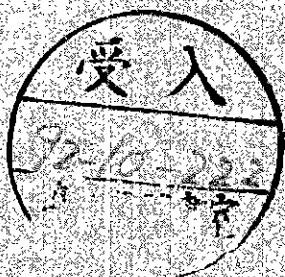




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**An Algebraic Characterization of  
Vacuum States in Minkowski Space**

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## An Algebraic Characterization of Vacuum States in Minkowski Space

DETLEV BUCHHOLZ AND STEPHEN J. SUMMERS

**Abstract.** An algebraic characterization of vacuum states on nets of  $C^*$ -algebras over Minkowski space is given and space-time translations are reconstructed with the help of the modular structures associated with such states. The result suggests that a "principle of geometrical modular action" might hold in quantum field theories on a wider class of spacetime manifolds.

### I. INTRODUCTION

In the algebraic setting of quantum field theory one commonly characterizes vacuum states on an algebra of observables  $\mathcal{A}$  by their invariance and spectral properties with respect to the group of space-time translations [1],[2]. In this note we pose the question whether it is possible to characterize these states using only the net structure of the observables, i.e. the assignment  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  of spacetime regions  $\mathcal{O}$  to local subalgebras  $\mathcal{A}(\mathcal{O})$  of  $\mathcal{A}$ . The existence of spacetime symmetries will not be assumed from the outset.

This question is motivated by the following considerations. First of all there is a matter of principle: it is believed that in algebraic quantum field theory the physical information of a model is encoded in the relative positions of the local algebras in a given net. One should therefore be able to characterize the vacuum and to determine the spacetime symmetries using only this net structure. Secondly, the question is of relevance to the theory of quantum fields on curved spacetimes, where one deals with physical systems for which, in general, one has little in the way of spacetime symmetries to simplify matters. Hence the characterization of physical states by means of the net structure is also of practical interest.

In the present approach we start from the property of isotony of any net of observables. This property imposes special relations between the modular structures associated to the local algebras and a given faithful state. Only these data, entirely determined by the net and the particular state chosen, will be used here. A remarkable fact is known in this connection: the modular objects associated to a vacuum state and algebras over Minkowski space corresponding to wedge-shaped regions  $\mathcal{W}$ , bounded by two characteristic planes, contain geometric information about the spacetime itself, as well as dynamical information about the representation [3],[4]. They act in a geometrical manner on the net and can be used to reconstruct the symmetries of Minkowski space, in other words, the Poincaré group.

We will show in this note that this specific geometric action is a distinctive feature of the modular structure associated with vacuum states. It thereby leads to a characterization of these states in terms of the net structure. Moreover, the space-time translations can be reconstructed from the modular objects and shown to be unique. These results suggest that a "principle of geometrical modular action" might be suitable for an algebraic characterization of states of physical interest in quantum field theories on more general spacetime manifolds. We comment on this idea at the end of this note.

### II. THE VACUUM IN MINKOWSKI SPACE

For the characterization of vacuum states in Minkowski space we proceed from a net  $\mathcal{A}$  of  $C^*$ -algebras  $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{R}}$  satisfying the condition of isotony, with  $\mathfrak{R}$  equal to the set of all (open) wedges  $\mathcal{W}$  and double cones  $\mathcal{K}$  in  $\mathbb{R}^4$ . It is useful (and standard) to assume that the algebras  $\mathcal{A}(\mathcal{O})$ ,  $\mathcal{O} \in \mathfrak{R}$ , are continuous from the inside in the sense that they are the  $C^*$ -inductive limits of all double cone algebras  $\mathcal{A}(\mathcal{K})$  for which the closures of  $\mathcal{K}$  are contained in the interior of  $\mathcal{O}$ .

We look upon  $\mathcal{A}$  as some abstract net on  $\mathbb{R}^4$  without any *a priori* dynamical information. In particular, we assume neither the existence of space-time translations nor that of causal properties (i.e. locality). These more detailed features will be attributed to the GNS representations of  $\mathcal{A}$  induced by suitable states  $\omega$ . It is our aim to characterize within this general setting those states which can be regarded as vacuum states of some relativistic quantum field theory in Minkowski space.

Given any state  $\omega$  on  $\mathcal{A}$ , we consider its GNS representation  $(\mathcal{H}, \pi, \Omega)$  and the corresponding net  $\mathcal{R}$  of von Neumann algebras  $\mathcal{R}(\mathcal{O}) = \bigcap_{\mathcal{W} \supset \mathcal{O}} \pi(\mathcal{A}(\mathcal{W}))''$ ,  $\mathcal{O} \in \mathfrak{R}$ . Note that  $\mathcal{R}(\mathcal{O}) \supseteq \pi(\mathcal{A}(\mathcal{O}))$  because of isotony. Anticipating the Reeh-Schlieder property of vacuum states, we shall be interested only in states  $\omega$  whose GNS-vector  $\Omega$  is cyclic and separating for the von Neumann algebras  $\mathcal{R}(\mathcal{W})$  corresponding to arbitrary wedges  $\mathcal{W}$ . With this input the modular operators  $\Delta_{\mathcal{W}}$  and modular conjugations  $J_{\mathcal{W}}$  associated with  $(\mathcal{R}(\mathcal{W}), \Omega)$  are well defined, and we shall state our conditions on  $\omega$  in terms of these modular objects.

