

On the Gluon Plasmon Self-Energy at $O(g)$

Fritjof Flechsig and Hermann Schulz

*Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, D-30167 Hannover, Germany*

ABSTRACT

The next-to-leading order contribution $\delta\Pi^{\mu\nu}(\omega, \vec{q})$ to the polarization function of the hot gluon system is analysed at non-zero wave vectors \vec{q} . Using Braaten-Pisarski resummation and general covariant gauges, $\delta\Pi^{\mu\nu}$ is found to be gauge-fixing independent and transverse on the longitudinal mass-shell. The real part of the longitudinal component $\delta\Pi_\ell$ is UV and IR stable (for real q). At imaginary q it is IR singular, and at the point $\omega = 0$, $q^2 = -3m^2$ it coincides with the result of Rebhan for next-to-leading order Debye screening. When q approaches the lightcone, $\delta\Pi_\ell$ diverges like $1/\sqrt{\omega^2 - q^2}$, reflecting the breakdown of the Braaten-Pisarski decomposition scheme in this limit.

1. Introduction

Infinite temperature is the limit in which QCD can be solved. After the complete "zeroth approximation", or $O(1)$, was worked out [1, 2] and had been cast into the form of an effective action [3], there were several studies of the Braaten-Pisarski resummed perturbation theory in "true first order", or $O(g)$, whose results are (and must be) automatically gauge-fixing independent. The gluon plasmon damping rate at zero wave vector [4], the Debye screening length [5] and the first correction to the plasmon frequency at zero wave vector [6] (henceforth referred to as " \mathcal{I} ") are such $O(g)$ phenomena and do indeed exhibit the required independence. The notorious difficulties, which are caused by the non-abelian infra-red instabilities (magnetic mass) [7, 8], might be solved by a second stage of effective theories [9], thereby justifying all results whose derivation does not hit this perturbative barrier. Problems of another type, arising when the plasmon dispersion line intersects the lightcone, are solved for a toy-model [10] of the gluon plasma, which is hot scalar electrodynamics.

In this note, we concentrate on the extension to non-zero wave vector argument of the gluon self-energy (or polarization function) $\delta\Pi^{\mu\nu}(\omega, \vec{q})$ at $O(g)$. Special attention is paid to the real part of its longitudinal component $\delta\Pi_\ell \equiv \text{Tr}(B\Pi)$ (for B see § 2), which determines the spectrum of the collective mode (plasmon). For simplicity, quarks are kept out of our hot black body volume. Hence its Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\alpha} \left(\partial^\mu A_\mu^a \right)^2 + \bar{c}^a \partial^\mu D_\mu^{ab} c^b \quad . \quad (1.1)$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ and $D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c$. The diagrams relevant to the order $O(g)$ under study are shown in fig. 1. As the present work is intimately related to the earlier paper \mathcal{I} , we refer to it with respect to the basic philosophy, to the use of hard thermal loops (HTL) as well as to most of notations.

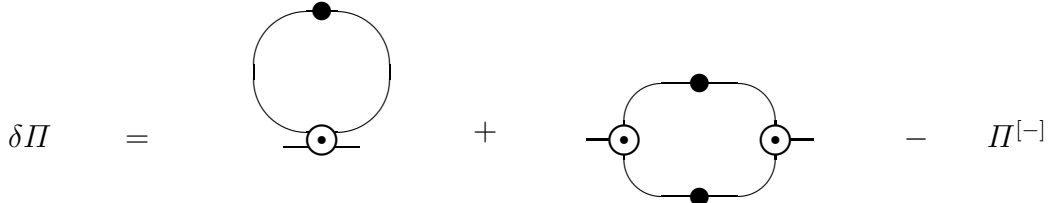


Figure 1: The actual next-to-leading order contributions to the polarization function. The gluon propagators are resummed (as indicated by a black bullet), and each vertex \odot is made up of a bare piece (dot) and a hard thermal loop.

Irrespective of $\vec{q} \neq \vec{0}$ there are the three possible origins of $O(g)$ terms as specified by Braaten and Pisarski in § 4.3 of [1]. The 'third' subset (1-loop soft) is determined by fig. 1 and will be seen in § 4 to form a separate gauge-fixing independent set. The 'second' subset (1-loop-hard minus leading) can be shown to be less than $O(g)$ in much the same way as in \mathcal{I} . Most probably, the 'first' subset (2-loop hard) remains below $O(g)$, too. Admittedly, we did not study these 2-loop diagrams at $q \neq 0$. Instead we trust in the argument (given at the end of § 3 of \mathcal{I}) that $O(g)$ can only be reached when the scale $m^2 = g^2 NT^2/9$ is involved in the loop integrals. We also refrained from numerical work. Rather, we focus on properties and structure. But note that if the plasmon frequency $\omega = mf(g, q/m)$ was known quantitatively one would be able to look at the line $\omega = 0$ in the g - q -plane and to learn about the phase transition. Lowering the temperature, the lowest value of g is reached first.

After collecting known details on the (true) leading order in § 2, we shall test the matrix $\delta\Pi^{\mu\nu}$, as determined by fig. 1, by comparing its properties such as transversality with known general predictions (§ 3). These are exploited in § 4 for the construction of a suitable scheme to calculate $\delta\Pi_\ell$. Several properties of the algebraic result are then deduced as listed in the abstract. The behaviour of $\delta\Pi_\ell$ near the lightcone is studied separately in § 5, followed by a short conclusion in § 6.

