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R. F. Wehrhahn
Theoretische Kernphysik, Universität Hamburg

A. O. Barut
Department of Physics and Astrophysics, University of Colorado, Boulder, USA

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Symmetry scattering for $SU(2,2)$ and its applications.

R.F. Wehrhahn and A.O. Barut*

Theoretical Nuclear Physics, University of Hamburg

Luruper Chaussee 149, 2000 Hamburg 50, Germany.

*Department of Physics and Astrophysics,

University of Colorado, Boulder, Colorado 80309-390, U.S.A.

In the framework of symmetry scattering, $SU(2,2)$-invariant differential equations on the homogeneous space $X = SU(2,2)/S(U(2)\otimes U(2))$ are studied. The radial Schrödinger equation for a family of one or two dimensional potentials or for two particles arise. From the symptotic behaviour of the solutions exact partial wave scattering amplitudes are derived.

I. Introduction. In [1]the possibility to study scattering problems of systems with internal structure using a group theoretical method called "symmetry scattering" for spaces of rank higher than one has been considered. In this paper we pursue this approach and study in detail the symmetry scattering for the Riemannian symmetric space $X = SU(2,2)/S(U(2)\otimes U(2)) \cong SO(4,2)/SO(4)\otimes SO(2)$. This space has already emerged several times in physics[2,3].

The space X is a non–compact Riemannian symmetric space of dimension eight and rank two. Hence, in this eighth dimensional space, the maximal flat subspace is two dimensional. X is generated by the orbits of $K = S(U(2)\otimes U(2))$ in $SU(2,2)$.

Symmetry scattering for this space is obtained from the asymptotic properties of the joint eigenfunctions defined on X of the $SU(2,2)$–invariant differential operators. These invariant differential operators are generated by the two Casimir operators C_2 and C_4 of second and fourth order, respectively. For physical applications we consider only the operator C_2, which is identical to the Laplace Beltrami operator on X. This is a second order differential operator on two variables x and y parametrizing A, and on the six angles parametrizing B defined below. The eigenfunctions of C_2 hence depend on eight variables, two of them with unbounded domains. As usual, in the symmetry scattering approach, the Casimir operator C_2 can be written as a sum of an operator depending only on the unbounded variables x and y, the radial part, and an operator acting on the remaining six variables, the transvers part. The eigenfunctions of C_2 then separate into the product of a function depending only on x and y

and an eigenfunction of the transvers part.
As an initial step in the $SU(2,2)$–symmetry scattering, we consider here the case where the eigenfunctions do not depend on the six angle parameters. Already for this situation a very rich class of scattering problems is obtained in the theory. Thus, we deal only with the radial part C_r of C_2 and hence with the scattering properties of systems described by the zonal spherical functions. The applications of symmetry scattering that we consider here result from identifying or relating the operator C_r with the Hamiltonian of some physical systems. Symmetry scattering then provides the scattering properties of such systems.

After setting up the formalism and obtaining explicit expressions for all required quantities like the radial Casimir C_r and the scattering operator, we discuss in detail the application of symmetry scattering to describe one dimensional potential scattering of a system with an internal degree of freedom. Potentials describing confined systems as well as purely repulsive potentials appear. After summarizing the results, future applications are indicated in the conclusion.
For a better understanding we have included an Appendix to the paper which illustrates, in the standard terminology of physics, the general constructions applied to $SO(4,2)$ which is better known to physicists than its isomorphic group $SU(2,2)$ discussed in the text. We also treat the well known case of rank–one space $SO(3,1)/SO(3)$.

II. General Theory.

This section is divided into two parts. In part A, the general decomposition of the Laplace Beltrami operator defined on a Riemannian symmetric space into a radial and a transvers part is recalled. For this purpose we first introduce some notation and the polar or Cartan decomposition of a semisimple Lie Group. We then apply this result to the operator C_2 of $X = SU(2,2)/S(U(2) \otimes U(2))$.
In part B the symmetry scattering operator for $SU(2,2)$ is derived.

Part A.

Let G be a semisimple Lie group of non–compact type with $\mathcal{G}$ its Lie algebra over $\mathbf{R}$. Let $\mathcal{G} = \mathcal{P} + \mathcal{K}$ be a fixed Cartan decomposition, $\mathcal{A} \subset \mathcal{P}$ be any maximal abelian subalgebra of $\mathcal{P}$ and denote by $\mathcal{M}$ the centralizer of $\mathcal{A}$ in $\mathcal{K}$. An element $H \in \mathcal{A}$ is called regular if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, the set of restricted roots. $\mathcal{A}'$ denotes the set of regular elements in $\mathcal{A}$.

Fix a Weyl chamber $\mathcal{A}^+$, we call a root positive if its restriction to $\mathcal{A}^+$ is positive. We have

$$\mathcal{A}^+ = \{H \in \mathcal{A} \,|\, \alpha(H) > 0\}, \qquad \text{where } \alpha \text{ runs over all simple roots.}$$

Let $K = e^{\mathcal{K}}$, $A = e^{\mathcal{A}}$, $A^+ = e^{\mathcal{A}^+}$, $\bar{A}^+$ its closure, $A' = e^{\mathcal{A}'}$ and $M = e^{\mathcal{M}}$.
We have the following polar or Cartan decomposition:

Theorem[4]

$(i) \qquad G = K\bar{A}^+K$

$(ii) \qquad X = K\bar{A}^+ \cdot \ell$

where ℓ is the identity coset in G/K.

Let $G' = KA'K$ and $X' = G' \cdot \ell$ we have,

$(iii) \qquad X' = (K/M) \times (\bar{A}^+ \cdot \ell)$

in the sense that $(kM, a) \to ka \cdot \ell$ is a diffeomorphism of $K/M \times A^+$ onto X'. Thus $B = K/M$ can be viewed as a boundary of $X = G/K$.

Corresponding to the above coordinates we now state the decomposition of the Laplace Beltrami operator on X into a radial and a transvers part.

Let $\mathcal{J}$ denote the orthogonal complement of $\mathcal{M}$ in $\mathcal{K}$. Diagonalizing the operators $ad(H)^2$, $H \in \mathcal{A}$, it follows that

$$\mathcal{J} = \sum_{\alpha \in \Sigma^+} \tau_\alpha, \quad \text{where} \quad \tau_\alpha := \{T \in \mathcal{J} \,|\, ad(H)^2 T = \alpha(H)^2 T\}.$$

Here $\Sigma^+ \subset \Sigma$ contains all positive restricted roots.

Now, $dim\, \tau_\alpha = m_\alpha$, the multiplicity of the root α, and we let $T_1^\alpha, \ldots T_{m_\alpha}^\alpha$ to be a basis of τ_α. Introducing

$$\omega_\alpha := \sum_{i=1}^{m_\alpha} T_i^\alpha T_i^\alpha,$$

where the operators T_i^α are viewed as differential operators on G, and with $\tilde{f} = f \circ \pi$ for $f \in C^\infty(G/K)$ where π is the canonical projection of G into G/K, it follows:

Theorem [5]

The Laplace Beltrami operator C_2 on G/K has the following form:

$$[C_2 f](ke^a \cdot \ell) = \left[C_A + \sum_{\alpha \in \Sigma^+} m_\alpha \coth(\alpha) A_\alpha\right]_{e^a} f(ke^a \cdot \ell)$$

$$+ \sum_{\alpha \in \Sigma^+} \sinh^{-2}(\alpha(a)) \left(Ad(e^{-a})\omega_\alpha \tilde{f}\right)(ke^a),$$

where $k \in K$, $a \in A^+$, and C_A is the Laplace Beltrami operator on $A \cdot \ell$, with again ℓ being the identity coset in G/K. Furthermore $A_\alpha \in \mathcal{A}$ is defined by $< A_\alpha, H >= \alpha(H)$ for all $H \in \mathcal{A}$.

