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Multipoint lightcone bootstrap from differential equations

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Abstract: One of the most striking successes of the lightcone bootstrap has been the perturbative computation of the anomalous dimensions and OPE coefficients of double-twist operators with large spin. It is expected that similar results for multiple-twist families can be obtained by extending the lightcone bootstrap to multipoint correlators. However, very little was known about multipoint lightcone blocks until now, in particular for OPE channels of comb topology. Here, we develop a systematic theory of lightcone blocks for arbitrary OPE channels based on the analysis of Casimir and vertex differential equations. Most of the novel technology is developed in the context of five- and six-point functions. Equipped with new expressions for lightcone blocks, we analyze crossing symmetry equations and compute OPE coefficients involving two double-twist operators that were not known before. In particular, for the first time, we are able to resolve a discrete dependence on tensor structures at large spin. The computation of anomalous dimensions for triple-twist families from six-point crossing equations will be addressed in a sequel to this work.

Keywords: Scale and Conformal Symmetries, Field Theories in Higher Dimensions, Space-Time Symmetries, Differential and Algebraic Geometry

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Contents

1 Introduction

Understanding the non-perturbative dynamics of quantum field theories is a central challenge of theoretical physics. This is particularly true for conformal field theories which are, through Maldacena's celebrated AdS/CFT correspondence, related to gravitational quantum theories in Anti-de Sitter space. The dynamics of conformal field theories turns out to be very strongly constrained by conformal symmetry. In the case of 2-dimensional theories, this has been exploited for several decades now, starting with the seminal work of Belavin, Polyakov, and Zamolodchikov [1] which was the first work to implement the conformal bootstrap program. But it took another 30 years until the constraining power of conformal symmetry was put to use in higher dimensions [2, 3]. The modern conformal bootstrap was originally based on a numerical analysis of the bootstrap constraints. Analytic solutions of the higher-dimensional crossing symmetry equations seem out of reach for now, mostly due to the complexity of the equations. But when pairs of insertion points in the correlator are light-like separated, the operator products for higher-dimensional conformal field theories are dominated by a finite set of leading-twist fields. This was first exploited in [4, 5] and initiated the development of the so-called lightcone or analytic bootstrap, see [6–12] and references therein.

One of the most striking early results of the lightcone bootstrap was to show that any conformal field theory must contain infinite double-twist families in which the conformal weights approach the values of a free-field theory when the spin becomes large [4, 5, 13].[1] Later, the analytic tools were sharpened such that it is now possible to systematically compute corrections to the free-field theory behavior for double-twist families in an expansion around large spin [15, 16]. In terms of the AdS dual theories, these results allow in particular to compute the mass spectrum of two-particle bound states [17, 18].

It is certainly highly desirable to obtain similar results for multi-twist families in conformal field theory or multi-particle states in the dual gravitational theories on AdS, see [19–21] for some previous discussions. But not much is actually known yet. While double-twist families already appear in the operator product of two fields in a generalized free-field theory, it requires at least three fields to build a triple-twist family. Hence, to analyze the behavior of triple-twist fields in free-field theory and beyond, it seems natural to study the crossing symmetry of six-point correlators. This turns out to be a rather formidable task. Nevertheless, it seems worth some effort. Indeed, in an interacting theory, triple- and higher-twist operators can be exchanged in the intermediate channel of a four-point function. As long as we remain agnostic about the behavior of higher-twist families, their exchange will contaminate four-point crossing symmetry equations and this contamination systematically limits the precision to which conformal data can be determined [22, 23]. Overcoming this limitation of existing bootstrap tools has recently motivated several groups to probe conformal field theories through multipoint correlation functions involving more than four scalar fields [24–28].

The main bottleneck of the conformal bootstrap in general and of the multipoint bootstrap in particular is our very limited knowledge of the relevant conformal blocks. For $N > 4$ insertion points, even partial results on conformal blocks are quite recent, see e.g. [29–37] and references therein. Even though conformal blocks are expected to simplify drastically when the insertion points lie on their respective lightcones (that is to say, the relevant regime in the lightcone bootstrap), there exist no universal results yet. The only cases that were understood recently concern blocks in which pairs of external fields can be expanded through the lightcone OPE of [38–40]. Through the use of this lightcone OPE, one can derive integral formulas for multipoint lightcone blocks for up to $N = 6$ external points in the snowflake channel. By taking appropriate limits of these integral formulas, the authors of [24–26] were able to analyze the crossing equation for a duality between two inequivalent snowflake channels in the lightcone limit and thereby derive new results on the leading large-spin behavior of OPE coefficients involving several double-twist families. But in order to effectively access triple-twist families, it is necessary to study crossing symmetry

[1] The original analysis was based on some natural assumptions regarding the behavior of sums of conformal blocks in the double lightcone limit. A rigorous proof was only given very recently in [14].

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relations in which at least one channel has comb topology. Unfortunately, lightcone limits for general comb channel blocks cannot be accessed through the lightcone OPE formula of Ferrara et al. Some new tools are required. Here we shall advocate the use of differential equations as a universal tool to study lightcone blocks for any channel topology and in particular the comb channels that can give access to multi-twist families.

The characterization of multipoint conformal blocks through differential equations is by now rather well understood, mostly through the integrability-based approach to conformal blocks that was initiated in [41] and developed in [42–44]. In the case of four-point blocks, Dolan and Osborn first proposed to study them through a set of Casimir differential equations. While the original equations did not look very inviting, they indeed allowed to efficiently analyze four-point blocks in dimension $d \geq 2$ [45]. Later, the Casimir differential operators of Dolan and Osborn were identified as the eigenvalue equations for the Hamiltonian of a two-particle integrable Calogero-Sutherland model [46]. This insight initiated an integrability-based approach to four-point function that was advanced in [47–51]. Very remarkably, this relation between integrable models of quantum mechanics and conformal blocks persists for correlators involving more than four external fields. More specifically, as shown in [44], one can identify the relevant differential operators that characterize blocks for any number of external fields with the commuting Hamiltonians of an integrable Gaudin model, in a certain homogeneous limit. Even though the complexity of these differential operators grows quickly with the number of cross-ratios, they can be worked out and possess very nice properties. As we shall see below, these properties make it possible to evaluate the differential operators in the lightcone limit. After taking sufficiently many insertion points to be lightlike separated, these differential operators lose much of their complexity and one can often find explicit solutions in closed form.

The main goal of this work is to develop a systematic theory of multipoint blocks in the lightcone limit, based on the study of the differential equations the blocks satisfy. In particular, our new approach does not rely on lightcone OPEs and hence applies to all channels including the important OPE channels highlighted above. For $N = 5$ and $N = 6$ external scalar fields, we will also apply our concrete results on blocks to the analysis of crossing symmetry equations, leading to a number of new results on OPE coefficients, see below.

1.1 Lightcone differential equations — an overview

In order not to get lost in technical details later on, we first want to give a bird's eye view of the new methods developed herein. We first describe the general setup, then explain some elements of constructing perturbative solutions to a set of differential equations, and finally turn to the multipoint lightcone bootstrap.

1.1.1 The general setup

Our setup starts with N external scalar fields that are inserted in points $x_i \in \mathbb{R}^d$ with $i = 1, \ldots, N$. As is common in the conformal field theory literature, we will represent the insertions points x_i through projective null lines $X_i \in \mathbb{R}^{1,d+1}$ in embedding space. Given X_i one can construct a certain number $n_{\text{cr}}(N, d)$ of conformally-invariant cross-ratios. In the case of $N = 5$ points and $d \geq 3$, for example, there exist five independent cross-ratios. For $N = 6$ points, the number of independent cross-ratios increases by three in $d = 3$ and by four in $d \geq 4$. This number of independent cross-ratios is easy to count in general, see e.g. [44, eq. (1.3)].

Let us now consider some correlation function G of N scalar fields. After some appropriate factor Ω is split off from $G = \Omega g$, the remaining function g depends on cross-ratios only. In principle, one can evaluate G by performing $N - 2$ operator products between fields until one ends up with a two-point function that is fixed by conformal symmetry. The precise sequence of operator products that are performed is known as the OPE channel. We shall denote OPE channels by $\mathcal{C}$. For $N = 6$ points, for example, there exist 90 different OPE channels with two different topologies, known as snowflake and comb topology, respectively, see the left-hand and right-hand sides of figure 5, respectively. Once all operator product expansions (OPEs) are performed, the correlator is written as a product of $N-2$ constant OPE coefficients and the kinematically-determined conformal blocks that carry all the dependence on the cross-ratios. These conformal blocks depend on certain quantum numbers that parametrize the conformal weights and spins of intermediate fields appearing in the operator products, as well as the choice of tensor structures. There are as many such quantum numbers as there are independent cross-ratios. In the case of $N = 5$ external scalars, for example, one has two intermediate fields that transform in symmetric traceless tensor representations (STTs) of the conformal group and hence carry two quantum numbers each, the conformal weight Δ and rank J of the tensor. In addition, there is a fifth label that determines the choice of the tensor structure in the OPE between a scalar and an STT field. So, indeed, the total number of labels coincides with the number of cross-ratios.

According to the approach advocated in [41], scalar N-point conformal blocks may be characterized by a system of $n_{\rm cr}(N, d)$ linear, higher-order differential equations. These arise as eigenvalue equations of a system of commuting operators $\mathcal{D}_A, A = 1, \ldots, n_{\rm cr}(N, d)$ that measure the quantum numbers of the conformal blocks. An explicit construction of such operators for any given OPE channel $\mathcal{C}$ was proposed in [44]. Even though these operators are quite complicated in general, it is possible to introduce a special set of coordinates in the space of cross-ratios such that all the coefficients in the differential operators become polynomials, see [42] for OPE channels with comb topology. These coordinates have been dubbed *polynomial cross-ratios*. The set of differential operators in polynomial cross-ratios is the starting point for the lightcone analysis in the present work.

1.1.2 Lightcone differential operators

In order to describe our analysis in a bit more detail, let us denote the polynomial cross-ratios for an N-point function by $w_Q, Q = 1 \ldots, n_{\rm cr}(N, d)$. The lightcone bootstrap analyzes crossing symmetry constraints in the limit where some pairs of insertion points x_i become lightlike separated, i.e. in which $X_{ij} := X_i \cdot X_j$ tends to zero for some set of pairs (i, j). How many and which pairs of points are selected to become lightlike depends on the pair of OPE channels that one relates through crossing symmetry, see below. We will denote the total number of independent lightlike pairs by m and enumerate the individual pairs (i_ν, j_ν) by $\nu = 1, \dots, m$. For each lightlike pair we shall introduce a formal parameter $\epsilon_\nu, \nu = 1, \dots, m$. The parameters ϵ_ν control the approach to the lightcone of the products $X_{i_\nu j_\nu}$. Since our cross-ratios w_Q are multi-homogeneous in each of the products X_{ij}, the substitution rules $X_{i_\nu j_\nu} \to \epsilon_\nu X_{i_\nu j_\nu}$ determine a map γ_ϵ that attaches formal variables to the coordinates w_Q,

$$\gamma_\epsilon(w_Q) = \epsilon^{s_Q} w_Q \ . \tag{1.1}$$

Here, $s = (s_\nu) \in \mathbb{Z}^m$ are multi-indices and we used the notation $\epsilon^s = \prod_{\nu=1}^m \epsilon_\nu^{s_\nu}$ as usual. We shall refer to the multi-index s_Q as the ϵ-scaling of the variable w_Q and to eq. (1.1) as scaling laws of the variables w_Q.

Next, let us consider the algebra $\mathcal{D}^{(n)}$ of differential operators $\mathcal{D}$ in the n variables w_Q. We can extend our scaling law (1.1) to a homomorphism γ_ϵ on differential operators by imposing the additional rule

$$\gamma_\epsilon(\partial_{w_Q}) = \epsilon^{-s_Q} \partial_{w_Q} \tag{1.2}$$

where we use $\epsilon^{-s} = \prod_{\nu=1}^m \epsilon_\nu^{-s_\nu}$. After application of γ_ϵ to some differential operator $\mathcal{D}$ we obtain an expression in the formal variables ϵ_ν that takes values in the space of differential operators. All operators we shall work with below have finite order and polynomial coefficients in some set of variables w_Q. Given any such operator, we can apply γ_ϵ and expand the result in a formal series in the variables ϵ_ν

$$\gamma_\epsilon(\mathcal{D}) \equiv \mathcal{D}^\epsilon = \sum_{k \in \mathbb{Z}^m} \epsilon^k \, \mathcal{D}^{(k)} \ . \tag{1.3}$$

Note that for differential operators of finite order with polynomial coefficients, the sum on the right-hand side involves a finite number of terms. From time to time we shall refer to the index k as the ϵ-scaling of the differential operator $\mathcal{D}^{(k)}$.

We will be studying differential operators in a limiting regime in which pairs of points become lightlike separated in some particular order, with the distance between the first pair becoming lightlike much faster than the second, which in turn becomes lightlike much faster than the third, etc. As we remarked above, the pairs are enumerated by our index $\nu = 1, \dots, m$. Because of this hierarchy of lightcone limits, the ordering of the formal parameters ϵ_ν does matter. Whenever we want to stress this ordering, we shall write

$$\vec{\epsilon} = (\epsilon_1, \dots, \epsilon_m) \ .$$

Similar notations are used for the multi-indices $s, k, \dots$. The ordering of the formal variables ϵ_ν introduces a lexicographic order among the scalings, i.e. we write $\vec{k} < \vec{k}'$ if the first non-vanishing entry in $\vec{k}' - \vec{k}$ is positive. Since the differential operators we are dealing with throughout this work have finite order, their expansion (1.3) contains a term that contains the leading term. The associated grade vector will be denoted by $\vec{k}_0$. We can then rewrite the expansion (1.3) in the form

$$\gamma_\epsilon(\mathcal{D}) \equiv \mathcal{D}^{\vec{\epsilon}} = \vec{\epsilon}^{\,\vec{k}_0} \sum_{\vec{p} \in \mathbb{N}^m} \vec{\epsilon}^{\,\vec{p}} \, \mathcal{D}^{(\vec{k}_0 + \vec{p})} = \vec{\epsilon}^{\,\vec{k}_0} \mathcal{D}^{(\vec{k}_0)} + \tilde{\mathcal{D}} \tag{1.4}$$

where the sum runs over a finite set of vectors $\vec{p}$ with non-negative integer components only. On the right-hand side we have split $\mathcal{D}$ into its leading term and the subleading remainder.

As we explained above, we will be interested in eigenfunctions of the differential operator $\mathcal{D}$. In this work we will only look at the leading terms of the eigenvalue equation, i.e. at eigenfunctions of the leading singular term $\mathcal{D}^{(\vec{k}_0)}$ that is selected by our choice of $\vec{\epsilon}$. For the associated eigenfunctions there are two cases that will occur, depending on the scaling behavior of the eigenvalues λ. In order to describe the scaling behavior of eigenvalues, we extend the map γ to the eigenvalues, i.e. we introduce $\lambda^{\vec{\epsilon}} = \gamma_{\vec{\epsilon}}(\lambda)$. The first case that turns out to be relevant for us is when the eigenvalue does not scale, i.e. when $\lambda^{\vec{\epsilon}} = \lambda$. If that is the case and if the grade vector $\vec{k}_0$ of the leading singular term is negative, then the differential equation forces the leading behavior of the eigenfunction ψ to lie in the kernel of the differential operator $\mathcal{D}^{(\vec{k}_0)}$. The other relevant case that we will encounter below is when the eigenvalue λ scales in the same way as the leading term of the differential operators, i.e. when $\lambda^{\vec{\epsilon}} = \vec{\epsilon}^{\,\vec{k}_0}\lambda + \tilde{\lambda}$. With this scaling of the eigenvalue, we are led to consider eigenfunctions of the differential operator $\mathcal{D}^{(\vec{k}_0)}$ for eigenvalue λ. While beyond the scope of this paper, this expansion of the differential equations actually defines a perturbative expansion of the eigenfunctions that can be used to calculate corrections to lightcone blocks away from the strict lightcone limit.

1.1.3 Multipoint lightcone bootstrap

After these more general comments on the perturbative study of differential eigenvalue equations, we now return to the study of lightcone blocks and lightcone bootstrap. The crossing equations we want to analyze are associated with a pair of OPE channels. We shall refer to one of these channels as the *direct channel* and denote it as $\mathcal{C}^{\mathrm{DC}}$. The second channel is referred to as *crossed channel* and denoted by $\mathcal{C}^{\mathrm{CC}}$. The choice of direct and crossed channel largely determines the choice of the vector $\vec{\epsilon}$ that featured in the previous subsection. In particular, the first $N-3$ entries of $\vec{\epsilon}$ must ensure that the lightcone limit of the direct channel receives its leading contributions from leading-twist operators. In order to remove higher-twist exchange from the crossed-channel contributions as well, one needs to take at least $N-3$ additional lightcone limits. Which pairs of points are required to become lightlike depends on the crossed OPE channel. Along with the first $N-3$ lightlike limits that removed direct-channel higher twists, the minimal number of lightlike limits thus amounts to $2N-6$. As a result, the vector $\vec{\epsilon}$ contains at least $2N-6$ components, of which the first $N-3$ entries are determined by the direct channel while the latter depends on the crossed channel. Obviously, taking additional lightcone limits further reduces the complexity of the crossing equations, but it also comes at the price of reducing resolution.

Once the vector $\vec{\epsilon}$ has been chosen, we can apply the general discussion from the previous subsection to the differential operators $\mathcal{D}_A$ that characterize multipoint conformal blocks, see above. As we have explained, this set of differential operators depends on the OPE channel. Since we have singled out a pair of such channels, namely the direct and the crossed channel, we shall denote the dependence on the channel by $\mathcal{D}_A^{\mathrm{DC}}$ and $\mathcal{D}_A^{\mathrm{CC}}$, respectively. With a sufficiently large number $m \geq N-3$ of lightcone limits performed, the original complexity of the differential operators is very much reduced and one obtains rather simple expressions for the leading terms $\mathcal{D}_A^{(\vec{k}_{A0})}$, as well as the subleading corrections. Many examples will be spelled out explicitly throughout the following sections.

Given explicit formulas for the leading terms of direct-channel differential operators, one can construct expressions for the lightcone limit of direct-channel blocks. In this case, the eigenvalues are chosen to scale trivially with the lightcone limit. Following our brief comments at the end of the previous section, the lightcone limit of direct-channel blocks must lie in the kernel of the leading terms of the differential operators $\mathcal{D}_A^{\mathrm{DC}}$. Given the simple expressions of the leading singular terms, the kernel can be constructed explicitly. The lightcone limit of the direct-channel blocks is some particular vector in this linear space. We will comment on how to identify this vector within the kernel in a moment.

Before we do so, we want to take a first look at the crossed channel. It turns out that, in order to reproduce the leading terms in the direct-channel expansion within the crossed channel, we need to take an appropriate scaling limit in the eigenvalues of the crossed-channel differential operators. To determine the scaling laws for crossed-channel eigenvalues, we adopt a procedure that was first described by David Simmons-Duffin in the context of the four-point lightcone bootstrap [22]. It involves applying the crossed-channel differential operators to the leading terms in the direct channel expansion. So, given the explicit expression for the crossed-channel differential operators, we are able to determine the scaling of the crossed-channel quantum numbers as soon as we have sufficient information about the direct-channel lightcone blocks. The scaling of crossed-channel eigenvalues in effect drives contributions from the crossed channel to large spins.

Given the relevant scaling of the quantum number/eigenvalues in the crossed-channel differential equations, we can now determine the lightcone limit of the crossed-channel blocks by solving the associated differential equations. In all the cases we have looked at, these solutions can be found explicitly in terms of exponentials and Bessel functions. Furthermore, the linear differential equations obtained from the complete set of commuting differential operators possess a finite- dimensional space of solutions. Once again, the lightcone limit of the crossed-channel conformal blocks corresponds to one particular vector within this space.