We denote by  $\mathcal{W}^{(0)}$  any wedge whose edge passes through the origin of  $\mathbb{R}^4$ , and by  $x^{(0)}$ ,  $y^{(0)}$ , etc. any translation in the two-dimensional subspace  $\mathbb{R}_{\mathcal{W}^{(0)}}^2$  generated by the two lightlike directions fixing the boundaries of  $\mathcal{W}^{(0)}$ . To simplify notation we denote the modular conjugations associated with  $(\mathcal{R}(\mathcal{W}^{(0)} + z^{(0)}), \Omega)$  by  $J_{x^{(0)}}$ . We distinguish a special class of states  $\omega$  on  $\mathcal{A}$  by making the following assumption on the action induced by the corresponding modular conjugations on the net  $\mathcal{R}$ .

**ASSUMPTION 1:** For each wedge  $\mathcal{W}^{(0)}$  and every  $\mathcal{O} \in \mathfrak{R}$  one has the action

$$(2.1) \quad J_{z^{(0)}} \mathcal{R}(\mathcal{O}) J_{x^{(0)}} = \mathcal{R}(\Lambda_{\mathcal{W}^{(0)}} \mathcal{O} + 2z^{(0)}),$$

where  $\Lambda_{\mathcal{W}^{(0)}}$  is a reflection which is equal to  $-1$  on  $\mathbb{R}_{\mathcal{W}^{(0)}}^2$  and equal to  $1$  on the two dimensional subspace of  $\mathbb{R}^4$  which forms the edge of  $\mathcal{W}^{(0)}$ .

It was shown in [3] that Assumption (1) holds in all irreducible vacuum representations of any net of local algebras that is locally associated with a quantum field, when one takes the modular objects associated with the vacuum state (see also [5]). Note that Assumption (1) entails the locality of the net  $\mathcal{R}$ . In fact, by the Tomita-Takesaki theory one has  $J_{\mathcal{W}} \mathcal{R}(\mathcal{W}) J_{\mathcal{W}} = \mathcal{R}(\mathcal{W})'$ , for any wedge  $\mathcal{W}$ , and (2.1) implies that  $J_{\mathcal{W}} \mathcal{R}(\mathcal{W}) J_{\mathcal{W}} = \mathcal{R}(\mathcal{W}')$ , where  $\mathcal{W}'$  denotes the spacelike complement of  $\mathcal{W}$ . Hence the net  $\mathcal{R}$  satisfies duality (thus, locality) on the wedges and it is standard to derive from this the locality of the net  $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{R}}$ . In this sense any state  $\omega$  on  $\mathcal{A}$  satisfying Assumption (1) induces on  $\mathbb{R}^4$ , through the action of the associated modular

conjugations, the causal structure of Minkowski space. By Assumption (1) we also have the following relation for each  $\mathcal{W}^{(0)}$  and any  $x^{(0)}, y^{(0)}$  as above

$$(2.2) \quad J_{x^{(0)}} J_{y^{(0)}} \mathcal{R}(\mathcal{O}) J_{y^{(0)}} J_{x^{(0)}} = \mathcal{R}(\mathcal{O} + 2x^{(0)} - 2y^{(0)}).$$

Since the modular conjugations are antiunitary involutions, the products  $J_{x^{(0)}} J_{y^{(0)}}$  are unitary operators. Relation (2.2) shows that these products induce translations of the local algebras, possibly accompanied by internal symmetry transformations. The latter possibility is excluded by our second assumption.

ASSUMPTION 2: For each fixed  $\mathcal{W}^{(0)}$  and any  $x^{(0)}, y^{(0)}$  as above, the unitaries  $J_{x^{(0)}} J_{y^{(0)}}$  depend only on the difference  $x^{(0)} - y^{(0)}$ .

Once again, this assumption follows from what has been proven by Bisognano and Wichmann for nets in vacuum representations locally associated with Wightman fields [3], cf. also [4]. With this assumption it is meaningful to set

$$(2.3) \quad V[2(x^{(0)} - y^{(0)})] \equiv J_{x^{(0)}} J_{y^{(0)}}.$$

Note that since the modular conjugations leave  $\Omega$  invariant, these unitary operators do so, as well. We shall construct a representation of the translation group out of these operators.

LEMMA 2.1. Under Assumption (2) the map  $x^{(0)} \rightarrow V(x^{(0)})$  gives a strongly continuous unitary representation of the additive group  $\mathbb{R}_{\mathcal{W}^{(0)}}^2$ .