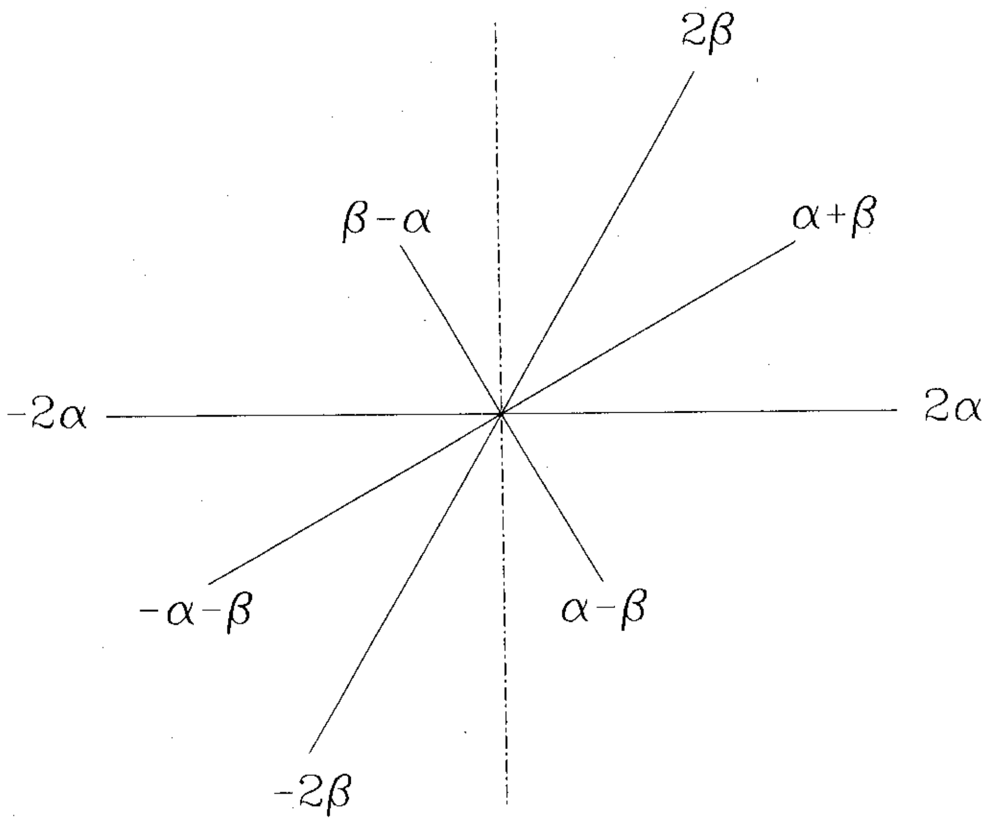

Figure 1.

The root space for $SU(2,2)/S(U(2)\otimes U(2))$. The root space is spanned by the two positive roots α and β. All restricted roots are indicated. The dotted-dashed line separates the positive from the negative roots.

In order to apply the above decomposition to the Laplace Beltrami operator of $SU(2,2)/S(U(2)\otimes U(2))$ we note that this is a rank two space. The two dimensional root–space A_2, shown in figure 1, is spanned by the roots α and β. The set $\sum^+$ of positive, restricted roots consists of the four roots 2α, 2β, $\alpha+\beta$ and $\alpha-\beta$ having multiplicities 1, 1,2 and 2, respectively. For the scalar products between the roots we have $<\alpha,\alpha>=<\beta,\beta>=1$ and $<\alpha,\beta>=\frac{1}{2}$. And in order to obtain the Laplace–Beltrami operator as a differential operator we take the following realization in the two dimensional Euclidean plane $e^{\mathcal{A}}\subset G/K$:

$$\alpha = x \qquad H_1 = \frac{\partial}{\partial x}$$
$$\beta = \frac{1}{2}x + \frac{\sqrt{3}}{2}y \qquad H_2 = \frac{1}{2}\frac{\partial}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial y}.$$

From the scalar product determined from the Cartan–Killing form, we have

$$<\alpha,\alpha> = \alpha(H_1) = 1,$$
$$<\beta,\beta> = \beta(H_2) = 1,$$
$$<\alpha,\beta> = \beta(H_1) = \frac{1}{2},$$
$$<\beta,\alpha> = \alpha(H_2) = \frac{1}{2},$$

as required.

Since $<A_\alpha, H>=\alpha(H)$, it follows that

$$A_\alpha = H_1 = \frac{\partial}{\partial x} \quad \text{and similarly} \quad A_\beta = H_2 = \frac{1}{2}\frac{\partial}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial y}.$$

The Laplace Beltrami operator on $A\cdot\ell$ becomes,

$$L_A = \sum_{i=1}^{2} H_i^2 = \frac{5}{4}\frac{\partial^2}{\partial x^2} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x\partial y} + \frac{3}{4}\frac{\partial^2}{\partial y^2}.$$

Further we have:

$$\left.\begin{aligned} m_{2\alpha} &= 1 \\ 2\alpha &= 2x \\ A_{2\alpha} &= 2\frac{\partial}{\partial x} \end{aligned}\right\} \quad \begin{aligned} &m_{2\alpha}\coth(2\alpha)A_{2\alpha} = \\ &\quad 2\coth(2x)\frac{\partial}{\partial x}. \end{aligned}$$

$$\left.\begin{aligned} m_{2\beta} &= 1 \\ 2\beta &= x+\sqrt{3}\,y \\ A_{2\beta} &= \frac{\partial}{\partial x}+\sqrt{3}\frac{\partial}{\partial y} \end{aligned}\right\} \quad \begin{aligned} &m_{2\beta}\coth(2\beta)A_{2\beta} = \\ &\quad 2\coth(x+\sqrt{3}\,y)\left(\frac{\partial}{\partial x}+\sqrt{3}\frac{\partial}{\partial y}\right). \end{aligned}$$

$$\left.\begin{aligned} m_{\alpha\pm\beta} &= 2 \\ \alpha\pm\beta &= \frac{3x\pm\sqrt{3}y}{2} \\ A_{\alpha\pm\beta} &= \frac{3}{2}\frac{\partial}{\partial x}\pm\frac{\sqrt{3}}{2}\frac{\partial}{\partial y} \end{aligned}\right\} \quad \begin{aligned} &m_{\alpha\pm\beta}\coth(\alpha\pm\beta)A_{\alpha\pm\beta} = \\ &\quad 2\coth(\frac{3x\pm\sqrt{3}y}{2})\left(\frac{3}{2}\frac{\partial}{\partial x}\pm\frac{\sqrt{3}}{2}\frac{\partial}{\partial y}\right). \end{aligned}$$

