag{4.10}$$

This reproduces the last term of the l.h.s. of Eq. (4.5), up to a factor of λ^2/Λ^2 which we set to one, keeping in mind the CCK rule of Eq. (3.7). [absorb instead] At the diagrammatic level, this implies that any GBAS diagram in which a gluon evolves with this dimension-six $FD\phi\phi$ interaction can be mapped to a dimension-eight diagram in the pure-gluon theory, where the scalar is interpreted as a field strength,



(4.11)

Besides contributions from this dimension-eight interaction, the solution for $\langle 0|A^a_\mu|0\rangle_J$ in the pure-gluon theory also involves diagrams with two dimension-six F^3 insertions. Therefore, the remaining terms in Eqs. (4.4–4.6) need to be compared as well. However, the F^3 interaction of the pure-gluon theory, which leads to a FDF term in the EOM, seems to have two counterparts in the gluon EOM of the GBAS theory, namely

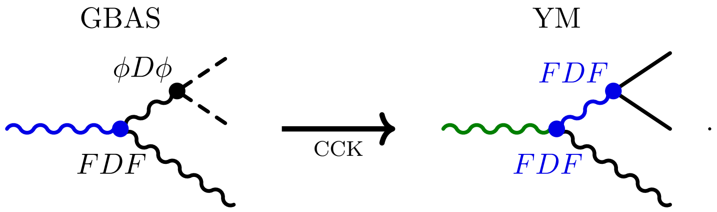
$$g f^{abc} \phi^{bA} D^\nu \phi^{cA} \qquad \text{and} \qquad \frac{g}{\Lambda^2} f^{abc} F^b{}_{\mu\rho} D^\nu F^{c\mu\rho}\,. \tag{4.12}$$

Two consecutive[6] insertions of FDF in the pure-gluon theory have an immediate analog in the GBAS theory. At the order we work, the first insertion of FDF can be written as an insertion of $F^{(6)}D^{(4)}F^{(4)} + F^{(4)}D^{(4)}F^{(6)} + F^{(4)}D^{(6)}F^{(4)}$, where by definition $F^{(6)}$ or $D^{(6)}$ creates the 'branch' in the diagram which contains the second FDF interaction.[7] As in Sec. 3, this branch is in one-to-one correspondence with a GBAS one where FDF is replaced by $\phi D\phi$. Therefore, the two FDF insertions in the pure-gluon theory are

[6]Since the diagrams to calculate $\langle 0|A^a_\mu|0\rangle_J$ from the EOM are read from left to right, there is a clear ordering in the interactions that occur on the same branch starting from the root leg towards the leaf legs (i.e. towards the sources).

[7]By $D^{(6)}$, we refer to the piece of the covariant derivative containing a gluon at order $\mathcal{O}(1/\Lambda^2)$.

equivalent in the GBAS theory to an insertion of FDF follow by that of $\phi D\phi$,

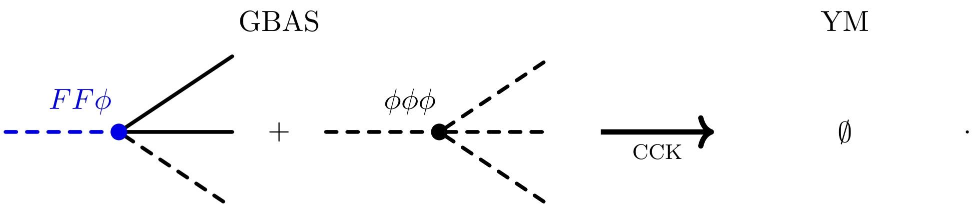


(4.13)

At four-point, Eqs. (4.11) and (4.13) capture all possibilities and therefore establish a map between the amplitudes. However, in general, the GBAS theory contains other diagrams involving the following terms of the scalar EOM of Eq. (4.6),

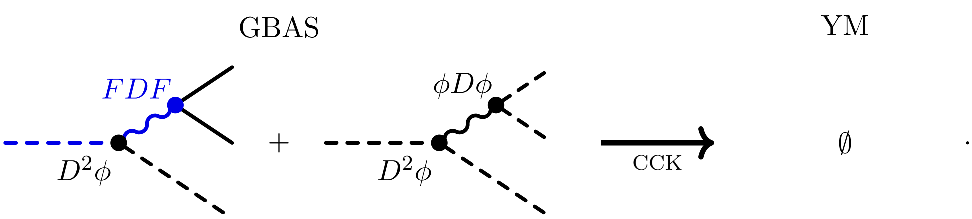
$$-\frac{g^2}{\Lambda^2} f^{abe} f^{ecd} F^b_{\mu\nu} F^{c\mu\nu} \phi^{dA} \quad \text{and} \quad -g^2 f^{abe} f^{ecd} \phi^{bB} \phi^{cB} \phi^{dA} \,. \tag{4.14}$$

Those have no direct interpretation in the pure-gluon EOM. However, we find that the tree amplitudes they give rise to, respectively at the dimension-six and double-trace levels, are related and can cancel each other. This possibility is suggested by the form of the terms in Eq. (4.14). In the first dimension-six $FF\phi$ term, the field strengths can again be taken to be dimension-four ones $F^{(4)}$ which are equivalent to ϕ scalars under the CCK duality. Up to a factor of $1/\Lambda^2$, the two terms therefore become identical. By including an additional relative sign between the single- and double-trace CCK replacement rules (made explicit in the next section), these two contributions can therefore be canceled against each other,