PROOF: From Assumption (2) it follows that

$$\begin{aligned} V(x^{(0)}) V(y^{(0)}) &= J_{\frac{1}{2}x^{(0)}} J_0 \cdot J_0 J_{-\frac{1}{2}y^{(0)}} = J_{\frac{1}{2}x^{(0)}} J_{-\frac{1}{2}y^{(0)}} = V(x^{(0)} + y^{(0)}), \\ V(x^{(0)})^* &= J_0 J_{\frac{1}{2}x^{(0)}} = V(-x^{(0)}) = V(x^{(0)})^{-1}. \end{aligned}$$

Thus  $x^{(0)} \rightarrow V(x^{(0)})$  is a unitary representation of the group  $\mathbb{R}_{\mathcal{W}^{(0)}}^2$ . If  $x^{(0)} \in \mathcal{W}^{(0)}$  tends to 0, we have  $J_{x^{(0)}} \rightarrow J_0$  in the strong operator topology (we may appeal to the technical lemmas in the Appendix, since  $\mathcal{W}^{(0)} + x^{(0)} \subseteq \mathcal{W}^{(0)}$  and consequently  $V(x^{(0)}) \rightarrow 1$ . Due to the identity  $\|(V(-x^{(0)}) - 1)\Phi\| = \|(1 - V(x^{(0)}))\Phi\|$ , it also follows that  $V(-x^{(0)}) \rightarrow 1$ . This establishes the continuity of the map  $\lambda \rightarrow V(\lambda \cdot e^{(0)})$ ,  $\lambda \in \mathbb{R}$ , for any  $e^{(0)} \in \mathcal{W}^{(0)}$ . Using the previously verified group structure, it follows that  $x^{(0)} \rightarrow V(x^{(0)})$  is continuous in the strong operator topology.  $\square$

For fixed  $\mathcal{W}^{(0)}$  we shall call the collection of wedges  $\{\mathcal{W}^{(0)} + z^{(0)} \mid z^{(0)} \in \mathbb{R}_{\mathcal{W}^{(0)}}^2\}$  a *coherent system of wedges*. As we have shown in Lemma 2.1, associated to any coherent system of wedges is a strongly continuous unitary representation of  $\mathbb{R}_{\mathcal{W}^{(0)}}^2$ . Introducing proper coordinates  $x = (x_0, x_1, x_2, x_3)$  on  $\mathbb{R}^4$  we pick the wedges  $\mathcal{W}_i^{(0)} = \{x \in \mathbb{R}^4 \mid x_i > |x_0|\}$ ,  $i = 1, 2, 3$ , and consider the corresponding representations  $V_i(\mathbb{R}_{\mathcal{W}_i^{(0)}}^2)$ ,  $i = 1, 2, 3$ .

PROPOSITION 2.2. Under Assumptions (1) and (2) the unitary operators  $\{V_1(x) \mid x = (x_0, x_1, 0, 0)\}$ ,  $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$ ,  $\{V_3(x) \mid x = (0, 0, 0, x_3)\}$ , with  $x_0, x_1, x_2, x_3 \in \mathbb{R}$  arbitrary, determine a strongly continuous, unitary representation  $V(\mathbb{R}^4)$  of the translations on  $\mathbb{R}^4$  that leaves  $\Omega$  invariant and acts geometrically correctly on the net, i.e.

$$V(x) \mathcal{R}(\mathcal{O}) V(x)^{-1} = \mathcal{R}(\mathcal{O} + x), \quad \forall \mathcal{O} \in \mathfrak{R}, x \in \mathbb{R}^4.$$

PROOF: By (2.2) and the preceding lemma, the proposition will follow as soon as one sees that the three sets of unitary operators above mutually commute. To illustrate the simple idea of the proof, consider the operators  $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$ . One has for the coherent system  $\{\mathcal{W}_1^{(0)} + z^{(0)} \mid z^{(0)} \in \mathbb{R}_{\mathcal{W}_1^{(0)}}^2\}$  the elementary fact that  $\mathcal{W}_1^{(0)} + z^{(0)} + x = \mathcal{W}_1^{(0)} + z^{(0)}$  for any  $x = (0, 0, x_2, 0)$ . Thus, by (2.2), one has  $V_2(x) \mathcal{R}(\mathcal{W}_1^{(0)} + z^{(0)}) V_2(x)^{-1} = \mathcal{R}(\mathcal{W}_1^{(0)} + z^{(0)})$  for each such translation  $x$ . Since, in addition,  $V_2(x) \Omega = \Omega$  for all such  $x$ ,  $\{V_2(x) \mid x = (0, 0, x_2, 0)\}$  is a collection of unitary operators leaving the pairs  $(\mathcal{R}(\mathcal{W}_1^{(0)} + z^{(0)}), \Omega)$  invariant. Hence  $V_2(x)$  must commute with the modular objects associated with these pairs (see [6, Thm.3.2.18]) and consequently with  $V_1(y)$ . The other cases are argued similarly.  $\square$

This result shows that in the GNS representation space of any state satisfying Assumptions (1) and (2) one has a representation of the translations. There is, of course, nothing unique about this construction, since many different choices of coherent systems of wedges can be made. We shall return to the question of uniqueness below.

