Topological features in a two-dimensional Higgs model

I. Montvay[a]

[a]Deutsches Elektronen-Synchrotron DESY,
Notkestr. 85, D-22603 Hamburg, FRG

Topological properties of the gauge field in a two-dimensional Higgs model are investigated. Results of exploratory numerical simulations are presented.

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1. INTRODUCTION

Fermion number, which is the sum of baryon number and lepton number $(B+L)$, is not conserved in the Standard Model [1]. This is due to the anomaly in the fermion current.

The lattice formulation of the anomalous fermion number non-conservation is problematic [2], because it has to do with chiral gauge couplings and, as is well known, there is a difficulty with chiral gauge theories on the lattice (see, for instance, the reviews [3,4]). There is, however, an approximation of the electroweak sector of the standard model which can be studied with standard lattice techniques, namely the limit when the $\mathrm{SU(3)_{colour}} \otimes \mathrm{U(1)_{hypercharge}}$ gauge couplings are neglected [5,6].

A simple prototype model is the standard $\mathrm{SU(2)_L}$ Higgs model coupled to an even number of fermion doublets. For a numerical study of the bounds on renormalized couplings in such a model without $\mathrm{SU(2)_L}$ gauge field see the contribution of Gernot Münster in this proceedings [7].

Before doing numerical simulations in the four dimensional $\mathrm{SU(2)_L}$ gauge model for anomalous fermion number violation, it is useful to study a simple U(1) toy model in two dimensions, which has often been studied in this context (see e. g. [8]). The lattice formulation of an appropriate two-dimensional U(1) Higgs model has been summarized in [9].

The lattice action depending on the compact U(1) gauge field $U_{x\mu} = \exp(iA_\mu(x))$, $(\mu = 1,2)$ and, for simplicity, fixed length Higgs scalar field $\phi(x)$, $|\phi(x)| = 1$ has two parameters: the inverse gauge coupling squared $\beta = 1/g^2$ and the hopping parameter of the scalar field κ:

$$S = \beta \sum_x \sum_{\mu=1,\nu=2} [1 - cos(F_{\mu\nu}(x))]$$

$$- 2\kappa \sum_x \sum_{\mu=1}^{2} \phi^*(x+\hat{\mu}) U_{x\mu} \phi(x) \ . \qquad (1)$$

Here the lattice gauge field strength is defined for $\mu,\nu = 1,2$ as

$$F_{\mu\nu}(x) =$$

$$A_\nu(x+\hat{\mu}) - A_\nu(x) - A_\mu(x+\hat{\nu}) + A_\mu(x) \ . \qquad (2)$$

Real angular variables $-\pi < \theta_{x\mu} \leq \pi$ on the links can be introduced by

$$U_{x\mu} \equiv \exp(i\theta_{x\mu}) \ ,$$

$$\theta_{x\mu} = A_\mu(x) - 2\pi \cdot \mathrm{NINT}(A_\mu(x)/2\pi) \ , \qquad (3)$$

where NINT() denotes nearest integer.

Fermions in this two dimensional model are introduced in the mirror fermion basis (ψ, χ). The continuum limit of the anomaly equation for the two dimensional current, which corresponds to the fermion number current in the four dimensional $\mathrm{SU(2)_L}$ model, is the following [9]:

$$\langle \partial_\mu J_\mu(x) \rangle = \frac{1}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x) = 2q(x) \ . \qquad (4)$$

Here $q(x)$ is the density of the topological charge

$$q(x) = \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x) \ . \qquad (5)$$

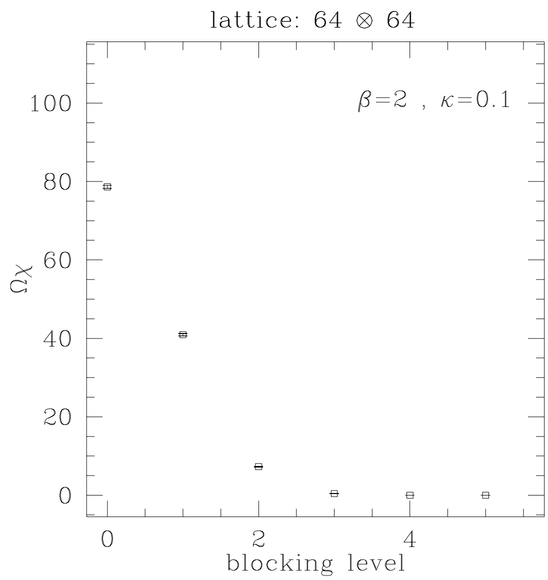

Figure 1. The topological susceptibility on blocked configurations as a function of blocking level at $\beta = 2.0$, $\kappa = 0.1$ on 64^2 lattice. $\Omega\chi$ is plotted.