The identification of the relevant vector turns out to be challenging. Even before taking any lightcone limits, the differential equations for conformal blocks possess finite-dimensional solution spaces. Within these spaces, the blocks are selected by boundary conditions that fix their behavior in the OPE limit, which we define in Lorentzian signature as the limit of two coincident points within lightlike separation. However, the OPE limit is not part of the lightcone regime for which we solve the lightcone differential equations. So, in order to select the limiting behavior of the blocks, both in direct and crossed channel, we must somehow connect the OPE limit with the lightcone limit. At least for the cases we have studied, it is indeed possible to obtain sufficient control along some curve in the space of cross-ratios that connects the OPE with the lightcone limit. To obtain this control for $N > 4$ is the main challenge within the approach we advance. Once this is overcome, the gate is open to analyze and solve the crossing equation in the lightcone limit. Before we describe more concretely the setups in which the above procedure is implemented, let us add one more comment. As we have reviewed above, the differential operators that are used to characterize conformal blocks fall in two classes. Casimir differential operators measure the weight and spin of the intermediate fields in a given OPE channel, while vertex differential operators measure the choice of tensor structure. But in order for these quantum numbers to be measurable simultaneously, we had to introduce a new basis of tensor structures that has not yet been well explored. It is certainly quite different from the choice of tensor structure that is commonly used in the conformal field theory literature. In our construction of lightcone blocks we will initially focus on the Casimir differential operators and stick to a more conventional basis of tensor structures. This also simplifies the comparison with previous work, most notably the results in lightcone blocks in [26]. Nevertheless, we shall have a look at the lightcone limit of the vertex differential operator once the lightcone blocks are constructed. This will provide interesting novel insight into the relation between eigenfunctions of vertex differential operators and the more conventional choice of tensor structures. There is an immediate payoff in the analysis of the crossing equations: crossed-channel lightcone vertex operators can be used much in the same way as the Casimir operators to determine the scaling of crossed-channel spins in the lightcone limit. Given our new understanding of the relation between eigenvalues of vertex operators and conventional tensor structure labels in the lightcone limit, we can apply the lightcone vertex operators to determine the scaling of tensor structures in the lightcone limit.

1.2 Summary of new results

Even though the main focus of this work is to sharpen universal tools for the multipoint lightcone bootstrap, and in particular for the analysis of multi-twist operators, see section 6 and [52], this paper does contain a few new results already, both on lightcone blocks and OPE coefficients. The purpose of this short subsection is to highlight these concrete new results before we dive into a broader outline of the paper.

Let us begin with the new results on five-point lightcone blocks. The most novel aspect of our analysis is that we study blocks in a partial lightcone limit where only four of the five cross ratios are sent to limiting values. That is, we only insist that four of the five distances between neighboring points become lightlike. In this restricted lightcone limit, we obtain new explicit formulas for the associated blocks, both in the direct and the crossed channel. For the direct channel, these are spelled out in eqs. (3.59), (3.61). In these expressions, it is also straightforward to send the fifth cross ratios to zero and thereby obtain formulas for direct channel blocks in the full lightcone limit, see eqs. (3.63), (3.64). We stress that even the latter are new for the regime in which the half-twists h_1, h_2 of the exchanged fields satisfy $h_1, h_2 > h_\phi$.

For the crossed channel, the lightcone blocks in the restricted lightcone limit can be found in eqs. (3.101), (3.102). The limiting behavior in the full lightcone limit is stated in eq. (3.81). In both the restricted and full lightcone limit, we obtain closed-form expressions for the blocks that were not known before. The results we have listed so far work with the usual basis of tensor structures at the central vertex of the five-point blocks. In section 4,

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we also derive the analogous results for the crossed channel blocks in the vertex operator eigenbasis, see eq. (4.11) and eqs. (4.13), (4.15), (4.16).

By exploiting our new results on five-point blocks in the restricted lightcone limit, we obtain new formulas for several OPE coefficients. These concern the OPE decompositions

$$[\phi\phi] \times \phi \;\sim\; [\phi\phi] \quad \textit{and} \quad [\phi\mathcal{O}_\star] \times \phi \;\sim\; [\phi\phi]\ .$$

In the second decomposition, the field $\mathcal{O}_\star$ is assumed to appear in the OPE of ϕ with itself and to have lower twist than ϕ, i.e. $h_\star < h_\phi$. In the limit of large tensor structures (and large spins, of course), the relevant OPE coefficients

$$C^{(\eta)}_{[\phi\phi]_{0,J_1}\phi[\phi\phi]_{0,J_2}} \qquad \textit{and} \qquad C^{(\eta)}_{[\phi\mathcal{O}_\star]_{0,J_1}\phi[\phi\phi]_{0,J_2}}$$

were first spelled out by Antunes et al. [26]. Here, we recover the same expressions, see eqs. (5.12) and (5.30), despite relying on different methods and assumptions. But we can do better. Our new results on five-point blocks in the restricted lightcone limit enable us to analyze crossing symmetry without sending the fifth cross ratio to zero. Using this technology, we bootstrap the above OPE coefficients in a new regime with large spins but discrete tensor structure, i.e. in a regime where the tensor structure label η takes values $\eta = J_1 - \delta n$ with $\delta n = 0, 1, 2, \ldots, \infty$. For the OPE coefficient that involves both $[\phi\phi]$ and $[\phi\mathcal{O}_\star]$, the relevant formula for the OPE coefficient is presented in eq. (5.52). For the other OPE coefficient that involves two double twist fields of the form $[\phi\phi]$, our new results are spelled out in eqs. (5.45) and (5.56). If the field ϕ appears in the OPE of ϕ with itself, eq. (5.56) represents a subleading correction to the leading term (5.45).

1.3 Detailed plan of part I

Let us now outline the plan of this work in some detail. In order to introduce our analytic tools and explore how to use them, we shall begin with the case of scalar four-point functions in section 2. In contrast to the usual treatment, however, we will derive all required results in lightcone blocks directly from the Casimir equations in the (restricted) lightcone limit. In the direct channel we first look at the Casimir equation in the restricted lightcone limit in which only one pair of points becomes lightlike separated, see section 2.2. This regime contains both the full lightcone limit and the OPE limit. It is well known that the associated Casimir equation can be solved in closed form. Then we look at the Casimir equations for both the direct and the crossed channel in the full lightcone limit and determine the finite-dimensional space of solutions. The existence of a closed-form solution in the restricted lightcone limit makes it easy to select the relevant solutions that are associated with the OPE boundary condition, including the overall normalization of the lightcone blocks. Once the lightcone blocks are constructed, we review the familiar lightcone bootstrap analysis in section 2.4. Some emphasis is put on how to determine the lightcone scaling behavior of the spin quantum numbers in the crossed channel by acting with crossed-channel Casimir operators on terms in the direct-channel expansion.

After this warm-up, we then turn to $N = 5$ in section 3. Scalar five-point functions provide an ideal framework to develop our new analytic tools. Indeed, in this case, lightcone blocks can be studied through the lightcone OPE, as was exploited in [24–26]. Hence, all the results that we shall obtain through the mere use of Casimir differential equations can be cross-checked with those standard techniques. In the full lightcone limit, five pairs of points become lightlike separated. Once again, we shall start in section 3.2 by looking at direct-channel blocks in a restricted lightcone limit where only two pairs become lightlike separated, and which still contains both the OPE and the full lightcone limit. The only difference with $N = 4$ is that lightcone blocks in the restricted lightcone limit are not written down in closed form but rather through an integral formula. As we shall show, the integral representation we obtain by solving lightcone Casimir equations is equivalent to the formula one obtains from the lightcone OPE. Then, we solve the direct- and crossed-channel Casimir equations in the (full) lightcone limit where additional pairs of points become lightlike separated. The correct solutions of the lightcone Casimir equations can be selected and normalized with the help of the integral formula for direct channel blocks in the restricted lightcone limit. As we shall show later in the text, see section 5, the analysis of crossing symmetry requires the construction of crossed-channel lightcone blocks for two different scaling laws of the eigenvalues. Our analysis of lightcone blocks results in a set of relatively simple formulas for direct- and crossed-channel blocks that are clearly marked, see section 3.3.

Section 4 is devoted to the study of the unique non-trivial vertex operator that exists for five external points. We find that this operator, being quite complicated for generic kinematics [43], simplifies considerably in the lightcone limit — so much so that we are able to find its joint eigenfunctions with the Casimir operators explicitly. This provides important new insight into the relation between the conventional tensor structure labels which are used in section 3 and the eigenvalues of vertex differential operators. In analogy to Simmons-Duffin's notion of Casimir singular terms we can use vertex differential operators to introduce the new notion of vertex singular behavior. In this context, the results of section 4 allow us to determine the scaling behavior of tensor structure labels in the lightcone regime, see section 5.

The explicit formulas for lightcone blocks and differential operators from sections 3 and 4 prepare us for the analysis of the crossing symmetry equation for five external scalars in the lightcone limit, see section 5. There we shall show how our results on blocks allow us to recover all of the known OPE coefficients and go beyond. In section 5.1 we spell out the leading terms within the direct channel. Then we address each of these terms in section 5.2 and show how to reproduce them from the crossed-channel expansion. While most of this analysis just reproduces results from [26], there is one important direct-channel contribution for which we can go much beyond the previous status. Our progress is related to the fact that we will be able to construct the lightcone blocks in both the direct and crossed channel with the minimal number $2N - 6 = 4$ of lightcone limits performed. This is one limit less than in [26] and will allow us for the first time to resolve the dependence of the OPE coefficients on the tensor structure. This motivates us to reanalyze the crossing equation in the fourfold lightcone limit in section 5.3, where we obtain the OPE coefficients in the new scaling regime of large spin and discrete tensor structures. Finally, we review some applications and checks of our results in section 5.4.

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Section 6 contains an outlook to the second part [52] which is devoted to the lightcone bootstrap for six-point functions. As we shall explain, the crossing equation we will address in the second part relates a direct snowflake channel to a crossed comb channel. While direct channel lightcone blocks in the snowflake channel can be studied through lightcone-OPE integral formulas, such tools no longer exist for the crossed comb channel, and here is where our new technology comes to shine. Indeed, it turns out that the program we described in section 3 carries over to six-point comb-channel blocks and for the first time provides explicit formulas for the relevant lightcone blocks. We shall show a few examples and also showcase some of the simplest results that can be extracted from the bootstrap analysis. Additional formulas for relevant lightcone blocks, detailed derivations, as well as applications to the six-point lightcone bootstrap will be given in the second part. This includes a discussion of the anomalous corrections to the conformal weights of triple-twist operators.

2 Lightcone bootstrap for four points

The purpose of this section is to provide a smooth introduction to the multipoint lightcone bootstrap. In order to do so we shall review the standard lightcone bootstrap analysis for scalar four-point functions. The lightcone bootstrap program requires a good knowledge of conformal blocks in certain limits. When dealing with four-point functions, the blocks and their limiting behavior are well known. Our discussion here will put some stress on the derivation of these properties from Casimir differential equations. We explain the usefulness of polynomial cross-ratios, the way we take (scaled) limits, and the resulting expressions for lightcone blocks. Special attention will be paid to the normalization of the blocks in the limit.

2.1 Preliminaries on blocks and lightcone limits

There are many ways to parametrize conformal invariants of four points. The most basic choice is the two cross-ratios (u, v) that are defined by

$$u = \frac{X_{12}X_{34}}{X_{13}X_{24}} = z\bar{z}, \quad v = \frac{X_{14}X_{23}}{X_{13}X_{24}} = (1-z)(1-\bar{z}). \tag{2.1}$$

Here, X_i denote the embedding space insertion points of the four scalar fields and $X_{ij} = X_i \cdot X_j$, as usual. Note that we have also introduced a second set of cross-ratios, $(z, \bar{z})$. For reasons we shall discuss below, we shall refer to u, v as *polynomial cross-ratios* and use the term *OPE cross-ratios* when dealing with $(z, \bar{z})$.

The usual conformal partial wave or conformal block expansions for correlation functions of four identical scalar fields ϕ with weight Δ_ϕ are given by

$$\langle \phi(X_1) \dots \phi(X_4) \rangle = (X_{14}X_{23})^{-\Delta_\phi} \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \psi^{(14)(23)}_{\mathcal{O}}(u, v) \tag{2.2}$$

$$= (X_{12}X_{34})^{-\Delta_\phi} \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \psi^{(12)(34)}_{\mathcal{O}}(u, v) \tag{2.3}$$

The central objects in these equations are the conformal blocks ψ. The second line is an expansion in terms of s-channel blocks $\psi^{(12)(34)}$ while in the first line we expanded the same correlator in terms of t-channel blocks $\psi^{(14)(23)}$. The two sets of blocks are related by a simple exchange of the two cross-ratios, $\psi^{(14)(23)}(u,v) = \psi^{(12)(34)}(v,u)$. The famous crossing symmetry equations for scalar four-point functions,

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \psi^{(14)(23)}_{\mathcal{O}}(u,v) = v^{\Delta_\phi} u^{-\Delta_\phi} \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \psi^{(12)(34)}_{\mathcal{O}}(u,v) \; . \tag{2.4}$$

are obtained by equating the two expansions in terms of t- and s-channel blocks, respectively.

As emphasized by Dolan and Osborn [45], the conformal blocks can best be characterized through the differential equations they satisfy. For s-channel blocks, these are given by

$$\mathcal{D}^2_{12} \psi^{(12)(34)}_{\mathcal{O}} = \left\{ h_{\mathcal{O}}(h_{\mathcal{O}} - d + 1) + \bar{h}_{\mathcal{O}}(\bar{h}_{\mathcal{O}} - 1) \right\} \psi^{(12)}_{\mathcal{O}} =: \lambda_{\mathcal{O}} \psi^{(12)}_{\mathcal{O}}, \tag{2.5}$$

$$\mathcal{D}^2_{12} := (X_{12}X_{34})^{\Delta_\phi} \frac{1}{4} \mathrm{tr}\, (\mathcal{T}_1 + \mathcal{T}_2)^2 \, (X_{12}X_{34})^{-\Delta_\phi}, \tag{2.6}$$

To write the eigenvalue $\lambda_{\mathcal{O}}$ on the right-hand side we defined the half-twist $h := \frac{\Delta - J}{2}$ and the half-anti-twist $\bar{h} := \frac{\Delta + J}{2}$ in terms of the weight Δ and the spin J of the intermediate field $\mathcal{O}$. In the second line we have introduced $\mathcal{T}_i$ to denote the usual action of the generators of the conformal Lie algebra on the primary field $\phi(X_i)$. The operator $\mathcal{D}^2_{12}$ was first computed by Dolan and Osborn [53]. When expressed in terms of the cross-ratios u, v, it takes the form

$$\mathcal{D}^2_{12} = (1 - u - v)\partial_v v \partial_v + u\partial_u \left(2u\partial_u - d\right) - (1 + u - v) \left(u\partial_u + v\partial_v\right)^2 . \tag{2.7}$$

We note that the coefficients of this second-order differential operator are polynomials in the cross-ratios u, v. This is why we refer to them as polynomial cross-ratios. Clearly, a similar discussion applies to the t-channel blocks, only that u and v are exchanged in passing from one channel to the other.

While the crossing symmetry equations of d-dimensional conformal field theories are too difficult to solve analytically for now, at least if $d > 2$, there exist certain limits in which they simplify drastically. The most interesting of these is the so-called lightcone limit that is reached after continuation to Lorentzian signature when x_4 is light-like separated from x_1, i.e. $X_{14} \sim 0$. In Lorentzian signature, this limit can be performed without making $x_1 = x_4$ and hence without imposing $u = 1$.[2] To perform the relevant limit, we shall assign appropriate orders to the cross ratios. Let us note that all the cross-ratios depend on the insertion points only through X_{ij}. We can keep track of how a given cross-ratio behaves as we make a pair of points x_i and x_j light-like separated by introducing a parameter ϵ_{ij} and performing the substitution $X_{ij} \mapsto \epsilon_{ij} X_{ij}$. In the case at hand, we want to make x_1

[2]**Caution:** contrary to the usual conventions of the four-point bootstrap and of [26], we will take the channel containing the (12) OPE as the *crossed channel*. This will simplify the computations of limits of blocks in the polynomial cross-ratios later on.

and x_4 light-like separated and hence introduce a parameter ϵ_{14}. Upon the substitution $X_{14} \to \epsilon_{14} X_{14}$, the cross-ratio v behaves as $v \to \epsilon_{14} v$ while the second cross-ratio u is invariant. We can therefore think of v as a quantity of ϵ_{14}-order $\mathcal{O}(\epsilon_{14})$ while the cross-ratio u has order $\mathcal{O}(\epsilon_{14}^0)$. Later in the analysis, we will furthermore make x_1 and x_2 light-like separated. This motivates us to study the order also with respect to ϵ_{12}. It is easy to see that u is of ϵ_{12} order $\mathcal{O}(\epsilon_{12}^1)$ while v is $\mathcal{O}(\epsilon_{12}^0)$. We note that the order extends from the polynomial cross-ratios to the Casimir differential operators. The operator (2.7), for example, has terms of ϵ_{12}-order $\mathcal{O}(\epsilon_{12}^n)$ with $n = 0, 1$. With respect to the ϵ_{14}-order, on the other hand, it contains terms of order $\mathcal{O}(\epsilon_{14}^m)$ with $m = -1, 0, 1$, i.e. there are also ϵ_{14}-singular terms. Often we want to keep track of several of these orders at the same time. In the case at hand, we introduce $\vec{\epsilon} = (\epsilon_{14}, \epsilon_{12})$ and then denote the $\vec{\epsilon}$-order by $\mathcal{O}(\epsilon_{14}^n \epsilon_{11}^m)$ or simply (n, m). With respect to this $\vec{\epsilon}$-order, our cross-ratios u and v have order $(0, 1)$ and $(1, 0)$, respectively. This implies that $\vec{\epsilon}$ is associated with the regime

$$\mathrm{LCL}_{\vec{\epsilon}}: \; v \ll u \ll 1$$

in the space of cross-ratios. Note that the vector $\vec{\epsilon}\,' = (\epsilon_{12}, \epsilon_{14})$ is associated with a different regime in which we take u and v to zero in the opposite order, i.e. in which $u \ll v \ll 1$.

2.2 Lightcone blocks in the direct channel

In order to study the behavior of the t-channel blocks in the lightcone regime $LCL_{\vec{\epsilon}}$, we shall study the limit of the Casimir eigenvalue equation. Let us note that the t-channel Casimir operator $\mathcal{D}_{14}^2$ is obtained from the s-channel operator we have displayed in eq. (2.7) by exchanging the cross-ratios u and v, i.e.

$$\mathcal{D}_{14}^2 = (1 - u - v)\partial_u u \partial_u + v\partial_v \left(2v\partial_v - d\right) - \left(1 + v - u\right)\left(u\partial_u + v\partial_v\right)^2 . \tag{2.8}$$

For reasons that will become clear later, we will implement the lightcone limit of the direct channel blocks in stages, starting with the restricted limit in which we only send v to zero while keeping u finite at first. For the associated regime, we shall write

$$LCL_{\epsilon}^{(1)}: \; v \ll 1 \; .$$

The superscript (1) reminds us that we only consider the first component ϵ_{14} of the order $\vec{\epsilon}$. The dependence of the Casimir operator $\mathcal{D}_{14}^2$ on ϵ_{14} gives rise to a split into a sum of two terms, i.e.

$$\mathcal{D}_{14}^{2,\epsilon_{14}} = \epsilon_{14}^0 \mathcal{D}_{14}^{(0)} + \epsilon_{14} \mathcal{D}_{14}^{(1)} \tag{2.9}$$

where the leading term is given by

$$\mathcal{D}_{14}^{(0)} = (1 - u)\,{}_2\mathcal{D}_1(v\partial_v, v\partial_v; 2v\partial_v; 1 - u; -\partial_u) + v\partial_v(2v\partial_v - d) \tag{2.10}$$

Here, we find it useful to introduce the symbol ${}_2\mathcal{D}_1(a, b; c; z, \partial)$ that will be used in later discussions:

$$ {}_2\mathcal{D}_1(a, b; c; w, \partial) := \partial(w\partial + c - 1) - (w\partial + a)(w\partial + b). \tag{2.11}$$

In general, the parameters a, b, c can be differential operators that commute with w, ∂, while w is some function of the cross-ratios. Since only v scales with ϵ_{14} and $\mathcal{D}_{14}^{(0)}$ is homogeneous of degree zero in ϵ_{14}, this operator commutes with the Euler operator $\vartheta_v \equiv v\partial_v$.

