Heat-kernel calculation of quark determinant and computer algebra

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Abstract

In this paper there we describe the calculational background of deriving a strong meson Lagrangian from the Nambu–Jona-Lasinio quark model using the computer algebra systems FORM and REDUCE in recursive algorithms, based on the heat-kernel method for the calculation of the quark determinant.

Computer Algebra Systems (CAS) such as FORM [1] or REDUCE [2] have been successfully used in the high energy physics for a long time, especially in the field of the perturbative theory calculations of higher order contributions in the standard model. Besides these "classical" fields of application of CAS, they can be also used for solving various problems connected with non-renormalizable models such as chiral meson models. In the following we describe the calculational background of the recent derivation [3] of meson Lagrangian describing strong interactions of scalar, pseudoscalar, vector and axial-vector fields.

The strong interactions of quarks at high energies ($E \gg 1\,\mathrm{GeV}$) are described by the Quantum Chromodynamics (QCD) [4]. The QCD Lagrangian has following form:

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \bar{q}^f (i\widehat{D} - m_f^0) q^f - \frac{1}{4} G^{\mu\nu} G_{\mu\nu}$$

with

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + ig[G_\mu, G_\nu],$$
$$\widehat{D} = \gamma^\mu(\partial_\mu - igG_\mu), \quad G_\mu = \sum_a G_\mu^a \frac{\lambda_C^a}{2},$$

where q is the quark field, G_μ is the gluon field, g is the coupling constant, m_f^0 is the current quark mass, γ^μ are the Dirac matrices, λ_C the $SU(3)_C$ matrices.

Some special properties of the QCD like the confinement of quarks and the momentum dependence of the strong coupling constant $\alpha_s(q^2)$ which becomes large at low momenta ($\alpha_s \approx 1$) make it impossible to use the QCD for an adequate description of meson processes at low energies.

Until now it was not possible to derive a meson theory from the QCD. On the other side, meson processes have been studied in a framework of chiral Lagrangians since 1960s. One possible way to built up a QCD-based chiral meson theory is to start with some effective quark Lagrangian, e.g. with the Nambu–Jona-Lasinio model (NJL) [5].

The NJL Lagrangian of the effective four-quark interaction has the form:

$$\mathcal{L}_{\mathrm{NJL}} = \bar{q}(i\widehat{\partial} - m_0)q + \mathcal{L}_{int} \tag{1}$$

with

$$\mathcal{L}_{int} = 2G_1 \left\{ (\bar{q}\frac{1}{2}\lambda_f^i q)^2 + (\bar{q}i\gamma^5\frac{1}{2}\lambda_f^i q)^2 \right\} - 2G_2 \left\{ (\bar{q}\gamma^\mu\frac{1}{2}\lambda_f^i q)^2 + (\bar{q}\gamma^\mu\gamma^5\frac{1}{2}\lambda_f^i q)^2 \right\},$$

G_1 and G_2 are some empirical constants.

Let us show that the NJL model can be motivated from the nonperturbative QCD using some assumption about the infrared behavior of gluons. The generating functional of QCD has the form [6]:

$$\mathcal{Z} = \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}G \exp\left\{ i \int d^4x \left[\bar{q}^f (i\widehat{\partial} - m_f^0) q^f + g\bar{q}^f \widehat{G}^a \frac{\lambda_C^a}{2} q^f - \frac{1}{4} G^{\mu\nu} G_{\mu\nu} \right] \right\}$$

After the integration over the gluon fields one will get

$$\mathcal{Z} = \int \mathcal{D}q\mathcal{D}\bar{q} \exp\left\{ i \int d^4x \bar{q}(i\widehat{\partial} - m_f^0) q \right\} \exp(W[j])$$

with

$$j^a_\mu(x) = \bar{q}(x)\gamma_\mu\lambda^a_C q(x)$$

as flavour-singlet quark current and W becomes

$$W[i] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^4x_1 ... d^4x_n D^{a_1...a_n}_{\mu_1...\mu_n}(x_1, ..., x_n) \prod_{i=1}^{n} j^{a_i}_{\mu_i}(x_i)$$

The functions $D^{a_1...a_n}_{\mu_1...\mu_n}$ contain all information about the gluons. An analytical integration is not possible because the behavior of the gluon propagator and the running coupling constant at long distances are unknown.

Neglecting higher order Green functions one can get some local approximation for the gluon propagator $D_{\mu\nu}(x, y) \propto g_{\mu\nu}\delta^4(x - y)$ and find some theoretical base for the approximation $\mathcal{L}_{\rm QCD} \to \mathcal{L}_{\rm NJL}$. In this approximation it is impossible to predict the values of the constants G_1 and G_2 from QCD principles. They are to be determined experimentally.