Taking this into account we obtain for the Laplace Beltrami operator of $SU(2,2)/S(U(2)\otimes U(2))$, C_2, expressed in polar coordinates

$$C_2\left[f(x,y,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)\right] =$$

$$\Big[C_r + \frac{4}{\sinh^2(2x)}T^{2\alpha}T^{2\alpha} + \frac{4}{\sinh^2(x+\sqrt{3}\,y)}T^{2\beta}T^{2\beta}+$$

$$\frac{3}{\sinh^2(\frac{3x+\sqrt{3}\,y}{2})}\left(T_1^{\alpha+\beta}T_1^{\alpha+\beta}+T_2^{\alpha+\beta}T_2^{\alpha+\beta}\right)+$$

$$\frac{3}{\sinh^2(\frac{3x-\sqrt{3}\,y}{2})}\left(T_1^{\alpha-\beta}T_1^{\alpha-\beta}+T_2^{\alpha-\beta}T_2^{\alpha-\beta}\right)\Big]\,f(x,y,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$$

where the operators $T^{2\alpha}, T^{2\beta}, T_1^{\alpha+\beta}, T_2^{\alpha+\beta}, T_1^{\alpha-\beta}$ and $T_2^{\alpha-\beta}$ act on the variables $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6, respectively. The radial part C_r is given by

$$C_r = \frac{5}{4}\frac{\partial^2}{\partial x^2} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x\partial y} + \frac{3}{4}\frac{\partial^2}{\partial y^2}+$$

$$2\coth(2x)\frac{\partial}{\partial x} + 2\coth(x+\sqrt{3}y)\left(\frac{\partial}{\partial x}+\sqrt{3}\frac{\partial}{\partial y}\right)+$$

$$\coth\left(\frac{3x+\sqrt{3}y}{2}\right)\left(3\frac{\partial}{\partial x}+\sqrt{3}\frac{\partial}{\partial y}\right)+\coth\left(\frac{3x-\sqrt{3}y}{2}\right)\left(3\frac{\partial}{\partial x}-\sqrt{3}\frac{\partial}{\partial y}\right).$$

The eigenvalue equation for the radial Casimir operator is thus a second order partial differential equation. This equation can immediately be interpreted either as the Schrödinger equation describing two particles interacting through a two dimensional potential, or as the Schrödinger equation for a particle in a two dimensional space. Symmetry scattering provides in both cases the scattering properties of such systems. However, since these potentials are not immediately related to known physical systems in the above parametrization, we may consider various coordinate transformations, or linear relations between the variables x and y, or equivalently, choose a fixed asymptotic direction for the scattering on the (x,y)–plane and thus obtain from the eigenvalue equation an ordinary second order differential equation that can be interpreted as the Schrödinger equation of a one dimensional system having one internal parameter which is scattered by a local potential.
Before going into the details of this application, we first recall the main features of the symmetry scattering method which will permit us to obtain the scattering properties for various applications.

Part B.

The symmetry scattering operator for a Riemannian symmetric space G/K is given by [6]

$$S_\chi(\lambda) = C^\chi_{Id}(\lambda)[C^\chi_{s^*}(\lambda)]^{-1},$$

where χ is a representation of K, s^* is a Weyl reflection sending $a \in \mathcal{A}^+$ into $-a$. The functions $C^\chi_{Id}(\lambda)$ and $C^\chi_{s^*}(\lambda)$ are generalized Harish–Chandra **c**–functions [7-8]. The argument of $C^\chi_{s^*}(\lambda)$ is related to the eigenvalues of the radial part of the Laplace Beltrami operator through [9]

$$C_r\phi_\lambda = \Lambda\phi_\lambda, \qquad \text{where} \qquad \Lambda = -(<\lambda,\lambda> + <\rho,\rho>)$$

$$\text{with,} \qquad \rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha.$$

The values of λ are not completely arbitrary, in fact they are determined by the series of representations of $SU(2,2)$. However, this constraint only applies to the imaginary part of λ which determines wether a scattering is possible. The vector $\Re e(\lambda)$ determines the asymptotic direction. It can take any arbitrary value on the plane A giving rise to different scattering problems. In the sequel with λ we will allways mean its real part.
As mentioned in the introduction, here we confine ourselves only to the zonal spherical functions, hence, χ corresponds to the trivial representation of K. The scattering operator becomes

$$S(\lambda) = c(\lambda)c(-\lambda)^{-1},$$

with [7,8,10]

$$c(\lambda) = c_0 \prod_{\alpha\in\Sigma_0^+} \frac{2^{-<i\lambda,\alpha_0>}\Gamma(<i\lambda,\alpha_0>)}{\Gamma(\frac{1}{2}(\frac{m_\alpha}{2}+1+<i\lambda,\alpha_0>))\,\Gamma(\frac{1}{2}(\frac{m_\alpha}{2}+m_{2\alpha}+<i\lambda,\alpha_0>))}$$

here c_0 is a constant, $\sum_0^+$ is the set of indivisible positive roots, $<,>$ is the scalar product induced by the Cartan–Killing form, and $\alpha_0 = \frac{\alpha}{<\alpha,\alpha>}$. For $SU(2,2)/S(U(2)\otimes U(2))$, $\sum_0^+$ has four elements α, β, $\alpha+\beta$ and $\alpha-\beta$ having multiplicities $m_\alpha = m_\beta = 0$, $m_{2\alpha} = m_{2\beta} = 1$, $m_{\alpha+\beta} = 2$ and $m_{\alpha-\beta} = 2$. Using $\lambda = \lambda_\alpha\alpha + \lambda_\beta\beta$ with λ_α, $\lambda_\beta \in$ C and $<\alpha,\alpha>= 1$ it follows

$$\frac{<\lambda,\alpha>}{<\alpha,\alpha>} = \lambda_\alpha + \frac{\lambda_\beta}{2}, \qquad \frac{<\lambda,\beta>}{<\beta,\beta>} = \lambda_\beta + \frac{\lambda_\alpha}{2},$$

$$\frac{<\lambda,\alpha+\beta>}{<\alpha+\beta,\alpha+\beta>} = \frac{\lambda_\alpha+\lambda_\beta}{2} \quad \text{and} \quad \frac{<\lambda,\alpha-\beta>}{<\alpha-\beta,\alpha-\beta>} = \frac{\lambda_\alpha-\lambda_\beta}{2}.$$

Further, with

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\alpha = 3\alpha+\beta$$

the eigenvalues of C_r become,

$$\Lambda = -(<\lambda,\lambda> + <\rho,\rho>) = -(\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\alpha\lambda_\beta + 13).$$

If we take these results into account and use the duplication formula for the Γ function,

$$2\sqrt{\pi}2^{-2z}\Gamma(2z) = \Gamma(z)\Gamma(z+\frac{1}{2}),$$

we obtain for Harish–Chandra's c function, the explicit form

$$c(\lambda) = \bar{c}_0 \frac{\Gamma\left(\frac{2\lambda_\alpha+\lambda_\beta}{4} i\right)}{\Gamma\left(\frac{1}{2} + \frac{2\lambda_\alpha+\lambda_\beta}{4} i\right)} \frac{\Gamma\left(\frac{2\lambda_\beta+\lambda_\alpha}{4} i\right)}{\Gamma\left(\frac{1}{2} + \frac{2\lambda_\beta+\lambda_\alpha}{4} i\right)} \frac{1}{(\lambda_\alpha^2 - \lambda_\beta^2)}$$

where $\bar{c}_0$ is some constant.

As shown in [6], the c–function corresponds to the Jost function of a scattering problem described by the Hamiltonian represented by the radial part of the Laplace Beltrami operator in which the numbers $\lambda_\alpha, \lambda_\beta$ are related to the momentum or to internal parameters of the system. In the next section we discuss in more details this identification and specific applications.