2. Notations and identities

To work with the Lagrangian (1.1) in thermal field theory [11, 12, 8] we introduce the number of gluons ($N^2 - 1$), the time contour (Matsubara), the metric (+ ---), the Bose function ($n(p) = 1/(e^{\beta p} - 1)$) and a hard-soft threshold (q^*). Let $Q = (i\omega_n, \vec{q})$ be the external momentum running through fig. 1, and P the loop momentum. Throughout *this* paper, the capital K has the fixed meaning $K \equiv Q - P$, and, correspondingly, $\vec{k} \equiv \vec{q} - \vec{p}$. If $\Delta(P)$ is any function of P , then Δ^- stands for $\Delta(K)$. The thermal propagator at $O(1)$ reads

$$G^{\mu\nu}(P) = \chi^{\mu\nu}(P) + \alpha\Delta_0^2(P)P^\mu P^\nu \quad \text{with} \quad \chi^{\mu\nu}(P) \equiv \Delta_t(P)A^{\mu\nu}(P) + \Delta_\ell(P)B^{\mu\nu}(P) \quad , \quad (2.1)$$

where $\Delta_0 = 1/P^2$ and $\Delta_{\ell,t} = 1/(P^2 - \Pi_{\ell,t}(P))$. The Lorentz matrices in (2.1) are members of the basis

$$A = g - B - D \quad , \quad B = \frac{V \circ V}{V^2} \quad , \quad C = \frac{P \circ V + V \circ P}{\sqrt{2}P^2 p} \quad , \quad D = \frac{P \circ P}{P^2} \quad (2.2)$$

$$\text{with} \quad V = P^2 U - (U \cdot P) P = (-p^2, -P_0 \vec{p}) \quad (2.3)$$

and $U = (1, \vec{0})$ the four-velocity of the thermal bath at rest. The functions $\Pi_{\ell,t}(P)$, related by $\Pi_\ell + 2\Pi_t = 3m^2$, derive through $\text{Tr}(B\Pi)$ and $\frac{1}{2}\text{Tr}(A\Pi)$, respectively, from the $O(1)$ polarization tensor $\Pi^{\mu\nu}$ [13, 2]

$$\Pi^{\mu\nu} = 3m^2 \left(U^\mu U^\nu - (UP) \int_\Omega \frac{Y^\mu Y^\nu}{YP} \right) \quad (2.4)$$

where $Y \equiv (1, \vec{e})$ with \vec{e} a unit vector. The angular integral \int_Ω over the directions of \vec{e} is normalized to one: $\int_\Omega Y = U$. To obtain the 3-leg and 4-leg HTL vertices in this nice "Y-language" (cf. (4.15) to (4.18) below) one conveniently starts off from the effective action [3].

Besides $\chi^{\mu\nu}$ in (2.1) there is another useful tensor

$$\wp^{\mu\nu}(P) \equiv P^\mu P^\nu - P^2 g^{\mu\nu} + \Pi^{\mu\nu}(P) = (\Pi_t - P^2) A^{\mu\nu} + (\Pi_\ell - P^2) B^{\mu\nu} . \quad (2.5)$$

Both, χ and \wp , are symmetric in the Lorentz indices and are projectors in the sense $\chi(P) \cdot P = 0$. The product of χ with \wp , used repeatedly in the sequel, has the neat property

$$-\chi^{\mu\rho}(P) \wp_{\rho\nu}(P) = g^\mu_\nu - D(P)^\mu_\nu . \quad (2.6)$$

In deriving (2.6), the properties $A^2 = A$, $B^2 = B$, $AB = 0$ have been exploited.

The following 'perturbative' Ward identities of hot thermal QCD are usually established by direct verification. However, they can also be derived [14] from BRS invariance of the effective action :

$$(Q_3)_3 {}^*\Gamma^{123}(Q_1, Q_2, Q_3) = \wp^{12}(Q_1) - \wp^{12}(Q_2) , \quad (2.7)$$

$$(Q_4)_4 {}^*\Gamma^{1234}(Q_1, Q_2, Q_3, Q_4) = {}^*\Gamma^{123}(Q_1, Q_2 + Q_4, Q_3) - {}^*\Gamma^{123}(Q_1 + Q_4, Q_2, Q_3) . \quad (2.8)$$

Here, the small numbers refer to Lorentz indices, while summations over colours are already done [1]. Momentum conservation $\sum Q_i = 0$ is understood. ${}^*\Gamma$ is " \odot " in fig. 1, i.e. the sum of the bare (3- or 4-) vertex and the HTL part: ${}^*\Gamma = {}^\circ\Gamma + \Gamma$, and Γ is the HTL. We avoid the common notation " $\delta\Gamma$ " to emphasize that the two elements of \odot are equal-rank partners. The symmetry properties of ${}^*\Gamma$ are

$${}^*\Gamma^{1,2,3}(Q_1, Q_2, Q_3) = {}^*\Gamma^{312}(Q_3, Q_1, Q_2) = -{}^*\Gamma^{132}(Q_1, Q_3, Q_2) = -{}^*\Gamma(-Q_i) , \quad (2.9)$$

$${}^*\Gamma^{1234}(Q_1, Q_2, Q_3, Q_4) = {}^*\Gamma^{3412}(Q_3, Q_4, Q_1, Q_2) = {}^*\Gamma^{2143}(Q_1, Q_2, Q_3, Q_4) . \quad (2.10)$$

All the relations so far listed are details of the true zeroth order. But the difference

$$\delta\Pi^{\mu\nu} = \frac{g^2 N}{2} \sum \left(G_{\lambda\rho} {}^*\Gamma^{\mu\nu\lambda\rho} + G_{\rho\sigma}^- G_{\tau\lambda} {}^*\Gamma^{\mu\sigma\tau} {}^*\Gamma^{\nu\rho\lambda} - \Delta_0^- \Delta_0 \left[10P^\mu P^\nu - 4P^2 g^{\mu\nu} \right] \right) , \quad (2.11)$$

is a true first order object (see (4.2) of \mathcal{I}). The three terms in the round bracket still correspond to the three elements of fig. 1. The arguments of the $4\text{-}\Gamma$ are $Q, -Q, -P, P$, those of both $3\text{-}\Gamma$'s are $Q, -K, -P$. The blank summation symbol means

$$\sum \equiv \sum_P^{\text{soft}} = \left(\frac{1}{2\pi}\right)^3 \int d^3p \sum_{P_0} \quad , \quad \sum_{P_0} = T \sum_n \quad . \quad (2.12)$$