One can see that the group structure of the QCD

$$SU(3)^{\rm local}_C \otimes SU(N_f)_L \otimes SU(N_f)_R \otimes U(1) \otimes S$$

and that of the NJL model

$$SU(3)^{\rm global}_C \otimes SU(N_f)_L \otimes SU(N_f)_R \otimes U(1) \otimes S$$

are very similar (S presents the set of discrete symmetries as C, P and T conjugation).

After this introduction of the NJL model we will now derive a meson Lagrangian from the NJL Lagrangian (1) using path integral techniques. First one has to introduce collective fields with the form:

$$S = \sum S_i \frac{1}{2}\lambda_i, \;\; P = \sum P_i \frac{1}{2}\lambda_i, \;\; V^\mu = -i \sum V^\mu_i \frac{1}{2}\lambda_i, \;\; A^\mu = -i \sum A^\mu_i \frac{1}{2}\lambda_i$$

which will correspond later to scalar (S), pseudoscalar (P), vector (V) and axial (A) colourless fields. The following notations have been used:

$$S_i = -4G_1\bar{q}\frac{1}{2}\lambda_i q, \;\; P_i = -4G_1\bar{q}i\gamma^5\frac{1}{2}\lambda_i q, \;\; V^\mu_i = -4G_2\bar{q}\gamma^\mu\frac{1}{2}\lambda_i q, \;\; A^\mu_i = -4G_2\bar{q}\gamma^\mu\gamma^5\frac{1}{2}\lambda_i q.$$

Now the Lagrangian (1) can be redefined:

$$\mathcal{L}_{\rm NJL} = -\frac{1}{4G_1}\,{\rm tr}\,\phi^+\phi - \frac{1}{4G_2}\,{\rm tr}\,(V^2_\mu + A^2_\mu) + \bar{q}\left\{i\widehat{\partial} + \widehat{V} + \widehat{A}\gamma^5 - P_R(\phi + m_0) + P_L(\phi^+ + m_0)\right\} q$$

with $\phi = S + iP$, $\widehat{V} = V^\mu\gamma_\mu$, $\widehat{A} = A^\mu\gamma_\mu$ and $P_{R/L} = \frac{1}{2}(1 \pm \gamma^5)$ as chiral projectors. This substitution transforms the Lagrangian (1) into a bilinear form in quark fields. Now the integration over quark fields becomes possible and the generating functional of the NJL model is

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^+\mathcal{D}V\mathcal{D}A \exp\left[i\int d^4x - \frac{1}{4G_1}\,{\rm tr}\,\phi^+\phi - \frac{1}{4G_2}\,{\rm tr}\,(V^2_\mu + A^2_\mu))\right] \cdot \mathcal{Z}_f$$

with

$$Z_f(\phi,\phi^+,V,A)=\int \mathcal{D}q\mathcal{D}\bar{q}\exp(i\int d^4x\bar{q}i\widehat{\mathbf{D}}q)=(\det i\widehat{\mathbf{D}}) \tag{2}$$

and

$$\begin{aligned} i\widehat{\mathbf{D}} &= i\widehat{\partial}+\widehat{V}+\widehat{A}\gamma^5-P_R(\phi+m_0)+P_L(\phi^++m_0) \\ &= i(\widehat{\partial}+\widehat{A}_R)-(\phi+m_0)P_R+i(\widehat{\partial}+\widehat{A}_L)-(\phi^++m_0)P_L \end{aligned}$$

as Dirac operator. $\widehat{A}_{R/L}=\widehat{V}\pm\widehat{A}$ are right- and left-handed vector fields. The quark determinant (2), describing the interactions of mesons, can be evaluated in quark loops [7, 8].

At this place it makes sense to refer to some different ways in the calculation of the quark-determinant (2). In contrary to the "straight" method of calculating the quark loops [7] we have used the heat-kernel method [6]. Of course, the physical results must not depend on the method of calculation but there are some technical reasons for using this method here. The main advantage of this method is that the recursive algorithms of this method can be adapted on CAS quite effectively. The infinite contributions can be conveniently separated too.