The t-channel conformal blocks are eigenfunctions of the Casimir differential operator $\mathcal{D}_{14}^2$. Hence, if we assume the eigenvalues not to scale with ϵ_{14}, their limits in the regime $LCL^{(1)}$ where we send v to zero must solve the differential equation

$$\mathcal{D}_{14}^{(0)} \psi_{\mathcal{O}}^{(14)(23);(0)}(u,v) = \left\{ h_{\mathcal{O}}(h_{\mathcal{O}} - d + 1) + \bar{h}_{\mathcal{O}}(\bar{h}_{\mathcal{O}} - 1) \right\} \psi_{\mathcal{O}}^{(14)(23);(0)}(u,v) \ . \tag{2.12}$$

Since the differential operator $\mathcal{D}_{14}^{(0)}$ commutes with the Euler operator $v\partial_v$, we can make the following Ansatz for a complete set of solutions

$$\psi_{(h,\bar{h})}^{(14)(23);(0)}(u,v) = v^h g_{h,\bar{h}}(u) \ . \tag{2.13}$$

We insert this Ansatz into the leading order eigenvalue equation using

$$v^{-h} \left(\mathcal{D}_{14}^{(0)} - h(h-d+1) \right) v^h = (1-u)\, {}_2\mathcal{D}_1(h,h;2h;1-u,-\partial_u). \tag{2.14}$$

It is now easy to see that eigenfunctions take the form

$$\psi_{(h,\bar{h})}^{(14)(23)}(u,v) \overset{LCL^{(1)}}{=} \psi_{(h,\bar{h})}^{(14)(23);(0)}(u,v) = v^h (1-u)^J {}_2F_1\left(\bar{h},\bar{h};2\bar{h};1-u\right). \tag{2.15}$$

with $J = \bar{h} - h$, as before. The formula we wrote requires some more comments. Obviously, the differential equation (2.12) is of second order and, hence, possesses two linearly independent solutions. But in the regime $LCL^{(1)}$ it is not difficult to determine which one is relevant. Let us recall that the relevant solution of the four-point Casimir equation is uniquely determined by its behavior in the OPE limit, which corresponds to $v \sim 0$ and $u \sim 1$[3] with $v \ll (1-u)$, in accordance with its Lorentzian definition. In this limit, the conformal block behaves as

$$\psi_{(h,\bar{h})}^{(14)(23)}(u,v) = v^h (1-u)^J \left(1 + \mathcal{O}(v, 1-u)\right) \ . \tag{2.16}$$

With this normalization condition for the blocks stated, we now note a fortunate fact: the regime $LCL^{(1)}$ in which we were able to solve the Casimir equation through eq. (2.15) does contain the OPE limit point at which blocks are selected and normalized. So, all we need to do is verify that the solution we proposed in eq. (2.15) satisfies the boundary condition (2.16) at $u \sim 1$. This is obviously the case.

After discussing conformal blocks in the partial lightcone limit, i.e. in the regime $LCL^{(1)}$, we now want to address the full lightcone limit in which u is sent to zero as well. Obviously, we could easily find the limiting behavior of the blocks from our formula (2.15), using standard properties of hypergeometric functions. But here we want to go back to the study of differential equations instead and see how much we can deduce from them. In order to study the full lightcone limit we reinstate the parameter ϵ_{12} that determines

[3] Recall that our direct channel is what is often referred to as t-channel and hence the cross-ratios u and v are exchanged with respect to the usual s-channel discussion.

the behavior of u as we go in the lightcone regime LCL. The leading term of the Casimir operator (2.8) in the order $\vec{\epsilon} = (\epsilon_{14}, \epsilon_{12})$ reads

$$\mathcal{D}_{14}^{2,\vec{\epsilon}} = \epsilon_{12}^{-1}\mathcal{D}_{14}^{(0,-1)} + \mathcal{O}(\epsilon_{12}^{0}), \tag{2.17}$$

where the superscript reminds us that the leading term of the quadratic Casimir operator is of order $\mathcal{O}(\epsilon_{14}^{0}\epsilon_{12}^{-1})$. Explicitly one finds that

$$\mathcal{D}_{14}^{(0,-1)} = \partial_u u \partial_u. \tag{2.18}$$

Let us also note that $\mathcal{O}(\epsilon_{12}^{0})$ consists of two terms, one that is constant in ϵ_{12} and one that is linear. Their precise form is easily found but irrelevant for us. If we now assume that the eigenvalue $\bar{h}(\bar{h}-1) + h(h-d+1)$ of the Casimir operator does not depend on ϵ_{12}, we conclude that the leading contribution of our conformal blocks in the limit in which ϵ_{12} is sent to zero must be in the kernel of the operator $\mathcal{D}_{14}^{0,-1}$,

$$\ker(\mathcal{D}_{14}^{(0,-1)}) = \mathrm{Span}(v^h, v^h \log u). \tag{2.19}$$

Here the eigenvalue h of the Euler operator $v\partial_v$ can assume any non-negative real value. This behavior of the lightcone blocks is indeed consistent with our previous equation (2.15) for the normalized lightcone blocks. As one can verify with the help of standard results in the behavior of hypergeometric functions, lightcone blocks behave as

$$\psi_{(h,\bar{h})}^{(14)(23)}(u,v) \overset{LCL_{\vec{\epsilon}}}{\sim} \begin{cases} v^h & \text{if } \bar{h}=0 \\ -\mathrm{B}_{\bar{h}}^{-1} v^h \log u + \mathcal{O}(1) & \text{if } \bar{h}>0 \end{cases}, \tag{2.20}$$

where $\mathrm{B}_{\bar{h}} := \Gamma(2\bar{h})^{-1}\Gamma(\bar{h})^2$. In the first line we have kept the prefactor v^h even though $\bar{h}=0$ actually implies $h=0$ in a unitary theory. The solution we displayed is consistent with the statement (2.19). But the analysis of the Casimir equation in the full lightcone limit LCL is not sufficient to fully determine the solution. Recall that the solution is determined through the behavior of the blocks in the OPE limit. But the latter is not contained in LCL. The only way in which we can determine the precise vector in the kernel (2.19) that is chosen by the OPE boundary condition is to go back to our solution (2.15) which we argued to satisfy the OPE boundary condition. We can then take the limit $u \to 0$ to obtain our results in eq. (2.20) with the correct normalization. This follows from standard limiting formulas for hypergeometric functions.

The analysis of direct channel blocks we presented here serves as a good model for our discussion of lightcone limits for multipoint blocks. It will turn out that in all the relevant cases we will be able to solve for blocks in a partial lightcone regime which contains both the OPE and the full lightcone limit. Though the asymptotics of these solutions are not well studied in comparison with standard hypergeometric functions, we will obtain integral formulas that provide access to the limiting behavior in both the OPE and the lightcone limit. In the case at hand, for example, we can also write the solution to the Casimir equation (2.12) in the form

$$\psi_{(h,\bar{h})}^{(14)(23)}(u,v) \overset{LCL^{(1)}}{\sim} \frac{v^h(1-u)^J}{\mathrm{B}_{\bar{h}}} \int_0^1 \mathrm{d}t\, (t(1-t))^{\bar{h}-1}(1-(1-u)t)^{-\bar{h}}\,. \tag{2.21}$$

It is easy to see that the integral in the last expression has a logarithmic divergence for $u \sim 0$ which stems from the integration near $t = 1$. In order to determine the leading term of the integral as we send u to zero, we can write

$$\int_0^1 \mathrm{d}t t^{\bar{h}-1}(1-t)^{\bar{h}-1}(1-(1-u)t)^{-\bar{h}} \overset{u\ll 1}{\sim} \int_0^1 \mathrm{d}t \frac{1-u}{1-(1-u)t} = \int_u^1 \frac{\mathrm{d}v}{v} \sim -\log u \,. \tag{2.22}$$

In sum, we indeed reproduce the expected $\log u$ divergence with a coefficient $\mathrm{B}_{\bar{h}}^{-1}$ of the integral in eq. (2.24). This matches our result in eq. (2.20).

While the integral formula (2.21) can be found through the analysis of the Casimir differential equations, it can also be obtained from the lightcone OPE. Let us note that even in generic kinematics, i.e. before sending any of the cross-ratios to zero, conformal blocks possess integral representations. But these are difficult to evaluate. They simplify significantly, however, when some of the external insertion points are lightlike separated so that one can use the lightcone OPE. In the case of $N = 4$ external points, the integral formula for blocks becomes

$$\psi^{(14)(23)}_{(h,\bar{h})}(u,v) = (X_{14}X_{23})^h (X_4 \wedge X_1 \cdot X_2 \wedge X_3)^J \int_{\mathbb{R}^2_+} \frac{\mathrm{d}s_1 \mathrm{d}s_4 (s_1 s_4)^{\bar{h}-1}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}}} X_{a2}^{-\bar{h}} X_{a3}^{-\bar{h}} \left(1 + \mathcal{O}(X_{14})\right). \tag{2.23}$$

In this expression, $X_a := s_1 X_1 + s_4 X_4$ is a point on the projective lightcone when $X_{14} = 0$, $\mathrm{B}_{\bar{h}} := \Gamma(\bar{h})^2 \Gamma(2\bar{h})^{-1}$ is the diagonal of the Euler Beta function, and $\mathbb{R}^\times := \int_0^\infty r^{-1} \mathrm{d}r$ is the volume of the dilation group $\{(s_1, s_2) \to (rs_1, rs_2)\}$. Formula (2.23) is obtained by inserting the lightcone OPE — see eq. (A.1) of appendix A.1 — into the four-point function, before evaluating the resulting three-point function in the integrand using standard formulas for the three-point function of two scalar fields ϕ and one spinning field with weight $\Delta_{\mathcal{O}} = h + \bar{h}$ and spin $J = \bar{h} - h$. Since the integral is homogeneous of degree zero in X_1, X_2, X_3, X_4 and manifestly $\mathrm{SO}(1, d+1)$ invariant, it can be written as a function of the cross-ratios (u, v). After the change of variables $(s_1, s_4) = (rt, r(1-t))$, it indeed takes the form

$$\begin{aligned}
\psi^{(14)(23)}_{(h,\bar{h})}(u,v) &\overset{v\ll 1}{\sim} \frac{(x_{14}^2 x_{23}^2)^h}{\mathrm{B}_{\bar{h}}} \int \mathrm{d}t \frac{(x_{24}^2 x_{13}^2 - x_{34}^2 x_{12}^2)^J}{(x_{12}^2 t + (1-t)x_{24}^2)^{\bar{h}} (x_{13}^2 t + (1-t)x_{34}^2)^{\bar{h}}} \\
&= \frac{v^h (1-u)^J}{\mathrm{B}_{\bar{h}}} \int_0^1 \mathrm{d}t \, (t(1-t))^{\bar{h}-1} (1-(1-u)t)^{-\bar{h}} \,.
\end{aligned} \tag{2.24}$$

The expression in the second line gives the integral representation of the hypergeometric solution (2.15) that we used in the previous paragraph to evaluate the limiting behavior in the full lightcone limit. With these comments stated, we close the discussion of the direct channel and turn to the analysis of lightcone blocks in the crossed channel.

2.3 Lightcone blocks in the crossed channel

Let us now discuss the lightcone limit of conformal blocks in the crossed s-channel. We recall that the Casimir differential operator $\mathcal{D}^2_{12}$ in the crossed channel has been displayed in eq. (2.7). To analyze its eigenfunctions in the full lightcone limit LCL, we expand the

Casimir operator $\mathcal{D}_{12}^2$ as

$$\mathcal{D}_{12}^{2,\vec{\epsilon}} = \epsilon_{14}^{-1} \left(\partial_v v \partial_v - \epsilon_{12} u \partial_v v \partial_v\right) + \mathcal{O}(\epsilon_{14}^0). \tag{2.25}$$

Since the leading contribution commutes with the Euler operator $\vartheta_u \equiv u\partial_u$, we can without loss of generality choose a basis of functions whose leading behavior takes the form

$$\psi_{(h,\bar{h})}^{(12)(34)}(u,v) \sim u^h g_{(h,\bar{h})}(v)(1+\mathcal{O}(u^{h+1})) \tag{2.26}$$

as we send the cross-ratio u to zero. The leading term eigenvalue equation then becomes

$$\partial_v v \partial_v g_{(h,\bar{h})}(v) = \lambda_{(h,\bar{h})} g_{(h,\bar{h})}(v), \quad \lambda_{(h,\bar{h})} = h(h-d+1) + \bar{h}(\bar{h}-1) \ . \tag{2.27}$$

Contrary to the direct t-channel, the $v \to 0$ limit does not restrict the twist of blocks appearing in the s-channel. Thus, the Casimir eigenvalue can vary over all positive numbers, and we can distinguish three regimes

$$g(v) \overset{v \lll 1}{\sim} \mathcal{N} \begin{cases} 1 \text{ or } \log v & \text{if } \lambda = \mathcal{O}(\epsilon_{14}^0), \\ K_0(2\sqrt{\lambda v}) & \text{if } \lambda = \mathcal{O}(\epsilon_{14}^{-1}), \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.28}$$

In the first case, the eigenfunctions must be in the kernel of $\partial_v v \partial_v$. If the eigenvalue $\lambda^{\vec{\epsilon}} = \epsilon_{14}^{-1}\lambda$ scales with ϵ_{14} as stated in the second line, the right-hand side of the eigenvalue equation contributes and we obtain the equation

$$\left(\partial_v v \partial_v - \lambda\right) g = 0 \tag{2.29}$$

which can be transformed into Bessel's differential equation by a simple change of variables. This goes a long way toward explaining the behavior of $g(v)$ we have displayed. But as in the direct channel, we still need to ensure that the blocks satisfy the correct boundary conditions and in particular determine the normalization $\mathcal{N}$.

In order to so, we exploit the fact that the leading term $\partial_v v \partial_v$ in the Casimir operator $\mathcal{C}_{12}^2$ at order $\mathcal{O}(\epsilon_{14}^{-1}\epsilon_{12}^0)$ is the same regardless of whether we take ϵ_{14} or ϵ_{12} to zero first, i.e.

$$\lim_{\epsilon_{14}\to 0} \lim_{\epsilon_{12}\to 0} \epsilon_{14} \left(\mathcal{D}_{12}^{2,\vec{\epsilon}} - \lambda^{\vec{\epsilon}}\right) = \lim_{\epsilon_{12}\to 0} \lim_{\epsilon_{14}\to 0} \epsilon_{14} \left(\mathcal{D}_{12}^{2,\vec{\epsilon}} - \lambda^{\vec{\epsilon}}\right) \ . \tag{2.30}$$

We can thus retrieve these asymptotics of the crossed channel lightcone blocks by taking ϵ_{12} to zero first. In this case, the analysis is identical to the one we described in our discussion of the direct channel. Consequently, in the limit $\epsilon_{12} \to 0$ (with v kept finite), our blocks take the exact form of (2.15) with $u \leftrightarrow v$ and with h corresponding to a fixed half-twist,

$$\psi_{(h,\bar{h})}^{(12)(34)}(u,v) \overset{u \lll 1}{\approx} u^h (1-v)^J {}_2F_1\left(\bar{h},\bar{h};2\bar{h};v\right) \ . \tag{2.31}$$

Here we have already inserted the normalization that we had determined previously. Consequently, given that the eigenvalue is $\lambda = h(h-d+1)+\bar{h}(\bar{h}-1)$, and that we have fixed h already, our three regimes correspond to

$$\psi_{(h,\bar{h})}^{(12)(34)}(u,v) \overset{LCL_{\vec{\epsilon}}}{\sim} \begin{cases} 1 \text{ or } \mathrm{B}_{\bar{h}}^{-1} \log v^{-1} & \text{if } \bar{h} = \mathcal{O}(\epsilon_{14}^0) \\ \mathcal{N}_{(h,\bar{h})}^{\mathrm{LS}} u^h K_0(2\bar{h}\sqrt{v}), & \text{if } \bar{h} = \mathcal{O}(\epsilon_{14}^{-1/2}) \\ 0 \quad \text{otherwise,} \end{cases} \tag{2.32}$$

with

$$\mathcal{N}^{\rm LS}_{(h,\bar{h})} = 4^{\bar{h}}\sqrt{\bar{h}/\pi} \tag{2.33}$$

and $\mathrm{B}_x := \mathrm{B}(x,x) = \Gamma(x)^2/\Gamma(2x)$ given by the diagonal of the beta function as before. In order to determine the normalization $\mathcal{N}$ in the second case we have used that $K_0(2x) \sim -\log(x)$ near $x \sim 0$ along with the Stirling formula to evaluate $\mathrm{B}_{\bar{h}}^{-1}$ as $\bar{h}$ tends to infinity. This concludes our discussion of the lightcone limit of four-point blocks for identical scalars in the direct and the crossed channel. We stress once again that in order to obtain the two key results (2.20) and (2.32) for the lightcone limits of blocks in the direct and crossed channel, respectively, we only needed the Casimir differential equations. This applies in particular to the dependence of the lightcone blocks on the cross-ratios. The normalizations could also be determined from the Casimir equations, though for this purpose we had to solve them outside the lightcone limit to connect with the OPE limit.

2.4 Review of the four-point lightcone bootstrap

We can now exploit the results on lightcone blocks to analyze the crossing symmetry equation (2.4). Here, our discussion will closely follow some of the original literature [4, 5]. Let us first evaluate the direct channel, i.e. the left-hand side of the crossing equation, in the regime $v \ll u \ll 1$. Using the result (2.20) for the limit of the blocks one finds

$$\sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}\psi^{(14)(23)}_{\mathcal{O}}(u,v) \sim 1+C^2_{\phi\phi\mathcal{O}_*}\frac{\Gamma(2\bar{h}_*)}{\Gamma(\bar{h}_*)^2}\left[v^{h_*}\left(\log u^{-1} + \mathcal{O}(u^0)\right) + \mathcal{O}(v^{h_*+1})\right] + \mathcal{O}(v^{h>h_*}). \tag{2.34}$$

Here, we denote the leading-twist operator in the operator product of ϕ with itself by $\mathcal{O}_\star$, and we assume it is unique. Note that uniqueness does not apply to generalized free-field (GFF) theory, where the leading twist is infinitely degenerate and the operator product takes the form

$$\phi(X_1)\phi(X_4) = 1 + \sum_{n=0}^{\infty} X_{14}^n \sum_{J\in 2\mathbb{Z}_+} C_{\phi\phi[\phi\phi]_{n,J}} f_{\phi\phi[\phi\phi]_{n,J}}(X_1 - X_4, \partial_{X_4})[\phi\phi]_{n,J}(X_4). \tag{2.35}$$

Here we have introduced the standard notation $[\phi\phi]_{n,J}$ for the double-twist operators of the form

$$[\phi\phi]_{n,J;\mu_1,\ldots\mu_J}(X) = \phi(X)\,\Box^n\,\partial_{\mu_1}\cdots\partial_{\mu_J}\phi(X)\ . \tag{2.36}$$

The label J denotes the spin of the field, while $\tau_{n,J} = 2\Delta_\phi + 2n$ is the twist in GFF theory. In terms of our parameters h and $\bar{h}$ this means

$$h_{[\phi\phi]_{n,J}} = \tau_{n,J}/2 = \Delta_\phi + n \quad , \quad \bar{h}_{[\phi\phi]_{n,J}} = \Delta_\phi + n + J\ . \tag{2.37}$$

Note that if ϕ is a real scalar field, then only *even* spins will appear in the $\phi\times\phi$ OPE, see e.g. [2, footnote 12]. As one can read off from these formulas, the operator product (2.35) contains an infinite tower of operators $\{[\phi\phi]_{n,J}\}_{n,J\in 2\mathbb{Z}_+}$ at leading twist. There are degeneracies that may be lifted by interactions.