3. Transversality

3.1 KNOWN EXACT PROPERTIES OF Π

As is well known, the BRS-invariance of the gauge-fixed Lagrangian (1.1) may be exploited to derive some exact relations. The one for the (full = overlined) two-point function \overline{G} reads $Q^\mu \overline{G}_{\mu\nu}(Q) Q^\nu = \alpha$ [15] and can be extended to other gauges [16, 17]. This relation, if rewritten with the expansion $\overline{G} = \overline{\Delta}_t A + \overline{\Delta}_\ell B + \overline{\Delta}_c C + \overline{\Delta}_d D$, turns into the first equation in (3.1) below. On the other hand, one may use Dyson's equation $\overline{G} = G_0 + G_0 \overline{\Pi} \overline{G}$, plug in the expansions of G_0 , \overline{G} and $\overline{\Pi}$ and derive the equation to the right :

$$\overline{\Delta}_d = \frac{\alpha}{Q^2} \quad , \quad \overline{\Delta}_d = \frac{\alpha (Q^2 - \overline{\Pi}_\ell)}{(Q^2 - \alpha \overline{\Pi}_d) (Q^2 - \overline{\Pi}_\ell) + \frac{1}{2} \alpha \overline{\Pi}_c^2} \quad . \quad (3.1)$$

Equating the two versions in (3.1), one arrives at the following exact relation [16], made explicit and discussed by Kunstatter [17] :

$$2 (Q^2 - \overline{\Pi}_\ell) \overline{\Pi}_d = \overline{\Pi}_c^2 \quad . \quad (3.2)$$

We exclude a hypothetical term $\sim 1/(Q^2 - \overline{\Pi}_\ell)$ in $\overline{\Pi}_d$, since it would make the propagator $\overline{\Delta}_\ell = (Q^2 - \alpha \overline{\Pi}_d)/[Q^2(Q^2 - \overline{\Pi}_\ell)]$ quadratic singular. Then, from (3.2), the exact coefficient $\overline{\Pi}_c$ vanishes on the longitudinal mass-shell. Hence, all terms of the small- g asymptotics of $\overline{\Pi}_c$ do so as well. To utilize this fact for the (true) first-order term $\delta \Pi$, we insert $\overline{\Pi}_i = \Pi_i + \delta \Pi_i + \dots$ into (3.2), remember that $\Pi_d = 0$ and $\Pi_c = 0$, read (3.2) at order $O(g)$ to get $(Q^2 - \Pi_\ell) \delta \Pi_d = 0$, i.e. $\delta \Pi_d = 0$ (since $\delta \Pi$ cannot develop a delta-function perturbatively), notice that $\delta \Pi = \delta \Pi_t A + \delta \Pi_\ell B + \delta \Pi_c C$ and infer that

$$Q^\mu \delta \Pi_{\mu\nu}(Q) = 0 \quad \text{if} \quad Q^2 - \Pi_\ell(Q) = 0 \quad . \quad (3.3)$$

Thus, $Q \delta \Pi$ vanishes on the longitudinal mass-shell. Whatever a detailed calculation of $\delta \Pi^{\mu\nu}$ brings about, it must respect the restricted transversality (3.3).

3.2 TESTING $\delta\Pi$

We shall study $Q\delta\Pi$, with $\delta\Pi$ given by (2.11), to see how the longitudinal mass-shell condition makes this expression vanish. We write $Q^\mu\delta\Pi_{\mu\nu} = Q\delta\Pi_\nu^{\text{tadpole}} + Q\delta\Pi_\nu^{\text{loop}} - Q\delta\Pi_\nu^{[-]}$. The last term can be simplified as

$$Q\delta\Pi_\nu^{[-]} = g^2 N \sum \Delta_0(K) Q^\sigma (g_{\sigma\nu} - D_{\sigma\nu}(P)) \quad . \quad (3.4)$$

For the tadpole contribution the identities (2.8), (2.9) lead to

$$Q\delta\Pi_\nu^{\text{tadpole}} = g^2 N \sum G(P)^{\rho\sigma} {}^*\Gamma_{\nu\sigma\rho}(Q, -K, -P) \quad , \quad (3.5)$$

where in a last step the sign of P has been reversed (leaving G invariant) in one of two terms. Remember that $K = Q - P$. In the loop contribution,

$$Q\delta\Pi_\nu^{\text{loop}} = \frac{g^2 N}{2} \sum G(K)^{\rho\sigma} Q^\mu {}^*\Gamma_{\mu\sigma\tau}(Q, -K, -P) G(P)^{\tau\lambda} {}^*\Gamma_{\nu\rho\lambda}(Q, -K, -P) \quad , \quad (3.6)$$

the product $Q {}^*\Gamma$ may be replaced by $\wp(K) - \wp(P)$. Since the remaining three factors $GG {}^*\Gamma$ are odd under the shift $P \rightarrow K$, this \wp -difference may be replaced by twice of one term. Hence, using (2.6), $(Q {}^*\Gamma G)_\sigma^\lambda$ may be replaced by $(-\wp G)_\sigma^\lambda = g_\sigma^\lambda - D(P)_\sigma^\lambda$. Obviously, the g -term cancels when the tadpole- and loop-results are added. The factor P^λ in the D -term allows to use the identity (2.7) once more:

$$Q\delta\Pi_\nu^{\text{tadpole+loop}} = g^2 N \sum \Delta_0(P) G(K)^{\rho\sigma} P_\sigma \left(\wp_{\nu\rho}(Q) - \wp_{\nu\rho}(K) \right) \quad . \quad (3.7)$$

For the term containing $\wp(K)$ we remember (2.6), i.e. $G(K) \cdot \wp(K) = D(K) - g$. By a shift $P \rightarrow K$ we see that it equals the subtraction term (3.4). The term in (3.7) containing $\wp(Q)$ may be written as

$$g^2 N \wp(Q)_{\nu\rho} \sum \Delta_0(K) \left(\chi(P)^{\rho\sigma} Q_\sigma + \alpha \Delta_0^2(P) (PQ) P^\rho - \alpha \Delta_0(P) P^\rho \right) \quad . \quad (3.8)$$