The absolute value of the quark determinant (2) in "proper-time" regularization is defined as

$$\log|\det i\widehat{\mathbf{D}}| =-\frac{1}{2}\,\mathrm{Tr}'\,\log(\widehat{\mathbf{D}}^+\widehat{\mathbf{D}})=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}d\tau\frac{1}{\tau}\,\mathrm{Tr}'\,\exp(-\widehat{\mathbf{D}}^+\widehat{\mathbf{D}}\tau) \tag{3}$$

with Λ as regularization parameter. The trace Tr' is to be understood as a space-time integration and a "normal" trace over Dirac, colour and flavour indices:

$$\mathrm{Tr}'=\int d^4x\,\mathrm{tr}'\,,\quad \mathrm{tr}'=\mathrm{tr}_C\cdot\mathrm{tr}_f\cdot\mathrm{tr}_\gamma.$$

The operator $\widehat{\mathbf{D}}^+\widehat{\mathbf{D}}$ can be presented as

$$\widehat{\mathbf{D}}^+\widehat{\mathbf{D}}=d_\mu d^\mu+a(x)+\mu^2$$

with

$$d_\mu=\partial_\mu+\Gamma_\mu,\quad \Gamma_\mu=V_\mu+A_\mu\gamma^5,\quad a(x)=i\widehat{\nabla}H+H^+H+\frac{1}{4}[\gamma^\mu,\gamma^\nu]\Gamma_{\mu\nu}-\mu^2.$$

Here μ plays the role of some free parameter which will fix the regularization in the region of low momenta. Below it will be identified to the constituent quark mass. We used the following notations:

$$H=P_R(\phi+m_0)+P_L(\phi^++m_0)=S+i\gamma_5P,$$

$$\Gamma_{\mu\nu}=[d_\mu,d_\nu]=\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu+[\Gamma_\mu,\Gamma_\nu]=F^V_{\mu\nu}+\gamma^5F^A_{\mu\nu},$$

with $F^{V,A}_{\mu\nu}$ as field strength tensors

$$F^V_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu+[V_\mu,V_\nu]+[A_\mu,A_\nu],$$

$$F^A_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[V_\mu,A_\nu]+[A_\mu,V_\nu]$$

and

$$\nabla_\mu H = \partial_\mu H + [V_\mu, H] - \gamma^5\{A_\mu, H\}$$

as covariant derivative.

The main idea of the heat-kernel method is to evaluate

$$< x \mid \exp(-\widehat{\mathbf{D}}^+\widehat{\mathbf{D}}\tau) \mid y >$$

around its nonperturbated part

$$< x \mid \exp(-(\Box + \mu^2)\tau) \mid y >= \frac{1}{(4\pi\tau)^2} e^{-\mu^2\tau + (x-y)^2/(4\tau)}$$

in powers of proper-time τ with the so-called Seeley–deWitt coefficients $h_k(x, y)$

$$< x \mid \exp(-\widehat{\mathbf{D}}^+\widehat{\mathbf{D}}\tau) \mid y >= \frac{1}{(4\pi\tau)^2} e^{-\mu^2\tau + (x-y)^2/(4\tau)} \sum_k h_k(x, y) \cdot \tau^k.$$

After integration over τ in (3) one gets the following expression for $\log \mid \det i\widehat{\mathbf{D}} \mid$

$$\frac{1}{2}\log(\det \widehat{\mathbf{D}}^+\widehat{\mathbf{D}}) = -\frac{1}{2}\frac{\mu^4}{(4\pi)^2}\sum_k \frac{\Gamma(k-2, \mu^2/\Lambda^2)}{\mu^{2k}} \operatorname{Tr}' h_k$$

with

$$\Gamma(\alpha, x) = \int_x^\infty dt\, e^{-t} t^{\alpha-1}$$

as incomplete gamma function. Using the definition of the gamma function one can separate the divergent and finite parts:

$$\frac{1}{2}\log(\det \widehat{\mathbf{D}}^+\widehat{\mathbf{D}}) = B_{\text{pol}} + B_{\log} + B_{\text{fin}}.$$

Here

$$B_{\text{pol}} = \frac{1}{2}\frac{e^{-x}}{(4\pi)^2}\left[-\frac{\mu^4}{2x^2}\operatorname{Tr}' h_0 + \frac{1}{x}(\frac{\mu^4}{2}\operatorname{Tr}' h_0 - \mu^2 \operatorname{Tr}' h_1)\right]$$

has a pole at $x = \mu^2/\Lambda^2 = 0$,

$$B_{\log} = -\frac{1}{2}\frac{1}{(4\pi)^2}\Gamma(0, x)\left[\frac{1}{2}\mu^4 \operatorname{Tr}' h_0 - \mu^2 \operatorname{Tr}' h_1 + \operatorname{Tr}' h_2\right].$$

is logarithmic divergent. The finite part has the form:

$$B_{\text{fin}} = -\frac{1}{2}\frac{1}{(4\pi)^2}\sum_{k=2}^\infty \mu^{4-2k}\Gamma(k-2, x) \operatorname{Tr}' h_k.$$