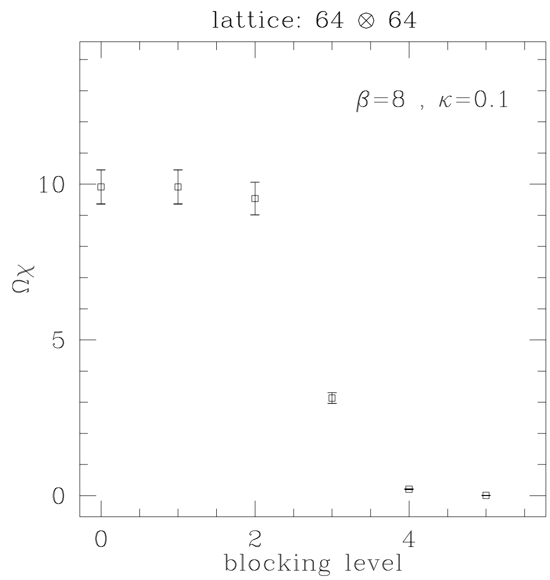

Figure 2. The topological susceptibility on blocked configurations as a function of blocking level at $\beta = 8.0$, $\kappa = 0.1$ on 64^2 lattice. $\Omega\chi$ is plotted.

Since according to (4) the non-conservation of the fermion current $J_{x\mu}$ is proportional to the topological charge density, the first step in understanding the fermion number anomaly on the lattice is to understand the topological features of the two dimensional U(1) lattice gauge fields.

2. TOPOLOGICAL CHARGE

The topological charge of U(1) lattice gauge field configurations can be defined as a sum over the contributions of plaquettes. The basic assumption is the existence of a piecewise continuous interpolation of the gauge field [10,11]. The gauge invariant topological charge on the torus corresponding to periodic boundary conditions is obtained either from the "transition functions" or from the "sections" of this interpolated gauge field.

Introducing the plaquette angles $-4\pi < \Theta_{x\mu\nu} \leq 4\pi$ and $-\pi < \theta_{x\mu\nu} \leq \pi$ by

$$\Theta_{x\mu\nu} \equiv \theta_{x\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x\nu} - \theta_{x+\hat{\nu},\mu} \ ,$$

$$\theta_{x\mu\nu} \equiv \Theta_{x\mu\nu} - 2\pi \cdot \mathrm{NINT}(\Theta_{x\mu\nu}/2\pi) \ , \qquad (6)$$

one can show that the total topological charge Q is given by

$$Q = \frac{1}{2\pi} \sum_x \theta_{x12} = -\sum_x \mathrm{NINT}(\Theta_{x12}/2\pi) \ . \qquad (7)$$

An interesting physical quantity is the topological susceptibility

$$\chi \equiv \Omega^{-1}(\langle Q^2 \rangle - \langle Q \rangle^2) \ , \qquad (8)$$

where $\Omega \equiv L_1 \cdot L_2$ denotes the number of lattice points. (Note that in [6] Q has been defined with opposite sign and χ without the normalization factor Ω^{-1}.)

2.1. Blocking and cooling

The total topological charge is a long distance property of the gauge configuration, therefore it should remain stable under smoothing the short range fluctuations, for instance, by blocking. This indeed happens in a few blocking steps. Defining the 2^n blocked gauge fields in the usual way by averaging over the two "staples" and a straight line connecting next nearest neighbour sites, one obtains, for instance, figs. 1, 2. As one can see,