Symmetrizing the third term of (3.8) with respect to $P \rightarrow K$ it becomes proportional to Q^ρ and drops out via $\wp \cdot Q = 0$. Thus,

$$Q^\mu \delta\Pi_{\mu\nu}(Q) = g^2 N \wp_{\nu\rho}(Q) \zeta^\rho(Q) \quad \text{with} \quad \zeta^\rho(Q) = \sum \Delta_0(K) G(P)^{\rho\sigma} Q_\sigma \quad . \quad (3.9)$$

The sum ζ^ρ "knows" of only Q_0 and \vec{q} . Therefore, it must be a linear combination of Q^ρ and U^ρ , or, equivalently, of Q^ρ and V^ρ : $\zeta^\rho(Q) = c_q Q^\rho + c_v V^\rho$. Using this in (3.9), and $\wp \cdot Q = 0$, $Q \cdot V = 0$ as well as $\Pi_{\nu\rho} V^\rho = \Pi_\ell V_\nu$, we end up with

$$Q^\mu \delta\Pi_{\mu\nu}(Q) = V_\nu g^2 N \left(\Pi_\ell(Q) - Q^2 \right) c_v \quad . \quad (3.10)$$

(3.10) is the desired result. The coefficient $c_v = V_\rho \sum \Delta_0(K) G(P)^{\rho\sigma} Q_\sigma / V^2$ is non-zero. Thus, $\delta\Pi^{\mu\nu}$ is transverse only on the longitudinal mass-shell. This agreement with Kunstatter [17] confirms our set-up of contributions to $\delta\Pi^{\mu\nu}$.

There is a bit more of information in (3.10). Multiplication with V^ν gives $\delta\Pi_c = g^2 N \sqrt{2} q (Q^2 - \Pi_\ell) c_v$. But completing the sandwich by Q^ν we get $\delta\Pi_d = 0$ (all Q). Hence, Π_d starts with a term below $O(g)$, which we call $\delta^2\Pi_d$. Now, equating the first non-vanishing terms on both sides of the exact relation (3.2) and cancelling a factor $(Q^2 - \Pi_\ell)$ on both sides, we obtain

$$\delta^2\Pi_d = \left(g^2 N q c_v\right)^2 (Q^2 - \Pi_\ell) \quad . \quad (3.11)$$

Thus, even the less-than- $O(g)$ term in Π_d vanishes on the longitudinal mass-shell.

4. The longitudinal next-to-leading order term $\delta\Pi_\ell$

To evaluate the sandwich $\delta\Pi_\ell = (V\delta\Pi V)/V^2$ on the longitudinal mass-shell, the restricted transversality (3.10) may be exploited to simplify each of the sandwiching vectors as $V = Q^2 U - (QU)Q \rightarrow Q^2 U$:

$$\delta\Pi_\ell = -\frac{Q^2}{q^2} \delta\Pi_{00} \quad \text{at} \quad Q^2 = \Pi_\ell(Q) \quad , \quad (4.1)$$

with $\delta\Pi_{00}$ to be read off from (2.11).

There is no gauge-fixing dependence of $\delta\Pi_{00}$ on the longitudinal mass-shell. To check this, consider for example the α^2 term. It derives from the loop contribution (at $\mu = \nu = 0$), with G reduced to $\alpha\Delta_0^2(P)P^\mu P^\nu \equiv f(P)P^\mu P^\nu$. Using the Ward identity (2.7) twice, and $P^\tau \wp_{\mu\tau}(P) = 0$, one arrives at

$$\delta\Pi_{00}|_{f^2} = \wp_{0\rho}(Q) \wp_{0\tau}(Q) \frac{g^2 N}{2} \sum f(K) f(P) P^\rho P^\tau \quad . \quad (4.2)$$

But, with view to (2.5), the matrix A has no 0-elements, hence $\wp_{0\tau} = (\Pi_\ell - Q^2) B_{0\tau}(Q)$, and (4.2) vanishes on the mass-shell. Note that this procedure working is much more convenient than that in \mathcal{I} . In a similar manner one can verify that the term linear in f cancels that of the tadpole (the subtraction term has no f). Thus, $\delta\Pi_{00}$ is independent of α . Moreover, it is an *invariant* under changes of the (even) function $f(P)$.

We shall exploit this invariance by the following special choice,

$$f(P) = \Delta_0(P) \Delta_{\ell t}(P) \frac{P^2}{p^2} + \Delta_0(P) \Delta_\ell(P) \quad (\Delta_{\ell t} \equiv \Delta_\ell - \Delta_t) \quad , \quad (4.3)$$

which leads, via $A = g - B - D$ and $-P^2 p^2 B(P) = (P^2 U - P_0 P) \circ (P^2 U - P_0 P)$, to the replacement

$$G^{\mu\nu} \rightarrow \Delta_t g^{\mu\nu} - \Delta_{\ell t} \frac{P^2}{p^2} U^\mu U^\nu + \Delta_{\ell t} \frac{P_0}{p^2} [U^\mu P^\nu + P^\mu U^\nu] \quad . \quad (4.4)$$

The four-vectors P in (4.4) multiply ${}^*\Gamma$'s and reduce them through the Ward identities (2.7), (2.8). But with the U -vectors, we force more Lorentz indices to zero. The corresponding analysis is straightforward (somewhat lengthy, but not tedious) and leads to the following "algebraically final result" :

$$\delta\Pi_\ell = \left(1 - \frac{\omega^2}{q^2}\right) g^2 N \sum \left(c_0 + \Delta_\ell^- \Delta_\ell c_{\ell\ell} + \Delta_\ell^- \Delta_t c_{\ell t} + \Delta_t^- \Delta_t c_{tt}\right) \quad (4.5)$$

with the "coefficients" c given by

$$c_0 = -2\Delta_0 - 4\Delta_0^- \Delta_0 p^2 \quad (4.6)$$

$$c_{\ell\ell} = \frac{P^2 K^2}{2p^2 k^2} \mathcal{X}^2 + \frac{1}{4} \mathcal{W} \left(\frac{K^2}{k^2} \delta_\ell - \frac{P^2}{p^2} \delta_\ell^- \right) \quad (4.7)$$