The representation  $V(\mathbb{R}^4)$  fulfills every desideratum of the translations in a vacuum representation except possibly one - the spectrum condition. And, in fact, there are simple examples of nets and states satisfying Assumptions (1) and (2) which generate representations of the translations that violate the spectrum condition [7]. In order for the representation  $V(\mathbb{R}^4)$  to satisfy this condition, it is necessary and sufficient that the modular groups  $\Delta_{\mathcal{W}}^{i\lambda}$  act upon the translations  $V(\mathbb{R}_{\mathcal{W}}^2)$  in the geometrical manner of Lorentz boosts. More precisely, the following assumption has to be satisfied.

ASSUMPTION 3: For every wedge  $\mathcal{W}^{(0)}$  and every positive lightlike vector  $e^{(0)}$  such that  $\mathcal{W}^{(0)} + e^{(0)} \subset \mathcal{W}^{(0)}$  there holds

$$(2.4) \quad \Delta_{\mathcal{W}^{(0)}}^{i\lambda} V(e^{(0)}) \Delta_{\mathcal{W}^{(0)}}^{-i\lambda} = V(e^{-2\pi\lambda} e^{(0)}), \quad \forall \lambda \in \mathbb{R}.$$

(From (2.4) it follows that also  $\Delta_{\mathcal{W}^{(0)}}^{i\lambda} V(-e^{(0)}) \Delta_{\mathcal{W}^{(0)}}^{-i\lambda} = V(-e^{-2\pi\lambda} e^{(0)})$ .)

PROPOSITION 2.3. Let  $V(\mathbb{R}^4)$  be the representation of the translations on Minkowski space constructed in Proposition 2.2. Then  $V(\mathbb{R}^4)$  satisfies the relativistic spectrum condition, i.e.<sup>1</sup>  $sp(V) \subset \bar{V}_+$ , if and only if relation (2.4) holds for all wedges.<sup>2</sup>

REMARK: At the cost of a significantly longer argument [7], this result can be established requiring Equation (2.4) only for the wedges in the three coherent systems used to construct the representation  $V(\mathbb{R}^4)$ .

<sup>1</sup>  $V_+$  (resp.  $V_-$ ) denotes the forward (resp. backward) lightcone.

<sup>2</sup> An analogous result that yields the spectrum condition for a given representation of the translations is given under somewhat more restrictive conditions in [8].

PROOF: From the work of Bisognano and Wichmann [3] and Borchers [4] it follows that in a vacuum representation (i.e. a translation-covariant representation having an invariant vector and satisfying the spectrum condition) equation (2.4) holds. Hence we need only give a proof of the implication in the other direction.

Choose a wedge  $\mathcal{W}^{(0)}$  and a positive lightlike vector  $e^{(0)}$  such that  $\mathcal{W}^{(0)} + e^{(0)} \subset \mathcal{W}^{(0)}$ . For simplicity of notation, the subscripts  $\mathcal{W}^{(0)}$  on the modular objects  $\Delta, J$  associated to the pair  $(\mathcal{R}(\mathcal{W}^{(0)}), \Omega)$  will be suppressed in the following. First note that by assumption,  $V(e^{(0)})\mathcal{R}(\mathcal{W}^{(0)})V(e^{(0)})^{-1} \subset \mathcal{R}(\mathcal{W}^{(0)})$ , which entails  $V(e^{(0)})\mathcal{R}(\mathcal{W}^{(0)})'V(e^{(0)})^{-1} \supset \mathcal{R}(\mathcal{W}^{(0)})'$  and therefore  $\mathcal{R}(\mathcal{W}^{(0)})' \supset V(e^{(0)})^{-1}\mathcal{R}(\mathcal{W}^{(0)})'V(e^{(0)})$ . Then, with  $X(e^{(0)}) \equiv JV(e^{(0)})^{-1}J$ , one has  $V(e^{(0)})^{-1}J = JX(e^{(0)})$  and

$$(2.5) \quad X(e^{(0)})\mathcal{R}(\mathcal{W}^{(0)})X(e^{(0)})^{-1} = JV(e^{(0)})^{-1}\mathcal{R}(\mathcal{W}^{(0)})'V(e^{(0)})J \subset J\mathcal{R}(\mathcal{W}^{(0)})'J = \mathcal{R}(\mathcal{W}^{(0)}).$$