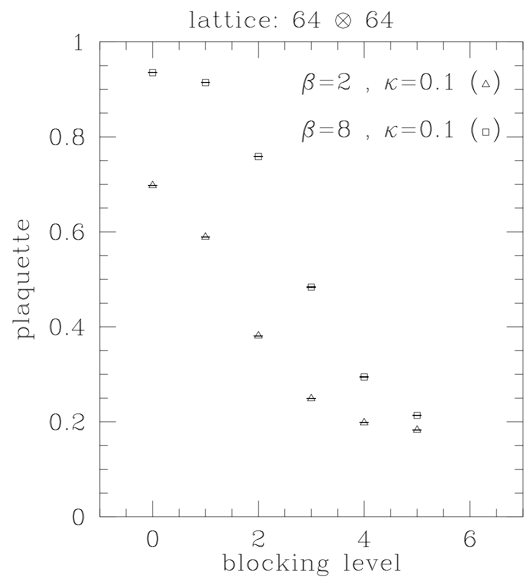

Figure 3. The average plaquette on blocked configurations as a function of blocking level at $\beta = 2.0$, $\kappa = 0.1$ (triangles) and $\beta = 8.0$, $\kappa = 0.1$ (squares). The original lattice is 64^2.

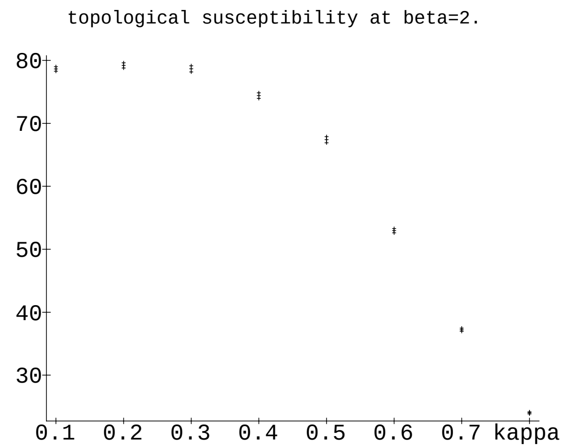

Figure 4. The topological susceptibility as function of κ at $\beta = 2.0$ on 64^2 lattice. $\Omega\chi$ is plotted.

at $\beta = 8.0$ the topological charge is kept in two blocking steps but at $\beta = 2.0$ already one blocking step is enough to decrease it substantially. The important variable is the plaquette expectation value $P \equiv \langle \cos \Theta_{x12} \rangle$ (see fig. 3). Below $P \simeq 0.6$ the topological charge is erased by blocking. This means that for gauge configurations with smaller average plaquette value the lattice definition of the topological charge is problematic (impossible?).

One can also investigate the effect of locally minimizing the gauge field action by *cooling* sweeps. It turns out that at larger gauge couplings, e. g. at $\beta = 2.0$, one can observe a slight decrease in the topological susceptibility but at $\beta = 8.0$, after several tens of cooling sweeps, there is no change at all.

2.2. Topological susceptibility

In a series of Monte Carlo runs the topological susceptibility has been measured on 64^2 lattices at $\beta = 2.0$ and $\beta = 8.0$ as a function of the scalar hopping parameter κ (see figs. 4, 5). At every point 60000 to 90000 measurements were collected, separated by one Metropolis and 12 overrelaxation updating steps. The expectation value of Q turns out to be small, as it should. For instance, at $\beta = 8.0$, $\kappa = 0.2$ one gets $\langle Q \rangle = -0.5(6)$. As one can see, χ is considerably smaller at $\beta = 8.0$, and at this small gauge coupling there is a maximum near $\kappa = 0.4$, which presumably corresponds to the continuation of the Kosterlitz-Thouless transition at $\beta = \infty$ for finite β.

3. GAUGE FIXING AND MODIFIED ACTION

A plaquette contributing by a non-zero integer to the right hand side of (7) can be imagined to carry a *"Dirac string"* or *"gauge field kink"* [12]. These local contributions to the topological charge play an important rôle in the dynamics of lattice U(1) gauge fields. However, it is important to keep in mind that the individual terms on the right hand side of (7) are not gauge invariant, in contrast to the total topological charge Q (and the terms in the first form of Q in (7)). Namely, performing large gauge transformations on the two ends of a link can create or annihilate a kink-antikink pair on the two plaquettes which both contain the link.

In order to give a physical meaning to the number of gauge kinks one has to fix the gauge. Since the plaquettes with $\Theta_{12} \simeq \pm 2\pi$ and $|\theta_\mu| \simeq \pi/2$ contain large link angles, one possibility is to minimize the sum of squares of the link angles: $\sum_x \sum_{\mu=1}^{2} \theta_{x\mu}^2$ [13]. In this way one can define a *coarse grained* topological charge density by taking the sum on the right hand side of (7) in this "maximally smooth" gauge over an arbitrary domain of plaquettes. This gives information on the size and distribution of topological objects.