$$c_{\ell t} = -\frac{P^2 K^2}{p^2 k^2} \mathcal{X}^2 - \frac{K^2}{k^2} {}^*\mathcal{Y}^2 + \delta_\ell^- \frac{P^2}{2p^2} \mathcal{W} \quad (4.8)$$

$$c_{tt} = \frac{P^2 K^2}{2p^2 k^2} \mathcal{X}^2 + \frac{K^2}{2k^2} {}^*\mathcal{Y}^2 + \frac{P^2}{2p^2} {}^*\mathcal{Y}_-^2 + \frac{1}{2} {}^*\mathcal{Z}^2 + \frac{P_0 K_0}{P^2 K^2} \delta_\ell^- \delta_\ell - \frac{3\delta_t}{2} - \frac{3\delta_t^-}{2} . \quad (4.9)$$

The terms in (4.5) are grouped such that they have neat properties at small q (see point (iii) below). (4.6) includes the subtraction term $\delta\Pi^{[-]}$. The objects δ , which originate from $P^2 \wp_{00} = p^2 \delta_\ell$, are inverse propagators,

$$\delta_t \equiv P^2 - \Pi_t(P) \quad , \quad \delta_\ell \equiv P^2 - \Pi_\ell(P) = 3P^2 - 3m^2 - 2\delta_t . \quad (4.10)$$

Clearly, (4.5) may be rewritten immediately by (further) cancellations $\Delta\delta = 1$. As before, an index minus refers to the substitution $P \rightarrow Q - P \equiv K$. The (full) vertex functions are hidden in the capital letters

$$\mathcal{W} = \Gamma_{0000}(Q, -Q, -P, P) \quad , \quad \mathcal{X} = \Gamma_{000}(Q, -K, -P) \quad , \quad (4.11)$$

$${}^*\mathcal{Y}^2 = {}^*\Gamma_{00}{}^\mu(Q, -K, -P) {}^*\Gamma_{00\mu}(Q, -K, -P) \quad , \quad {}^*\mathcal{Z}^2 = {}^*\Gamma_0{}^{\mu\nu}(\dots) {}^*\Gamma_{0\mu\nu}(\dots) \quad , \quad (4.12)$$

There is no bare part in \mathcal{W} and \mathcal{X} (therefore no star). To separate the HTL-pieces, $\mathcal{Y}^2 = \Gamma_{00}{}^\mu \Gamma_{00\mu}$ and $\mathcal{Z}^2 = \Gamma_0{}^{\mu\nu} \Gamma_{0\mu\nu}$, in ${}^*\mathcal{Y}^2$ and ${}^*\mathcal{Z}^2$, respectively, one is led into further analysis with further use of the Ward identity (2.7) :

$${}^*\mathcal{Y}^2 = \mathcal{Y}^2 + (2P_0 - 4Q_0) \mathcal{X} - 4\frac{p^2}{P^2} \delta_\ell + 2\frac{k^2}{K^2} \delta_\ell^- + 5p^2 - 4k^2 - 4q^2 \quad (4.13)$$

$${}^*\mathcal{Z}^2 = \mathcal{Z}^2 - 2\frac{p^2}{P^2} \delta_\ell - 2\frac{k^2}{K^2} \delta_\ell^- + 3(2P_0 - Q_0)^2 + p^2 + k^2 - 8q^2 . \quad (4.14)$$

\mathcal{W} and \mathcal{X} change sign under $P \rightarrow K$. \mathcal{Y}^2 and \mathcal{Z}^2 and ${}^*\mathcal{Z}^2$ are invariants under this transformation, but ${}^*\mathcal{Y}^2$ is not.

Using the Y -language (cf. (2.4)), the above HTL-parts can be dealt with as

$$\mathcal{W} = -6m^2 P_0 \partial_{Q_0} H - \text{ditto}^- \quad , \quad \mathcal{X} = 3m^2 P_0 H - \text{ditto}^- \quad , \quad (4.15)$$

$$\mathcal{Y}^2 = \mathcal{X}^2 - \vec{y}^2 \quad , \quad \vec{y} = 3m^2 P_0 \vec{I} - \text{ditto}^- \quad , \quad (4.16)$$

$$\mathcal{Z}^2 = \mathcal{X}^2 - 2\vec{y}^2 + \text{Tr} \left(\underline{\underline{z}} \underline{\underline{z}} \right) \quad , \quad \underline{\underline{z}} = 3m^2 P_0 \underline{\underline{J}} - \text{ditto}^- \quad (4.17)$$

with the following angular integrals involved

$$H = \int_{\Omega} \frac{1}{(YQ)(YP)} \quad , \quad \vec{I} = \int_{\Omega} \frac{\vec{e}}{(YQ)(YP)} \quad , \quad \underline{\underline{J}} = \int_{\Omega} \frac{\vec{e} \circ \vec{e}}{(YQ)(YP)} \quad . \quad (4.18)$$

Compared to \mathcal{I} , the whole trouble of the present non-zero wave vector analysis stems from these three integrals, see also (5.5) below. In order to do the summations over P_0 , the angular integrations in (4.5) may be simply shifted to the left and commuted with \sum .

So far, the result (4.5) is only recognized to be a pretty lengthy expression. Next, we enumerate some of its general properties and limiting cases.

(i) UV-convergence. The expression (4.5) is restricted to soft-momentum contributions automatically, i.e. it does not depend on the cutoff q^* , which bounds the soft scale from above. For an immediate check, one may reduce the dressed propagators to bare ones and omit the HTL-parts \mathcal{W} to \mathcal{Z}^2 . By retaining only UV-dangerous terms (those occurring in c_0), (4.5) becomes $\sum (c_0 - c_0)$, as expected. In passing, the sum over P_0 in front of (4.5) is convergent, of course. There is danger only in the terms containing $\mathcal{W} \sim 1/P_0$. Adding them together gives a finite difference.