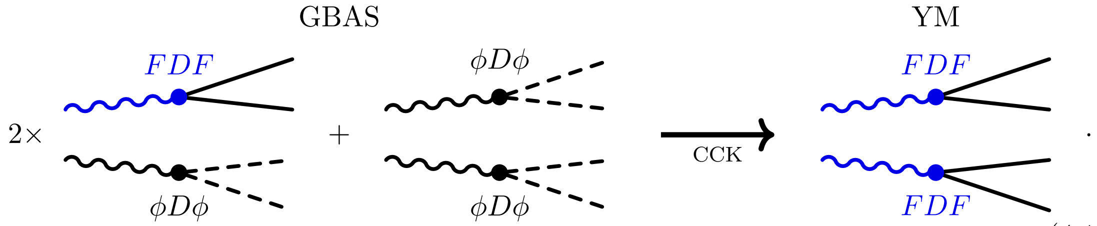
(4.15)

This pattern of cancellations between certain dimension-six single-trace and dimension-four double-trace contributions turns out to be general. They then also occur in diagrams where a gluon is emitted from a scalar and branches through the term $\frac{g}{\Lambda^2} f^{abc} F^b{}_{\mu\rho} D^\nu F^{c,\mu\rho}$ in its EOM. As seen in Sec. 3, this is equivalent to using the term $g f^{abc} \phi^{bA} D^\nu \phi^{cA}$, leading to a double-trace diagram,



(4.16)

Furthermore, when the interactions appear on different 'branches' emerging from the root leg gluon, all outgoing particles satisfy the dimension-four single-trace EOM at the order that we consider. The double-trace diagrams then cancel an overcounting that arises from exchanging the distinguishable vertices of the $\phi D\phi$ and FDF interactions of the pure-gluon EOM, leading to an exact equivalence with the pure-gluon diagrams involving a double insertion of the $\frac{g}{\Lambda^2} f^{abc} F^b_{\ \mu\rho} D^\nu F^{c,\mu\rho}$ term,



(4.17)

Eventually, using Eq. (4.3) and the appropriate extension of the CCK duality to double traces, one can effectively retain diagrams in which one branch contains first the interaction $\frac{g}{\Lambda^2} f^{abc} F^b_{\mu\rho} D^\nu F^{c\mu\rho}$ and then $g f^{abc} \phi^{bA} D^\nu \phi^{cA}$ as in Eq. (4.13), as well as diagrams in which the two interactions occur on different branches as in Eq. (4.17), without degeneracy.

The different cases discussed above correspond to all possibilities at any multiplicity, proving the validity of our CCK procedure at dimension eight. For illustration, we display all the five-point diagrams of both YM+F^3+F^4 and GBAS theories in App. A.

4.2 Explicit CCK replacement rules for scattering amplitudes

As argued above, the CCK duality at dimension eight requires the cancellation of contributions from the $\mathcal{O}(1/\Lambda^0)$ double-trace sector against some of the $\mathcal{O}(1/\Lambda^2)$ single-trace ones. At the level of the amplitudes, a relative factor of $-1/\Lambda^2$ is therefore necessary between the single- and double-trace replacements rules,

$$\begin{aligned} \lambda^{n-2}\, \mathrm{Tr}(T^{A_1}...T^{A_n}) &\to \frac{1}{\Lambda^2}\, \mathrm{Tr}(\mathsf{F}_1...\mathsf{F}_n)\,, \\ \lambda^{n+m-4}\, \mathrm{Tr}(T^{A_{i_1}}...T^{A_{i_n}})\, \mathrm{Tr}(T^{A_{j_1}}...T^{A_{j_m}}) &\to -\frac{1}{\Lambda^4}\, \mathrm{Tr}(\mathsf{F}_{i_1}...\mathsf{F}_{i_n})\, \mathrm{Tr}(\mathsf{F}_{j_1}...\mathsf{F}_{j_m})\,, \end{aligned} \tag{4.18}$$

This generalizes the dimension-six rule of Eq. (3.7) to dimension eight and leads to the following formula for YM+F^3+F^4 amplitudes:

$$\mathcal{A}^{(8)}_{\mathrm{YM},n} = \frac{1}{\Lambda^2} \sum_{\phi\in\mathbb{P}^{++}(1...n-1)} \sum_{\sigma\in S(\phi)/Z_{|\phi|}} A^{(6)}_{\mathrm{GBAS}}[\sigma]\ \mathsf{F}[\sigma] - \frac{1}{\Lambda^4} \sum_{\phi,\bar\phi} \sum_{\sigma,\bar\sigma} A^{(4)}_{\mathrm{GBAS}}[\sigma|\bar\sigma]\ \mathsf{F}[\sigma]\ \mathsf{F}[\bar\sigma]\,, \tag{4.19}$$

where the sums in the second term run over $(\phi,\bar\phi) \in \mathbb{P}^{++}(1...n-1)$ with $\phi \cap \bar\phi = \emptyset, \phi < \bar\phi$ (in some ordering to avoid double counting) and $\sigma \in S(\phi)/Z_{|\phi|}$ and similarly for $\bar\sigma$. In words, these simply span all different double-trace amplitudes with the nth particle being a gluon. It is then relevant to note that the double-trace amplitudes $A^{(4)}_{\mathrm{GBAS}}[\sigma|\bar\sigma]$ require a minimum of four scalar particles. Similarly, the amplitudes $A^{(6)}_{\mathrm{GBAS}}[\sigma]$ are zero when there is only one external gluon. We have explicitly confirmed Eq. (4.19) up to six points against Feynman diagram calculations.