The main technical problem now is to calculate the Seeley–deWitt coefficients h_k. One can find the calculated heat coefficients up to $k = 3$ [7] and with simplifications up to $k = 6$ in [6]. In this paper we have calculated the full coefficients up to the order $n = 4$ and and also present the minimal parts of the heat-coefficients $h_{5,6}$. Let us show the general heat-kernel techniques in detail: The determinant

of the positive definite operator $\mathbf{A}$ is defined in proper-time regularization by the following integral relation

$$\log(\det \mathbf{A}) = -\int_{1/\Lambda^2}^{\infty} \frac{d\tau}{\tau} \operatorname{Tr}' K(\tau),$$

where $K(\tau) = e^{-\mathbf{A}\tau}$ is the so-called "heat kernel" which satisfies the equation

$$\frac{\partial}{\partial\tau} K(\tau) + \mathbf{A}K(\tau) = 0$$

with the boundary condition

$$K(\tau = 0) = 1.$$

In the case discussed in the paper the operator $\mathbf{A}$ has the structure

$$\mathbf{A} = d_\mu d^\mu + a(x) + \mu^2,$$

where

$$d_\mu = \partial_\mu + \Gamma_\mu$$

contains an differential operator and $a(x)$ does not.

The asymptotic behavior of $\mathbf{A}$ at long distances is corresponding with the infinite part of $\log(\det A)$, which is defined by the "free" part

$$\mathbf{A}_0 = \Box + \mu^2, \quad \Box \equiv \partial_\mu \partial^\mu.$$

Using the ansatz

$$K = K_0 H$$

it is convenient to separate the "free" part K_0 from the heat kernel. In coordinate representation K_0 reads

$$K_0(x, y; \tau) \equiv < x \mid K_0 \mid y >= \frac{1}{(4\pi\tau)^2} e^{-\mu^2\tau + (x-y)^2/(4\tau)}$$

and also satisfies the equation

$$\frac{\partial K_0}{\partial\tau} + \mathbf{A}_0 K_0 = 0,$$

with the boundary condition

$$K_0(\tau = 0) = 1.$$

The "interaction" part H of the heat kernel satisfies the equation

$$(\frac{\partial}{\partial\tau} + \frac{1}{\tau} z_\mu d^\mu + d^\mu d_\mu + a) H(x, y; \tau) = 0,$$

$$H(x, y = x; \tau = 0) = 1, \tag{4}$$

where $z_\mu = x_\mu - y_\mu$ and the differential operator d_μ acting only on x. Using now an expansion for $H(\tau)$ in powers of τ

$$H(x, y; \tau) = \sum_{k=1}^{\infty} h_k(x, y) \cdot \tau^k$$

one will get from (4) the recursive relation

$$(n+1+z_\mu d^\mu)h_{n+1}+(a+d_\mu d^\mu)h_n=0$$

with boundary condition

$$z_\mu d^\mu h_0=0. \tag{5}$$

The heat coefficients $h_k(x)=h(x,y=x)$ are defined by the diagonal part of the recursive relation (4)

$$nh_n(x)=-[d_\mu d^\mu h_{n-1}(x,y)]\mid_{y=x}-ah_{n-1}(x) \tag{6}$$

which gives for $n=0$

$$h_1(x,x)=-(d_\mu d^\mu+a)h_0(x,x).$$

Here we use the more convenient notations

$$d_\alpha\dots d_\beta h_n(x,x)\equiv[d_\alpha\dots d_\beta h_n(x,y)]\mid_{y=x}.$$

The expression (6) can be trivially calculated once the $(n-1)^{th}$ order heat coefficient is known. To find the derivative terms $d_\mu d^\mu h_{n-1}$ one applies the operator $d_\mu d^\mu$ on the recursion relation (6) for $n-1$ and puts afterwards $y=x$. This introduces derivatives up to the order of h_{n-2}, and in addition derivatives of a. One continues to apply repeatedly operators d_μ on the recursion relation (6) for smaller and smaller n until the desired n^{th} order heat coefficient is completely expressed in terms differing only by a permutation of the d_μ's. To get an explicit expression for the multiple derivatives of h_n one has to bring the d_μ's in all of these terms into the same order. The reordering of the d_μ's introduces multiple commutators of the form:

$$K_{\mu\nu}=[d_\mu,d_\nu]=\Gamma_{\mu\nu},\quad K_{\lambda\mu\nu}=[d_\lambda,K_{\mu\nu}],\quad K_{\kappa\lambda\mu\nu}=[d_\kappa,K_{\lambda\mu\nu}]\quad\text{, etc.} \tag{7}$$