Let now  $A' \in \mathcal{R}(\mathcal{W}^{(0)})'$  and  $B \in \mathcal{R}(\mathcal{W}^{(0)})$ . By the Tomita-Takesaki theory, the function

$$z \rightarrow f(z) \equiv \langle \Delta^{-iz} A' \Omega, V(-e^{(0)}) \Delta^{-iz} B \Omega \rangle$$

is continuous and bounded on the strip  $0 \leq \text{Im}(z) \leq 1/2$  and analytic in its interior. Moreover, for  $\lambda \in \mathbb{R}$  the following bounds hold:

$$|f(\lambda)| \leq \|A' \Omega\| \cdot \|B \Omega\|$$

and

$$\begin{aligned} |f(\lambda + i/2)| &= |\langle J \Delta^{-i\lambda} A'^* \Omega, V(-e^{(0)}) J \Delta^{-i\lambda} B^* \Omega \rangle| \\ &= |\langle J \Delta^{-i\lambda} A'^* \Omega, J X(e^{(0)}) \Delta^{-i\lambda} B^* \Omega \rangle| \\ &= |\langle X(e^{(0)}) \Delta^{-i\lambda} B^* \Delta^{i\lambda} X(e^{(0)})^{-1} \Omega, \Delta^{-i\lambda} A'^* \Delta^{i\lambda} \Omega \rangle| \\ &= |\langle \Delta^{-i\lambda} A' \Omega, X(e^{(0)}) \Delta^{-i\lambda} B \Omega \rangle| \leq \|A' \Omega\| \cdot \|B \Omega\|, \end{aligned}$$

where the fourth equality follows from equation (2.5) and  $\Delta^{-i\lambda} \mathcal{R}(\mathcal{W}^{(0)}) \Delta^{i\lambda} = \mathcal{R}(\mathcal{W}^{(0)})$ . Hence by the Three-Line-Theorem one has

$$|f(z)| \leq \|A' \Omega\| \cdot \|B \Omega\|, \quad 0 \leq \text{Im}(z) \leq 1/2.$$

Now pick any two vectors  $\Phi, \Psi$  in  $\mathcal{H}$  and let  $A'_n \in \mathcal{R}(\mathcal{W}^{(0)})'$  and  $B_n \in \mathcal{R}(\mathcal{W}^{(0)})$  be such that  $A'_n \Omega \rightarrow \Phi$  and  $B_n \Omega \rightarrow \Psi$  strongly. Then it follows from the preceding estimate that the sequence of analytic functions

$$z \rightarrow f_n(z) \equiv \langle \Delta^{-iz} A'_n \Omega, V(-a) \Delta^{-iz} B_n \Omega \rangle$$

converges uniformly on the strip  $0 \leq \text{Im}(z) \leq 1/2$ . Thus the limit  $f_\infty(z)$  is continuous and bounded on  $0 \leq \text{Im}(z) \leq 1/2$ , analytic in the interior, and  $|f_\infty(z)| \leq \|\Phi\| \|\Psi\|$ . Moreover, when  $\lambda$  is real, it follows from relation (2.4) that

$$f_\infty(\lambda) = \langle \Phi, V(-e^{-2\pi\lambda} e^{(0)}) \Psi \rangle.$$

Since  $\Phi$  and  $\Psi$  are arbitrary, one may conclude that the operator function  $z \rightarrow V(-e^{-2\pi z} e^{(0)})$  is weakly continuous on  $0 \leq \text{Im}(z) \leq 1/2$ , analytic on the interior, and bounded in norm by 1. Setting  $z = i/4$  one finds in particular that  $\|V(i e^{(0)})\| \leq 1$ , and consequently  $P \cdot e^{(0)} \geq 0$ , where  $P$  is the generator of  $V(\mathbb{R}^4)$ . This inequality holds for all positive lightlike vectors  $e^{(0)}$ , hence  $\text{sp}(V) \subseteq \bigcap_{e^{(0)}} \{p \in \mathbb{R}^4 \mid p \cdot e^{(0)} \geq 0\} = \bar{V}_+$ .  $\square$

The representations  $V(\mathbb{R}^4)$  of the translations constructed above are uniquely fixed by the properties established so far, as can be extracted from the literature (cf. the remarks below). For completeness we give here a proof of this fact.

PROPOSITION 2.4. Let Assumptions (1) to (3) be satisfied and let  $U(\mathbb{R}^4)$  be a continuous unitary representation of the translations on  $\mathcal{H}$  which acts geometrically correctly on the net  $\mathcal{R}$ , leaves  $\Omega$  invariant, and satisfies the spectrum condition. Then  $U(\mathbb{R}^4) = V(\mathbb{R}^4)$ , where  $V(\mathbb{R}^4)$  is the representation of translations constructed above.

PROOF: 1. Let  $\Gamma(x) \equiv V(x)U(x)^{-1}$ ; then  $\Gamma(x)\Omega = \Omega$  and  $\Gamma(x)\mathcal{R}(\mathcal{O})\Gamma(x)^{-1} = \mathcal{R}(\mathcal{O})$  for all  $\mathcal{O} \in \mathfrak{R}$ . Appealing once again to [6, Thm. 3.2.18], it follows that  $\Gamma(x)$  must commute with the modular objects associated with the pair  $(\mathcal{R}(\mathcal{O}), \Omega)$  for all regions  $\mathcal{O}$ , in particular for the wedges. Thus by (2.3)  $\Gamma(x)$  must commute with  $V(\mathbb{R}^4)$ , which entails that  $V(\mathbb{R}^4)$  and  $U(\mathbb{R}^4)$  must commute.