(ii) IR-convergence. The real part of $\delta\Pi_\ell$ is finite in the infra-red along the whole real q -axis. However, there are mass-shell singularities in the imaginary part for real q as well as in the real part for imaginary q , which are investigated in a forthcoming paper with A. K. Rebhan [18]. See also point (iv).

(iii) Check at $q = 0$. As we know from \mathcal{I} , $\delta\Pi_\ell$ is regular when $q \rightarrow 0$. Thus, the singular prefactor q^{-2} in (4.5) must be compensated by a vanishing sum $\sum(c_0 + \dots)$. Indeed, its expansion in powers of q starts with a q^2 term. To show this, we consider the sum over c_0 first. It can be evaluated at arbitrary q as

$$\sum c_0 = -\frac{4q^2}{3m^2} \sum_P^{\text{soft}} \Delta_0(P) = q^2 \frac{2T}{3m^2 \pi^2} \int_0^{q^*} dp \quad . \quad (4.19)$$

Here, a term less than $O(g)$ had been neglected (see (5.35) of \mathcal{I}), and the mass-shell condition was used. For the last step in (4.19) see (5.9) of \mathcal{I} . The prefactor q^2 in (4.19) is to our likings. The cut-off q^* just tells us that there are also other q^2 contributions which compensate for it. For the other three sums in (4.5), we need the integrals (4.18)

at $q = 0$, i.e. $YQ = Q_0$:

$$\begin{aligned} H_0 &= \frac{1}{Q_0 P_0} (1 + \mathcal{V}) \quad , \quad \underline{J}_0 = \frac{1}{2Q_0 P_0} \left(1 - \frac{P^2}{p^2} \mathcal{V} + \frac{\vec{p} \circ \vec{p}}{p^2} \left[3 \frac{P^2}{p^2} \mathcal{V} + 2\mathcal{V} - 1 \right] \right) \quad , \\ \vec{I}_0 &= \frac{1}{Q_0} \frac{\vec{p}}{p^2} \mathcal{V} \quad \text{with} \quad \mathcal{V} \equiv \frac{p^2}{P^2} \frac{1}{3m^2} \Pi_\ell(P) \quad . \end{aligned} \quad (4.20)$$

From (4.20), the limiting values of \mathcal{W} , \mathcal{X} , $^*\mathcal{Y}^2$ and $^*\mathcal{Z}^2$ may be obtained (keep bare and HTL-parts together!). Through shifts $P \rightarrow K$ and $P_0 \rightarrow -P_0$ under the sum, one finally obtains

$$\sum \Delta_\ell^- \Delta_\ell c_{\ell\ell} \rightarrow 0 \quad \text{and} \quad - \sum \Delta_\ell^- \Delta_t c_{\ell t} \quad , \quad \sum \Delta_t^- \Delta_t c_{tt} \rightarrow \frac{1}{Q_0^2} \sum \Delta_t (\delta_\ell - \delta_\ell^-) \quad (4.21)$$

which exhibits the desired compensation.

When the whole expression (4.5) is regarded in the limit $q \rightarrow 0$, it must turn into the final result of \mathcal{I} , which is eq. (6.1) there. To verify this, the integrals (4.18) were to be expanded including q^2 . For the simplest term, which is $c_{\ell\ell}$ in (4.5), one arrives straightforwardly at \mathcal{M}_3 in (6.2) of \mathcal{I} . But to handle the immense total number of terms, we had several sessions with Miss MAPLE [19]. She produced the desired answer with all details.

(iv) Check at $\omega = 0$. If we leave the real q -half-axis and go to purely imaginary q , the longitudinal mass-shell condition $Q^2 = \Pi_\ell(Q)$ may be followed down to zero-frequency. There, $\delta\Pi_\ell$ (real part) determines the Debye screening length [5]. Along this line, $\delta\Pi_\ell$ always stays an UV-convergent expression (see **(i)**), while its IR-singularity is hidden in (4.5) with (4.8), see [18]. With $Q_0 \rightarrow 0$, all terms in (4.5) which contain HTL-vertices like \mathcal{X} vanish. To derive this we performed the frequency sums first and the analytical continuation $Q_0 \rightarrow \omega + i\varepsilon \rightarrow 0$ afterwards. It is then a rather easy task to do the remaining sums (along the lines given in \mathcal{I}). The result is

$$\delta\Pi_\ell(0, q) = g^2 NT \left(\frac{1}{2\pi} \right)^3 \int d^3p \left(\frac{1}{3m^2 + p^2} - \frac{1}{p^2} + \frac{2(3m^2 - q^2)}{p^2(3m^2 + k^2)} \right) \quad (4.22)$$

and agrees essentially with eq. (13) of Rebhan [5] when restricted to the longitudinal mass-shell on which we stayed from the outset and which is at $q^2 = -3m^2$, $\omega = 0$. In [5] this result was obtained in the simplified resummation scheme of Arnold and Espinosa [20], which differs in that the second term in (4.22) comes with a different sign, making the whole expression UV divergent. With dimensional regularization, which is required in the method of [20], this leads to exactly the same result as (4.22) whose manifest UV finiteness comes from the explicit subtraction of the one-loop bare piece in (2.11)¹. Note that m^2 of Rebhan is $3m^2$ in our notation.

¹A. K. Rebhan, private communication

5. Approaching the lightcone

With increasing (real) wave vector q , the plasmon spectrum $\omega(q)$ approaches the lightcone. Hence, there is a small parameter $\varepsilon^2 \equiv (\omega^2 - q^2)/q^2$. In this section we discuss the behaviour of the next-to-leading order term $\delta\Pi_\ell$ as a function of ε for $\varepsilon \rightarrow 0$. We expect order $1/\varepsilon$ -contributions to $\delta\Pi_{00} = -\delta\Pi_\ell q^2/Q^2$ from an earlier study [10] of scalar electrodynamics as a toy-model of the gluon plasma. As in the scalar theory, it turns out that these singularities signal the need of further resummations beyond the scheme of Braaten and Pisarski and that $\omega(q)$ intersects the lightcone at some finite value q^{crit} as given in [10] at $O(1)$. Here we shall *not* go up to this point. Instead, we remain interested in properties of $\delta\Pi_\ell$ only, stay within the methodology of Braaten and Pisarski, and, thus, defer the construction of a (new) consistent perturbative scheme for possible future work.