Similarly one has to bring the a's to the left of all d_μ's. This introduces the multiple commutators

$$S_\mu=[d_\mu,a],\quad S_{\mu\nu}=[d_\mu,S_\nu],\quad S_{\lambda\mu\nu}=[d_\lambda,S_{\mu\nu}]\quad\text{, etc.} \tag{8}$$

In this way the heat coefficients can be completely expressed in terms of multiple derivatives of h_0, of multiple commutators (7) and (8), of lower-order heat coefficients, and of a's, with the latter ones being always on the left of the derivative operators d_μ. The multiple derivatives

$$d_\alpha\dots d_\nu h_0(x)\equiv d_\alpha\dots d_\nu h_0(x,y)\mid_{y=x}$$

are finally calculated by applying the d_μ's repeatedly on the differential equation (5) defining h_0.

Following the strategy outlined above the calculation of the heat coefficients is straightforward but cumbersome. The very lengthy calculations can be performed only by computer support. The calculation of the heat-coefficients is a recursive process which can be done by CAS very conveniently.

For the calculations of the heat coefficients we used the CAS REDUCE and FORM. The reason for the use of different CAS for the solution of one problem is connected with the special features of the

systems. FORM is designed for doing fast arithmetic with large formulae and a special attention was to implement there structures commonly used in high energy physics, e.g. summation over dummy indices. It easy easy to control even difficult calculation processes step by step since operations are only done on request. However, the scope of FORM operations is limited and if some particular sphere of symbolic calculations is somewhat biased from the generally used area of multiloop calculations or Dirac algebra, then a need for unimplemented operations can arise. In our case of calculations with nonrenormalisable chiral models in low-energy meson physics there is a need not only for summation over some dummy indices but also for the following operations.

First of all one needs the operation of cyclic shifting of non-commuting operators products under the trace sign. This operator is required since the following parts of Lagrangian are equivalent:

$$\mathcal{L}_1 = \mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi) \quad \text{and} \quad \mathcal{L}_2 = \mathrm{tr}\ (\Phi\ \partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi)$$

and one should transform them to some unique form, either the former or the latter one.

Secondly, one way of testing the obtained physical results is to test their symmetry properties. In our case we have Hermitian conjugation and the transformation of intrinsic parity. One needs some means to test automatically the symmetry properties of the results.

Both FORM and REDUCE lack the described operations. FORM has no possibility of adding new non-trivial operations. For this purpose we used CAS REDUCE which is a completely open system allowing the user to access directly the internal data structure and implement new operations and data types by the "symbolic" style programming. We implemented the required operations absent in FORM and also such as summation over dummy indices. In such a way the whole work described in this paper could be done only with the help of the single REDUCE system. Let us briefly describe the corresponding REDUCE functions from our package [9] for applications in meson chiral models. Each REDUCE example below follows the usual mathematical notation.

`traceshift(X)` – transforms expression to the unique form using the possibility to perform cyclic permutation of the products of matrix symbols under the trace operation, e.g.

$$\begin{aligned}
\mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi) &\longrightarrow \mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi)\\
\texttt{traceshift(fi(mu)*fi()*fi(mu)*fi())} &\longrightarrow \texttt{fi(mu)*fi()*fi(mu)*fi()}\\
\mathrm{tr}\ (\Phi\ \partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi) &\longrightarrow \mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi)\\
\texttt{traceshift(fi()*fi(mu)*fi()*fi(mu))} &\longrightarrow \texttt{fi(mu)*fi()*fi(mu)*fi()}
\end{aligned}$$

`orderind(X)` – transforms expression to the unique form using the possibility to redesignate the dummy indices, e.g.

$$\begin{aligned}
\mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi) &\longrightarrow \mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi)\\
\texttt{orderind(fi(mu)*fi()*fi(mu)*fi())} &\longrightarrow \texttt{fi(mu)*fi()*fi(mu)*fi()}\\
\mathrm{tr}\ (\partial_\nu\Phi\cdot\Phi\ \partial_\nu\Phi\cdot\Phi) &\longrightarrow \mathrm{tr}\ (\partial_\mu\Phi\cdot\Phi\ \partial_\mu\Phi\cdot\Phi)\\
\texttt{orderind(fi(nu)*fi()*fi(nu)*fi())} &\longrightarrow \texttt{fi(mu)*fi()*fi(mu)*fi()}.
\end{aligned}$$