Now, analytic bootstrap teaches us that for a general conformal field theory, with an isolated operator contribution like $\mathcal{O}_\star$ in eq. (2.34), there exist towers of operators whose

OPE coefficients and twists approach those of GFF double-twist operators at large spin. The main goal is to find precise expressions for how their OPE data is corrected. Note also that $C^2_{\phi\phi\mathcal{O}_*}\mathcal{O}(v^{h_*+1})$ in eq. (2.34) corresponds to the contributions of descendants of the leading-twist operator, whereas the next-to-leading-twist operator $\mathcal{O}'$ would give $C^2_{\phi\phi\mathcal{O}'}\mathcal{O}(v^{h'})$. In many cases of interest, we can expect several $h' < h_\star + 1$ primaries to dominate over the first descendants. In the 3D Ising model, for example, ϵ and $T_{\mu\nu}$ give important contributions [22].

We will now re-derive two of the important results of the lightcone bootstrap, the computation of the OPE coefficients $C_{\phi\phi[\phi\phi]_{0,J}}$ and the anomalous dimensions $h_{[\phi\phi]_{0,J}}-\Delta_\phi$, to leading order in the large J limit. Here $[\phi\phi]_{0,J}$ is the family of operators whose twists approach the GFF double twist tower for $n=0$. Given our results (2.32) on the lightcone limit of crossed-channel blocks, we can write the s-channel sum on the right-hand side of the crossing symmetry equation (2.4) in the $v \ll u \ll 1$ limit as

$$\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\psi^{(12)(34)}(u,v) =$$
$$1+\int_{\mathcal{O}(\epsilon^0_{14})}^{\infty}\frac{\mathrm{d}\lambda_{\mathcal{O}}}{4\sqrt{\lambda_{\mathcal{O}}}}\mathcal{N}^{\mathrm{LS}}C^2_{\phi\phi\mathcal{O}}u^{h^{\min}_{\mathcal{O}}}K_0(2\sqrt{\lambda_{\mathcal{O}}v})\left(1+\mathcal{O}(v^{\frac{\tau}{2}})\right)+\sum_{\lambda_{\mathcal{O}}=\mathcal{O}(\epsilon^0_{14})}\frac{\log v^{-1}}{\mathrm{B}_{\bar{h}}}\left(1+\mathcal{O}(v^0)\right). \tag{2.38}$$

In the integral over large Casimir eigenvalues, $h^{\min}_{\mathcal{O}}$ is the minimal twist that appears in the operator product of ϕ with itself for which the Casimir $\lambda_{\mathcal{O}}$ is not bounded above. If this $h^{\min}_{\mathcal{O}}$ is finite (which it usually is), then this implies that $\mathcal{O}$ has large spin

$$\bar{h}_{\mathcal{O}} = J_{\mathcal{O}} + h^{\min}_{\mathcal{O}} = \sqrt{\lambda_{\mathcal{O}}}+\mathcal{O}(\epsilon^0_{12}). \tag{2.39}$$

Comparing the two expressions (2.34) and (2.38) for the limiting behaviors of the two sides of the crossing equation (2.4) we deduce to leading order

$$1+\mathcal{O}(v^{h_\star}) = v^{\Delta_\phi}u^{-\Delta_\phi}\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\psi^{(12)(34)}(u,v). \tag{2.40}$$

One immediately observes that the s-channel sum must yield a divergent power law $v^{-\Delta_\phi}$ to leading order — this eliminates the finite sum over primaries with $\lambda_{\mathcal{O}} = \mathcal{O}(v^0)$, since these contributions diverge *logarithmically* at most. We thus need to solve

$$1+\dots = v^{\Delta_\phi}\int_{\mathcal{O}(1)}^{\infty}\frac{\mathrm{d}\lambda_{\mathcal{O}}}{4\sqrt{\lambda_{\mathcal{O}}}}\mathcal{N}^{\mathrm{LS}}C^2_{\phi\phi\mathcal{O}}u^{h^{\min}_{\mathcal{O}}-\Delta_\phi}K_0(2\sqrt{\lambda_{\mathcal{O}}v})+\dots \tag{2.41}$$

This equation requires us to impose $h^{\min}_{\mathcal{O}} = \Delta_\phi = 2h_\phi$, from which we deduce that the left-hand side of the crossing symmetry equations is obtained from the exchange of the double-twist operators $[\phi\phi]_{0,J}$ with spins J that scale as

$$\sqrt{\lambda} = J+\mathcal{O}(1) = \mathcal{O}(v^{-1/2})\ .$$

If we make such a scaling Ansatz, we then find

$$\mathcal{N}^{\mathrm{LS}}C^2_{\phi\phi\mathcal{O}} = c_0\frac{8J^{\beta-1}}{\Gamma(\beta/2)^2}\left(1+\mathcal{O}(J^{-\tau})\right), \tag{2.42}$$

and using the auxiliary formula

$$\int_{\mathcal{O}(\sqrt{v})}^{\infty} \frac{\mathrm{d}y}{y} \left(\frac{y}{2}\right)^{\beta} K_{\alpha}(y) = \frac{1}{4}\Gamma\left(\frac{\beta+\alpha}{2}\right)\Gamma\left(\frac{\beta-\alpha}{2}\right) + \mathcal{O}(v^{\frac{\beta-\alpha}{2}}), \quad \beta > \alpha, \tag{2.43}$$

we reduce the leading order terms of the crossing symmetry equation to

$$1 + \dots = c_0 v^{\Delta_\phi - \frac{\beta}{2}} + \ldots \Rightarrow \mathcal{N}^{\mathrm{LS}} C^2_{\phi\phi\mathcal{O}} = \frac{8J^{2\Delta_\phi - 1}}{\Gamma(\Delta_\phi)^2}\left(1 + \mathcal{O}(J^{-\tau})\right). \tag{2.44}$$

This is the same as the large spin limit of the OPE coefficients of double-twist operators in a GFF theory [54], obtained for a more general conformal field theory using the lightcone bootstrap.

Moving on to the next-to-leading order in the crossing equation, we have

$$\begin{aligned} &1 + \frac{C^2_{\phi\phi\mathcal{O}}}{\mathrm{B}(\bar{h}_\star)} v^{h_\star}\left(\log u^{-1} + \mathcal{O}(u^0)\right) + \mathcal{O}(v^{h>h_\star}) = \\ &v^{\Delta_\phi}\int_{\mathcal{O}(1)}^{\infty} \frac{\mathrm{d}J}{J}\frac{2J^{2\Delta_\phi}}{\Gamma(\Delta_\phi)^2} u^{h_{[\phi\phi]_{0,J}} - \Delta_\phi} K_0(2J\sqrt{v})(1 + \mathcal{O}(J^{-\tau})). \end{aligned} \tag{2.45}$$

Here we once again discarded the finite spin contributions with $\mathcal{O}(\log v)$ divergence at most. To reproduce $v^{h_\star}\log u$ asymptotics, one makes the following ansatz for leading anomalous dimension corrections to double-twist operators at large spin:

$$h_{[\phi\phi]_{0,J}} = \Delta_\phi + \frac{\gamma}{2J^{2h_\star}} + \mathcal{O}\left(J^{-\tau}\right), \quad \tau > 2h_\star. \tag{2.46}$$

Expanding $v^{h_\star}$ in the t-channel, we obtain

$$\frac{C^2_{\phi\phi\mathcal{O}_\star}}{\mathrm{B}_{\bar{h}_\star}} = -\frac{\gamma}{2}\frac{\Gamma(\Delta_\phi - h_\star)^2}{\Gamma(\Delta_\phi)^2}. \tag{2.47}$$

Note that the anomalous dimension correction vanishes when $h_\star = \Delta_\phi$. This is the signature of GFF: the leading-twist operators are $[\phi\phi]_{0,J}$, and their scaling dimensions are not anomalous. Analogous results for the $n > 0$ families in eq. (2.37) can be obtained by focusing on higher powers of u in eqs. (2.41) and (2.45) [8, 9].

In the previous analysis, we have seen that we can only reproduce the leading terms on the left-hand side of the crossing symmetry equations by summing over large spin double-twist operators on the right-hand side, i.e. with the help of crossed channel lightcone blocks (2.32) for which the Casimir eigenvalue λ scales as $\lambda \sim \epsilon_{14}^{-1}$. In [22], Simmons-Duffin gave a simple proof of this fact. The argument is based on the observation that the Casimir operator $\mathcal{D}^2_{12}$ has a stable action on any finite spin subspace in the (12) channel, whereas its action on $v^{-\Delta_\phi}$ produces higher-order divergences in v. To see this in more detail, let us first define the vector space

$$\mathbb{V}_L := \mathrm{Span}\left\{\psi^{(12)(34)}_{\mathcal{O}} : h_{\mathcal{O}} = \Delta_\phi, \bar{h}_{\mathcal{O}} < L\right\}, \quad L < \infty. \tag{2.48}$$

We rewrite the crossing equation as

$$u^{\Delta_\phi} v^{-\Delta_\phi} + \dots = \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \psi^{(12)(34)}_{\mathcal{O}}(u, v). \tag{2.49}$$

Acting repeatedly with $\mathcal{D}_{12}^2$ on the identity contribution, we find

$$\begin{aligned}
&\mathcal{D}_{12}^2 u^{\Delta_\phi} v^{-\Delta_\phi} = \Delta_\phi(\Delta_\phi+1) u^{\Delta_\phi} v^{-(\Delta_\phi+1)} + \mathcal{O}(v^{-\Delta_\phi}) \\
\longrightarrow\ &\mathcal{D}_{12}^2 u^{\Delta_\phi} v^{-(\Delta_\phi+1)} = (\Delta_\phi+1)(\Delta_\phi+2) u^{\Delta_\phi} v^{-(\Delta_\phi+1)} + \mathcal{O}(v^{-(\Delta_\phi+1)}) \\
\longrightarrow\ &\ldots
\end{aligned}$$

As long as $\Delta_\phi \notin \mathbb{Z}_{<0}$, this series never truncates and $\mathcal{D}_{14}^2$ stabilizes only on an infinite dimensional vector space $\mathrm{Span}(u^{\Delta_\phi} v^{-(\Delta_\phi+n)})_{n\in\mathbb{Z}_{>\geq 0}}$ — we say that $(u/v)^{\Delta_\phi}$ is $\mathcal{D}_{12}^2$-singular. It follows that the expansion of $(u/v)^{\Delta_\phi}$ in eigenvectors of $\mathcal{D}_{12}^2$ has *infinite* support. In short,

$$(u/v)^{\Delta_\phi} \notin \mathbb{V}_L, \quad \forall L < \infty. \tag{2.50}$$

In general, a direct channel contribution is reproduced by large spin operators in the crossed channel if and only if it is $\mathcal{D}$-singular for some Casimir differential operator $\mathcal{D}$ in the crossed channel. In five- and six-point crossing equations, where we obtain more unconventional asymptotics in the lightcone limit of the direct channel, we will demonstrate that they yield large spin conformal field theory data by virtue of being Casimir-singular.

3 Lightcone blocks for five points

The aim of this section is to develop the technology that is needed to determine the lightcone limits of multipoint conformal blocks from the differential equations they satisfy. Our constructions are carried out in the case of $N = 5$, for which lightcone blocks have been studied through the lightcone OPE, so that we can compare many of our findings with previous results, most notably those obtained in [26]. On the other hand, the approach we follow here can be carried out for arbitrary topology of the OPE channel, and in particular for comb channels where no other approach exists. Even for $N = 5$ insertion points, our approach adds to the study of [26] in two respects. On the one hand, for any exchanged twists h_1, h_2, we are able to construct lightcone blocks explicitly in a limit where only four of the five cross-ratios are taken to zero. This will give us a much higher resolution in the bootstrap analysis later on. In addition, we can put to a stringent test and provide evidence for one of the central assumptions of [26], namely that it is possible to compute crossed-channel lightcone blocks by taking limits of an integral formula that is valid only in the direct channel.

In the first subsection, we review some basic notations that are needed in dealing with five-point functions and their blocks. We will also introduce the crossing symmetry equation to be studied in the next section. This equation provides us with a precise understanding of the relevant limit and a list of blocks whose lightcone behavior we need to determine. The relevant lightcone blocks are then computed in the remainder of the section, first for the direct channel and then for the crossed one. The special case of three-dimensional lightcone blocks with parity-odd tensor structures is treated in appendix E, and is based on the results present in the main text. As we already saw in our discussion of four insertions, the differential equations are a powerful tool to determine the dependence of the lightcone blocks on the cross-ratios, but they leave an overall normalization undetermined. To find the normalizations, we need to connect the lightcone limit to the OPE limit. In order to do so, we shall use a strategy that is closely modeled after the strategy we explained for four-point blocks in the previous section.

3.1 Preliminaries on blocks and lightcone limits

Five-point functions in $d \geq 3$ are parametrized by five cross-ratios. The following polynomial cross-ratios for five-point functions were first constructed in [42],

$$u_1 = \frac{X_{12}X_{34}}{X_{13}X_{24}}, \qquad v_1 = \frac{X_{14}X_{23}}{X_{13}X_{24}}, \qquad U_1^{(5)} = \frac{X_{15}X_{23}X_{34}}{X_{24}X_{13}X_{35}}. \tag{3.1}$$
$$u_2 = \frac{X_{23}X_{45}}{X_{24}X_{35}}, \qquad v_2 = \frac{X_{25}X_{34}}{X_{24}X_{35}},$$

It will often be convenient to use the "snowflake" cross-ratios $u_{si}, i = 1, \ldots, 5$ used in [24, 26]. In terms of our polynomial cross-ratios, the cross-ratios u_{si} read

$$u_{s1} = u_1/v_2 = \frac{X_{12}X_{35}}{X_{13}X_{25}}, \qquad u_{s2} = v_1 = \frac{X_{14}X_{23}}{X_{13}X_{24}}, \tag{3.2}$$
$$u_{s3} = v_2 = \frac{X_{25}X_{34}}{X_{24}X_{35}}, \qquad u_{s4} = u_2/v_1 = \frac{X_{45}X_{13}}{X_{35}X_{14}}, \tag{3.3}$$
$$u_{s5} = \frac{U^5}{v_1 v_2} = \frac{X_{15}X_{24}}{X_{14}X_{25}}. \tag{3.4}$$

These cross-ratios have the simple shift properties $u_{s(i+1)} = u_{si}|_{X_i \to X_{i+1}}$ as we shift the label i of the external scalar fields by one unit.

While we shall mostly work with the snowflake cross-ratios, there is one more set of cross-ratios that will play some role below, namely the so-called OPE cross-ratios $z_1, z_2, \bar{z}_1, \bar{z}_2$ and $\mathcal{X} := 1 - w$ that were introduced in [42]. These are the five-point analog of the cross-ratios z and $\bar{z}$ that Dolan and Osborn introduced to study the OPE limit. In the case of five-point functions, the OPE limit corresponds to the regime

$$\mathrm{OPE}^{(12)3(45)} : \bar{z}_1, \bar{z}_2 \ll z_1, z_2 \ll 1, \tag{3.5}$$

For the purposes of this paper, it suffices to record the map from the snowflake cross ratios to the OPE cross-ratios near the region where $\bar{z}_{1,2}$ are small,

$$u_{s1} = \frac{\bar{z}_1 z_1}{1 - z_2}(1 + \mathcal{O}(\bar{z}_2)), \qquad u_{s2} = 1 - z_1 + \mathcal{O}(\bar{z}_1), \tag{3.6}$$
$$u_{s4} = \frac{\bar{z}_2 z_2}{1 - z_1}(1 + \mathcal{O}(\bar{z}_1)), \qquad u_{s3} = 1 - z_2 + \mathcal{O}(\bar{z}_2), \tag{3.7}$$
$$u_{s5} = 1 - \frac{z_1 z_2 (1 - w)}{(1 - z_1)(1 - z_2)} + \mathcal{O}(\bar{z}_1, \bar{z}_2). \tag{3.8}$$

The behavior of conformal blocks in this OPE limit provides boundary conditions that are used to select a particular solution of the differential equations.

For five points there exist a total number of 15 different channels which are all of a unique (comb-channel) topology. Here we shall look at one particular pair of channels that

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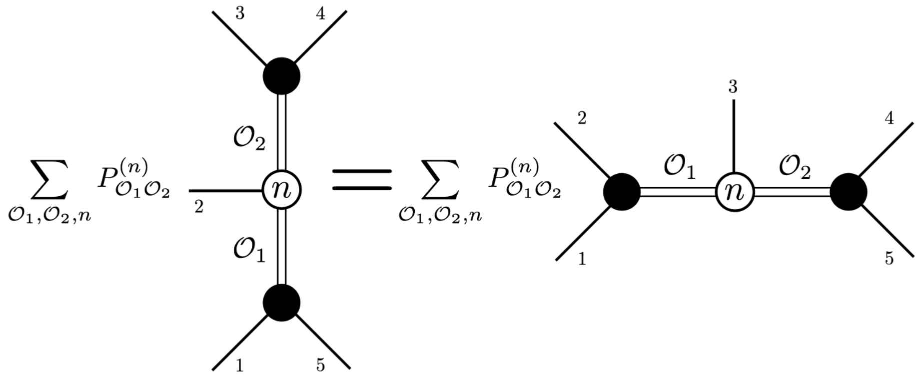

Figure 1. Graphical representation of the planar five-point crossing symmetry equation studied in this paper. The crossed channel is obtained from the direct channel by cyclic permutation of points, $i \to i+1$ mod 5.

is obtained by shifting the label of the external fields $i \mapsto 1 + i \bmod 5$ by one unit, see figure 1. Let us stress that in the case of five points, this duality can be used to generate all others by iterated application since all OPE channels have the same topology, as in the case of four points. We will fix the cyclic enumeration of external points to be such that

$$\mathrm{DC} = (51)2(34) \quad \textit{and} \quad \mathrm{CC} = (12)3(45) \tag{3.9}$$

as shown in figure 1. To fix our conventions for the conformal blocks let us display the expansion in the crossed channel

$$\langle \phi(X_1) \dots \phi(X_5) \rangle = \left(X_{12} X_{45} \sqrt{\frac{X_{23} X_{34}}{X_{24}}} \right)^{-\Delta_\phi} \sum_{\mathcal{O}_1, \mathcal{O}_2, n} P^{(n)}_{\mathcal{O}_1 \mathcal{O}_2} \psi^{\mathrm{CC}}_{\mathcal{O}_1 \mathcal{O}_2; n}(u_{si}), \tag{3.10}$$

where the coefficients P are determined by the coefficients C of the involved operator products as

$$P^{(n)}_{\mathcal{O}_1 \mathcal{O}_2} := C_{\phi\phi\mathcal{O}_1} C^{(n)}_{\mathcal{O}_1 \phi \mathcal{O}_2} C_{\phi\phi\mathcal{O}_2}, \tag{3.11}$$