2. It follows from Step 1 that  $\Gamma(x)$ ,  $x \in \mathbb{R}^4$ , is a strongly continuous unitary representation of  $\mathbb{R}^4$ . It shall next be shown that the spectrum  $\text{sp}(\Gamma)$  of  $\Gamma$  is symmetric about the origin. Let  $\tilde{h}(k)$  be any test function with support about  $\zeta \in \text{sp}(\Gamma)$ . Then there exists an operator  $A \in \mathcal{R}(\mathcal{K})$ ,  $\mathcal{K}$  sufficiently large, such that  $\gamma_h(A) \equiv \int h(y)\Gamma(y)A\Gamma(y)^{-1}dy \neq 0$ , because  $\Omega$  is cyclic for  $\mathcal{R}(\mathcal{W})$  and  $\mathcal{R}(\mathcal{W})$  is generated by double cone algebras  $\mathcal{R}(\mathcal{K})$ . Hence one also has  $\gamma_{\tilde{h}}(A^*) = \gamma_h(A)^* \neq 0$ , where  $\tilde{h}(k) = \tilde{h}(-k)$  has support about  $-\zeta$ . Since  $\gamma_{\tilde{h}}(A^*) \in \mathcal{R}(\mathcal{K})$  and  $\Omega$  is separating for  $\mathcal{R}(\mathcal{K})$  it follows that also  $-\zeta \in \text{sp}(\Gamma)$ .

3. Let  $E(\cdot)$  (resp.  $F(\cdot)$ ) be the spectral resolution of  $U(\mathbb{R}^4)$  (resp.  $\Gamma(\mathbb{R}^4)$ ). Since both  $V(\mathbb{R}^4)$  and  $U(\mathbb{R}^4)$  satisfy the spectrum condition, one has  $E(C_1) \cdot F(C_2) = 0$  for all compact sets  $C_1, C_2$  with  $(C_1 + C_2) \cap \bar{V}_+ = \emptyset$ . Now if  $\Gamma(x) \neq 1$  there exists an element  $\zeta \in \text{sp}(\Gamma)$  such that  $\zeta \notin \bar{V}_+$ , by Step 2. Then pick a test function  $\tilde{h}(k)$  with support in a small neighborhood  $\mathcal{O}$  of  $\zeta$  satisfying  $\bar{\mathcal{O}} \cap \bar{V}_+ = \emptyset$  and an operator  $A \in \mathcal{R}(\mathcal{K})$  such that  $B \equiv \gamma_h(A) \neq 0$ , as in Step 2. The spectral support of the vector function  $x \rightarrow U(x)B\Omega$  is thus contained in  $\bar{V}_+ \cap (\bar{V}_+ - \mathcal{O})$ .

Consider now the function  $x \rightarrow K(x) \equiv \langle \Omega, [B^*, B(x)]\Omega \rangle$ , where  $B(x) \equiv U(x)BU(x)^{-1}$ , and set  $K_\pm(x) \equiv \pm \Theta(\pm x_0)K(x)$ , where  $\Theta(x_0)$  is the unit step function at 0. Because of locality, the Fourier transforms  $\tilde{K}_\pm$  are analytic in the tubes  $T_\pm$ <sup>3</sup>. Hence, using the spectral support properties of  $B\Omega$ , it follows from the Edge-of-the-Wedge-Theorem that  $\tilde{K}$  is the discontinuity on the reals of some function analytic in  $T_+ \cup T_- \cup \mathcal{N}$ , where  $\mathcal{N}$  is a complex neighborhood of the complement of the region of discontinuities

<sup>3</sup>With  $T_+$  (resp.  $T_-$ ) we denote the forward (resp. backward) complex tube  $\{z \in \mathbb{C}^4 \mid \text{Im}(z) \in V_\pm\}$ .

$[\bar{V}_+ \cap (\bar{V}_+ - C)] \cup \bar{V}_-$ . This situation is encountered frequently in applications of the Jost-Lehmann-Dyson technique. And since no complete (double shelled) hyperboloid fits into the region of discontinuities, one must conclude that the analytic function is identically zero. Because of the spectrum condition for  $U(\mathbb{R}^4)$  it follows that  $B\Omega = E(\{0\})B\Omega = 0$ , where the latter equality is a consequence of the support properties of  $\tilde{h}$ . But as  $\Omega$  is separating for  $\mathcal{R}(K)$ , this implies  $B = 0$ , which is a contradiction. Hence  $\Gamma(x) = 1$ .  $\square$

REMARK: In Step 1 of the proof of Proposition 2.4, we have used the connection of the representation  $V(\mathbb{R}^4)$  with the modular involutions in order to show that  $U(\mathbb{R}^4)$  and  $V(\mathbb{R}^4)$  commute. Combining results in [9] and [10] this property can in fact be established for any two representations of the translations which act geometrically correctly on the net  $\mathcal{R}$ , leave  $\Omega$  invariant, and satisfy the spectrum condition. Proceeding then as in Steps 2 and 3 it follows also in this general case that the translations are uniquely fixed by a vacuum state.