After voluminous computations one gets the complex expressions for heat-coefficients $h_1, \dots h_4$:

$$
\begin{aligned}
h_0(x) &= 1, \\
h_1(x) &= -a, \\
\mathrm{tr}'\left[h_2(x)\right] &= \mathrm{tr}'\left\{\frac{1}{12}(\Gamma_{\mu\nu})^2 + \frac{1}{2}a^2\right\}, \\
\mathrm{tr}'\left[h_3(x)\right] &= -\frac{1}{12}\,\mathrm{tr}'\left\{2a^3 - S_\mu S^\mu + a(\Gamma_{\mu\nu})^2 - \frac{2}{45}(K_{\alpha\beta\gamma})^2 - \frac{1}{9}(K^\alpha{}_{\alpha\beta})^2\right\}, \\
\mathrm{tr}'\left[h_4(x)\right] &= \mathrm{tr}'\left\{\frac{1}{24}a^4 + \frac{1}{12}a^2 S^\mu{}_\mu + aS_\mu S^\mu + \frac{1}{720}7(S^\mu{}_\mu)^2 - (S_{\mu\nu})^2\right. \\
&\quad + \frac{1}{30}a^2(\Gamma_{\mu\nu})^2 + \frac{1}{120}(a\Gamma_{\mu\nu})^2 + \frac{1}{180}a(K^\alpha{}_{\alpha\mu})^2 + \frac{1}{75}a\Gamma_{\mu\nu}K_\beta{}^{\beta\mu\nu} + \frac{7}{900}\Gamma_{\mu\nu}S^\mu K_\alpha{}^{\alpha\nu} \\
&\quad + \frac{1}{50}aK_\beta{}^{\beta\mu\nu}\Gamma_{\mu\nu} - \frac{1}{300}\Gamma_{\mu\nu}K_\alpha{}^{\alpha\mu}S^\nu + \frac{1}{3600}K^\alpha{}_{\alpha\mu}\left(S_\beta{}^{\beta\mu} + S_\beta{}^{\mu\beta}\right) + \frac{1}{72}S_\mu{}^\mu(\Gamma_{\alpha\beta})^2 \\
&\quad + \frac{1}{180}S^{\mu\nu}\{\Gamma_{\mu\alpha}, \Gamma_\nu{}^\alpha\} + \frac{1}{40}a\left(\Gamma_{\mu\nu}S^{\mu\nu} + \frac{11}{9}S_{\mu\nu}\Gamma^{\mu\nu}\right) + \frac{1}{144}a\left[K^\mu{}_{\mu\nu}, S^\nu\right] \\
&\quad + \left(\frac{2}{135}aK_{\beta\mu\nu} + \frac{11}{900}\Gamma_{\mu\nu}S_\beta + \frac{1}{100}S_\beta\Gamma_{\mu\nu} + \frac{1}{4725}\left[\Gamma_{\mu\nu}, K^\alpha_{\alpha\beta}\right]\right)(K^{\beta\mu\nu} - K^{\mu\nu\beta}) \\
&\quad + \frac{1}{1260}\Gamma_{\mu\nu}K_\alpha{}^{\alpha\mu}K_\beta{}^{\beta\nu} - \frac{1}{12600}\left(29\Gamma^\beta{}_\alpha\Gamma^{\mu\alpha} + 27\Gamma^{\mu\alpha}\Gamma^\beta{}_\alpha\right)\left(K^\nu_{\mu\beta\nu} + K^\nu_{\mu\nu\beta}\right) \\
&\quad + \Gamma_{\alpha\beta}\Gamma_{\mu\nu}\left(\frac{83}{25200}K^{\mu\nu\alpha\beta} + \frac{4}{1575}K^{\alpha\beta\mu\nu} - \frac{127}{5040}K^{\alpha\mu\nu\beta} - \frac{1}{600}K^{\mu\alpha\beta\nu}\right) \\
&\quad + \frac{13}{12600}\Gamma_{\mu\beta}\Gamma^\beta{}_\nu\Gamma^\nu{}_\alpha\Gamma^{\alpha\mu} + \frac{47}{16800}(\Gamma_{\mu\nu})^2(\Gamma_{\alpha\beta})^2 + \frac{17}{25200}(\Gamma_{\mu\nu}\Gamma_{\alpha\beta})^2 \\
&\quad + \frac{4}{1575}(\Gamma_{\mu\alpha}\Gamma^\alpha{}_\nu)^2 + \frac{19}{25200}K^\alpha{}_{\alpha\mu\nu}K^\mu{}_\beta{}^{\beta\nu} - \frac{1}{12600}(K^\alpha{}_{\mu\nu\alpha})^2 + \frac{1}{1575}(K_\mu{}^\alpha{}_{\alpha\nu})^2 \\
&\quad + \frac{1}{6300}K_\mu{}^\alpha{}_{\alpha\nu}K^{\beta\mu\nu}{}_\beta + \frac{1}{5600}(K^\alpha{}_{\alpha\mu\nu})^2 - \frac{1}{5040}K^\alpha{}_{\alpha\mu\nu}K_\beta{}^{\mu\nu\beta} + K_{\mu\nu\alpha\beta}K^{\alpha\mu\nu\beta} \\
&\quad \left. - \frac{1}{1800}K_{\mu\alpha}{}^\alpha{}_\nu K^{\mu\beta}{}_\beta{}^\nu - \frac{1}{25200}K_{\mu\nu\alpha\beta}\left[3\left(K^{\mu\nu\alpha\beta} + K^{\nu\alpha\beta\mu}\right) + 2\left(K^{\mu\alpha\beta\nu} + K^{\alpha\nu\mu\beta}\right)\right]\right\}
\end{aligned}
$$