Here $\mathcal{O}_a, a = 1, 2$, denote two STT primary fields of weight $\Delta_a = h_a + \bar{h}_a$ and spin $J_a = \bar{h}_a - h_a$. Given the two spin labels J_1 and J_2, the parameter $n = 0, \dots, \min(J_1, J_2)$ labels the basis of tensor structures of the three-point function in the expansion

$$\langle \mathcal{O}_1(X_1, Z_1) \phi(X_3) \mathcal{O}_2(X_2, Z_2) \rangle = \sum_{n=0}^{\min(J_1, J_2)} C^{(n)}_{\mathcal{O}_1 \phi \mathcal{O}_2} \prod_{i<j} X_{ij}^{n_{ij}} J_{1,32}^{J_1} J_{2,31}^{J_2} \mathcal{X}^n, \quad \mathcal{X} := \frac{H_{12} X_{13} X_{23}}{J_{1,32} J_{2,31}}, \tag{3.12}$$

where $\mathcal{X}$ is a three-point cross-ratio that corresponds precisely to the OPE limit reduction of the five-point cross-ratio $(1-w)$ when projecting to exchanged STT primaries. The objects H_{ij} and $J_{i,jk} \equiv V_{i,jk} X_{jk}$ are tensor structures used to parametrize three-point functions, see [55]. This choice of basis in the space of three-point tensor structures implies that the five-point blocks in eq. (3.10) with expansion coefficients (3.11) satisfy the normalization condition

$$\lim_{z_1, z_2 \to 0} \lim_{\bar{z}_1, \bar{z}_2 \to 0} \frac{\psi^{(12)3(45)}_{\mathcal{O}_1 \mathcal{O}_2; n}(z_a, \bar{z}_a, w)}{\prod_{a=1}^2 \bar{z}_a^{h_a} z_a^{\bar{h}_a} (1-w)^n} = 1, \tag{3.13}$$

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see [42] for notations and the derivation. With all our conventions fixed, the crossing equation we want to analyze takes the form

$$\sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\rm DC}_{\mathcal{O}_1\mathcal{O}_2;n} = \left(\frac{u_{s5}\sqrt{u_{s3}}}{u_{s4}\sqrt{u_{s1}}}\right)^{\Delta_\phi} \sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\rm CC}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{3.14}$$

Up to relabeling, this is the same planar crossing equation as in [26, section (3.1)]. To evaluate the constraints that arise from this crossing symmetry equation we shall consider a regime in which

$$X_{15} \ll X_{34} \ll X_{12} \ll X_{45} \ll X_{23} \ll 1 \tag{3.15}$$

i.e. we make pairs of neighboring points light-like separated in the order that is given by reading the previous line from left to right. While the first two limits favor internal leading-twist exchanges in the direct channel, the following two have the same effect on crossed-channel exchanges. The last limit in which we place x_3 on the lightcone of x_2 is not that fundamental and in fact it will be important for some of our results to include corrections to limiting behavior as we send X_{23} to zero. The limit we consider here amounts to introducing the following order, see previous section,

$$\vec{\epsilon} = (\epsilon_{15}, \epsilon_{34}, \epsilon_{12}, \epsilon_{45}, \epsilon_{23})$$

With respect to this prescription, the cross-ratios u_{si} are of order $\mathcal{O}(\epsilon_{i,i+1})$. At the same time, the transition between direct and crossed channel is also easily expressed in terms of these cross-ratios u_{si}.

$$\psi^{(12)3(45)}(u_{si}) = \psi^{(51)2(34)}(u_{s,i-1}) \tag{3.16}$$

We conclude that for the crossing equation we are about to study, the cross-ratios u_{si} are the perfect generalization of the cross-ratios u, v we used in the discussion of four-point functions. Once again, the duality between the two channels corresponds to a cyclic permutation of the cross-ratios and each of the cross-ratios has unit order with respect to exactly one of the order parameters ϵ_{ij} we are taking to zero. On the other hand, the cross-ratios u_{si} are simple rational functions of our polynomial cross-ratios (3.1). Hence, when expressed in terms of the u_{si}, all terms in the Casimir differential operators possess some definite $\vec{\epsilon}$-order.

3.2 Lightcone blocks in the direct channel

As in the four-point case that began this discussion, we shall perform the lightcone limit of the five-point blocks for the direct channel in two stages, starting with only that part of the limit that exposes leading-twist contributions in the direct channel

$$X_{15} \ll X_{34} \ll 1\,. \tag{3.17}$$

As we shall discuss in the first subsection, going to the partial lightcone regime is sufficient for the blocks to possess a relatively simple integral representation that is amenable to further evaluation. Since the OPE-limit point at which we normalize our blocks lies within

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the regime (3.17), see discussion in the previous subsection, the limiting blocks are also easy to normalize. For our bootstrap analysis we will only need the lightcone blocks in the (almost) full lightcone limit (3.15). These are derived in the second and third subsections by sending the remaining three cross-ratios to zero one by one.

3.2.1 Partial lightcone limit

In terms of the cross-ratios u_{si}, the partial lightcone limit (3.17) amounts to considering the order $\vec{\epsilon}^{(2)} = (\epsilon_{15}, \epsilon_{34})$. In terms of cross-ratios, the associated regime is given by

$$LCL^{(2)}_{\vec{\epsilon}}: \quad u_{s5} \ll u_{s3} \ll 1 \ .$$

Here, we use the superscript (2) to signal that we perform only the first two limits. The following analysis and in particular the normalization of the limiting blocks turn out to be simplest when written in terms of OPE cross-ratios. In general, the relation between the snowflake and OPE cross-ratios is somewhat complicated, but in the regime $LCL^{(2)}$, i.e. after sending u_{s5} and u_{s3} to zero, the relation simplifies. We had seen this for the crossed channel already in the eqs. (3.8). In order to treat the direct channel we recall that after the shift of the index i that brings us from the crossed to the direct channel, the snowflake cross-ratios are related to the OPE cross-ratios of the direct channel via the relation

$$(u_{s5}, u_{s1}, u_{s2}, u_{s3}, u_{s4}) = \left(\frac{z_1}{1-z_2}\bar{z}_1, 1-z_1, 1-z_2, \frac{z_2}{1-z_1}\bar{z}_2, 1-\frac{(1-w)z_1z_2}{(1-z_1)(1-z_2)}\right) + \mathcal{O}(\bar{z}_a). \tag{3.18}$$

The inverse map is

$$(\bar{z}_1, \bar{z}_2, z_1, z_2, w) = \left(\frac{u_{s5}u_{s2}}{(1-u_{s1})}, \frac{u_{s3}u_{s1}}{(1-u_{s2})}, 1-u_{s1}, 1-u_{s2}, 1-\frac{u_{s1}u_{s2}(1-u_{s4})}{(1-u_{s1})(1-u_{s2})}\right) + \mathcal{O}(u_{s5,3}). \tag{3.19}$$

In comparison to the OPE cross-ratios we have discussed in the previous subsection, the index of the snowflake cross-ratios is now shifted by one unit, i.e. the OPE cross-ratios in this subsection are the ones relevant for the OPE limit in the direct channel.

After these brief comments on coordinates, we are now ready to display and study the Casimir differential equations in the lightcone regime $LCL^{(2)}$. If we use the OPE cross-ratios, the Casimir operators take the form

$$\mathcal{D}^{2,\vec{\epsilon}^{(1)}}_{15} = \epsilon^0_{15}\epsilon^0_{34}\left(z_1\, {}_2\mathcal{D}_1\left(\hat{A}_1, 0; 0; z_1, \partial_{z_1}\right) + \bar{z}_1\partial_{\bar{z}_1}(\bar{z}_1\partial_{\bar{z}_1} - d + 1)\right) + \mathcal{O}(\epsilon_{15}), \tag{3.20}$$

$$\hat{A}_1 = \vartheta_{z_2} - h_\phi + w(\vartheta_{\bar{z}_2} - \vartheta_{z_2} + \vartheta_w) \tag{3.21}$$

$$\mathcal{D}^{2,\vec{\epsilon}^{(1)}}_{34} = \epsilon^0_{15}\epsilon^0_{34}\left(z_2\, {}_2\mathcal{D}_1\left(\hat{A}_2, 0; 0; z_2, \partial_{z_2}\right) + \bar{z}_2\partial_{\bar{z}_2}(\bar{z}_2\partial_{\bar{z}_2} - d + 1)\right) + \mathcal{O}(\epsilon_{34}), \tag{3.22}$$

$$\hat{A}_2 = \vartheta_{z_1} - h_\phi + w(\vartheta_{\bar{z}_1} - \vartheta_{z_1} + \vartheta_w), \tag{3.23}$$

where the operator ${}_2\mathcal{D}_1$ was defined in eq. (2.11). We will now study the associated Casimir eigenvalue equations and determine their solutions in order to show that

$$\psi^{\text{DC}}_{(h_a,\bar{h}_a;n)}(u_{si}(\bar{z}_a, z_a, w)) \overset{LCL^{(2)}}{\sim} \prod_{a=1}^{2} \bar{z}_a^{h_a} z_a^{\bar{h}_a}(1-w)^n \tilde{F}_{(h_a,\bar{h}_a;n)}(z_1, z_2, w), \tag{3.24}$$

where $\tilde{F}$ is given by the following double integral over two integration variables t_1 and t_2

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = \prod_{a\neq b}\int_0^1 \frac{\mathrm{d}t_a}{\mathrm{B}_{\bar{h}_a}} \frac{(t_a(1-t_a))^{\bar{h}_a-1}(1-wz_at_a)^{J_b-n}}{(1-z_1t_1-z_2t_2+wz_1z_2t_1t_2)^{\bar{h}_{12;\phi}}}. \tag{3.25}$$

Recall that the cross-ratios z_i, w can be considered as functions of the snowflake cross-ratios u_{s1}, u_{s2} and u_{s4} that are not sent to zero in the lightcone regime we consider. Let us note that indeed the right hand side of eq. (3.24) is correctly normalized since one infers from eq. (3.25) that

$$\lim_{z_1,z_2\to 0}\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = 1 \tag{3.26}$$

and the factor in front of $\tilde{F}$ gives the usual behavior of the block in the OPE limit. The remainder of this subsection is devoted to the derivation of eqs. (3.24), (3.25).

Intuitively, it is rather clear that the two second-order Casimir equations suffice to reconstruct the dependence of the blocks on z_1, z_2 from the behavior in the OPE limit where the z_a are sent to zero. In order to give a formal proof, we first note that our second-order Casimir equations for the lightcone blocks imply

$$\left(\mathcal{D}^{2\vec{\epsilon}}_{15} - \bar{h}_1(\bar{h}_1-1) - h_1(h_1-d+1)\right)\psi_{(h_a,\bar{h}_a;n)}(u_{si}) \propto \mathcal{D}_1\cdot\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = 0 \tag{3.27}$$

$$\left(\mathcal{D}^{2\vec{\epsilon}}_{45} - \bar{h}_2(\bar{h}_2-1) - h_2(h_2-d+1)\right)\psi_{(h_a,\bar{h}_a;n)}(u_{si}) \propto \mathcal{D}_2\cdot\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = 0, \tag{3.28}$$

where $\propto$ means that we dropped some non-vanishing overall prefactor. The operators $\mathcal{D}_a, a = 1,2$ that act on the function $\tilde{F}$ are given by

$$\begin{aligned}\mathcal{D}_a =& \vartheta_{z_a}(2\bar{h}_a+\vartheta_{z_a}-1) - z_a(\bar{h}_a+\vartheta_{z_a})(\bar{h}_{12;\phi}+\vartheta_{z_1}+\vartheta_{z_2}-\vartheta_w)\\ &- wz_a(\bar{h}_a+\vartheta_{z_a})(n-J_b-\vartheta_{z_b}+\vartheta_w),\end{aligned} \tag{3.29}$$

where $b\in\{1,2\}$ is chosen such that $b\neq a$. We will now determine a unique solution to this system of differential equations subject to the OPE limit boundary condition (3.26).

Our strategy will be to solve these differential equations for some special 2-dimensional submanifolds first and then to reconstruct the whole function using the various solutions as boundary condition. Let us first note that the two differential equations can be solved very explicitly if we set either $w = 0$ or $w = 1$. For $w = 1$, the solution can easily be obtained in terms of Gauss' hypergeometric functions as

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w=1) = \prod_{a\neq b} {}_2F_1\left[\begin{matrix}\bar{h}_a, \bar{h}_a+h_{b\phi}+n\\ 2\bar{h}_a\end{matrix}\right](z_a). \tag{3.30}$$

When we set $w = 0$, the set of differential equations may be seen to coincide with the equations that characterize Appell's hypergeometric function F_2. After matching all the parameters we find

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w=0) = F_2\left(\bar{h}_{12;\phi};\bar{h}_1,\bar{h}_2;2\bar{h}_1,2\bar{h}_2;z_1,z_2\right). \tag{3.31}$$

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Similarly, we can solve our system of differential equations if we set one of the variables $z_b = 0$. Once again, we can construct the solution in terms of Appell's hypergeometric functions, only this time it is in terms of F_1,

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_a,w)|_{z_b=0} = F_1\left(\bar{h}_c;\bar{h}_{12;\phi},n-J_b;2\bar{h}_c;z_c,wz_c\right). \tag{3.32}$$

Here $c \in \{1,2\}$ with $c \neq b$. In order to be completely explicit about our conventions concerning the Appell functions F_1 and F_2 we used in the last two formulas, let us state their series expansions,

$$F_1(b;a_1,a_2;c;z_1,z_2) := \sum_{m_1,m_2=0}^{\infty} \frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}(a_1)_{m_1}(a_2)_{m_2}\frac{(b)_{m_1+m_2}}{(c)_{m_1+m_2}} \tag{3.33}$$

$$F_2(a;b_1,b_2;c_1,c_2;z_1,z_2) := \sum_{m_1,m_2=0}^{\infty} \frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!}(a)_{m_1+m_2}\frac{(b_1)_{m_1}}{(c_1)_{m_1}}\frac{(b_1)_{m_1}}{(c_1)_{m_1}}. \tag{3.34}$$

Note that all of these solutions may be obtained by rewriting the action of the Casimir operators on a power series in z and/or wz as a recursion relation with initial condition governed by the OPE limit.

To find a general expansion of $\tilde{F}_{(h_a,\bar{h}_a;n)}$ outside of the $z_b = 0$ limit, we can make use of the relations

$$\mathcal{D}_c\vartheta_{z_b} = \vartheta_{z_b}\mathcal{D}_c, \quad \mathcal{D}_c(wz_b)^k = (wz_b)^k\mathcal{D}_c, \quad \mathcal{D}_c z_b^k = z_b^k\mathcal{D}_c|_{J_b\to J_b+k}, \tag{3.35}$$

where we assumed $c \neq b$ and in the last relation we take $\bar{h}_b = h_b + J_b$ while keeping h_b fixed. In the vicinity of $z_b = 0$ we can conclude that the general solution admits an expansion of the form

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = \sum_{m=0}^{\infty} f_{b,m}(wz_b)z_b^m F_1\left(\bar{h}_c;\bar{h}_{12;\phi}+m,n-J_b-m;2\bar{h}_c;z_c,wz_c\right) \tag{3.36}$$

with some coefficients $f_{b,m}$ that need to be determined. Note that we have introduced two different expansions here which depend on which of the two variables $z_b, b = 1,2$ we set to zero in the leading term. Each of the two distinct expansions for $b = 1,2$ are explicitly in the kernel of $\mathcal{D}_c$. At this stage, the functions $f_{b,m}(wz_b)$ can be solved explicitly by recasting $\mathcal{D}_b f = 0$ as a recursion relation in m. However, we can find the general solution more efficiently by making use of the integral representation of the Appell F_1:

$$F_1\left(\bar{h}_c;\bar{h}_{12;\phi}+m,n-J_b-m;2\bar{h}_c;z_c,wz_c\right) = \int_0^1 \frac{\mathrm{d}t_c}{\mathrm{B}_{\bar{h}_c}}\frac{(t_c(1-t_c))^{\bar{h}_c-1}}{(1-z_a t_c)^{\bar{h}_{12;\phi}+m}}(1-wz_c t_c)^{m+J_b-n}, \tag{3.37}$$

which is equivalent to the convergent power series (3.33) for $0 \leq z_c, wz_c < 1$. Assuming we can commute the sum over m and the integral over t_c, we can rewrite the expansion (3.36) as

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(z_1,z_2,w) = \int_0^1 \frac{\mathrm{d}t_c}{\mathrm{B}_{\bar{h}_c}}\frac{(t_c(1-t_c))^{\bar{h}_c-1}}{(1-z_c t_c)^{\bar{h}_{12;\phi}}}(1-wz_c t_c)^{J_b-n} f_b\left(\frac{1-wz_c t_c}{1-z_c t_c}z_b, wz_b\right), \tag{3.38}$$

where $f_b(x,y) = \sum_m f_{b,m}(y)x^m$ is a power series in each variable. At $z_c = 0$, where we know that $\tilde{F}_{(h_a,\bar{h};n)}$ is an Appell F_1, the above integral simplifies to $f_b(z_b, wz_b)$, such that

$$f_b(x,y) = F_1(\bar{h}_b; \bar{h}_{12;\phi}, n - J_c; 2\bar{h}_b; x, y) \ .$$

Plugging this in the integral representation for the Appell function F_1 we obtain the formula (3.25). We have thus established that in the lightcone regime $LCL^{(2)}$, the direct channel blocks are given by eqs. (3.24) and (3.25). Note that all we used were the limiting expressions (3.27) and (3.28) for the Casimir operators.

Generalized Euler transformation. In our derivation, the double integral of eq. (3.25) appeared merely as a tool to make the power series in z_1, z_2, w that solves the Casimir equations more compact. To write the latter, one can expand the integrand explicitly into a power series in z_a, wz_a in the domain $0 \leq z_a, wz_a < 1$ before integrating each summand to a product of Euler Beta functions. However, in our applications to the lightcone bootstrap, we must analyze the asymptotics of blocks near certain edges $z_a, w \to 1$ of this domain, where the formal hypergeometric series need not converge. Here, the integral provides a useful tool to analyze the convergence properties near the singular regions, as is well-known from the classical theory of hypergeometric functions. In particular, two singular regions relevant to lightcone bootstrap take the form of a double scaling limit in OPE cross-ratios:

$$z_a = 1 + \mathcal{O}(\epsilon_a), \quad w = 1 + \mathcal{O}(\epsilon_a), \quad \epsilon_a \to 0, \tag{3.39}$$

for $a = 1$ or 2. When restricted to two-dimensional submanifolds $z_b = 0$ or $w = 1$, the asymptotics of the blocks (${}_2F_1$ or Appell F_1) in the above limit depend on the parameters h_b, h_ϕ, n. In particular, on the $w = 1$ submanifold, the parameter dependence of $z_a \to 1$ asymptotics is captured by the Euler transformation of the Gauss hypergeometric function,

$${}_2F_1\left[\begin{matrix}\bar{h}_a + h_{b\phi} + n, \bar{h}_a \\ 2\bar{h}_a\end{matrix}\right](z_a) = (1-z_a)^{-(h_{b\phi}+n)}{}_2F_1\left[\begin{matrix}\bar{h}_a - h_{b\phi} - n, \bar{h}_a \\ 2\bar{h}_a\end{matrix}\right](z_a). \tag{3.40}$$

The series on the left-hand side diverges while the series on the right-hand side converges if $h_{b\phi+n} > 0$, and vice versa when $h_{b\phi} + n < 0$.[4] In the integral representation (3.25) evaluated at $w = 1$, this relation is obtained from the change of variables

$$\tilde{t}_a = \frac{1 - t_a}{1 - z_a t_a}.$$

We would now like to find an appropriate generalization of the Euler transformation (3.40) away from the $w = 1$ submanifold that controls the parameter-dependence of the lightcone limits $(z_a, w) = (1,1) + \mathcal{O}(\epsilon_a)$, $a = 1, 2$. To this end, we first introduce a new set of variables $(v_a(z_a), x(z_1, z_2, w))$ defined as

$$v_a(z_a) := 1 - z_a, \quad x(z_1, z_2, w) := \frac{z_1 z_2 (1-w)}{(1-z_1)(1-z_2)}. \tag{3.41}$$

[4]In the case where $h_{b\phi} + n = 0$ we saw in section 2 that the hypergeometric function admits a logarithmic divergence at $z_a = 1$.