This formula is best exemplified at lowest multiplicities:

$$\mathcal{A}^{(8)}_{\mathrm{YM},4} = \frac{1}{\Lambda^2}\left(A^{(6)}_{\phi\phi gg}[12]\ \mathsf{F}[12] + A^{(6)}_{\phi g\phi g}[13]\ \mathsf{F}[13] + A^{(6)}_{g\phi\phi g}[23]\ \mathsf{F}[23]\right) \tag{4.20}$$

$$
\begin{aligned}
\mathcal{A}^{(8)}_{\text{YM},5} = \frac{1}{\Lambda^2}\Bigg(& A^{(6)}_{\phi\phi ggg}[12]\ \mathsf{F}[12] + \ldots + A^{(6)}_{gg\phi\phi g}[34]\ \mathsf{F}[34] \\
& + A^{(6)}_{\phi\phi\phi gg}[123]\ \mathsf{F}[123] + \ldots + A^{(6)}_{g\phi\phi\phi g}[243]\ \mathsf{F}[243]\Bigg) \\
& - \frac{1}{\Lambda^4}\left(A^{(4)}_{\phi\phi\phi\phi g}[12|34]\ \mathsf{F}[12]\ \mathsf{F}[34] + \ldots + A^{(4)}_{\phi\phi\phi\phi g}[14|23]\ \mathsf{F}[14]\ \mathsf{F}[23]\right),
\end{aligned} \tag{4.21}
$$

where we have suppressed some permutations of the displayed terms, noting again that the nth particle is always a gluon. We also emphasize that the orderings refer to the flavor structures: no color ordering is taken.

4.3 Dimension eight from dimension four

Although high-multiplicity expressions become lengthy, the strategy is simple: compute all GBAS amplitudes with $2, \ldots, n-1$ scalars and replace the flavor traces by traces of the linearized field-strength tensors F. In fact, we can further leverage the CCK duality between dimension-four double-trace GBAS amplitudes and dimension-six single-trace GBAS amplitudes, as depicted in Eq. (4.15) and Eq. (4.16). Let us consider a dimension-six single-trace GBAS amplitude. All relevant terms can be found in Eq. (4.6), in particular the amplitude is computed through

$$
D^2\phi^{aA} + \lambda f^{abc} f^{ABC}\phi^{bB}\phi^{cC} - \frac{g^2}{\Lambda^2} f^{abe} f^{ecd} F^b_{\mu\nu} F^{c\mu\nu}\phi^{dA} = J^{aA}\,, \tag{4.22}
$$

where in GBAS theory the source J^{aA} is independent of the gluon source J^a_μ. At the order considered, it suffices that the gluon field strength F solves the dimension-four pure-gluon EOM. Then, CCK for dimension-four YM theory implies that the same result can be obtained from

$$
\begin{aligned}
& D^2\phi^{aA} + \lambda f^{abc} f^{ABC}\phi^{bB}\phi^{cC} - \frac{g^2\tilde{\lambda}^2}{\Lambda^2} f^{abe} f^{ecd}\tilde{\phi}^{b\tilde{B}}\tilde{\phi}^{c\tilde{B}}\phi^{dA} = J^{aA}\,, \\
& D^2\tilde{\phi}^{a\tilde{A}} + \tilde{\lambda} f^{abc}\tilde{f}^{\tilde{A}\tilde{B}\tilde{C}}\tilde{\phi}^{b\tilde{B}}\tilde{\phi}^{c\tilde{C}} = \tilde{J}^{a\tilde{A}}\,,
\end{aligned} \tag{4.23}
$$

where $\tilde{f}$ and $\tilde{J}$ are given by Eq. (2.17) and Eq. (2.18), respectively. The amplitude which now arises is “twice single-trace”, i.e. it features one trace of ϕ flavor and one trace of $\tilde{\phi}$ flavor. Now, since $\tilde{\phi}$ verifies the same EOM as ϕ, we notice that the diagrams relevant for a given amplitude would precisely be found in the double-trace sector arising from the following EOM,

$$
D^2\phi^{aA} + \lambda f^{abc} f^{ABC}\phi^{bB}\phi^{cC} - g^2 f^{abe} f^{ecd}\phi^{bB}\phi^{cB}\phi^{dA} = J^{aA}\,, \tag{4.24}
$$

which is nothing but the double-trace part of in Eq. (4.6). However, the resulting double-trace amplitudes are such that the CCK rule of Eq. (3.7) should only be applied on the second trace. At the level of the amplitudes, this implies[8]

$$\mathcal{A}^{(6)}_{\text{GBAS},n}[\phi \in \mathbb{P}^{++}(1...n-1)] = \frac{1}{\Lambda^2} \sum_{\substack{\bar\phi \in \mathbb{P}^{++}(1...n) \\ \phi \cap \bar\phi = \emptyset}} \sum_{\bar\sigma \in S(\bar\phi)/Z_{|\bar\phi|}} A^{(4)}_{\text{GBAS},n}[\phi|\bar\sigma]\ \mathsf{F}[\bar\sigma]\,. \tag{4.25}$$