A very different argument to this effect can be deduced from [4]. In the language established above, Borchers has shown that for any representation  $U(\mathbb{R}^4)$  of the translations acting geometrically correctly on the net, having an invariant vector and satisfying the spectrum condition, and for any coherent system of wedges  $\{\mathcal{W}^{(0)} + z^{(0)} \mid z^{(0)} \in \mathbb{R}^2_{\mathcal{W}^{(0)}}\}$ , one has

$$J_0 J_{z^{(0)}} = U(-2z^{(0)}).$$

Since the left side of this equation is fixed by the net and the choice of state, the right side likewise depends only on these data. Thus there is only one representation of the translations satisfying all requirements.

Summarizing our results, we have established the following theorem.

**THEOREM 2.5.** *Let  $\mathcal{A}$  be a net of  $C^*$ -algebras over  $\mathbb{R}^4$ , let  $\omega$  be a state on  $\mathcal{A}$  which satisfies Assumptions (1) to (3), and let  $(\mathcal{H}, \pi, \Omega)$  be the corresponding GNS representation. Then there exists on  $\mathcal{H}$  a continuous unitary representation  $V(\mathbb{R}^4)$  of the translations which acts geometrically correctly on the net  $\mathcal{R}$ , leaves  $\Omega$  invariant, and satisfies the spectrum condition. These properties uniquely fix the translations.*

### III. THE PRINCIPLE OF GEOMETRICAL MODULAR ACTION

In the preceding section we have characterized the vacuum states on a given net of algebras over Minkowski space by conditions on the associated modular operators and conjugations. These conditions were of a purely geometrical nature and it seems therefore natural to apply this approach to quantum field theories on other spacetime manifolds  $\mathcal{M}$  with the intent to characterize physically significant states. In this more general setting one still deals with nets of  $C^*$ -algebras  $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathfrak{R}}$ , where  $\mathfrak{R}$  is a suitable family of subregions of  $\mathcal{M}$  [2]. In order to subsume this larger class of theories we propose the following generalization of Assumption (1) for the characterization of special states  $\omega$  analogous to the vacuum. Again, this condition is expressed in terms of the GNS representation induced by  $\omega$ .

**CONDITION:** There exists a distinguished subset  $\mathfrak{S}$  of regions  $\mathcal{G} \in \mathfrak{R}$ , whose intersections generate  $\mathfrak{R}$ , such that the net comprised of the algebras  $\mathcal{R}(\mathcal{O}) \equiv \bigcap_{\mathcal{S} \supset \mathcal{O}} \pi(\mathcal{A}(\mathcal{S}))''$ ,  $\mathcal{O} \in \mathfrak{R}$ , is stable under the action of the modular conjugations  $J_{\mathcal{G}}$  affiliated with  $(\mathcal{R}(\mathcal{G}), \Omega)$ ,  $\mathcal{G} \in \mathfrak{S}$ . More precisely, for every  $\mathcal{O} \in \mathfrak{R}$  and  $\mathcal{G} \in \mathfrak{S}$  there is some  $\mathcal{O}_{\mathcal{G}} \in \mathfrak{R}$  such that

$$(2.6) \quad J_{\mathcal{G}} \mathcal{R}(\mathcal{O}) J_{\mathcal{G}} = \mathcal{R}(\mathcal{O}_{\mathcal{G}}).$$

The manifolds  $\mathcal{M}$  and the families  $\mathfrak{R}$  of subregions are, in principle, arbitrary. In fact, the condition is meaningful for nets based on any partially ordered index set  $\mathfrak{R}$ . One should neither assume from the outset that the maps from  $\mathcal{O}$  to  $\mathcal{O}_{\mathcal{G}}$ , induced by  $J_{\mathcal{G}}$ , are point transformations, nor should one in general expect that there is an analogue to Assumption (2) or (3). Nevertheless, the above condition imposes stringent conditions on a theory, respectively the underlying states. It is an interesting problem to classify the manifolds, the causal structures, and the symmetries (if any) which are compatible with this condition. We view this as a step towards an algebraic characterization of quantum field theories on arbitrary spacetime manifolds and their respective elementary systems.

As a first exercise in this program we have tried to recover the results of the preceding section, starting from this more general point of view. To this end we have reconsidered the case of a net over the manifold  $\mathbb{R}^4$  which is based on the set  $\mathfrak{R}$  of all double cones and wedges. If this net satisfies the condition given above, it follows without further specification of the relation between  $\mathcal{O}$  and  $\mathcal{O}_{\mathcal{G}}$  that the geometrical action induced on  $\mathbb{R}^4$  by the modular involutions is given by point transformations which are elements of the extended Poincaré group (cf. [11]). From this one infers that the causal structure induced on  $\mathbb{R}^4$  (which, in view of relation (2.6), is fixed by setting  $\mathcal{G}' \equiv \mathcal{G}_{\mathcal{G}}$ ) coincides with that of Minkowski space. We note that the modular structure can also be used to induce a metric  $d$  on  $\mathbb{R}^4$ ,

$$d(x, y) \equiv \sup \sum_n 2^{-n} \| (J_{\mathcal{W}_x} - J_{\mathcal{W}_y}) \Phi_n \|,$$

where  $\{\Phi_n\}_{n \in \mathbb{N}}$  is some orthonormal basis in  $\mathcal{H}$  and the supremum is to be taken with respect to all pairs  $(\mathcal{W}_x, \mathcal{W}_y)$  of coherent wedges whose edges pass through  $x$  and  $y$ , respectively.