In v, x cross-ratios, the double scaling limit of OPE cross-ratios defined in eq. (3.39) is equivalent to the limit $v_a \to 0$ at v_b, x fixed. The change of variables is simple to invert,

$$z_a(v_a) = 1 - v_a, \quad w(v_1, v_2, x) = 1 - \frac{v_1 v_2 x}{(1-v_1)(1-v_2)}. \tag{3.42}$$

Now, if we apply the same change of variables $\tilde{t}_a = (1 - z_a t_a)^{-1}(1 - t_a)$ to the double integral in eq. (3.25) and express z_a, w in terms of v_a, x using eq. (3.42), then we obtain the desired "generalized Euler transformation" of the form

$$\tilde{F}_{(h_a,\bar{h}_a;n)}(1-v_1, 1-v_2, w(v_1,v_2,x)) = \prod_{a=1}^{2} v_a^{-(h_{b\phi}+n)} F_{(h_a,\bar{h}_a;n)}(v_1, v_2, x), \tag{3.43}$$

where the function F on the right-hand side is given by the integral

$$F_{(h_a,\bar{h}_a;n)}(v_1,v_2,x) = \prod_{b\neq c} \int_0^1 \frac{\mathrm{d}t_b}{\mathrm{B}_{\bar{h}_b}} \frac{(t_b(1-t_b))^{\bar{h}_b - 1}(1-(1-v_b)t_b)^{h_{c\phi}+n-\bar{h}_b}}{(1+\frac{v_b x}{1-v_b}(1-t_c))^{n-J_b}(1-x(1-t_1)(1-t_2))^{\bar{h}_{12;\phi}}}. \tag{3.44}$$

For $h_{b\phi} + n > 0$ and $0 \leq x < 1$, this expression evaluates at $v_a = 0$ to a convergent integral that can be expanded into a power series in x.

We conclude this subsection with a couple of important comments. One nice feature of the integral formulas (3.25) and (3.44) we have derived is that the two integrations decouple in the limit in which we send w to one (see eq. (3.30) for the function $\tilde{F}$) or, equivalently, x to zero. For the function F, one has

$$\lim_{x\to 0} F_{(h_a,\bar{h}_a;n)}(v_1,v_2,x) = f_1^{\mathrm{dec}}(v_1) f_2^{\mathrm{dec}}(v_2) \tag{3.45}$$

where each factor f_a^{dec} depends only on a single cross-ratio and is given by

$$f_a^{\mathrm{dec}}(v) = {}_2F_1\left[\begin{matrix}\bar{h}_a, \bar{h}_a - h_{b\phi} - n \\ 2\bar{h}_a\end{matrix}\right](1-v) \quad a \neq b = 1,2. \tag{3.46}$$

Finally, recall that we derived these integrals with the help of the Casimir equations. It is actually possible to obtain the same integral formulas through the lightcone OPE, in close analogy to what we explained in our discussion of the four-point lightcone blocks. The details for five external points can be found in appendix A.1. In order to obtain the first integral representation (3.25) one needs to choose the gauge,

$$(s_1, s_2, s_4, s_5) = (r_1 t_1, r_1(1-t_1), r_2(1-t_2), r_2 t_2), \quad r_1, r_2 \in \mathbb{R}^\times, \tag{3.47}$$

see appendix A.1 for notations. The derivation of the second integral formula (3.44) from the lightcone OPE is carried out in some detail in the same appendix.

3.2.2 The full lightcone limit

From the integral formulas for blocks in the regime $LCL^{(2)}$ that we obtained in the previous subsection, we now want to move towards the full lightcone limit by sending the remaining cross-ratios to zero, one by one.

Sending u_{s1} to zero. Let us start with the cross ratio u_{s1} so that we pass from $LCL^{(2)}$ to the new regime

$$LCL^{(3)}_{\vec{\epsilon}} : u_{s5} \ll u_{s3} \ll u_{s1} \ll 1. \tag{3.48}$$

Our plan is to derive the asymptotics of the blocks by evaluating the integral expressions in eqs. (3.25) or (3.44) at $u_{s1} = 0$. In the case of the expression (3.24), the blocks formally evaluate to

$$\begin{aligned}\frac{\psi^{\mathrm{DC}}_{(h_a,\bar{h}_a;n)}(u_{si})}{u_{s5}^{h_1} u_{s2}^{h_1+n}(1-u_{s2})^{J_2-n}} &\overset{LCL^{(3)}}{\sim} u_{s1}^{h_2+n} u_{s3}^{h_2}(1-u_{s4})^n \tilde{F}_{(h_a,\bar{h}_a;n)}(1, 1-u_{s2}, 1) \\ &= u_{s1}^{h_2+n} u_{s3}^{h_2}(1-u_{s4})^n {}_2F_1\left[\begin{matrix}\bar{h}_1, \bar{h}_1 + h_{2\phi} + n \\ 2\bar{h}_1\end{matrix}\right](1)\, {}_2F_1\left[\begin{matrix}\bar{h}_2, \bar{h}_2 + h_{1\phi} + n \\ 2\bar{h}_1\end{matrix}\right](1-u_{s2}).\end{aligned} \tag{3.49}$$

Note, however, that the first Gauss hypergeometric series with argument $z_1 = 1 - u_{s1} = 1$ does not converge when $h_\phi \leq h_2 + n$. On the other hand, it evaluates to the following finite combination of Γ functions in the complementary case:

$${}_2F_1\left[\begin{matrix}\bar{h}_1, \bar{h}_1 + h_{2\phi} + n \\ 2\bar{h}_1\end{matrix}\right](1) = \frac{\Gamma(2\bar{h}_1)\Gamma(h_\phi - h_2 - n)}{\Gamma(\bar{h}_1 - h_{2\phi} - n)\Gamma(\bar{h}_1)} \quad \mathit{for} \quad h_\phi > h_2 + n\ . \tag{3.50}$$

Let us now look at the second integral formula (3.43). Evaluating the limit $u_{s1} = 0$ yields

$$\frac{\psi^{\mathrm{DC}}_{(h_a,\bar{h}_a;n)}(u_{si})}{u_{s5}^{h_1} u_{s2}^{h_\phi}(1-u_{s2})^{J_2-n}} \overset{LCL^{(3)}}{\sim} u_{s1}^{h_\phi} u_{s3}^{h_2}(1-u_{s4})^n F_{(h_a,\bar{h}_a;n)}(0, u_{s2}, 1-u_{s4}). \tag{3.51}$$

In this case, one can show that $F_{(h_a,\bar{h}_a;n)}(0, u_{s2}, x)$ evaluates to a convergent power series in $1 - u_{s2}$ and $x = 1 - u_{s4}$ *only* if $h_2 + n > h_\phi$, see eq. (A.41) in appendix A.3. This is precisely the regime in which the functions $\tilde{F}$ from the first integral expression were singular. We can understand this parameter dependence of the $u_{s1} \to 0$ asymptotics in terms of the Casimir differential operators. More specifically, for the second-order Casimir $\mathcal{D}^2_{15}$ we find

$$\mathcal{D}^{2,\vec{\epsilon}^{(3)}}_{15} = \epsilon^0_{15}\left\{\epsilon^{-1}_{12}\mathcal{D}^{(0,0,-1)}_{15} + \mathcal{O}(\epsilon^0_{12})\right\} + \mathcal{O}(\epsilon_{15}), \tag{3.52}$$

$$\mathcal{D}^{(0,0,-1)}_{15} = \left(\partial_{u_{s1}} - \frac{\vartheta_{u_{s3}} + \vartheta_{1-u_{s4}}}{u_{s1}}\right)(\vartheta_{u_{s1}} - h_\phi). \tag{3.53}$$

Note that the operator contains a single singular term. As long as we are interested in direct channel contributions that arise from fields with finite weight and spin, the eigenvalues of the quadratic Casimir operators do not scale and hence, the leading term of the relevant lightcone blocks must lie in the kernel of the singular term. At the same time, any second-order differential equation admits two independent solutions to the kernel condition. But our equation is second order only in the derivative $\partial_{u_{s1}}$ with respect to the variable that we are sending to zero. Hence, depending on the parameter h_ϕ in the right factor the operator (3.53), there is a unique solution that dominates the behavior at small u_{s1}. For $h_\phi > h_2 + n$ we see that the function in the second line of eq. (3.49) is indeed annihilated by the linear operator in the left factor of the Casimir operator (3.53) (after commuting it to the right) and hence it lies in the kernel. On the other hand, the function on the right-hand side of equation (3.51) lies in the kernel of $\vartheta_{u_{s1}} - h_\phi$ and hence also in the kernel of the Casimir operator.

It remains to discuss the case $h_\phi = h_2 + n$. When this equality holds, $\tilde{F}_{(h_a,\bar{h}_a;n)}(1-u_{s1}, 1-u_{s2}, 1-u_{s4}) = F_{(h_a,\bar{h}_a;n)}(u_{s1}, u_{s2}, 1-u_{s4})$ and we expect both expressions to diverge as u_{s1} is sent to zero. More specifically, in the case where $\vartheta_{u_{s3}}\psi = h_2\psi$ and $\vartheta_{1-u_{s4}}\psi = n\psi$, the Casimir equation admits a logarithmic solution

$$\left(\partial_{u_{s1}} - \frac{\vartheta_{u_{s3}} + \vartheta_{1-u_{s4}}}{u_{s1}}\right)(\vartheta_{u_{s1}} - h_2 - n)u_{s3}^{h_2}(1-u_{s4})^n u_{s1}^{h_2+n}\log u_{s1} = 0. \tag{3.54}$$

In accordance with this solution, a power series expansion of $F_{\mathcal{O}_1\mathcal{O}_2;n}(u_{s1}, u_{s2}, x)$ around $x = 0$ shows that only the x^0 term exhibits a logarithmic divergence, while all higher order terms x^m are $\mathcal{O}(1)$ at most, i.e.

$$F_{\mathcal{O}_1\mathcal{O}_2;n}(u_{s1}, u_{s2}, 0) = \frac{\log u_{s1}^{-1}}{\mathrm{B}_{\bar{h}_1}} f_2^{\mathrm{dec}}(u_{s2}) + \mathcal{O}(u_{s1}^0), \quad \partial_x^m F_{\mathcal{O}_1\mathcal{O}_2;n}(u_{s1}, u_{s2}, x)|_{x=0} = \mathcal{O}(u_{s1}^0), \tag{3.55}$$

for all integers $m > 0$. Here we have used the symbol f_2^{dec} as a shorthand for the hypergeometric function (3.46) that we introduced above. A more detailed derivation of this result can be found in appendix A.3. In summary, we obtain the following asymptotics in the $u_{s1} \ll 1$ limit of our partial lightcone blocks,

$$\frac{\psi^{\mathrm{DC}}_{(h_a,\bar{h}_a;n)}(u_{si})}{u_{s5}^{h_1}u_{s3}^{h_2}(1-u_{s4})^n u_{s2}^{h_\phi}(1-u_{s2})^{J_2-n}} \overset{LCL^{(3)}}{\sim} \begin{cases} \mathrm{B}_{\bar{h}_1}^{-1}\mathrm{B}_{\bar{h}_1,|h_{2\phi}+n|}u_{s1}^{h_2+n}f_2^{\mathrm{dec}}(u_{s2}), & h_2+n<h_\phi, \\ \mathrm{B}_{\bar{h}_1}^{-1}u_{s1}^{h_\phi}\log u_{s1} f_2^{\mathrm{dec}}(u_{s2}), & h_2+n=h_\phi, \\ F_{(h_a,\bar{h}_a;n)}(0,u_{s2},1-u_{s4}), & h_2+n>h_\phi. \end{cases} \tag{3.56}$$

Sending u_{s4} to zero. We can now move on the next level by sending the cross-ratio u_{s4} to zero, i.e. we study lightcone blocks in the regime

$$LCL_{\vec{\epsilon}}^{(4)} : u_{s5} \ll u_{s3} \ll u_{s1} \ll u_{s4} \ll 1 \,. \tag{3.57}$$

Unlike the $u_{s1} \to 0$ limit, there is no sign of a divergence when evaluating each expression in our formula (3.56) at $u_{s4} = 0$. Once again, this convergent behavior is corroborated by the analysis of Casimir differential operators. Indeed, in the regime $LCL_{\vec{\epsilon}}^{(4)}$, we find the leading order singular terms to be

$$\mathcal{D}_{15}^{(0,0,-1,-1)} = \left(\partial_{u_{s1}} - \frac{h_\phi}{u_{s1}}\right)\partial_{u_{s4}}, \quad \mathcal{D}_{34}^{(0,0,0,-1)} = \partial_{u_{s4}}\left(\vartheta_{u_{s4}} - \vartheta_{u_{s1}} - \vartheta_{1-u_{s2}} - \frac{1-u_{s2}}{u_{s2}}h_\phi\right). \tag{3.58}$$

The lightcone block in our regime $LCL_{\vec{\epsilon}}^{(4)}$ must lie in the kernel of both operators. As one can immediately notice, any function that is independent of u_{s4} lies in the common kernel of the two leading order singular terms. We conclude that the asymptotics of blocks in the lightcone limit is given by

$$\boxed{\frac{\psi^{\mathrm{DC}}_{(h_a,\bar{h}_a;n)}(u_{si})}{u_{s5}^{h_1}u_{s3}^{h_2}(1-u_{s4})^n u_{s2}^{h_\phi}(1-u_{s2})^{J_2-n}} \overset{LCL^{(4)}_{\vec{\epsilon}}}{\sim} \begin{cases} \mathrm{B}^{-1}_{\bar{h}_1}\mathrm{B}_{\bar{h}_1,|h_{2\phi}+n|}u_{s1}^{h_2+n}f_2^{\mathrm{dec}}(u_{s2}), & h_2+n<h_\phi, \\ \mathrm{B}^{-1}_{\bar{h}_1}u_{s1}^{h_\phi}\log u_{s1}^{-1}f_2^{\mathrm{dec}}(u_{s2}), & h_2+n=h_\phi, \\ (1-u_{s2})^{n-J_2}g^{\mathrm{fin}}_{(h_a,\bar{h}_a;n)}(1-u_{s2}), & h_2+n>h_\phi, \end{cases}} \tag{3.59}$$

where the new functions $g^{\mathrm{fin}}(z)$ can be written as the following double integral

$$\begin{aligned} g^{\mathrm{fin}}_{(h_a,\bar{h}_a;n)}(z) &:= z^{J_2-n}F_{(h_a,\bar{h}_a;n)}(0,1-z,1) = \int_0^1 \frac{\mathrm{d}t_2}{t_2(1-t_2)}\frac{(t_2(1-t_2))^{\bar{h}_2}}{\mathrm{B}_{\bar{h}_2}}(1-zt_2)^{h_{1\phi}+n-\bar{h}_2}. \\ &\int_0^1 \frac{\mathrm{d}t_1}{t_1(1-t_1)}\frac{t_1^{\bar{h}_1}(1-t_1)^{h_{2\phi}+n}}{\mathrm{B}_{\bar{h}_1}}(1-(1-z)t_1)^{J_2-n}(1-(1-t_1)(1-t_2))^{-\bar{h}_{12;\phi}}. \end{aligned} \tag{3.60}$$

While the convergence of this integral is not immediately clear, note that it converges at $z=0,1$ to the following quantities:

$$\begin{aligned} g^{\mathrm{fin}}_{(h_a,\bar{h}_a;n)}(0) &= \frac{\Gamma(2\bar{h}_1)\Gamma(2\bar{h}_2)\Gamma(\bar{h}_2-h_\phi)\Gamma(h_\phi)}{\Gamma(\bar{h}_1)\Gamma(\bar{h}_2)\Gamma(\bar{h}_2+h_\phi)\Gamma(\bar{h}_{12;\phi})}, \\ g^{\mathrm{fin}}_{(h_a,\bar{h}_a;n)}(1) &= \prod_{a\neq b=1}^{2}\frac{\Gamma(2\bar{h}_a)\Gamma(h_{b\phi}+n)}{\Gamma(\bar{h}_a)\Gamma(\bar{h}_a+h_{b\phi}+n)} \\ &\quad {}_3F_2\left(\bar{h}_{12;\phi}, h_{1\phi}+n, h_{2\phi}+n; h_{1\phi}+n+\bar{h}_2, h_{2\phi}+n+\bar{h}_1; 1\right). \end{aligned}$$

Consequently, we can expand the integrand of eq. (3.60) and integrate each term to obtain a convergent power series for $0\leq z\leq 1$ of the form

$$\begin{aligned} g^{\mathrm{fin}}_{(h_a,\bar{h}_a;n)}(z) =& \frac{\Gamma(2\bar{h}_2)\Gamma(h_{1\phi}+n)}{\Gamma(\bar{h}_2)\Gamma(\bar{h}_2+h_{1\phi}+n)}\sum_{k=0}^{\infty}\frac{(\bar{h}_{12;\phi})_k}{k!}\frac{(\bar{h}_1)_k}{(2\bar{h}_1)_k}\frac{(h_{1\phi}+n)_k}{(h_{1\phi}+n+\bar{h}_2)_k} \\ & {}_2F_1\left[\begin{matrix}\bar{h}_1-h_{2\phi}-n,\bar{h}_1\\ 2\bar{h}_1+k\end{matrix}\right](z)\, {}_2F_1\left[\begin{matrix}n-J_1,\bar{h}_2\\ \bar{h}_2+h_{1\phi}+n+k\end{matrix}\right](1-z). \end{aligned} \tag{3.61}$$

Sending u_{s2} to zero. Finally, to reach the full lightcone regime $LCL_{\vec{\epsilon}}$, we take the leading $u_{s2}\to 0$ asymptotics of eq. (3.59) and the expansion (3.61) of g^{fin}. Before we spell out the final result, let us introduce the following family of functions

$$f_\beta(v) = \begin{cases} v^\beta, & \text{if} \quad \beta<0, \\ -\log v, & \text{if} \quad \beta=0, \\ 1, & \text{if} \quad \beta>0. \end{cases} \tag{3.62}$$

It is defined such that we are able to write the dependence of the direct channel blocks on the cross-ratios u_{s1},u_{s2} in a single line

$$\boxed{\psi^{\mathrm{DC}\,(0)}_{(h_a,\bar{h}_a;n)}(u_{si}) \overset{LCL_{\vec{\epsilon}}}{\sim} \mathcal{N}^{\mathrm{DC}\,(0)}_{(h_a,\bar{h}_a;n)}(u_{s1}u_{s2})^{h_\phi}u_{s5}^{h_1}u_{s3}^{h_2}\,f_{n+h_{2\phi}}(u_{s2})f_{n+h_{1\phi}}(u_{s1}).} \tag{3.63}$$

Note that f_β depends on whether β is negative, zero, or positive and hence the single line we have displayed is indeed capable of summarizing the limiting behavior of the three lines in eq. (3.59). It remains to work out the normalization $\mathcal{N}$. From the normalizations we listed in eq. (3.59) as well as the evaluation of $g^{\text{fin}}_{(h_a,\bar{h}_a;n)}(1)$ we obtain

$$\boxed{\mathcal{N}^{\text{DC}\,(0)}_{(h_a,\bar{h}_a;n)} = \begin{cases} \prod\limits_{a\neq b} \frac{\Gamma(2\bar{h}_a)}{\Gamma(\bar{h}_a)\Gamma(\bar{h}_a+|h_{b\phi}+n|)} \mathring{\Gamma}(|h_{a\phi}+n|), & \text{iff} \quad h_{1\phi}+n \leq 0 \textit{ or } h_{2\phi}+n \leq 0 \\ \prod\limits_{a\neq b} \frac{\Gamma(2\bar{h}_a)\Gamma(h_{b\phi}+n)}{\Gamma(\bar{h}_a)\Gamma(\bar{h}_a+h_{b\phi}+n)} \, {}_3F_2 \begin{bmatrix} \bar{h}_{12;\phi}, & h_{1\phi}+n, & h_{2\phi}+n \\ & \bar{h}_1+h_{2\phi}+n, & \bar{h}_2+h_{1\phi}+n \end{bmatrix} (1) \, . \end{cases}} \tag{3.64}$$

The second line applies whenever the conditions in the first line are not satisfied, i.e. if $h_{1\phi}+n$ and $h_{2\phi}+n$ are both positive. There, note that the ${}_3F_2(1)$ converges for any $h_\phi > 0$. In the first line, we used the 'regularized' Γ function $\mathring{\Gamma}$ which coincides with $\Gamma(x) = \mathring{\Gamma}(x)$ for $x > 0$ but is defined as $\mathring{\Gamma}(0) := \text{Res}_{x=0}\Gamma(x) = 1$ at $x = 0$ where $\Gamma(x)$ has a pole. As a final comment, we would like to emphasize the improved analytical control we have built through the differential equations in comparison with previous works on five-point lightcone bootstrap [24, 26]. First, we extend the asymptotics of leading-twist blocks to the regime of twists $h_1, h_2 \geq h_\phi$ on the second line of eq. (3.64). Second, we are granted access to less restrictive limits such as $LCL^{(4)}_{\vec{\epsilon}}$ in eq. (3.59). The latter will allow us to bootstrap double-twist OPE coefficients with finite tensor structure labels in section 5.3. This concludes our discussion of the direct channel lightcone blocks.