For example,

$$A^{(6)}_{\phi\phi gg}[12] = A^{(4)}_{\phi\phi\phi\phi}[12|34]\ \mathsf{F}[34] \tag{4.26}$$

$$A^{(6)}_{\phi\phi\phi gg}[123] = A^{(4)}_{\phi\phi\phi\phi\phi}[123|45]\ \mathsf{F}[45] \tag{4.27}$$

$$\begin{aligned} A^{(6)}_{\phi\phi ggg}[12] = {} & A^{(4)}_{\phi\phi\phi\phi g}[12|34]\ \mathsf{F}[34] + A^{(4)}_{\phi\phi\phi g\phi}[12|35]\ \mathsf{F}[35] + A^{(4)}_{\phi\phi g\phi\phi}[12|45]\ \mathsf{F}[45] \\ & + A^{(4)}_{\phi\phi\phi\phi\phi}[12|345]\ \mathsf{F}[345] + A^{(4)}_{\phi\phi\phi\phi\phi}[12|354]\ \mathsf{F}[354] \end{aligned} \tag{4.28}$$

where we again emphasize that only flavor orderings are explicitly shown. Such relations, together with the results of previous sections, lead to the conclusion that any amplitude of the considered YM EFT up to mass dimension eight can be obtained from *renormalizable* GBAS amplitudes using the CCK duality. The general formula, which we explicitly confirmed through Feynman diagrammatic computations to six-point, reads

$$\mathcal{A}^{(8)}_{\text{YM}} = \frac{1}{\Lambda^4} \sum_{\phi,\bar\phi} \sum_{\sigma,\bar\sigma} A^{(4)}_{\text{GBAS}}[\sigma|\bar\sigma]\ \mathsf{F}[\sigma]\ \mathsf{F}[\bar\sigma]\,, \tag{4.29}$$

where the sums run over $\phi, \bar\phi \in \mathbb{P}^{++}(1...n)$ with $\phi \cap \bar\phi = \emptyset$, $\phi < \bar\phi$ and $\sigma \in S(\phi)/Z_{|\phi|}$; $\bar\sigma \in S(\bar\phi)/Z_{|\bar\phi|}$. In words, we sum over all different double-trace amplitudes where, in contrast to before, the nth particle can be of any type.

It is now tempting to speculate that the relations between the GBAS theory and the YM theory extend to even higher orders in their EFT expansions, although the cancellations between single- and higher-trace are not a priori obvious. We will explore this in the next section.

4.4 Derivation of YM+F^3+F^4 numerators

Similar to Sections 2.3 and 3.2, the RHS of Eq. (4.29) can also be expressed in terms of (single-trace) BAS amplitudes, allowing for a derivation of the BCJ numerators. In fact, with the closed form of Yang-Mills numerators at hand [66], the procedure is straightforward. Starting from the dimension-eight amplitude,

$$\mathcal{A}^{(8)}_{\text{YM}} = \frac{1}{\Lambda^4} \left(A^{(4)}_{\phi\phi\phi\phi}[12|34]\ \mathsf{F}[12]\ \mathsf{F}[34] + A^{(4)}_{\phi\phi\phi\phi}[13|24]\ \mathsf{F}[13]\ \mathsf{F}[24] + A^{(4)}_{\phi\phi\phi\phi}[14|23]\ \mathsf{F}[14]\ \mathsf{F}[23] \right), \tag{4.30}$$

[8]We stress again that this formula does not require that one acts using the CCK rule on the first trace. In particular, one is free to act with *any* CCK rule on that first trace, not necessarily with Eq. (3.7) to be consistent with the action on the second trace. This implies that cancellations between dimension-six single-trace and dimension-four double-trace GBAS amplitudes are generic, a fact which we will use in Sec. 5, when a scalar is treated as a root leg and Eq. (2.26) is applied on the first trace.

and using the fact that [73]

$$A^{(4)}_{\phi\phi\phi\phi}[12|34] = 4\,\partial_{\epsilon_1\cdot\epsilon_2}\partial_{\epsilon_3\cdot\epsilon_4}\,\mathcal{A}^{(4)}_{\text{YM},4}\,, \tag{4.31}$$

it follows that

$$K^{(8)}[1234] = \frac{4}{\Lambda^4}\Big(\mathsf{F}[12]\mathsf{F}[34]\,\partial_{\epsilon_1\cdot\epsilon_2}\partial_{\epsilon_3\cdot\epsilon_4}+\mathsf{F}[13]\mathsf{F}[24]\,\partial_{\epsilon_1\cdot\epsilon_3}\partial_{\epsilon_2\cdot\epsilon_4}+\mathsf{F}[14]\mathsf{F}[23]\,\partial_{\epsilon_1\cdot\epsilon_4}\partial_{\epsilon_2\cdot\epsilon_3}\Big)K^{(4)}[1234]\,. \tag{4.32}$$

We leave the derivation of a closed form formula for arbitrary multiplicity at dimension eight for future work.

4.5 Comments on restricting to dimension eight only

From Eq. (4.11), and associated study at the level of the EOMs, it might seem that the covariant CK duality can be applied separately to the dimension-eight vertex, even though this vertex does not satisfy the (traditional) CK duality by itself. It is however important to realize that the dimension-eight interaction in the EOM of Eq. (4.4) is not in one-to-one correspondence with the dimension-eight operator in the Lagrangian of Eq. (4.1). Instead, iterations of the dimension-six terms in the EOM are necessary to bring the interaction in this form. It would therefore not be consistent to consider the dimension-eight term separately at the level of the EOM. This suggests that the traditional CK duality is necessary for the covariant CK duality.

5 Effective-field-theory extension beyond dimension eight

The above EFT analysis suggests that gluon amplitudes at increasingly high mass dimension can be obtained from lower order GBAS amplitudes using the covariant color-kinematics duality. This provides a map from the GBAS EFT into the YM EFT, where both theories consist of a tower of operators that satisfy the color-kinematics duality. It was previously found in the literature that such towers are encoded by the so-called $(DF)^2$+YM and $(DF)^2$+YM+ϕ^3 theories [62, 63]. The double copy of these theories has also been studied in [78, 79]. In this section, we will explore the correspondence between these theories and the EFTs that we considered above, as well as the covariant color-kinematics duality between them.