It seems unlikely that there always exist space-time translations in a theory satisfying these general conditions. Yet if one presumes that Assumption (2) is satisfied, one can exhibit once again a continuous unitary representation of the translations in the GNS representation induced by  $\omega$ . And note that Assumption (2) implies that the metric  $d$  above is invariant under these space-time translations. As before, Assumption (3) implies that the translations so constructed satisfy the relativistic spectrum condition. A detailed account of these results, including also a discussion of Lorentz transformations, will be published elsewhere [7].

## APPENDIX

We present here some technical lemmas required in the main text of this paper.

**LEMMA A.1 [12].** Let  $A$  be a closed, densely defined linear (or antilinear) operator on a Hilbert space  $\mathcal{H}$  with domain  $D(A)$ . Let  $D_n \subset D(A)$ ,  $n \in \mathbb{N}$ , determine an increasing sequence of dense vector spaces such that  $D \equiv \bigcup_{n=1}^{\infty} D_n$  is a core for  $A$ . Let  $A_n \equiv \overline{A|D_n}$  and let  $A = VH$ ,  $A_n = V_n H_n$  be the corresponding polar decompositions. Then  $V_n \rightarrow V$  in the strong operator topology and  $H_n \rightarrow H$  in the strong resolvent sense.

**COROLLARY A.2.** Let  $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$  be an increasing sequence of von Neumann algebras on a Hilbert space  $\mathcal{H}$  such that  $\mathcal{M} = (\cup \mathcal{M}_n)''$ , and let  $\Omega$  be a cyclic and separating vector for  $\mathcal{M}$  and  $\mathcal{M}_n$ ,  $\forall n \in \mathbb{N}$ . If  $\Delta^{1/2}, \Delta_n^{1/2}, J, J_n$  are the corresponding modular objects, then  $J_n \rightarrow J$  and  $\Delta_n^{\lambda} \rightarrow \Delta^{\lambda}$  strongly for any  $\lambda \in \mathbb{R}$ .

**PROOF:** Let  $S = \overline{S_0}$ , with  $S_0 A \Omega \equiv A^* \Omega$ ,  $\forall A \in \mathcal{M}$ . One needs only to show that if  $D_n \equiv \mathcal{M}_n \Omega$ ,  $n \in \mathbb{N}$ , then the set  $D \equiv \bigcup_{n=1}^{\infty} D_n$  is a core for  $S$ . Let  $\Psi \in D(S)$ . Then since  $\mathcal{M} \Omega$  is a core for  $S$ , there exists a sequence  $\{M_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$  such that  $M_n \Omega \rightarrow \Psi$  and  $S M_n \Omega = M_n^* \Omega \rightarrow S \Psi$  strongly. On the other hand, since  $\mathcal{M} = (\cup \mathcal{M}_n)''$ , it is also true that  $\cup \mathcal{M}_n$  is dense in  $\mathcal{M}$  in the strong \*-topology (cf. Theorem 2.6 in [13]). In other words, there exist sequences  $\{M_{nm}\}_{m \in \mathbb{N}} \subset \cup \mathcal{M}_n$  such that  $M_{nm} \Omega \rightarrow M_n \Omega$  and  $M_{nm}^* \Omega \rightarrow M_n^* \Omega$  strongly, for every  $n \in \mathbb{N}$ . Thus, for suitable subsequences  $\{N_n = M_{nm(n)}\} \subset \cup \mathcal{M}_n$  one has  $N_n \Omega \rightarrow \Psi$  and  $S N_n \Omega = N_n^* \Omega \rightarrow S \Psi$ .  $\square$

In the main text we make use of these facts as follows: according to our assumption the wedge algebras  $\mathcal{A}(\mathcal{W})$  are the inductive limits of double cone algebras  $\mathcal{A}(\mathcal{K})$ , where the closures of  $\mathcal{K}$  are contained in the interior of  $\mathcal{W}$ . Hence if  $\mathcal{W}_z \subset \mathcal{W}$  determines an increasing sequence of wedges such that  $\bigcup_z \mathcal{W}_z = \mathcal{W}$ , then  $\{\cup \mathcal{R}(\mathcal{W}_z)\}'' = \mathcal{R}(\mathcal{W})$ . From the corollary it then follows that  $J_{\mathcal{W}_z}$  converges strongly to  $J_{\mathcal{W}}$ .

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