3.3 Lightcone blocks in the crossed channel

Our goal in the remainder of this section is to compute those crossed channel blocks that we will later need for the analysis of the crossing symmetry equation. In order to do so, we shall study the crossed-channel Casimir equations in the lightcone limit. As we have seen before, however, these equations cannot fully determine the lightcone blocks since they are linear second-order equations with multiple independent solutions. So some work is required to select solutions that describe the lightcone limit of actual conformal blocks, which are uniquely determined by their behavior in the OPE limit. In the first subsection, we will take a first look at the leading terms in the Casimir equations and sketch the strategy we use in order to find the relevant solution that describes the limiting behavior of blocks. This strategy is then implemented in detail in the two subsequent subsections.

3.3.1 Crossed channel Casimir equations

Let us now use the Casimir operators $\mathcal{D}^2_{12}$ and $\mathcal{D}^2_{45}$ to study the behavior of crossed channel blocks in the lightcone limit. To leading $\vec{\epsilon}$-order these operators read

$$\mathcal{D}^{2,\vec{\epsilon}}_{12} = \frac{1}{\epsilon_{15}}\left(\epsilon^0_{34}\left(\frac{1}{\epsilon_{23}}\left(\epsilon^0_{12}\epsilon^0_{45}\mathcal{D}^{(-1,0,-1)}_{12} + \mathcal{O}(\epsilon_{12})\right) + \mathcal{O}(\epsilon^0_{23})\right) + \mathcal{O}(\epsilon^0_{15})\right), \tag{3.65}$$

$$\mathcal{D}^{(-1,0,-1)}_{12} = \left(\partial_{u_{s2}} - \frac{h_\phi}{u_{s2}}\right)\partial_{u_{s5}}, \tag{3.66}$$

$$\mathcal{D}^{2,\vec{\epsilon}}_{45} = \frac{1}{\epsilon_{15}}\left(\frac{1}{\epsilon_{34}}\epsilon^0_{23}\left(\epsilon^0_{12}\epsilon^0_{45}\mathcal{D}^{(-1,-1,0)}_{12} + \mathcal{O}(\epsilon_{56})\right) + \mathcal{O}(\epsilon^0_{34})\right) + \mathcal{O}(\epsilon^0_{15}), \tag{3.67}$$

$$\mathcal{D}^{(-1,-1,0)}_{45} = \left(\partial_{u_{s3}} - \frac{h_\phi}{u_{s3}}\right)\partial_{u_{s5}}. \tag{3.68}$$

Since all the terms we spelled out have vanishing grade with respect to the formal variables $\epsilon_{12}, \epsilon_{45}$ we shortened the label $(\vec{k})$ and only displayed three entries, i.e. the superscript on the right-hand side only gives the grades with respect to $(\epsilon_{15}, \epsilon_{34}, \epsilon_{23})$.

We are going to analyze the Casimir eigenvalue equations for the crossed channel in two different cases that will turn out to be relevant below. But as we explained in the introduction, the analysis depends crucially on whether the eigenvalue of the operator scales in the same way as the most singular term or not. Correspondingly, we are going to study two very different cases that will turn out to be relevant for the discussion of the bootstrap constraints. We call these case I and case II respectively.

Case I. In this case, we assume the eigenvalues λ_1 and λ_2 to scale such that the products $\lambda_1 u_{s2} u_{u_{s5}}$ and $\lambda_1 u_{s3} u_{s5}$ stay finite in the lightcone limit. This means that

$$\text{LS}_\text{I}: \quad \lambda^{\vec{\epsilon}}_1 = \epsilon^{-1}_{15}\epsilon^{-1}_{23}\lambda_1, \quad \lambda^{\vec{\epsilon}}_2 = \epsilon^{-1}_{15}\epsilon^{-1}_{34}\lambda_2, \tag{3.69}$$

i.e. both eigenvalues scale in the same way as the most singular term of the associated Casimir operator.

Case II. In this case, we assume that only one of the eigenvalues scales like the singular term. For definiteness, let this be the eigenvalue λ_2 and assume that we scale λ_1 such that $\lambda_1 u_{s5}$ remains finite, i.e.

$$\text{LS}_\text{II}: \quad \lambda^{\vec{\epsilon}}_1 = \epsilon^{-1}_{15}\lambda_1, \quad \lambda^{\vec{\epsilon}}_2 = \epsilon^{-1}_{15}\epsilon^{-1}_{34}\lambda_2. \tag{3.70}$$

This list is by no means complete. Clearly, we could also consider the case in which the scaling behavior of both eigenvalues λ_1 and λ_2 is subleading. Of course, we could change the scaling behavior of the subleading eigenvalue, e.g. keep them finite, etc. But fortunately, case I and case II are (almost) all we need.

Before we dive into the computation of these wave functions, first for case I and then for case II, we want to explain our general strategy for how to select and normalize solutions of the Casimir equations to describe the lightcone limit of the actual blocks. The procedure we shall use mimics to a certain extent the strategy we outlined for four points, but the additional fifth variable u_{s5} (or w in terms of OPE cross-ratios) that arises from the need to

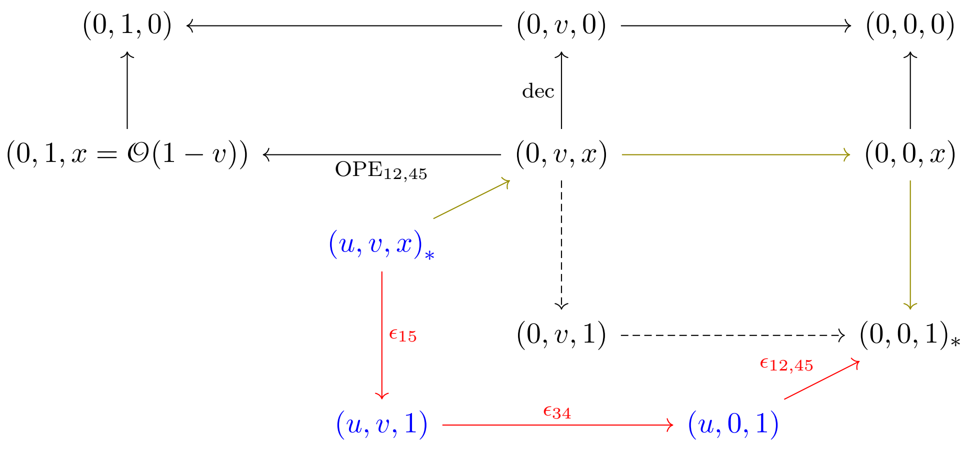

Figure 2. Diagrammatic representation of the space of five-point cross-ratios. To make this plot 3-dimensional, we have combined u_{s1} and u_{s4} into a single letter u and we omitted u_{s2}. The variable $v = u_{s3}$ increases from right to left, while $x = 1 - u_{s5}$ increases from top to bottom. Our letter u corresponds to the direction that leaves the plane of the paper. From the generic point $(u, v, x)_*$ in the center of the diagram we can reach the lightcone limit $(0, 0, 1)_*$ in the bottom right corner in the direct (green) and the crossed (red) channel. The top half of the figure contains the OPE limit $(0, 1, x)$ on the left which provides boundary conditions for blocks.

parametrize the choice of tensor structures at the middle vertex causes some new challenges, in particular when it comes to normalizing our lightcone blocks.

In order for the discussion to apply to both case I and case II blocks, we will keep the variable u_{s2} separate and group the remaining four variables into three groups u, v, x, where u denotes the pair $u = \{u_{s1}, u_{s4}\}$ while $v = u_{s3}$ and $x = 1 - u_{s5}$. Once we have formed these three groups, we can draw a three-dimensional map for the space of cross-ratios, see figure 2. The black vertices in this diagram label points in the $u = 0$ plane while the blue vertices have $u \neq 0$. Moreover, v varies along the horizontal direction while x varies along the vertical direction. We have marked the generic point with cross-ratios (u, v, x) by an index $*$. The lightcone limit we explore below corresponds to the vertex $(0, 0, 1)_*$ in the figure. In the crossed channel, we reach this point from generic cross-ratios by going along the red arrows. We want to compute the lightcone blocks by studying the Casimir differential equations at the limit point. As we shall see, it is actually not that difficult to solve the limiting differential equations. The only issue is, however, that we have to pick the solution that satisfies the OPE boundary conditions. Now, the OPE limit is actually the point with coordinates $(0, 1, 0)$ in the upper left corner of figure 2, very far away from the lightcone limit. The only thing the two points have in common is that they both lie in the $u = 0$ plane.

Fortunately, within this $u = 0$ plane, the blocks are under relatively good control — one can infer this from our discussion of the direct channel blocks in the previous subsection. It implies that we can actually write down an integral representation that is quite amenable to evaluation. By adapting our formulas (3.24) and (3.43) to the crossed-channel setup, we have

$$\psi^{\mathrm{CC}}_{(h_a,\bar h_a;n)}(u_{si}) \overset{u=0}{\sim} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi - n} z_1^{\bar h_1 - h_1} z_2^{\bar h_2 - h_2} (1-w)^n F_{(h_a,\bar h_a;n)}(u_{s2}, u_{s3}, 1-u_{s5}), \tag{3.71}$$

where the function F is given by the integral (3.44), only that now the arguments are $v_1 = u_{s2}, v_2 = u_{s3}$ and $x = 1 - u_{s5}$. In our regime $u = \{u_{s1}, u_{s4}\} = 0$ the variables z_i, w are computed from the non-zero snowflake cross-ratios through

$$z_1 = 1 - u_{s2}\,, \quad z_2 = 1 - u_{s3}\,, \quad 1 - w = \frac{u_{s2}u_{s3}(1-u_{s5})}{(1-u_{s2})(1-u_{s3})} \tag{3.72}$$

Compared to the previous formula, we have just shifted the index of the snowflake cross-ratios by one to account for the different labeling of external points we use in the crossed channel.

As we have seen before, we can write a closed-form expression for the function F all along the upper edge $x = 0$ of figure 2:

$$\lim_{x\to u}\lim_{u\to 0} F_{(h_a,\bar h_a;n)}(u_{si}) = f_1^{\mathrm{dec}}(u_{s2}) f_2^{\mathrm{dec}}(u_{s3}) \tag{3.73}$$

where

$$f_a^{\mathrm{dec}}(v) = {}_2F_1\begin{bmatrix}\bar h_a, \bar h_a - h_{b\phi} - n \\ 2\bar h_a\end{bmatrix}(1-v), \quad a \neq b = 1,2. \tag{3.74}$$

In some sense, this formula is the analog of eq. (2.31) in our discussion of four-point blocks, only that now it holds merely for $x = 0$. In order to move from the upper edge of figure 2 down to the lightcone limit, we can start from the decoupling limit in the region where v is small (upper right corner) and then expand the integral formula in the variable x. Up to subleading corrections in the small u_{s3}, large J_2 limit (see appendix A.4), this expansion can be written in the form

$$F_{(h_a,\bar h_a;n)}(u_{si}) \;\sim\; \mathcal{Q}\left(\left(1 - x\mathcal{S}_1(\vartheta_{u_{s2}})\mathcal{S}_2(\vartheta_{u_{s3}})\right)^{h_\phi - \bar h_1 - \bar h_2}\right) f_1^{\mathrm{dec}}(u_{s2}) f_2^{\mathrm{dec}}(u_{s3}), \tag{3.75}$$

$$\text{where} \quad \mathcal{Q}\left(\mathcal{S}_a(\vartheta)^k\right) := \frac{(h_{b\phi} + n + \vartheta)_k}{(h_{b\phi} + n + \bar h_a)_k}, \quad a \neq b = 1,2. \tag{3.76}$$

and $\vartheta_u = u\partial_u$ is the Euler operator, as before. In order to evaluate the first line we first evaluate the argument of $\mathcal{Q}$ as a power series in $x\mathcal{S}_1\mathcal{S}_2$ and then apply the 'quantization map' $\mathcal{Q}$ defined in the second line in order to express powers of the commuting objects $\mathcal{S}_1$ and $\mathcal{S}_2$ as differential operators that act on functions of $v_1 = u_{s2}$ and $v_2 = u_{s3}$. It is this formula that will allow us to determine the normalization of the crossed channel lightcone blocks below.

3.3.2 Lightcone blocks for case I

We will begin our analysis of crossed channel lightcone blocks with the case in which the eigenvalues scale as in eq. (3.69). Looking back at the expressions (3.66) and (3.68), we note that both leading terms commute with the Euler operators $\vartheta_{u_{s1}}$ and $\vartheta_{u_{s4}}$. Consequently, we can expand the lightcone blocks in eigenfunctions of these two Euler operators. But there is another simple differential operator that commutes with the operators (3.66) and (3.68) as well as the two Casimir operators $\vartheta_{u_{s1}}$ and $\vartheta_{u_{s4}}$: the operator $\partial_{u_{s5}}$. Hence we can expand the lightcone limit of the crossed channel blocks as[5]

$$\psi^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)}(u_{si}) \overset{LCL_{\vec{\epsilon}}}{\sim} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi} \sum_\eta \langle \eta | n \rangle \, e^{-\eta u_{s5}} g_{h_a,\bar{h}_a;\eta}(u_{s2}, u_{s3}). \tag{3.77}$$

Here $\langle \eta | n \rangle$ is just a symbol for the unknown expansion coefficients. As in our discussion of the direct channel blocks, the OPE boundary condition

$$\psi^{\mathrm{CC}}_{(\bar{h}_a,h_a;n)}(u_{si}) \overset{\mathrm{OPE}}{\propto} u_{s1}^{h_1} u_{s4}^{h_2} + \ldots$$

implies that the eigenvalues of the Euler operators $\vartheta_{u_{s1}}$ and $\vartheta_{u_{s4}}$ are fixed and coincide with the quantum numbers h_1 and h_2 of the exchanged fields in the crossed channel. Giving an interpretation of the eigenvalue η and finding the precise coefficients $\langle \eta | n \rangle$ is however a bit more difficult.

Before we go into the details of this, let us provide a rather suggestive argument. To begin with, let us recall that the operator that measures the quantum number n that labels tensor structures in the decoupling limit is given by the Euler operator

$$\vartheta_x = \vartheta_{1-u_{s5}} = (u_{s5} - 1)\partial_{u_{s5}} \overset{LCL_{\vec{\epsilon}}}{\sim} -\partial_{u_{s5}} \ .$$

this may lead us to suspect that the eigenvalue η of the operator $\partial_{u_{s5}}$ in the lightcone limit is directly related to the quantum number n. In addition, let us also observe that the argument ηu_{s5} of the exponential can only be finite when u_{s5} goes to zero if the eigenvalue η scales as

$$\eta^{\vec{\epsilon}} = \epsilon_{15}^{-1} \eta \tag{3.78}$$

If we combine this fact with the expectation that $\eta = n$, then we are led to conclude that the lightcone limit of our blocks can only have a nontrivial dependence on the cross-ratio u_{s5} if we scale n such that nu_{s5} remains finite. Note that this scaling ansatz $n = \mathcal{O}(\epsilon_{15}^{-1})$ is consistent with the scaling of the vertex operator at leading order in the lightcone limit, see section 4 and the argument after equation (5.9). As an attentive reader may have noticed, the regime we are probing here is such that the tensor structure label $n = \mathcal{O}(\epsilon_{15}^{-1})$ scales faster than the spins $J_i = \sqrt{\lambda_i} = \mathcal{O}\left(\epsilon_{15}^{-\frac{1}{2}} \epsilon_{(i+1)(i+2)}^{-\frac{1}{2}}\right)$. This seemingly unphysical regime can nonetheless be defined in the continuum limit for the spins J_i that is used when replacing the sum over large spins with an integral, see appendix C for its construction based on the vertex differential operator.

Let us now turn to discuss the non-trivial functions $g(u_{s2}, u_{s3})$ of the two remaining cross-ratios $v_1 = u_{s2}$ and $v_2 = u_{s3}$. For these functions, the two Casimir equations reduce to the following set of linear differential equations

$$\eta \partial_{u_{s2,3}} g_{h_a,\bar{h}_a;\eta}(u_{s2}, u_{s3}) = -\lambda_{1,2} g_{h_a,\bar{h}_a;\eta}(u_{s2}, u_{s3}) \ . \tag{3.79}$$

[5] The summation over η is symbolic. Of course, in general one might expect a continuum of eigenvalues η to contribute in which case one would replace the sum with an integral. As we shall see, however, in this case the leading contribution arises from a single term, namely $\eta = n$.

These equations are solved by

$$g_{h_a,\bar{h}_a;\eta}(u_{s2},u_{s3}) = e^{-\frac{\lambda_1 u_{s2}+\lambda_2 u_{s3}}{\eta}}. \tag{3.80}$$

Let us stress that, by our assumption on the scaling behavior (3.69) of the eigenvalues λ_a, the argument of the exponential function is finite as we send the cross-ratios u_{s5}, u_{s2} and u_{s3} to zero.

Putting things together we can now conclude that the lightcone limit of our crossed channel blocks in case I is given by

$$\boxed{\psi^{\mathrm{CC},I}_{(h_a,\bar{h}_a;n)}(u_{si}) \overset{LCL_{\vec{\epsilon}}}{\sim} \mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi} e^{-nu_{s5}-\frac{\bar{h}_1^2 u_{s2}+\bar{h}_2^2 u_{s3}}{n}}}. \tag{3.81}$$

Let us stress again that the validity of this formula requires scaling the quantum numbers $\bar{h}_a$ and n such that $\bar{h}_1^2 u_{s2}/n, \bar{h}_2^2 u_{s3}/n$ and nu_{s5} stay finite as we sent the cross-ratios into the regime $LCL_{\vec{\epsilon}}$.

We still need to determine the normalization $\mathcal{N}^{\mathrm{CC,I}}$ and to justify the expectation $\eta = n$. In order to do so we note that the exponential term in the crossed channel lightcone blocks (3.81) drops out in the limit in which $\bar{h}_1^2 u_{s2}, \bar{h}_2^2 u_{s3} \ll n \ll u_{s5}^{-1}$. After taking the limit, the remaining dependence on cross-ratios coincides with that of the direct channel blocks (3.63) in case $h_{a\phi}+n > 0$ for both $a = 1, 2$, of course up to the obvious replacement $u_{si} \to u_{s,i-1}$. This suggests that we can simply compute the normalization of our crossed channel lightcone blocks by taking an appropriate limit of the direct channel normalization, i.e.

$$\mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)} = \lim_{n\to\infty} \lim_{\bar{h}_a\to\infty} \mathcal{N}^{\mathrm{DC}\,(0)}_{(h_a,\bar{h}_a;n)} = \lim_{n\to\infty} \lim_{\bar{h}_a\to\infty} \prod_{a\neq b} \frac{\Gamma(2\bar{h}_a)\Gamma(h_{b\phi}+n)}{\Gamma(\bar{h}_a)\Gamma(\bar{h}_a+h_{b\phi}+n)}. \tag{3.82}$$

The direct channel normalizations can be found in eq. (3.64), and lim is a shorthand for taking the leading term in the Stirling formula for the Gamma functions[6]. The evaluation with the help of Stirling's formula gives

$$\boxed{\mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)} = \frac{1}{2n}\left(\frac{n}{e}\right)^{2n} n^{h_1+h_2-2h_\phi} \prod_{a\neq b} 4^{\bar{h}_a} \bar{h}_a^{\frac{1}{2}-(h_{b\phi}+n)}}. \tag{3.83}$$

The two formulas (3.81) with the normalization (3.83) are indeed correct, even though our derivation was based on the identification that we did not derive rigorously yet.