We shall show here, that there are (at least) two origins of $1/\varepsilon$. One is the same as in the scalar theory (index SED). The other is in the HTL-vertex pieces that remained in $\delta\Pi_\ell$, (4.5) (remember that parts of the HTLs have been converted by Ward identities and annihilated through the mass-shell condition or by cancellations $\Delta\delta = 1$). Let us split (4.5) into the parts just mentioned:

$$\delta\Pi_{00} = -\frac{q^2}{Q^2}\delta\Pi_\ell \equiv g^2 N \sum \quad \text{with} \quad \sum = \sum (c_0 + \dots) = \sum^{\text{SED}} + \sum^{\text{HTL}} \quad . \quad (5.1)$$

For the SED-part, put $\mathcal{W} = \mathcal{X} = \mathcal{Y} = \mathcal{Z} = 0$ in (4.5). \sum^{HTL} is simply the rest.

The two parts form separate UV-convergent sets. If, with view to the lightcone, the replacements $Q_0^2 \rightarrow q^2$, $\Pi_\ell \rightarrow 0$ and $\Pi_t \rightarrow 3m^2/2 \equiv \mu$ are consequently performed in \sum^{SED} , which implies δ_ℓ , Δ_ℓ , $P^2\Delta_t \rightarrow P^2$, Δ_0 , $1 + \mu^2\Delta_t$, respectively, then the sum turns into

$$\sum^{\text{SED}} \rightarrow \sum \left(2 [\Delta - \Delta_0] + 4 [\Delta^- \Delta - \Delta_0^- \Delta_0] + 4\mu^2 \Delta^- \Delta \right) \quad (5.2)$$

with $\Delta = 1/(P^2 - \mu^2)$. In the course of this, terms $\sum \Delta^- \Delta$ have been neglected since they contain no $1/\varepsilon$ (see eqs. (6.5), (6.10) in [10]). The last term in (5.2), which is of this type, is included only for better identification of the above result with eqs. (3.3), (3.8) in [10]. Clearly, for the extraction of $1/\varepsilon$ from (5.2) we may simply refer to [10].

The above argument was based on "replacements". Its justification, however, runs into (non-abelian) difficulties. Note that inner momenta P were taken at the lightcone, but actually only the outer Q is placed there. For a rough argument, consider the formula

$$\begin{aligned} \sum \Delta_\ell^- \Delta_t f(\vec{p}) = \\ T \left(\frac{1}{2\pi} \right)^3 \int d^3p f(\vec{p}) \int dx \frac{1}{x} \rho_t(x, p) \frac{1}{x - Q_0} [Q_0 \Delta_\ell(Q_0 - x, k) - x \Delta_\ell(0, k)] \quad , \end{aligned} \quad (5.3)$$

which is valid if by virtue of $f(\vec{p})$ the p -momenta are restricted to be soft. (5.3) generalizes (6.5) of \mathcal{I} to non-zero \vec{q} . The square bracket becomes large at $x = Q_0 \pm k$, or, if $Q^2 = 0$, at $\vec{p} = x\vec{q}/q$. In this region of the x - p -plane, at larger (but still soft) p , ρ_t is dominated by its pole-contribution $r_t(p) [\delta(x - \omega_t(p)) - \delta(x + \omega_t(p))]$ (see Appendix B of \mathcal{I}), which indeed gives the transversal propagator the "scalar" form $\Delta_t = c/[P^2 - \Pi_t(\omega_t(p), p)]$ with $c = 2r_t(p)\omega_t(p)$. We add the general definition of a spectral density: $\Delta(P) = \int dx \rho(x, \vec{p}) / (P_0 - x)$.

The idea that even the HTL-part in (5.1) could diverge at the lightcone comes to mind if the above rough argument (Q at the cone enforces P to be there too) is applied to the hard loop-integrations inside a HTL-vertex, too. To exhibit the corresponding mechanism consider a typical but simple term Υ :

$$\Upsilon \equiv \sum f(p, \vec{p}\vec{q}) H^2(P_0) = T \left(\frac{1}{2\pi} \right)^3 \int d^3p f(p, \vec{p}\vec{q}) H^2(0) , \quad (5.4)$$

where H is the integral in (4.18) viewed as a function of P_0 , and f is any function restricting the integration to soft p . Υ occurs in the last term of (5.10) below. The summation over P_0 is performed to the right in (5.4), while Q_0 stays an imaginary Matsubara frequency. $H(P_0)$ is real (symmetrize (4.18) with respect to $\vec{e} \rightarrow -\vec{e}$), and its sign is dominated by the hard parts Q_0, P_0 . If $Q_0 P_0 \neq 0$, $\text{sign}(H) = \text{sign}(Q_0 P_0) \equiv -\eta$. Now, using Feynman parametrization (and avoiding vanishing denominators), the angular integration can be done:

$$\frac{1}{ab} = \int_0^\infty dv \frac{1}{(a + vb)^2} \Rightarrow H(P_0) = \int_0^\infty dv \frac{\eta}{Q^2 + 2v\eta PQ + v^2 P^2} . \quad (5.5)$$