In the same way the next orders of heat expansion coefficients h_i can be obtained using the developed calculation technique based on the usage of computer algebra. For simplicity we present below expressions only for minimal parts of heat-coefficients, i.e. only for the parts which do not vanish in the pseudoscalar region of the theory when $V_\mu = A_\mu = 0$:

$$
\begin{aligned}
\mathrm{tr}'\left[h_5(x)^{min}\right] &= -\,\mathrm{tr}'\left\{\frac{1}{120}a^2(a^3 + S_\mu S^\mu) + \frac{1}{180}a^3 S_\mu{}^\mu + 2(aS_\mu)^2\right. \\
&\quad + \frac{1}{6300}\left[10aS_\mu(S^{\mu\nu}{}_\nu + S_\nu{}^{\nu\mu}) - 2a(S_{\mu\nu})^2 + 17a(S^\mu_\mu)^2 + S^{\mu\nu}{}_\nu S_\mu + 3aS_\mu{}^{\mu\nu}S_\nu\right] \\
&\quad + \frac{11}{1008}S_\mu S^\mu S_\nu{}^\nu + \frac{19}{2800}S_\mu S_\nu S^{\mu\nu} + \frac{2}{225}S_\mu S_\nu S^{\nu\mu} \\
&\quad \left. + \frac{1}{25200}\left[3(S^\mu{}_{\mu\nu})^2 - 2(S_{\mu\nu\alpha})^2 - 23(S^{\mu\nu}{}_\nu)^2 + 7S^\mu{}_{\mu\nu}S^{\nu\alpha}{}_\alpha\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}'\left[h_6(x)^{min}\right] \;=\; & \mathrm{tr}'\left\{\frac{1}{720}a^2(a^4+4S_\mu aS^\mu)+\frac{1}{420}a^3S_\mu S^\mu\right.\\
&+\frac{1}{20160}a^2\left[20a^2S_\mu{}^\mu+5S_\mu(S^{\mu\nu}{}_\nu+S_\nu{}^{\nu\mu})+S^\mu{}_{\mu\nu}S^\nu-(S_{\mu\nu})^2+11(S_\mu{}^\mu)^2+9S^{\mu\nu}{}_\nu S_\mu\right]\\
&+\frac{1}{25200}a\left[S_\mu a(73S^{\mu\nu}{}_\nu+37S_\nu{}^{\nu\mu})+5S^{\mu\nu}(S_\mu S_\nu+4S_\nu S_\mu)\right]+\frac{1}{2016}aS^\mu{}_\mu S^\nu S_\nu\\
&+\frac{1}{9450}aS_\mu\left(37S^{\mu\nu}+23S^{\nu\mu}\right)S_\nu+\frac{1}{9072}aS_\mu\left(S^{\mu\nu}{}_\nu{}^\alpha{}_\alpha+S_\nu{}^{\nu\mu\alpha}{}_\alpha+S_\nu{}^\nu{}_\alpha{}^{\alpha\mu}\right)\\
&+aS^\mu\left(\frac{23}{4800}S_\mu S_\nu{}^\nu+\frac{937}{302400}S^\nu S_{\mu\nu}+\frac{23}{10800}S^\nu S_{\nu\mu}\right)+\frac{1}{252}aS_\mu S^\nu{}_\nu S^\mu\\
&+\frac{1}{352800}aS^\mu{}_{\mu\nu}\left(52S_\alpha{}^{\alpha\nu}+53S^{\nu\alpha}{}_\alpha\right)+aS^{\mu\nu}{}_\nu\left(\frac{1}{3360}S^\alpha{}_{\alpha\mu}-\frac{1}{11340}S_{\mu\alpha}{}^\alpha\right)\\
&+\frac{17}{226800}a(S_{\mu\nu\alpha})^2+\frac{1}{317520}a\left(23S^\mu{}_\mu{}^{\nu\alpha}{}_\alpha+5S^\mu{}_{\mu\alpha}{}^{\alpha\nu}+77S^{\nu\mu}{}_\mu{}^\alpha{}_\alpha\right)S_\nu\\