III. Applications.

In this section we apply the $SU(2,2)$ symmetry scattering to describe the scattering of a one dimensional system with an internal degree of freedom. First we recall that the systems that can be treated are described by the eigenfunction of the radial Casimir C_r. Secondly, the symmetry scattering operator gives the asymptotic behaviour of these functions when translated along a given geodesics, here determined by a ray in the two dimensional flat space A. The general idea of symmetry scattering is to interpret the scattering process geometrically due to the symmetry of the system, i.e., to the non–commutativity of certain of the generators of the underlying Lie algebra. Since these commutators are related to the sectional curvatures of the space, we can attribute symmetry scattering to the curvatures of the subspaces transvers to the geodesics

which we choose to follow for the asymptotic translation of the system. Because each asymptotic geodesics will determine a different sequence of transvers manifolds, each having its own curvature, we see that with each geodesics, a different one dimensional scattering problem can be associated. Now as mentioned above the geodesics can be related to rays in the flat space A [1], hence, by fixing a unit vector in $\mathcal{A}$ we have chosen the interaction that the one dimensional system will experience.

As a first application of this approach we consider asymptotic directions parallel to α. Along such an asymptotic direction there is no change in the β–component, hence, $A_\beta = 0$ which also implies in the realization considered here, that $\frac{x+\sqrt{3}y}{2} := y_0$ is constant. The eigenvalue equation for the radial Casimir operator becomes:

$$\left[\frac{\partial^2}{\partial x^2} + 2[(\coth(2x) + \coth(x+y_0) + \coth(x-y_0)]\frac{\partial}{\partial x}\right]\Psi_\lambda = \Lambda\Psi_\lambda$$

where $\lambda = \lambda_\alpha\alpha + \lambda_\beta\beta$ yields $\Lambda = -(\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\alpha\lambda_\beta + 13)$.

After a similarity transformation, this eigenvalue equation results

$$\left[-\frac{\partial^2}{\partial x^2} + \frac{4\cosh(2x)}{\sinh(x+y_0)\sinh(x-y_0)} - \frac{1}{\sinh^2(2x)} - 8\right]\Phi_\lambda = -(\Lambda+9)\Phi_\lambda .$$

where $\Psi_\lambda = F\,\Phi_\lambda$. The function $F = e^{-\frac{1}{2}\int Q(x)dx}$, with $Q(x) = 2(\coth(2x) + \coth(x+y_0) + \coth(x-y_0))$ becomes

$$F(x) = [\sinh(2x) + 2\sinh(x)\cosh(y_0)]^{-\frac{1}{2}}.$$

The above equation corresponds to the one dimensional Schrödinger equation with potential

$$V(x) = \frac{4\cosh(2x)}{\sinh(x+y_0)\sinh(x-y_0)} - \frac{1}{\sinh^2(2x)} - 8$$

and momentum

$$k = \sqrt{\lambda_\alpha^2 + \lambda_\beta^2 + \lambda_\alpha\lambda_\beta + 4}.$$

The term -8 has been included to the potential to adjust its value at infinity equal to zero.

Since the asymptotic direction is parallel to α, the real part of λ_α is related in a natural manner to the momentum while λ_β acts as an internal parameter of the system. Note that since the real part of $\lambda = \lambda_\alpha\alpha + \lambda_\beta\beta$ is arbitrary, λ_β and y_0 are independent quantities.

Using the general expression for the Jost function of the the SU(2,2)–symmetry scattering,

$$c(\lambda_\alpha, y_0) = \bar{c}_0 \frac{\Gamma\left(\frac{2\lambda_\alpha+\lambda_\beta}{4}i\right)}{\Gamma\left(\frac{1}{2}+\frac{2\lambda_\alpha+\lambda_\beta}{4}i\right)} \frac{\Gamma\left(\frac{2\lambda_\beta+\lambda_\alpha}{4}i\right)}{\Gamma\left(\frac{1}{2}+\frac{2\lambda_\beta+\lambda_\alpha}{4}i\right)} \frac{1}{(\lambda_\alpha^2-\lambda_\beta^2)},$$

we obtain the phase shift:

$$\delta(\lambda_\alpha, \lambda_\beta) = \arg\left\{ \frac{\Gamma\left(\frac{2\lambda_\alpha+\lambda_\beta}{4}i\right)}{\Gamma\left(\frac{1}{2}+\frac{2\lambda_\alpha+\lambda_\beta}{4}i\right)} \frac{\Gamma\left(\frac{2\lambda_\beta+\lambda_\alpha}{4}i\right)}{\Gamma\left(\frac{1}{2}+\frac{2\lambda_\beta+\lambda_\alpha}{4}i\right)} \frac{1}{(\lambda_\alpha^2-\lambda_\beta^2)} \right\},$$

with

$$\lambda_\alpha = \frac{-\lambda_\beta + \sqrt{4E - 3\lambda_\beta^2 - 16}}{2}$$

where $E = k^2$ is the energy.

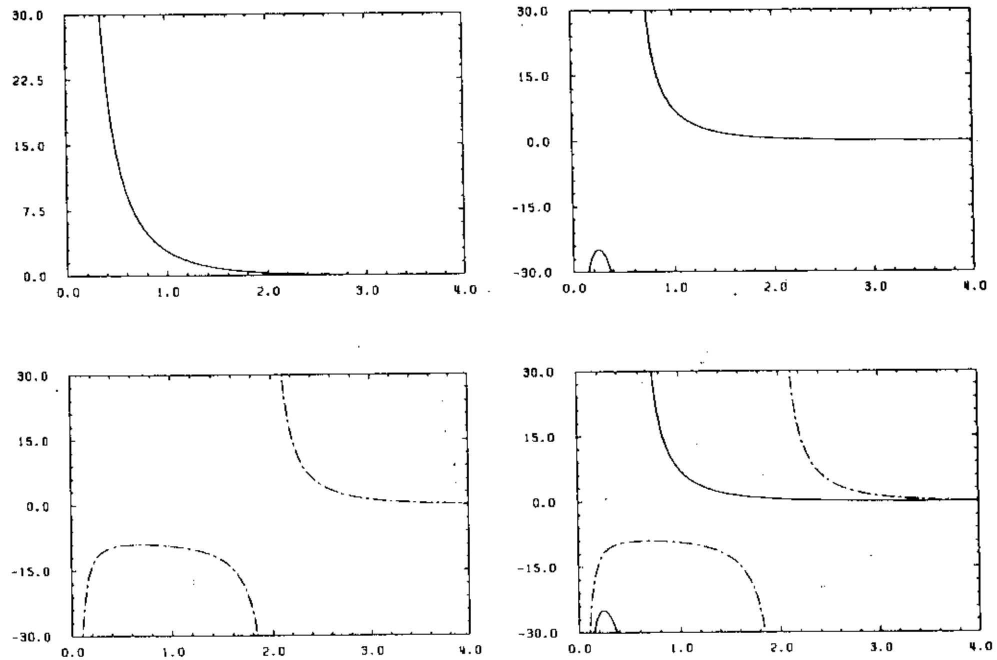

Figure 2.

The scattering potential for the $SU(2,2)/S(U(2)\otimes U(2))$ symmetry in the cartesian coordinates realization for fixed value of the variable y_0. Figure 2a corresponds to the value $y_0 = 0$. Figure 2b corresponds to the value $y_0 = 0.4$. Figure 2c the point–dotted line corresponds to the value $y_0 = 1.0$. Figure 2d is a synthesis of figures 2b and 2c. The axes are in arbitrary units.

In figure 2, we have plotted for several values of the parameter y_0 the resulting scattering potential. We find, that the parameter y_0 can be associated with the "size" of the system. At the value of $x = y_0$ the potentials have an infinite repulsion. For values bigger than y_0 the potentials drop exponentially while for values smaller than y_0 the potential is negative having two infinite traps, one at zero, the other immediately left from y_0. This potential may be used for example, to describe a nucleus consisting of two equally charged particles. The phase shift above would then describe the scattering of spinless particles by mesons or by alpha particles. Using other symmetries more traps in the inner part of the system would be available opening the application to more complicated systems.