If $P_0 = 0$, the symmetric version $\frac{1}{2} [\int_{\text{at } \eta=+1} + \int_{\text{at } \eta=-1}]$ of (5.5) must be used in accord with a principal value prescription of (4.18) at $P_0 = 0$. The result of integrating over v ,

$$H(0) = \frac{i}{2p\sqrt{\gamma^2 - Q_0^2}} \ln \left(\frac{-ip\sqrt{\gamma^2 - Q_0^2} - \vec{p}\vec{q}}{-ip\sqrt{\gamma^2 - Q_0^2} + \vec{p}\vec{q}} \right) , \quad \gamma^2 = \frac{n}{p^2} , \quad n = p^2 q^2 - (\vec{p}\vec{q})^2 , \quad (5.6)$$

is the right place to do the analytic continuation. In the complex Q_0 -plane, there are cuts on the real axis ranging from γ to ∞ and $-\gamma$ to $-\infty$. Thus, through $Q_0 \rightarrow \omega + i0$ with $\omega \gtrsim q$, we arrive at

$$H(0) = -\frac{1}{2qp} \frac{1}{\sqrt{\varepsilon^2 + u^2}} \ln \left(\frac{\sqrt{\varepsilon^2 + u^2} + u}{\sqrt{\varepsilon^2 + u^2} - u} \right) , \quad u = \frac{\vec{p}\vec{q}}{pq} . \quad (5.7)$$

Note that with $\varepsilon \rightarrow +0$, the square of the above expression becomes $1/\varepsilon$ times a representation of the delta function, by means of which our toy term Υ is easily evaluated:

$$H^2(0) \rightarrow \frac{1}{\varepsilon} \frac{\pi^3}{4q^2 p^2} \delta(u) \Rightarrow \Upsilon = \frac{1}{\varepsilon} \frac{T\pi}{16q^2} \int_0^\infty dp f(p, 0) . \quad (5.8)$$

From the above we are led to a handy method for treating Σ^{HTL} . Apparently, terms linear in \mathcal{X} etc. can lead to logarithms of ε only (consider (5.5) at $Q^2 = 0$). Hence we restrict Σ^{HTL} to the terms quadratic in \mathcal{X} , \mathcal{Y} , \mathcal{Z} . For a further simplification we observe that the latter two HTLs, \mathcal{Y}^2 and \mathcal{Z}^2 , can be expressed by \mathcal{X}^2 [21]. For example, the integral \vec{I} in (4.18) "knows" of only two spatial directions and is therefore a linear combination of \vec{q} and \vec{p} , and so on. The procedure is rather tedious and leads (omitting further terms linear in \mathcal{X}) to the surprisingly simple relations

$$\mathcal{Y}^2 = \left(1 - \frac{r}{n}\right) \mathcal{X}^2 \quad , \quad \mathcal{Z}^2 = 2 \left(1 - \frac{r}{n}\right)^2 \mathcal{X}^2 \quad \text{with} \quad r = (P_0 \vec{q} - Q_0 \vec{p})^2 \quad (5.9)$$

and n see (5.6). Note that both, n and r are invariants under $P \rightarrow K$. Using (5.9), the HTL-term can be written conveniently as:

$$\Sigma^{\text{HTL}} = \frac{1}{4} \sum \mathcal{X}^2 \frac{1}{p^2 k^2} \left(\Omega^- \Omega + \Lambda^- \Lambda + 4q^2 \frac{k^2}{n} \Lambda + 4q^4 \frac{p^2 k^2}{n^2} \right) . \quad (5.10)$$

The "propagators" in (5.10) were introduced by $\Omega = P^2 (\Delta_\ell - \Delta_t)$ and $\Lambda = \Omega - 2p^2(1 - r/n)\Delta_t - 2q^2 p^2/n$. But we actually need only their spectral densities

$$\rho_\Omega = (x^2 - p^2) (\rho_\ell - \rho_t) \quad , \quad \rho_\Lambda = \rho_\Omega + \frac{2p^2}{n} (qx - \vec{q} \vec{p})^2 \rho_t . \quad (5.11)$$

For Ω see the table 1 in \mathcal{I} . It remains to evaluate sums of three types, $\sum \mathcal{X}^2 \Delta^- \Delta / p^2 k^2$, $\sum \mathcal{X}^2 \Delta / p^2 k^2$ and $\sum \mathcal{X}^2 / p^2 k^2$, along the lines (5.4) to (5.8). After some analysis [21] and repeated omission of less-divergent terms (less than $1/\varepsilon$), we arrive at

$$\begin{aligned} \Sigma^{\text{HTL}} = & -\frac{1}{\varepsilon} \frac{9m^4 T}{32q^2} \int d^3 p \frac{1}{p^2 k^2} n \sqrt{n} \\ & \left[\frac{1}{\vec{p} \vec{q}} \rho_t \left(\frac{\vec{p} \vec{q}}{q}, p \right) - \frac{1}{\vec{p} \vec{q}} \rho_\ell \left(\frac{\vec{p} \vec{q}}{q}, p \right) \right] \left[\Delta_t \left(\frac{\vec{k} \vec{q}}{q}, k \right) - \Delta_\ell \left(\frac{\vec{k} \vec{q}}{q}, k \right) \right] . \quad (5.12) \end{aligned}$$

The expression (5.12) is UV-convergent and IR-finite. The first square bracket is positive in the whole range of integration (the densities reduce to their cut-parts, and $\rho_t^{\text{cut}}/\vec{p} \vec{q} \geq 0$, $\rho_\ell^{\text{cut}}/\vec{p} \vec{q} \leq 0$, [22]). But the real part of the second square bracket does not have such a nice property. The propagators contain Landau damping (see (B.4) and (B.5) of \mathcal{I}), and Δ_ℓ , though dominating, changes sign. At this point, we abstain from a more detailed (numerical) analysis. In short, HTL vertices *do* contribute to the singularity at the lightcone.

6. Conclusions

The next-to-leading order calculations on the gluon plasmon dispersion known so far are extended to arbitrary wave vectors \vec{q} . The real part of the plasmon self-energy $\delta\Pi_\ell$ (although remaining a lengthy, still algebraic, expression) is found to have all the expected properties, such as gauge-fixing independence, convergence in the UV and IR, and the correct limiting behaviour at $q \rightarrow 0$ as well as at $\omega \rightarrow 0$ along the longitudinal mass-shell line. Close to the lightcone, two mechanisms are detected which violate the common $O(g)$ -scheme in this limit, since they let $\delta\Pi_\ell$ diverge as $1/\varepsilon$, i. e. stronger than the $\ln(\varepsilon)$ in the leading order. The study of the longitudinal dispersion near the lightcone needs a new consistency scheme which is still unknown.

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