&-\frac{1}{30240}\left[53(S_\mu S^\mu)^2+(S_\mu S_\nu)^2\right]+\frac{1}{352800}S_\mu S_\nu\left(157S^\mu{}_\alpha{}^{\alpha\nu}+298S^{\mu\nu\alpha}{}_\alpha\right)\\
&+\frac{1}{70560}S_\mu S_\nu\left(31S_\alpha{}^{\alpha\nu\mu}+58S^{\nu\mu\alpha}{}_\alpha+47S^{\nu\alpha}{}_\alpha{}^\mu\right)+\frac{1}{720}S^\mu S_\mu S^\nu{}_\nu{}^\alpha{}_\alpha\\
&+\frac{5}{14112}S_\mu S_\nu S_\alpha{}^{\alpha\mu\nu}+\frac{1}{105840}S_\mu\left[S^{\mu\nu}\left(37S^\alpha{}_{\alpha\nu}+70S_{\nu\alpha}{}^\alpha\right)+35S^{\nu\mu}S^\alpha_{\nu\alpha}\right]\\
&+\frac{1}{21168}S_\mu\left[S^{\nu\mu}S^\alpha{}_{\alpha\nu}+S_{\nu\alpha}(5S^{\nu\mu\alpha}+2S^{\nu\alpha\mu})\right]+\frac{1}{2880}S_\mu\left(S^{\mu\nu}{}_\nu+S_\nu{}^{\nu\mu}\right)S_\alpha{}^\alpha\\
&+S_\mu S^\nu{}_\nu\left(\frac{83}{141120}S^{\mu\alpha}{}_\alpha+\frac{1}{9408}S_\alpha{}^{\alpha\mu}\right)+\frac{1}{30240}S_\mu\left(17S_{\nu\alpha}{}^\alpha+13S^\alpha{}_{\alpha\nu}\right)S^{\nu\mu}\\
&+\frac{1}{7560}S_\mu\left[2\left(S^{\mu\nu\alpha}+S^{\nu\mu\alpha}+S^{\nu\alpha\mu}\right)S_{\nu\alpha}+\left(S_{\nu\alpha}{}^\alpha+2S^\alpha{}_{\alpha\nu}\right)S^{\mu\nu}\right]+\frac{1}{2160}S_\mu S_{\nu\alpha}S^{\mu\nu\alpha}\\
&-\frac{1}{635040}\left(701(S_\mu{}^\mu)^3+583S^{\mu\nu}S_{\mu\alpha}S^\alpha_\nu\right)-\frac{689}{316386}S^\alpha{}_\alpha(S^{\mu\nu})^2\\
&-\frac{2}{2835}S^\mu{}_\mu S^{\nu\alpha}S_{\alpha\nu}-\frac{1}{952560}S^{\mu\nu}\left(619S^\alpha{}_\nu S_{\mu\alpha}+190S_{\mu\alpha}S^\alpha{}_\nu\right)\\
&+\frac{1}{151200}\left[11(S^\mu{}_\mu{}^\nu{}_\nu)^2-2(S_{\mu\nu}{}^\alpha{}_\alpha)^2\right]+\frac{1}{176400}\left((S^\mu{}_{\mu\nu\alpha})^2+S^\mu{}_{\mu\nu\alpha}S^{\nu\beta}{}_\beta{}^\alpha\right)\\
&-\frac{1}{226800}(S_{\mu\nu\alpha\beta})^2-\frac{103}{12700800}(S_\mu{}^\alpha{}_{\alpha\nu})^2+\frac{1}{66150}S^\mu{}_{\mu\nu\alpha}S^{\nu\alpha\beta}{}_\beta\\
&\left.-\frac{1}{52920}S^\mu{}_\mu{}^\nu{}_\nu S^{\alpha\beta}{}_{\beta\alpha}-\frac{13}{604800}S^{\mu\nu\alpha}{}_\alpha S_\mu{}^\beta{}_{\beta\nu}\right\}.
\end{aligned}
$$

To obtain these expressions for the heat coefficients we have extensively made use of the cyclic properties of the trace and of the Jacobi identity. To obtain effective meson Lagrangians in terms of collective fields one should calculate in $\mathrm{tr}'\,h_i(x)$ the trace over Dirac indices.

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