As a second application of the $SU(2,2)$–symmetry scattering to treat one dimensional scattering problems with internal structure, we consider next asymptotic directions determined by a fixed angle. Using polar coordinates on the plane $e^{\mathcal{A}} \subset G/K$, we note that along such a trajectory, the polar angle θ is constant, $\theta := \theta_0$, it follows,

$$\frac{\partial}{\partial x} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} = \cos(\theta_0)\frac{\partial}{\partial r}$$
$$\frac{\partial}{\partial y} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta} = \sin(\theta_0)\frac{\partial}{\partial r}.$$

The eigenvalue equation for the radial Casimir becomes:

$$\Bigg(\left[\frac{5}{4}\cos^2(\theta_0) + \frac{\sqrt{3}}{2}\cos(\theta_0)\sin(\theta_0) + \frac{3}{4}\sin^2(\theta_0)\right]\frac{\partial^2}{\partial r^2} +$$
$$[2\cos(\theta_0)\coth(2r\cos(\theta_0))]\frac{\partial}{\partial r} +$$
$$[2(\cos(\theta_0) + \sqrt{3}\sin(\theta_0))\coth(r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0))]\frac{\partial}{\partial r} +$$
$$[\left(3\cos(\theta_0) + \sqrt{3}\sin(\theta_0)\right)\coth(\frac{3r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0)}{2})]\frac{\partial}{\partial r} +$$
$$[\left(3\cos(\theta_0) - \sqrt{3}\sin(\theta_0)\right)\coth(\frac{3r\cos(\theta_0) - \sqrt{3}\,r\sin(\theta_0)}{2})]\frac{\partial}{\partial r}\Bigg)\Psi_\lambda = \Lambda\Psi_\lambda$$

where, for the physical applications that follow, $\lambda = \lambda_\alpha\alpha + \lambda_\beta\beta$, $\lambda \in \mathbf{R}$ is written in polar coordinates:

$$\lambda_\alpha = \lambda_r\cos(\lambda_0) \qquad \text{and} \qquad \lambda_\beta = \lambda_r\frac{\sqrt{3}\sin(\lambda_0) - \cos(\lambda_0)}{2},$$

and the eigenvalue Λ becomes $\Lambda = -\frac{3\lambda_r^2+52}{4}$.
After a similarity transformation, we obtain the eigenvalue equation,

$$\left[-\frac{\partial^2}{\partial x^2} + M(\theta_0)\,V(r,\theta_0)\right]\Phi_\lambda = -(M(\theta_0)\Lambda)\Phi_\lambda$$

where

$$M(\theta_0) = \frac{4}{3 + 2\cos(\theta_0)(\cos(\theta_0) + \sqrt{3}\,\sin(\theta_0))}, \qquad \Psi_\lambda = F\,\Phi_\lambda \quad \text{and}$$

$$-M(\theta_0)\Lambda = M(\theta_0)(\lambda_\alpha^2 + \lambda_0^2 + \lambda_\alpha\lambda_0 + 13) = \frac{3\lambda_r^2 + 52}{3 + 2\cos(\theta_0)(\cos(\theta_0) + \sqrt{3}\,\sin(\theta_0))}.$$

$$T(\theta_0) = \begin{cases} T1 = 3 + 10\sqrt{3}\,\sin(\theta_0)\cos(\theta_0) + 22\cos^2(\theta_0) & \text{for } -30 < \theta_0 < 60 \\ T2 = 4 + 8\sqrt{3}\,\sin(\theta_0)\cos(\theta_0) + 8\sin^2(\theta_0) & \text{for } \quad 60 < \theta_0 < 90 \\ T3 = 12\sin^2(\theta_0) & \text{for } \quad 90 < \theta_0 < 120 \\ T4 = 3 - 6\sqrt{3}\,\sin(\theta_0)\cos(\theta_0) + 6\cos^2(\theta_0) & \text{for } \ 120 < \theta_0 < 150 \end{cases}$$

and with $n \in \mathbf{N}$

$$T(\theta_0) = \begin{cases} T1 & \text{for } -30 + 180n < \theta_0 < 60 \quad + 180n \\ T2 & \text{for } \quad 60 + 180n < \theta_0 < 90 \quad + 180n \\ T3 & \text{for } \quad 90 + 180n < \theta_0 < 120 + 180n \\ T4 & \text{for } \ 120 + 180n < \theta_0 < 150 + 180n. \end{cases}$$

Again the function F is defined by $F = e^{-\frac{1}{2}\int Q(x)dx}$, where

$$Q(x) = [2\cos(\theta_0)\coth(2r\cos(\theta_0))] +$$

$$[2(\cos(\theta_0) + \sqrt{3}\sin(\theta_0))\coth(r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0))] +$$

$$[(3\cos(\theta_0) + \sqrt{3}\sin(\theta_0))\coth(\frac{3r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0)}{2})] +$$

$$[(3\cos(\theta_0) - \sqrt{3}\sin(\theta_0))\coth(\frac{3r\cos(\theta_0) - \sqrt{3}\,r\sin(\theta_0)}{2})].$$

The above equation corresponds to the one dimensional Schrödinger equation of a system having a momentum

$$k = \sqrt{\frac{3\lambda_r^2 + 52 - 4T(\theta_0)}{3 + 2\cos(\theta_0)(\cos(\theta_0) + \sqrt{3}\,\sin(\theta_0))}}$$

and being scattered by the potential $M(\theta_0)\,V(r,\theta_0)$, where the function $V(r,\theta_0)$ is given by

$$V(r,\theta_0) =$$

$$2\cos(\theta_0)[\cos(\theta_0) + \sqrt{3}\sin(\theta_0)]$$

$$[\coth(2r\cos(\theta_0))\coth(r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0))] +$$

$$\cos(\theta_0)[3\cos(\theta_0) + \sqrt{3}\sin(\theta_0)]$$

$$\left[\coth(2r\cos(\theta_0))\coth(\frac{3r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0)}{2})\right] +$$

$$\cos(\theta_0)[3\cos(\theta_0) - \sqrt{3}\sin(\theta_0)]$$

$$\left[\coth(2r\cos(\theta_0))\coth(\frac{3r\cos(\theta_0) - \sqrt{3}\,r\sin(\theta_0)}{2})\right] +$$

$$[3 + 4\sqrt{3}\sin(\theta_0)\cos(\theta_0)]$$

$$\left[\coth(r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0))\coth(\frac{3r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0)}{2})\right] +$$

$$[3(\cos^2(\theta_0) - \sin^2(\theta_0)) + 2\sqrt{3}\sin(\theta_0)\cos(\theta_0)]$$

$$\left[\coth(r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0))\coth(\frac{3r\cos(\theta_0) - \sqrt{3}\,r\sin(\theta_0)}{2})\right] +$$

$$[\frac{9\cos^2(\theta_0) - 3\sin^2(\theta_0)}{2}]$$

$$\left[\coth(\frac{3r\cos(\theta_0) + \sqrt{3}\,r\sin(\theta_0)}{2})\coth(\frac{3r\cos(\theta_0) - \sqrt{3}\,r\sin(\theta_0)}{2})\right] +$$

$$\left[-\frac{\cos^2(\theta_0)}{\sinh^2(2r\cos(\theta_0))} - 2\sin^2(\theta_0) + 2\sqrt{3}\cos(\theta_0)\sin(\theta_0) + \frac{13}{2}\right] - T(\theta_0).$$

The term $-T(\theta_0)$ has been added to the potential to adjust its value at infinity equal to zero.