The rôle of the gauge kinks can be important for quantities sensitive to the topological charge, or to the smoothness of the gauge field. Therefore it is instructive to study the model with some modified actions suppressing kinks. One way to push in the $\beta \to \infty$ limit all link angles to zero is to introduce a modified gauge field action like

$$S_4 = 16\beta_4 \sum_x \sum_{\mu=1,\nu=2} [1 - cos(\Theta_{x\mu\nu}/4)] \ . \qquad (9)$$

The factor 16 in front of β_4 is introduced in order to have in the continuum limit of the action the same normalization as for β.

Numerical simulations show that at $\beta_4 = 2.0$ the topological susceptibility is considerably suppressed by the action (9): compare figs. 4 and 6. At $\beta_4 = 8.0$ in a long run it turned out to be impossible to create even a single configuration with non-zero topological charge.

An important question for future studies is to investigate the continuum limit of the anomaly equation (4) and compare it to different definitions of the topological charge density.

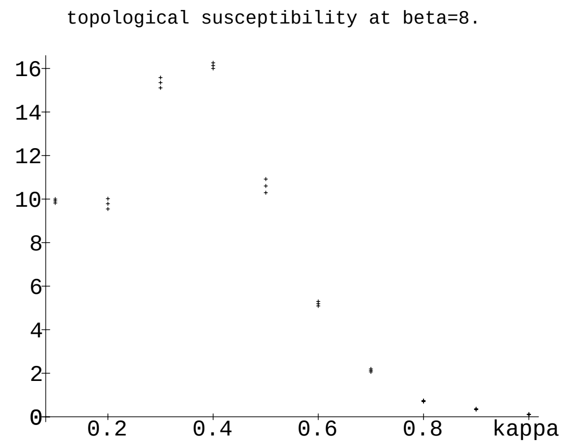

Figure 5. The topological susceptibility as function of κ at $\beta = 8.0$ on 64^2 lattice. $\Omega\chi$ is plotted.

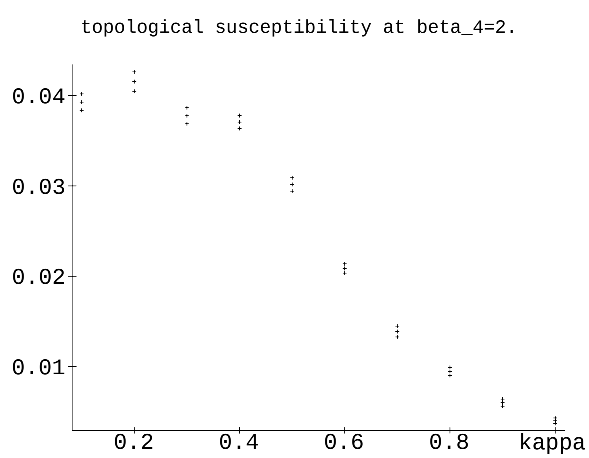

Figure 6. The topological susceptibility with the modified action S_4 as function of κ at $\beta_4 = 2.0$ on 64^2 lattice. $\Omega\chi$ is plotted.

REFERENCES

1. G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8; Phys. Rev. D14 (1976) 3432.
2. T. Banks, Phys. Lett. B272 (1991) 75.
3. I. Montvay, Nucl. Phys. B (Proc. Suppl.) 26 (1992) 57.
4. D.N. Petcher, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 50.
5. I. Montvay, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 621.
6. I. Montvay, DESY preprint 93-134 (1993), to appear in *Proc. of the 17th Johns Hopkins Workshop*, Budapest, July 1993.

7. G. Münster, contribution to Lattice '93; L. Lin, I. Montvay, G. Münster, M. Plagge, H. Wittig, Phys. Lett. B317 (1993) 143.
8. A.I. Bochkarev, M.E. Shaposhnikov, Mod. Phys. Lett. A2 (1987) 991.
9. I. Montvay, contribution to the 2nd IMACS Conference, St. Louis, October, 1993.
10. M. Lüscher, Comm. Math. Phys. 85 (1982) 39.
11. M. Göckeler, A.S. Kronfeld, G. Schierholz, U.-J. Wiese, Nucl. Phys. B404 (1993) 839.
12. T.A. DeGrand, D. Toussaint, Phys. Rev. D22 (1980) 2478.
13. I. Montvay, to be published.