$(DF)^2$+YM. In four space-time dimensions, the $(DF)^2$+YM Lagrangian can be written as [62]

$$\begin{aligned}\mathcal{L}_{(DF)^2+\text{YM}} = &-\frac{1}{4}(F^a_{\mu\nu})^2+\frac{1}{2\,m^2}(D^\mu F^a_{\mu\nu})^2+\frac{1}{2}(D_\mu\varphi^\alpha)^2-\frac{m^2}{2}(\varphi^\alpha)^2\\ &+\frac{m\,g}{3!}d^{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma+\frac{g}{2\,m}C^{\alpha ab}\varphi^\alpha F^a_{\mu\nu}F^{b\mu\nu}-\frac{g}{3\,m^2}f^{abc}F^a{}_\mu{}^\nu F^b{}_\nu{}^\rho F^c{}_\rho{}^\mu\,,\end{aligned} \tag{5.1}$$

where φ^α is a real scalar with mass m in a real representation of the SU(N) gauge group. The Clebsch-Gordan coefficients $C^{\alpha ab}$ and $d^{\alpha\beta\gamma}$ satisfy the following relations [62],

$$C^{\alpha ab}C^{\alpha cd} = f^{ace}f^{edb}+(c\leftrightarrow d) \tag{5.2}$$

$$C^{\alpha ab}d^{\alpha\beta\gamma} = (T^a)^{\beta\alpha}(T^b)^{\alpha\gamma}+C^{\beta ac}C^{\gamma cb}+(a\leftrightarrow b)\,, \tag{5.3}$$

where $(T^a)^{\alpha\beta}$ are the generators of the representation of φ^α. The $(DF)^2$ term gives corrections to the gluon propagator, which (after gauge fixing) can be written as

μ  ν (p)

$$= \frac{-i\,\eta_{\mu\nu}}{p^2 - \frac{p^4}{m^2}} = -i\,\eta_{\mu\nu}\left(\frac{1}{p^2} - \frac{1}{p^2-m^2}\right) . \qquad (5.4)$$

This theory therefore propagates a ghost of mass m. It was found in [62] that the $(DF)^2$+YM theory satisfies the color-kinematics duality at tree level for any value of the mass m.

To compare with the Lagrangian of Eq. (4.1), we take the heavy mass limit and integrate out the scalar at tree level by replacing it (recursively) with its classical solution, which solves its equation of motion,

$$\varphi^\alpha_{\rm cl} = \frac{g}{2\,m^3}\,C^{\alpha ab}F^a_{\mu\nu}F^{b\mu\nu} - \frac{g}{2\,m^5}\,C^{\alpha ab}D^2(F^a_{\mu\nu}F^{b\mu\nu}) + \mathcal{O}\big(1/m^7\big) . \qquad (5.5)$$

This gives the effective field theory Lagrangian

$$\begin{aligned}\mathcal{L}^{\rm EFT}_{(DF)^2+{\rm YM}} \overset{\rm FR}{=} & -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{g}{3\,m^2} f^{abc}F^{a\ \nu}_{\ \mu}F^{b\ \rho}_{\ \nu}F^{c\ \mu}_{\ \rho} - \frac{g^2}{4\,m^4}\,f^{abe}f^{ecd}F^a_{\mu\nu}F^b_{\rho\sigma}F^{c\mu\nu}F^{d\rho\sigma} \\ & - \frac{g^2}{m^6}\,f^{abe}f^{ecd}F^a_{\mu\nu}D_\tau F^b_{\rho\sigma}D^\tau F^{c\mu\nu}F^{d\rho\sigma} + \mathcal{O}\big(1/m^8\big) . \end{aligned} \qquad (5.6)$$

We emphasized that we have also performed performed a field redefinition (FR) in order to replace $(DF)^2$ by operators with more fields and higher mass dimension. Indeed, $(DF)^2$ can be treated perturbatively in the EFT limit of small $1/m$. In other words, we integrate out the massive ghost at tree level. Besides showing the correspondence with Eq. (4.1), the above Lagrangian makes explicit which operator satisfies the CK duality at the next order in $1/m$. This is a natural candidate operator for the CCK duality as well.

$(DF)^2$+YM+ϕ^3. The $(DF)^2$+YM+ϕ^3 theory is defined by the Lagrangian

$$\mathcal{L}_{(DF)^2+{\rm YM}+\phi^3} = \mathcal{L}_{(DF)^2+{\rm YM}} + \frac{1}{2}(D_\mu\phi^{aA})^2 - \frac{g\,\lambda}{3} f^{abc}f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC} + \frac{m\,g}{2}C^{\alpha ab}\varphi^\alpha\phi^{aA}\phi^{bA} , \qquad (5.7)$$

and also satisfies the color-kinematics duality at tree level [62]. Similarly to before, the heavy scalar can be integrated out, to give the EFT Lagrangian