To have a better view of the potentials that appear in this realization we give the explicit expression for $\theta_0 = 0$ and for $\theta_0 = \pi/2$,

$$M(0)V(r,0) = \frac{4}{5}\left[\coth^2(2r) + \frac{2\cosh(2r)}{\sinh^2(2r)} + 6\frac{\cosh(3r)+1}{\sinh(2r)\sinh(r)}\right] + \frac{18}{5}\left[\frac{\cosh(3r)+1}{\sinh(3r)}\right]^2 - 14$$

and

$$M(\pi/2)V(r,\pi/2) = \left[\frac{36\cosh(\sqrt{3}\,r)+34}{3\sinh^2(\sqrt{3}\,r)}\right].$$

Since the asymptotic direction is radial, λ_r is related in a natural way to the momentum while λ_0 acts as an internal parameter of the system. Again note that since $\lambda = \lambda_\alpha \alpha + \lambda_\beta \beta$ is arbitrary, λ_0 and θ_0 are independent quantities.

Substituting

$$\lambda_\alpha = \lambda_r \cos(\lambda_0) \qquad \text{and} \qquad \lambda_\beta = \lambda_r \frac{\sqrt{3}\sin(\lambda_0) - \cos(\lambda_0)}{2}$$

into the general expression for the Jost function of the SU(2,2)–symmetry scattering, we obtain for the phase shift,

$$\delta(\lambda_r, \lambda_0) = \arg\left\{ \frac{1}{\left(\frac{3(\cos^2(\lambda_0)-\sin^2(\lambda_0))+2\sqrt{3}\sin(\lambda_0)\cos(\lambda_0)}{4}\, i\lambda_r^2\right)} \frac{\Gamma\left(\frac{\sqrt{24}\sin(\lambda_0)+3\cos(\lambda_0)}{8}\, i\lambda_r\right)}{\Gamma\left(\frac{1}{2} + \frac{\sqrt{24}\sin(\lambda_0)+3\cos(\lambda_0)}{8}\, i\lambda_r\right)} \frac{\Gamma\left(\frac{\sqrt{3}\sin(\lambda_0)}{4}\, i\lambda_r\right)}{\Gamma\left(\frac{1}{2} + \frac{\sqrt{3}\sin(\lambda_0)}{4}\, i\lambda_r\right)} \right\}.$$

In figure 3, we have plotted for several values of the parameter θ_0 the resulting scattering potential. We find in this realization only repulsive potentials.

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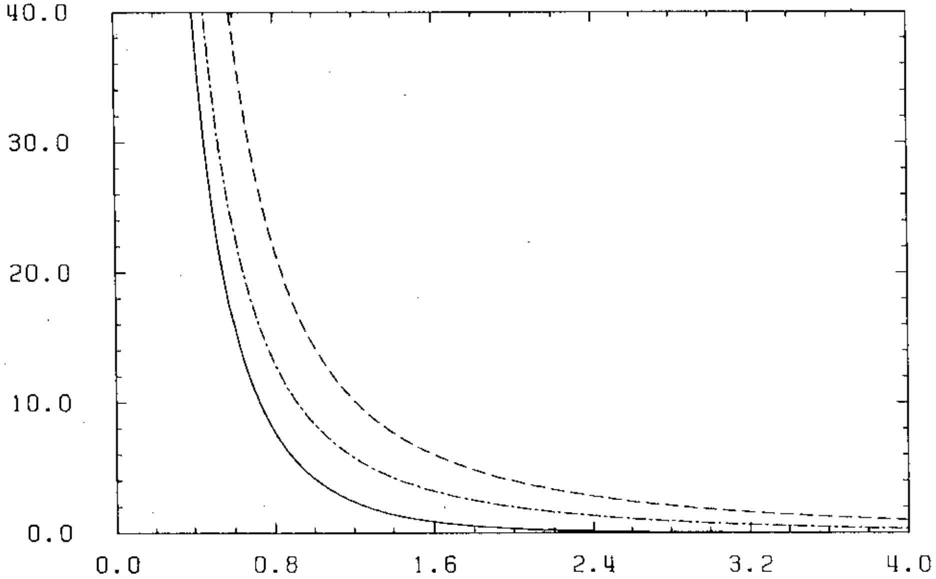

Figure 3.

The scattering potential for the $SU(2,2)/S(U(2) \otimes U(2))$ symmetry in the polar coordinates realization for fixed polar angle θ_0. The full line corresponds to the angle $\theta_0 = 0^o$, the dotted–dashed line corresponds to the angle $\theta_0 = 55^o$ and the dashed line to the angle $\theta_0 = 145^o$. The axes are in arbitrary units.

Appendix A

Illustration of the notions used in the paper

on the space $SO(3,1)/SO(3)$

The algebra $SO(3,1)$ is generated by the following six generators [2]

$$\begin{pmatrix} - & - & - & \vdots & + \\ 0 & L_{12} & L_{13} & \vdots & L_{15} \\ 0 & 0 & L_{23} & \vdots & L_{25} \\ 0 & 0 & 0 & \vdots & L_{35} \end{pmatrix},$$

or using the Pauli–matrices

$$\begin{pmatrix} - & - & - & \vdots & + \\ 0 & \sigma_3 & -\sigma_2 & \vdots & i\sigma_1 \\ 0 & 0 & \sigma_1 & \vdots & i\sigma_2 \\ 0 & 0 & 0 & \vdots & i\sigma_3 \end{pmatrix}.$$

The generators L_{ij} correspond to rotations in the i,j–plane hence, the non trivial commutators are given by

$$[L_{ij}, L_{ik}] = -ig_{ii}L_{jk}, \qquad L_{ij} = -L_{ji}$$

with the metric g having signature $(- - -+)$.

In the Cartan decomposition $\mathcal{G} = \mathcal{P} + \mathcal{K}$, $\mathcal{K}$ contains the generators

$$\begin{pmatrix} - & - & - & \vdots & + \\ 0 & L_{12} & L_{13} & \vdots & 0 \\ 0 & 0 & L_{23} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

and $\mathcal{P}$ contains the generators

$$\begin{pmatrix} - & - & - & \vdots & + \\ 0 & 0 & 0 & \vdots & L_{15} \\ 0 & 0 & 0 & \vdots & L_{25} \\ 0 & 0 & 0 & \vdots & L_{35} \end{pmatrix}.$$

A maximal commutative subalgebra $\mathcal{A} \subset \mathcal{P}$ is generated by L_{15}. We thus see that the space $SO(3,1)/SO(3)$ has rank one.

By inspecting the commutators between L_{15} and $\mathcal{K}$ we see that the centralizer $\mathcal{M} \subset \mathcal{K}$ of $\mathcal{A}$, contains only the generator L_{23}.

To determine the decomposition of the Laplace Beltrami operator on $SO(3,1)/SO(3)$ in a radial and a transvers part we require $\mathcal{J}$, the orthogonal complement of $\mathcal{M}$ in $\mathcal{K}$.