$$\begin{aligned}\mathcal{L}^{\rm EFT}_{(DF)^2+{\rm YM}+\phi^3} \overset{\rm FR}{=} & \mathcal{L}^{\rm EFT}_{(DF)^2+{\rm YM}} + \frac{1}{2}(D_\mu\phi^{aA})^2 - \frac{g\,\lambda}{3}\,f^{abc}f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC} - \frac{g^2}{4}f^{abe}f^{ecd}\phi^{aA}\phi^{bB}\phi^{cA}\phi^{dB} \\ & - \frac{g}{2\,m^2}f^{abe}f^{ecd}F^a_{\mu\nu}F^{c\,\mu\nu}\phi^{bA}\phi^{dA} - \frac{g^2}{m^2}f^{abe}f^{ecd}\phi^{aA}D_\mu\phi^{bB}D_\mu\phi^{cA}\phi^{dB} \\ & - 2\frac{g^2}{m^4}f^{abe}f^{ecd}F^a_{\mu\nu}D_\rho\phi^{bA}D_\rho F^{c\mu\nu}\phi^{dA} + \mathcal{O}\big(1/m^6\big) , \end{aligned} \qquad (5.8)$$

where we have neglected all terms that contribute beyond dimension six double trace and dimension eight single trace, because these relate to dimension twelve and higher through the covariant CK relations and are therefore not relevant to map to Eq. (5.6). This Lagrangian follows from dimensional reduction of Eq. (5.6).[9]

How can these two massive theories be related by the CCK duality? At large m, they generate the EFTs we encountered before and extend them to arbitrary mass dimension. Due to our power counting, see e.g. Eq. (4.18), we expect that an increasing number of flavor traces will be needed in the CCK replacement rule when we consider the amplitudes of the gluon/GBAS EFTs expanded to arbitrary order. We therefore expect any number of flavor traces to be needed to write a map between the amplitudes of $(DF)^2$+YM and $(DF)^2$+YM+ϕ^3 theories for general m. Finding the complete set of rules is beyond the scope of this paper. Nevertheless, restricting to low-multiplicity amplitudes, we can test CCK duality at the level of these two massive theories while avoiding the need for the CCK map of an arbitrary number of flavor traces. In particular, up to six-point, the amplitudes with at least one external gluon do not involve three factors of traces. Therefore the CCK map potentially extends to all orders in the EFT expansion.

We have indeed explicitly confirmed that the CCK replacement rule of Eq. (4.18) maps amplitudes from the $(DF)^2$+YM+ϕ^3 theory to the $(DF)^2$+YM theory for *any* value of the mass.[10] The corresponding formula reads

$$\mathcal{A}_{(DF^2)+\text{YM}} - \mathcal{A}^{(4)}_{\text{YM}} = \sum_{\phi,\sigma} A_{(DF^2)+\text{YM}+\phi^3}[\sigma]\frac{\mathsf{F}[\sigma]}{m^2} - \sum_{\phi,\sigma,\bar{\phi},\bar{\sigma}} A_{(DF^2)+\text{YM}+\phi^3}[\sigma|\bar{\sigma}]\frac{\mathsf{F}[\sigma]}{m^2}\frac{\mathsf{F}[\bar{\sigma}]}{m^2}\,, \tag{5.9}$$

which is valid for $n \leq 6$ and the sums are taken with a root leg gluon as in Eq. (4.19). We remind the reader of the fact that λ has been implicitly set to 1 on the r.h.s. This implies that the CCK relations extend to all orders in the EFT expansion for at least up to six external particles.

Beyond six-point, we expect that Eq. (5.9) receives triple-trace contributions. Indeed, at seven-point, we have confirmed that the dimension-ten EFT amplitudes of $\mathcal{L}^{\text{EFT}}_{(DF)^2+\text{YM}+\phi^3}$ and $\mathcal{L}^{\text{EFT}}_{(DF)^2+\text{YM}}$ are related by the following generalization of Eq. (4.18):

$$\prod_{i=1}^{n} \lambda^{|\sigma_i|-2}\,\text{Tr}(\sigma_i) \to (-1)^{n+1}\prod_{i=1}^{n}\frac{\mathsf{F}[\sigma_i]}{m^2}\,, \tag{5.10}$$

for products of n traces. We conjecture that this CCK replacement rule is valid at all mass dimensions and multiplicity.

************Unified picture************

For aesthetic purposes, it is possible to generalize the replacement rule of Eq. (5.10) such that $\mathcal{A}^{(4)}_{\text{YM}}$ appears on the right side of Eq. (5.9). Schematically,

$$\mathcal{A}_{(DF^2)+\text{YM},n} = \sum_{\text{all permutations}} \mathcal{A}_{(DF^2)+\text{YM}+\phi^3,n}\Big|_{\text{replace flavor traces}}\,, \tag{5.11}$$

[9] As before, the ϕ^3 vertex does not follow from dimensional reduction, but needs to be included by hand. In principle, this could be a source of ambiguity if it matters whether this interaction is included before or after performing field redefinitions. However, we find that the difference between these two cases is given by a term of the form $f^{abx}f^{ycd}f^{yex}f^{ABC}\phi^{aA}\phi^{bB}\phi^{cC}F^{d}_{\mu\nu}F^{e\mu\nu}$, which vanishes by the Jacobi identity.

[10] As we are interested in comparing with the EFTs, we did not consider amplitudes with external heavy scalars φ. We leave the discussion of such amplitudes to future work.

where we sum over all permutations of all $(DF^2) + \text{YM} + \phi^3$ amplitudes with $2 \leq m \leq n$ scalars. The replacement rule for flavor traces is $\lambda^{|\sigma_i|-2}\,\text{Tr}(\sigma_i) \to \frac{\mathsf{F}[\sigma_i]}{m^2}$ if $n \notin \sigma_i$ as well as $\lambda^{|\sigma_i|-2}\,\text{Tr}(\sigma_i n) \to \mathsf{F}[\sigma_i \tilde{n}]$, including the crucial minus sign for multiple traces as in Eq. (5.10). This relies on the CCK duality between GBAS amplitudes of different mass dimensions, from which we benefited in Sec. 4.3 at dimension eight. This requires a cancellation and hence is practically unnecessary We have explicitly confirmed Eq. (5.11) up to 6-point.
************Unified picture************

Finally, we have tested the extension of Eq. (4.29), which relates dimension-four GBAS to dimension-eight YM amplitudes, to the full tower of EFT operators. We find that such a relation does indeed hold for general m up to at least six-point. At six-point beyond dimension eight, there are contributions from triple trace amplitudes, because Eq. (4.29) does not require a root leg gluon. These are captured by the formula,

$$\begin{aligned}\mathcal{A}_{(DF^2)+\text{YM}} - \mathcal{A}^{(4)}_{\text{YM}} - \mathcal{A}^{(6)}_{\text{YM}} =& \frac{1}{m^4}\sum_{\phi,\bar{\phi}}\sum_{\sigma,\bar{\sigma}} A_{(DF^2)+\text{YM}+\phi^3}[\sigma|\bar{\sigma}]\ \mathsf{F}[\sigma]\ \mathsf{F}[\bar{\sigma}] \\ &- \frac{2}{m^6}\sum_{\phi_1,\phi_2,\phi_3} A_{(DF^2)+\text{YM}+\phi^3}[\phi_1|\phi_2|\phi_3]\ \mathsf{F}[\phi_1]\ \mathsf{F}[\phi_2]\ \mathsf{F}[\phi_3]\end{aligned} \tag{5.12}$$

where the sums in the first line are the same as in Eq. (4.29), while the triple trace sums satisfy $\phi_1, \phi_2, \phi_3 \in \mathbb{P}^{++}(1...6)$ with $\phi_i \cap \phi_j = \emptyset$ and $\phi_1 < \phi_2 < \phi_3$, referring again to some ordering to avoid overcounting. The fact that the $(DF)^2$+YM amplitudes can be decomposed in two different ways, namely according to Eqs. (5.9) and (5.12), requires an intricate self-duality of the $(DF)^2$+YM+ϕ^3 amplitudes which deserves to be better understood. Comparing the equations of motion of these theories for general mass would certainly shed light on this mapping and clarify how to go beyond it. We leave this for future work.

6 Conclusions

In this work, we have extended the covariant color-kinematics duality between GBAS and YM theories to the EFT regime, focusing on theories which are known to verify the usual color-kinematics duality. We first investigated the operators of lowest mass dimensions, then we proposed generalizations to a full tower of operators. We found that a pure gluon theory with operators of mass dimension $\leq 2n$ is mapped to a theory of gluons and cubic bi-adjoint scalars with operators of mass dimension $\leq 2(n-1)$. The latter follows the remarkable pattern that it can be obtained from the dimensional reduction of the operators of mass dimensions $\leq 2(n-1)$ of the former pure gluon theory considered in more spacetime dimensions. (This is up to the defining cubic interaction, which needs to be added by hand after dimensional reduction.)

Except for the dimension-six pure gluon theory, the duality that emerges does not only follow from a clear correspondence between all terms in the equations of motion: it also requires that one treats the flavor tensors (generated by the scalar on the GBAS side) corresponding to p flavor traces in a way which depends on p. For instance, at dimension eight, the CCK duality requires an intricate cancellation between double-trace dimension-four and single-trace dimension-six GBAS diagrams. With this insight, we established simple amplitude-level relations between GBAS and YM at all multiplicity up to dimension eight.

Specifically, dimension-six YM+F^3 amplitudes can be derived from single-trace dimension-four GBAS amplitudes by replacing single traces by local gauge-invariant functions of the kinematics. We leveraged this relation to derive closed-form YM BCJ numerators at mass dimension six, for any number of external particles, with manifest gauge invariance for all legs. We stress that those results do not depend on our assumption of regular CK duality, as F^3 is the only dimension-six operator in pure gluon theory, whose amplitudes happen to automatically display CK duality. That does not hold anymore beyond dimension six, where our assumption of the CK duality fixes otherwise free coefficients in the EFT. Regarding dimension-eight YM amplitudes, we found two ways of constructing them from GBAS input. Firstly, the same color-kinematics replacement rule acting on dimension-four double-trace —with a crucial minus sign— and dimension-six single-trace GBAS amplitudes, results precisely in purely gluonic amplitudes. Alternatively, the same amplitudes can be derived from double-trace GBAS amplitudes only, when more permutations of the external particles are included. These relations again prescribe a simple procedure to derive the dimension-eight YM BCJ numerators. This serves as a new proof of the standard CK duality up to dimension eight for all multiplicity at tree level. By confirming our relations in the $(DF)^2$+YM$\,(+\phi)$ theories at low multiplicity, we have obtained strong indications that the CCK relations extend to all orders in the EFT expansion.

There are several clear future directions that remain to be investigated. First, we postponed the systematic study of a couple of questions. We conjectured relations to all orders and multiplicity between the $(DF)^2$+YM$(+\phi)$ theories, which ought to be (dis)proved. We also showed that a CK duality is sufficient for the presence of a CCK duality, but we do not know if this is necessary. Hence, it would be very interesting to explore further the equations of motion generated by gluon operators which are not those considered in this work. Similarly, staying in the realm of CK-dual theories, there may be BCJ-compatible operators at high mass dimensions beyond those encoded in $(DF)^2$+YM theory. If so, understanding how they enter a CCK duality would be insightful. Finally, we found that all GBAS EFTs relevant for CCK duality with a gluon EFT of mass dimension $2n$ can be obtained from the dimensional reduction of the truncation of the pure gluon theory to operators of mass dimension $2(n-1)$. We have not explained that pattern, which therefore constitutes a key target for future exploration. If it holds to all orders, it implies that one can build the full $(DF)^2$+YM theory by applying dimensional reduction and CCK duality alternatively. Directly studying the relation between the $(DF)^2$+YM$(+\phi)$ theories through dimensional reduction would certainly be enlightening. Their relation through transmutation [73] is an interesting question too, which is also applicable to their EFTs truncated to a given mass dimension.