$\mathcal{J}$ is obtained by finding all elements L of $\mathcal{K}$ with $< \mathcal{M}, L >= 0$, with the scalar product determined from the Cartan–Killing form,

$$< \mathcal{M}, L > = \mathrm{trace}(\mathcal{M}L).$$

We have

$$\mathrm{trace}(L_{23}L_{12}) = \mathrm{trace}(\sigma_1\sigma_3) = 0$$

$$\mathrm{trace}(L_{23}L_{13}) = \mathrm{trace}(\sigma_1(-\sigma_2)) = 0.$$

Hence, $\mathcal{J}$ contains the generators L_{12} and L_{13}. Diagonalizing the operators $ad(H)^2$, $H \in \mathcal{A}$, i.e., L_{15}, it follows that

$$\mathcal{J} = \sum_{\alpha \in \Sigma+} \tau_\alpha, \quad \text{where} \quad \tau_\alpha := \{T \in \mathcal{J} \,|\, ad(L_{15})^2 T = \alpha(L_{15})^2 T\}.$$

Here we have only one root α with $\alpha(L_{15}) = i$ and multiplicity $m_\alpha = 2$. Further, since $\mathcal{J}$ is two dimensional, τ_α is generated by the following two elements:

$$T_1 = (L_{12} + L_{13}) = (\sigma_3 - \sigma_2) \qquad \text{and} \qquad T_2 = (L_{12} - L_{13}) = (\sigma_3 + \sigma_2),$$

as can be seen from

$$ad(L_{15})^2 T_i = [L_{15}[L_{15}, T_i]] = -T_i = \alpha(L_{15})^2 T_i \qquad (i = 1, 2).$$

Accordingly the Laplace Beltrami operator C_2 on $SO(3,1)/SO(3)$ can be expressed using only the generators L_{15}, $(L_{12}+L_{13})$ and $(L_{12}-L_{13})$. We explicitly have

$$[C_2 f](ke^a \cdot \ell) = \left[L_{15}^2 + 2\coth(\alpha)A_\alpha\right]_{e^a} f(ke^a \cdot \ell)$$

$$+ \sinh^{-2}(\alpha(a)) \left(Ad(e^{-a})(T_1^2 + T_2^2)\tilde{f}\right)(ke^a),$$

where $k \in SO(3)$, $e^a \in e^{rL_{15}}$, $r \in \mathbf{R}$ and ℓ is the identity coset in $SO(3,1)/SO(3)$. $A_\alpha \in \mathcal{A}$, is defined by $< A_\alpha, L_{15} >= i$.

To obtain the Laplace–Beltrami operator as a differential operator we take the following realization

$$\alpha = x \qquad\qquad L_{15} = i\frac{\partial}{\partial x}$$

For the scalar product determined from the Cartan–Killing form we have $< \alpha, \alpha >= \alpha(L_{15}) = i$. Since $< A_\alpha, L_{15} >= \alpha(L_{15}) = i$, it follows, $A_\alpha = \frac{\partial}{\partial x}$ and the Laplace–Beltrami takes the form:

$$C_2\left[f(x,y,\theta_1,\theta_2)\right] =$$

$$\left[\frac{\partial^2}{\partial x^2} + 2\coth x\frac{\partial}{\partial x} + \frac{1}{\sinh^2 x}\left(T_1^2 + T_2^2\right)\right] f(x,y,\theta_1,\theta_2).$$

Appendix B

Illustration of the notions used in the paper

on the space $SO(4,2)/(SO(4)\otimes SO(2))$

The algebra $SO(4,2)$ is generated by the following 15 generators [2]

$$\begin{pmatrix}
- & - & - & - & \vdots & + & + \\
0 & L_{12} & L_{13} & L_{14} & \vdots & L_{15} & L_{16} \\
0 & 0 & L_{23} & L_{24} & \vdots & L_{25} & L_{26} \\
0 & 0 & 0 & L_{34} & \vdots & L_{35} & L_{36} \\
0 & 0 & 0 & 0 & \vdots & L_{45} & L_{46} \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & L_{56}
\end{pmatrix}.$$

The generators L_{ij} correspond to rotations in the ij-plane hence, the non trivial commutators are given by

$$[L_{ij}, L_{ik}] = -ig_{ii}L_{jk}, \qquad L_{ij} = -L_{ji}$$

with the metric g having signature $(- - - - ++)$.

In the Cartan decomposition $\mathcal{G} = \mathcal{P} + \mathcal{K}$, $\mathcal{K}$ contains the generators

$$\begin{pmatrix} - & - & - & - & \vdots & + & + \\ 0 & L_{12} & L_{13} & L_{14} & \vdots & 0 & 0 \\ 0 & 0 & L_{23} & L_{24} & \vdots & 0 & 0 \\ 0 & 0 & 0 & L_{34} & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{56} \end{pmatrix}$$

and $\mathcal{P}$ contains the generators

$$\begin{pmatrix} - & - & - & - & \vdots & + & + \\ 0 & 0 & 0 & 0 & \vdots & L_{15} & L_{16} \\ 0 & 0 & 0 & 0 & \vdots & L_{25} & L_{26} \\ 0 & 0 & 0 & 0 & \vdots & L_{35} & L_{36} \\ 0 & 0 & 0 & 0 & \vdots & L_{45} & L_{46} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A maximal commutative subalgebra $\mathcal{A} \subset \mathcal{P}$ is generated by L_{35} and L_{46}. We thus see that the space $SO(4,2)/(SO(4) \otimes SO(2))$ has rank two. By inspecting the commutators between $\mathcal{A}$, i.e., L_{35} and L_{46}, and $\mathcal{K}$ we see that the centralizer $\mathcal{M} \subset \mathcal{K}$ of $\mathcal{A}$, contains only the generator L_{12}. In order to determine the decomposition of the Laplace Beltrami operator on $SO(4,2)/(SO(4) \otimes SO(2))$ in a radial and a transvers part we

require $\mathcal{J}$, the orthogonal complement of $\mathcal{M}$ in $\mathcal{K}$. $\mathcal{J}$ is obtained by finding all elements L of $\mathcal{K}$ with $< \mathcal{M}, L > = 0$, with the scalar product determined from the Cartan–Killing form,

$$< \mathcal{M}, L > = \text{trace}(\mathcal{M}L).$$

Using the representation

$$L_{ij} = E_{ij} - E_{ji}$$

where E_{ij} is the 6×6 matrix having zeros everywhere except one at the ij position, we obtain

$$\text{trace}(L_{12}L_{13}) = 0 \qquad \text{trace}(L_{12}L_{14}) = 0$$
$$\text{trace}(L_{12}L_{23}) = 0 \qquad \text{trace}(L_{12}L_{24}) = 0$$
$$\text{trace}(L_{12}L_{34}) = 0 \qquad \text{trace}(L_{12}L_{56}) = 0.$$

Hence, $\mathcal{J}$ contains the generators L_{13}, L_{14}, L_{23}, L_{24}, L_{34}, and L_{56}. Diagonalizing the operators $ad(H)^2$, $H \in \mathcal{A}$, i.e., L_{35} and L_{46}, it follows,

$$\mathcal{J} = \sum_{\alpha \in \Sigma^+} \tau_\alpha$$

$$\text{where} \quad \tau_\alpha := \{T \in \mathcal{J} \,|\, ad(H)^2 T = \alpha(H)^2 T, \quad H = L_{35}, L_{46}\}.$$

The following table indicating the eigenvalues of the operators $Ad(iL_{35})^2$ and $Ad(iL_{46})^2$ allows us to determine the above decomposition of $\mathcal{J}$ into eigenspaces.

Table of eigenvalues		
$\mathcal{J}$	$Ad(iL_{35})^2$	$Ad(iL_{46})^2$
L_{13}	1	0
L_{14}	0	1
L_{23}	1	0
L_{24}	0	1
L_{34}	1	1
L_{56}	1	1

Table

The eigenvalues of the operators $Ad(iL_{35})^2$ and $Ad(iL_{46})^2$

For the space $SO(4,2)/(SO(4)\otimes SO(2))$, Σ^+ consists of four roots α, β and $\alpha+\beta$ all having multiplicity 2, and $\alpha-\beta$ with multiplicity zero. Figure 4 shows the root space with α orthogonal to β.