Beyond the ideas touched upon in this paper, we have not studied the double copy to gravity. We derived BCJ numerators for the YM EFT which can directly be used in the traditional double copy, but it would be a natural extension of our work to manifest a CCK duality for higher-derivative corrections to gravity, as done in [66]. This reference also identified a CCK duality at the level of the EOMs of the BAS and NLSM theories, which is another direction of study that we plan to undertake in the future. We anticipate that the NLSM+ϕ theories found in [56, 80] are likely to play an important role. Another insight of [66] which we have not extended yet concerns the relation between conserved currents, which is probably affected by our enlarged CCK dualities. Finally, the ultimate amplitude relations that we find are simpler than could be expected from a first inspection of the equations of motion, due to intricate cancellations between multi-trace replacements. It would be worth exploring whether such relations extend to the level of loop integrands. The same applies to CCK duality more generally.

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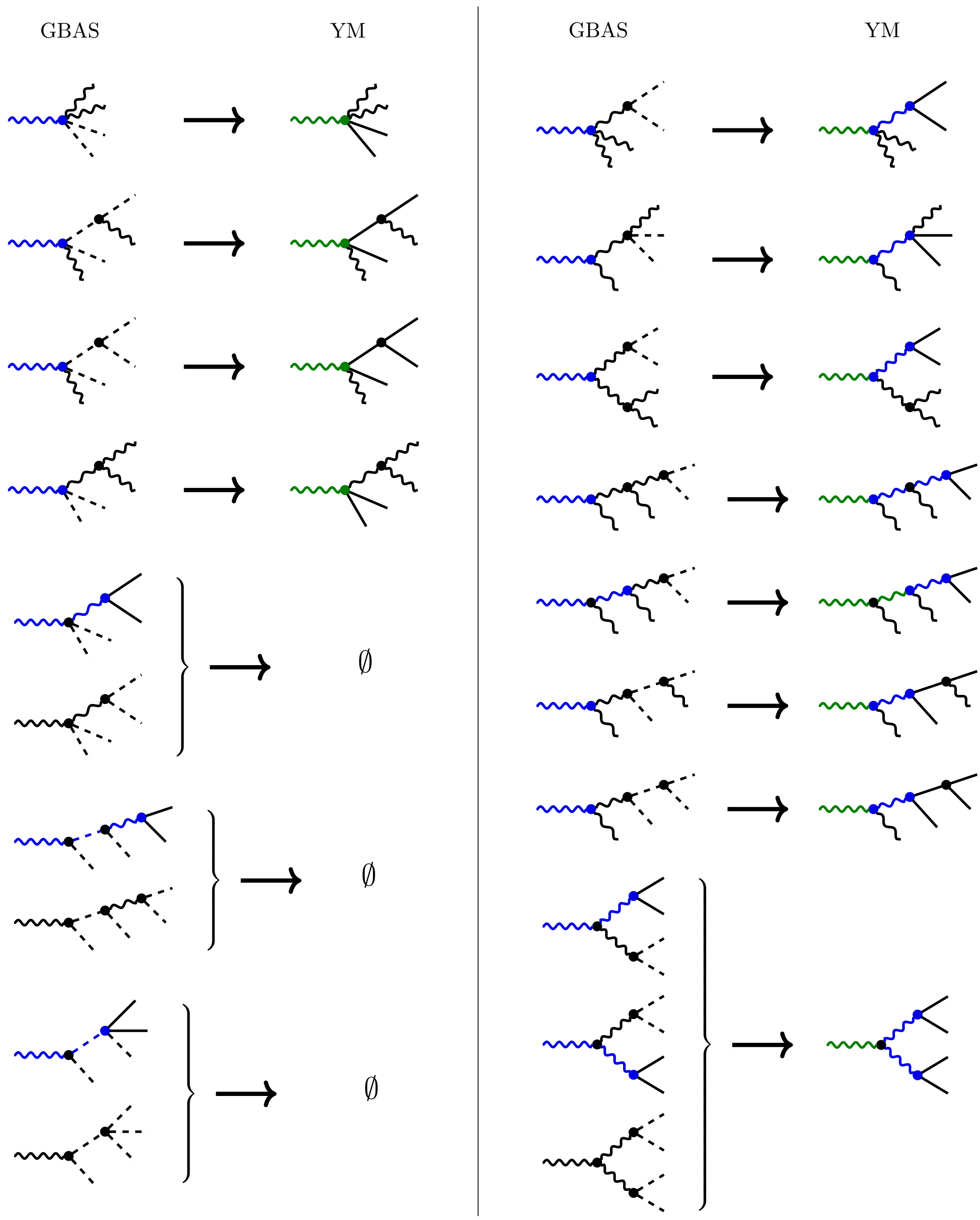
A Diagrammatic map at five points and dimension eight

In this appendix we exemplify the covariant CK duality between GBAS at mass dimension four (double trace) and six (single trace) and YM at mass dimension eight. Blue and green lines correspond to EOM solutions at dimension six and eight, respectively.



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