By choosing

$$\alpha(L_{35}) = 1 \quad \text{and} \quad \alpha(L_{46}) = 0$$

$$\beta(L_{35}) = 0 \quad \text{and} \quad \beta(L_{46}) = 1$$

we have the following bases for the three two dimensional eigenspaces:

$$\tau_\alpha = \{sL_{13} + tL_{23} \quad s,t \in \mathbf{R}\},$$

$$\tau_\beta = \{sL_{14} + tL_{24} \quad s,t \in \mathbf{R}\},$$

$$\tau_{\alpha+\beta} = \{sL_{34} + tL_{56} \quad s,t \in \mathbf{R}\}.$$

Accordingly the Laplace Beltrami operator C_2 on $SO(4,2)/(SO(4)\otimes SO(2))$ can be expressed using only the generators L_{35}, L_{46}, L_{13}, L_{23},

L_{14}, L_{24}, L_{34} and L_{56}. We explicitly have

$$[C_2 f](ke^a\cdot\ell) =$$

$$\left[L_{35}^2 + L_{46}^2 + 2\coth(\alpha)A_\alpha + 2\coth(\beta)A_\beta + 2\coth(\alpha+\beta)A_{\alpha+\beta}\right]_{e^a} f(ke^a\cdot\ell) +$$
$$\sinh^{-2}(\alpha(a))\left(Ad(e^{-a})(L_{13}^2 + L_{23}^2)\tilde{f}\right)(ke^a) +$$
$$\sinh^{-2}(\beta(a))\left(Ad(e^{-a})(L_{14}^2 + L_{24}^2)\tilde{f}\right)(ke^a) +$$
$$\sinh^{-2}((\alpha+\beta)(a))\left(Ad(e^{-a})(L_{34}^2 + L_{56}^2)\tilde{f}\right)(ke^a)$$

where $k \in SO(4)\otimes SO(2)$, $e^a \in e^{rL_{35}+sL_{46}}$, $r,s \in \mathbf{R}$ and ℓ is the identity coset in $SO(4,2)/(SO(4)\otimes SO(2))$. $A_\alpha \in \mathcal{A}$ is defined by $< A_\alpha, rL_{35} + sL_{46} >= \alpha(rL_{35} + sL_{46})$.

To obtain the Laplace–Beltrami operator as a differential operator we take the following realization in the two dimensional Euclidean plane $e^{\mathcal{A}} \subset G/K$:

$$\alpha = x \qquad L_{35} = \frac{\partial}{\partial x}$$

$$\beta = y \qquad L_{46} = \frac{\partial}{\partial y}.$$

For the scalar product determined from the Cartan–Killing form we have

$$< \alpha, \alpha > = \alpha(L_{35}) = 1,$$

$$< \beta, \beta > = \beta(L_{46}) = 1,$$

$$< \alpha, \beta > = \beta(L_{35}) = 0,$$

$$< \beta, \alpha > = \alpha(L_{46}) = 0,$$

as required. Further we have

$$A_\alpha = L_{35} = \frac{\partial}{\partial x} \qquad \text{and} \qquad A_\beta = L_{46} = \frac{\partial}{\partial y}.$$

The Laplace Beltrami operator of $SO(4,2)/(SO(4)\otimes SO(2))$ becomes,

$$
\begin{aligned}
&C_2\left[f(x,y,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)\right] = \\
&\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\right. \\
&2\coth(x)\frac{\partial}{\partial x}+2\coth(y)\left(\frac{\partial}{\partial y}\right)+2\coth(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+ \\
&\left.\frac{1}{\sinh^2(x)}\left(T_1^2+T_2^2\right)+\frac{1}{\sinh^2(y)}\left(T_3^2+T_4^2\right)+\frac{2}{\sinh^2(x+y)}\left(T_5^2+T_6^2\right)\right] \\
&f(x,y,\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)
\end{aligned}
$$

where T_1, T_2, T_3, T_4, T_5 and T_6 act on the variables $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6, respectively.

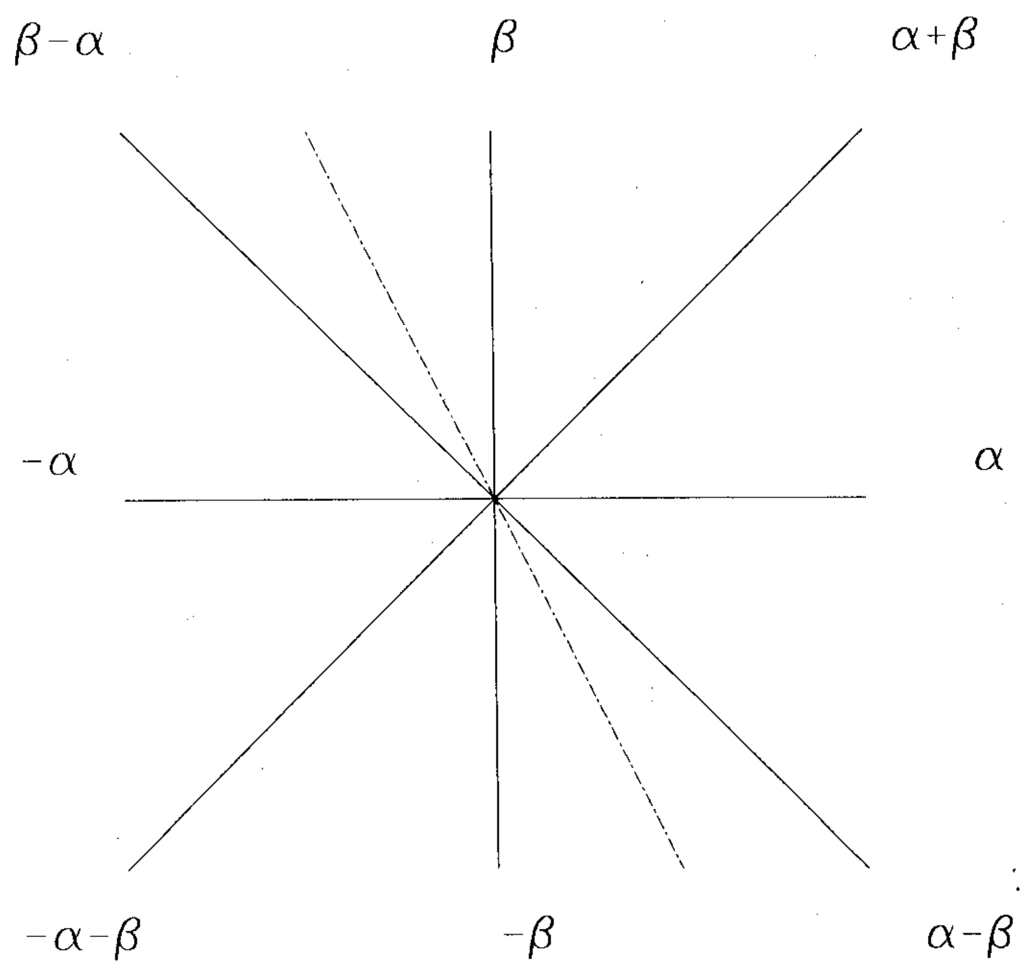

Figure 4.

The root space for $SO(4,2)/SO(4)\otimes SO(2)$. The root space is spanned by the two positive roots α and β. All restricted roots are indicated. The dotted-dashed line separates the positive from the negative roots.

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