In order to establish eqs. (3.81) and (3.83) rigorously, we go back to the expansion (3.77) and note that the formula (3.80) completely determines the dependence of each term in the expansion on the cross-ratios u_{s2} and u_{s3}. Hence, in order to determine the coefficients $\langle\eta|n\rangle$ and thereby the relation between η and n, it suffices to look at the limit where u_{s2} and u_{s3} tend to zero. In this limit, we now want to compare with the formula (3.75). Recall, that this formula computes the lightcone behavior from the decoupling limit in the direct channel.

[6] In particular, one can check that ${}_3F_2(1) \sim 1$ in the large $\bar{h}_1, \bar{h}_2$ limit of the normalization formula (3.64).

In the case of four-point functions, we pointed out that the leading terms of the crossed channel Casimir operators in the limit where $u \ll v$ coincide with the leading terms of the direct channel Casimir operators with u and v exchanged. A closely related statement holds true for five-point Casimirs. More specifically, the leading terms in the differential equations are the same regardless of whether we use the usual ordering prescription with $\vec{\epsilon} = (\epsilon_{15}, \epsilon_{34}, \epsilon_{12}, \epsilon_{45}, \epsilon_{23})$ or the one that is obtained by shifting all indices, $\vec{\epsilon}' = (\epsilon_{12}, \epsilon_{45}, \epsilon_{23}, \epsilon_{15}, \epsilon_{34})$,

$$\lim_{\vec{\epsilon}\to 0} \epsilon_{15}\epsilon_{23,34} \left(\mathcal{D}^{2,\vec{\epsilon}}_{12,45} - \lambda^{\vec{\epsilon}}_{1,2}\right) = \lim_{\vec{\epsilon}'\to 0} \epsilon_{15}\epsilon_{23,34} \left(\mathcal{D}^{2,\vec{\epsilon}}_{12,45} - \lambda^{\vec{\epsilon}}_{1,2}\right). \tag{3.84}$$

The left-hand side of this equation represents the red path in the diagram 2, while the right-hand side represents the green path, with the same scaling of λ_1, λ_2, n in both cases.

This suggests that we can indeed determine the matrix elements $\langle \eta | n \rangle$ in the expansion (3.77) from the formula (3.75). After re-instating the factor that was removed in order to pass from ψ to F, eq. (3.75) reads

$$\psi^{\mathrm{CC}}_{(h_a,\bar{h}_a;n)}(u_{si}) \sim u_{s1}^{h_1} u_{s2}^{h_2} (u_{s2}u_{s3})^{h_\phi} (1-u_{s5})^n \mathcal{Q}\left((1 - x\mathcal{S}_1(\vartheta_{u_{s2}})\mathcal{S}_2(\vartheta_{u_{s3}}))^{h_\phi - \bar{h}_1 - \bar{h}_2}\right)$$
$$f_1^{\mathrm{dec}}(u_{s2}) f_2^{\mathrm{dec}}(u_{s3}),$$

For definitions of the various objects, see eqs. (3.74), (3.76). In the limit where $u_{s2,3} \to 0$, we see that the only x-dependence comes from the factor $x^n = (1-u_{s5})^n$. In fact, since in case I the eigenvalues λ_a are sent to infinity with a scaling law (3.69), the operators $\mathcal{S}_a^k \sim \bar{h}_a^{-k}$ vanish in this regime. Hence, our task is to reproduce the factor $\exp(-\eta u_{s5})$ in eq. (3.77) from x^n. This requires that we scale n such that $n u_{s5}$ remains finite. Then, in the limit of large n, we find

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} \left(1 - \frac{n u_{s5}}{n}\right)^n = e^{-n u_{s5}}\ .$$

Comparing with the expansion (3.77) we obtain agreement provided that

$$\langle \eta | n \rangle = \delta_{\eta n} \mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)}.$$

In addition, we can compute the normalization $\mathcal{N}$ from the limiting behavior of the functions $f^{\mathrm{dec}}(v)$ as we send v to zero. The result coincides with our formula (3.83). This concludes our rigorous derivation of formulas (3.81) and (3.83).

3.3.3 Lightcone blocks for case II

In case II, one of the two eigenvalues is subleading and hence, in order to find eigenfunctions depending on the subleading eigenvalues, we must expand the corresponding Casimir operator to the desired order at least. Let us recall from eq. (3.70) that the eigenvalue λ_1 of the Casimir $\mathcal{D}^2_{12}$ is subleading in the last entry ϵ_{23} of our order $\vec{\epsilon}$ and we should include the next-to-leading order term in u_{23} which reads

$$\mathcal{D}_{12}^{(-1,0,0)} = \partial_{u_{s5}} \left(\vartheta_{u_{s5}} - \vartheta_{u_{s2}} - \vartheta_{u_{s3}} + h_\phi\right) \tag{3.85}$$

Once again, our notation suppresses the entries $\epsilon_{12}, \epsilon_{45} = 0$ in the superscript so that we only have three entries $(\epsilon_{15}, \epsilon_{34}, \epsilon_{23})$. We notice that this term still commutes with the Euler operators $\vartheta_{u_{s1}}$ and $\vartheta_{u_{s4}}$ for the cross-ratios u_{s1} and u_{s4}. As we explained before, when acting on lightcone blocks the value of these two Euler operators is fixed to h_1 and h_2, respectively. Hence, the lightcone blocks have the form

$$\psi^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;n)}(u_{si}) = u_{s1}^{h_1} u_{s4}^{h_2} g(u_{s2}, u_{s3}, u_{s5}) \; . \tag{3.86}$$

Before we continue our discussion of differential equations, let us pass from the blocks ψ to the functions F by splitting off the simple prefactor

$$\omega^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;n)} = u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2} u_{s3})^{h_\phi} (1 - u_{s2})^{J_1 - n} \; . \tag{3.87}$$

In passing from ψ to F, the simple dependence of the lightcone block on u_{s1} and u_{s4} is removed and one find

$$F^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;n)}(u_{si}) = F^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;n)}(u_{s2}, u_{s3}, u_{s5}). \tag{3.88}$$

To fix the dependence of the functions on the remaining variables we need to consider the eigenvalue equations

$$\mathcal{D}_{12}^{(-1,0,*)} : \partial_{u_{s5}} \left(\vartheta_{u_{s5}} - \vartheta_{1-u_{s2}} - \vartheta_{u_{s3}} - J_1 + n - h_\phi\right) F_{(h_a,\bar{h}_a;n)} = \lambda_1 F_{(h_a,\bar{h}_a;n)} \tag{3.89}$$

$$\mathcal{D}_{45}^{(-1,-1,0)} : \partial_{u_{s5}} \partial_{u_{s3}} F_{(h_a,\bar{h}_a;n)} = \lambda_2 F_{(h_a,\bar{h}_a;n)}. \tag{3.90}$$

The $*$ in the label of the first Casimir operator reminds us that the equation includes the term that is subleading in u_{s2}. Note that we have written these terms as differential operators for the function F, i.e. after removing ω. This needs to be taken into account when comparing with our expressions (3.66), (3.85) and (3.68) which were written for the action on ψ.

We observe that the two operators on the left-hand side of the differential equations (3.89) and (3.90) commute with the Euler operator $\vartheta_{1-u_{s2}} = (1 - u_{s2})\partial_{u_{s2}}$. This suggests to expand the functions F as

$$F_{(h_a,\bar{h}_a;n)}(u_{s2}, u_{s3}, u_{s5}) = \sum_\mu \langle \mu | n \rangle \, (1 - u_{s2})^\mu F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5}) \; . \tag{3.91}$$

Inserting this Ansatz into our eigenvalue equations and using the notation $J_1 - n := \delta n$ we obtain the following differential equations

$$\mathcal{D}_{12}^{(-1,0,*)} : \partial_{u_{s5}} \left(\vartheta_{u_{s5}} - \vartheta_{u_{s3}} - h_\phi - \delta n - \mu\right) F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5}) = \lambda_1 F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5}) \tag{3.92}$$

$$\mathcal{D}_{45}^{(-1,-1,0)} : \partial_{u_{s5}} \partial_{u_{s3}} F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5})(u_{s3}, u_{s5}) = \lambda_2 F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5}). \tag{3.93}$$

The solution can be derived by expanding F^μ into a basis of common eigenfunctions of $\mathcal{D}_{45}^{(-1,-1,0)}$ and $\partial_{u_{s5}}$,

$$F^\mu_{(h_a,\bar{h}_a;n)}(u_{s3}, u_{s5}) = \int_0^\infty \mathrm{d}k \, f^\mu_{(h_a,\bar{h}_a;n)}(k) \, e^{-\left(k u_{s5} + k^{-1} \lambda_2 u_{s3}\right)}.$$

Here, we assume that only the positive part of the $\partial_{u_{s5}}$ spectrum contributes to the superposition and that said spectrum is continuous. Acting with the Casimir operator $\mathcal{D}_{12}^{(-1,0,*)}$ on the integral and integrating by parts (assuming appropriate falloff at large k), we obtain a first-order differential equation for the function $f^{\mu}_{(h_a,\bar{h}_a;n)}(k)$ that fixes it up to a multiplicative constant f_0:

$$f^{\mu}_{(h_a,\bar{h}_a;n)}(k) = \frac{f_0}{2}\frac{e^{-\lambda_1/k}}{k^{1+h_\phi+\delta n+\mu}}. \tag{3.94}$$

Plugging this back into eq. (3.93) yields the integral representation (B.4) of the modified Bessel-Clifford function $\mathcal{K}_\alpha(x)$, see appendix B.1 for its definition and properties. Thus, up to a multiplicative constant, the solution with boundary conditions suitable for our problem is given by

$$F^{\mu}_{(h_a,\bar{h}_a;n)}(u_{s3},u_{s5}) = u_{s5}^{h_\phi+\delta n+\mu}\mathcal{K}_{h_\phi+\delta n+\mu}(\lambda_1 u_{s5}+\lambda_2 u_{s3}u_{s5}). \tag{3.95}$$

Inserting the solution (3.95) into the ansatz (3.91), we obtain

$$F_{(h_a,\bar{h}_a;n)}(u_{s2},u_{s3},u_{s5}) = \sum_\mu \langle\mu|n\rangle\,(1-u_{s2})^\mu u_{s5}^{h_\phi+\delta n+\mu}\mathcal{K}_{h_\phi+\delta n+\mu}(\lambda_1 u_{s5}+\lambda_2 u_{s3}u_{s5}). \tag{3.96}$$

To summarize, we have shown that any function of the form (3.96) satisfies our two Casimir equations to the desired order, regardless of the coefficients $\langle\mu|n\rangle$.

To determine the domain of the summation index μ, the scaling of the tensor structure label n and the coefficients $\langle\mu|n\rangle$, we follow the strategy we outlined in section 3.3.1 and executed in our discussion of case I blocks already. The procedure rests on the observation that the relevant terms of the two Casimir operators are the same if we take the limits with the ordering $\vec{\epsilon}$ (red path in figure 2) or choose the direct channel ordering $\vec{\epsilon}\,'$ instead (green path in figure 2),

$$\begin{aligned}\lim_{\vec{\epsilon}\to 0}\epsilon_{15}\left(\mathcal{D}_{12}^{2,\vec{\epsilon}}-\lambda_1^{\vec{\epsilon}}\right) &= \lim_{\vec{\epsilon}\,'\to 0}\epsilon_{15}\left(\mathcal{D}_{45}^{2,\vec{\epsilon}}-\lambda_1^{\vec{\epsilon}}\right),\\ \lim_{\vec{\epsilon}\to 0}\epsilon_{15}\epsilon_{34}\left(\mathcal{D}_{45}^{2,\vec{\epsilon}}-\lambda_2^{\vec{\epsilon}}\right) &= \lim_{\vec{\epsilon}\,'\to 0}\epsilon_{15}\epsilon_{34}\left(\mathcal{D}_{45}^{2,\vec{\epsilon}}-\lambda_2^{\vec{\epsilon}}\right).\end{aligned}$$

Thus, in direct analogy to case I blocks, we should retrieve the form (3.96) from the expansion (3.75) of blocks around the decoupling limit which is defined here at the $(0,0,x)$ node of figure 2 for $v=u_{s3}$.

Our first goal is to show that the only contribution to the sum (3.96) comes from $\mu=0$. In order to do this, we start by observing that the sum runs over $\mu\geq 0$ only. This can be shown by writing the leading contribution to $F_{\mathcal{O}_1\mathcal{O}_2;n}$ in eq. (3.75) as a power series in $1-u_{s2}$. Now, to deduce the scaling of n, we make the simple observation

$$\mathcal{D}_{12}^{(-1,0,*)} - J_1^2 = \partial_{u_{s5}}\vartheta_{1-u_{s2}} + \mathcal{O}(\epsilon_{15}^{-1}). \tag{3.97}$$

On the left-hand side, the Casimir operator and its eigenvalue $J_1^2+\mathcal{O}(J_1)$ scale like $\mathcal{O}(\epsilon_{15}^{-1})$ at leading order, while on the right-hand side, $\partial_{u_{s5}}$ also scales like $\mathcal{O}(\epsilon_{15}^{-1})$ and $\vartheta_{1-u_{s2}}$ has eigenvalues $\delta n+\mu$, $\mu\geq 0$. Assuming $n<\min(J_1,J_2)=J_1$, there is only one way to ensure that the right-hand side does not scale faster than the ϵ_{15}^{-1} to infinity, namely to keep δn finite,

$$\delta n = J_1 - n = \mathcal{O}(1). \tag{3.98}$$

More specifically, we will assume from now on that $\delta n = 0, 1, 2, \ldots$ is a positive integer. In section 4, we will further demonstrate that this discrete tensor structure labeling is consistent with the spectrum and eigenbasis of the vertex operator.

Let us now analyze the scaling behavior of the coefficients $\langle\mu|n\rangle$. Looking at eq. (3.96), we see that the function multiplied by $\langle\mu|n\rangle$ scales like ϵ_{15}^{μ} at order $(1-u_{s2})^{\mu}$. As a result, the μ-th term in the sum can only contribute to the leading behavior if its coefficient compensates for the aforementioned suppression, i.e. $\langle\mu|n\rangle \sim \epsilon_{15}^{-\mu}\,\langle\mu = 0|n\rangle$. Whether this is the case or not can be deduced from our formula (3.75). Upon expansion of the x-dependent part into a binomial sum, we find

$$(1-(1-u_{s5})\mathcal{S}_1\mathcal{S}_2)^{-\bar{h}_{12;\phi}} = \sum_{k=0}^{\infty} \frac{(1-u_{s5})^k}{k!}(\bar{h}_{12;\phi})_k \mathcal{S}_1^k \mathcal{S}_2^k. \tag{3.99}$$

In the limit where u_{s5} is of order $\mathcal{O}(\epsilon_{15})$, the region in which k is of order $\mathcal{O}(\epsilon_{15}^{-1})$ dominates this sum[7]. Now, we can expand each contribution $\mathcal{S}_1(\vartheta_{u_{s2}})^k f_1^{\text{dec}}(u_{s2})$ explicitly as a hypergeometric in $1-u_{s2}$ by equating it with the right-hand side of eq. (A.29) in appendix A.4 for $\nu = k$, $a = 1$, $v_1 = u_{s2}$. In this case, we find:

$$\mathcal{S}_1(\vartheta_{u_{s2}})^k f_1^{\text{dec}}(u_{s2}) = \frac{(\bar{h}_1)_k}{(2\bar{h}_1)_k}\left(1 + (\delta n + h_\phi - h_1 - h_2)\bar{h}_1 \frac{1-u_{s2}}{k} + \mathcal{O}\left(\bar{h}_1^2 k^{-2}(1-u_{s2})^2\right)\right). \tag{3.100}$$

Given $\bar{h}_1^2, k = \mathcal{O}(\epsilon_{15}^{-1})$, we can read off from this equation that $\langle\mu|n\rangle \sim \epsilon_{15}^{\mu/2}\,\langle\mu = 0|n\rangle$. Consequently, the coefficients in the sum (3.96) do not diverge fast enough in the $u_{s5} \to 0$ limit for $\mu \neq 0$ terms to compete with the $\mu = 0$ term, such that only the $\mu = 0$ term survives. Putting all this together, we finally arrive at the following expression for the lightcone blocks

$$\boxed{\begin{aligned}\psi^{\text{CC,II}}_{(h_a,\bar{h}_a;J_1-\delta n)}(u_{si}) \overset{LCL_{\epsilon}^{(4)}}{\sim} \mathcal{N}^{\text{CC,II}}_{(h_a,\bar{h}_a;n)} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi}(1-u_{s2})^{\delta n} u_{s5}^{h_\phi+\delta n} \\ \mathcal{K}_{h_\phi+\delta n}\left(\bar{h}_1^2 u_{s5} + \bar{h}_2^2 u_{s3}u_{s5}\right).\end{aligned}} \tag{3.101}$$

Here, the superscript (4) on the LCL reminds us that we keep the cross-ratios u_{s2} finite, i,e, we only send four of the five cross-ratios to zero. As we shall demonstrate below, the normalization is given by

$$\boxed{\mathcal{N}^{\text{CC,II}}_{(h_a,\bar{h}_a;J_1-\delta n)} = 4^{\bar{h}_1+\bar{h}_2}\sqrt{\frac{\bar{h}_2}{2\pi}}\, e^{-\bar{h}_1}\,\bar{h}_1^{\bar{h}_1}\,\bar{h}_2^{\bar{h}_1-h_\phi-\delta n}.} \tag{3.102}$$

The derivation of the normalization here is less intuitive than in case I — this is because even at all orders in u_{s2}, only the first term of the power series in $(1-u_{s2})$ survives the case II

[7]In analogy to the crossed channel of a crossing equation, one can think of k as the "spin" and $(u_{s5}-1)\partial_{u_{s5}}$ as the "crossed channel Casimir".

limit. In analogy with a derivative expansion, the sum over descendants $\mu > 0$ truncates, and this sum is precisely what is counted by the second factor in the normalization (3.64) coming from $f_1^{\mathrm{dec}}(0)$. At the same time, the descendants $\eta > 0$ coming from higher powers of $(1-u_{s5})$ counted in the ${}_3F_2(1)$ in eq. (3.64) is modified, because it depends itself the contribution of $\mu \geq 0$ descendants. With a more careful tracking of the sum over (μ, η) descendants, that is to say, powers of $1-u_{s2}$ and x in eq. (3.75), we can obtain the normalization via the limit

$$F_{(h_a,\bar{h}_a;n)} \overset{\bar{h}_2^2 u_{s2} u_{s5} \ll \bar{h}_1^2 u_{s5} \ll 1}{\sim} \mathcal{N}^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;n)} \bar{h}_1^{-2(h_\phi+\delta n)} \frac{\Gamma(h_\phi+\delta n)}{2}. \tag{3.103}$$

In the formula (3.75), this is equivalent to setting

$$(u_{s2}, u_{s3}) = (1,0), \quad \mathcal{S}_1^k = \frac{(\bar{h}_1)_k}{(2\bar{h}_1)_k}, \quad \mathcal{S}_2^k(\vartheta) = \mathcal{S}_2^k(0). \tag{3.104}$$

From this we obtain

$$\frac{1}{2}\mathcal{N}^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;J_1-\delta n)} = \lim_{\bar{h}_1\to\infty} \bar{h}_1^{2(h_\phi+\delta n)} \frac{\Gamma(2\bar{h}_1)}{\Gamma(\bar{h}_1)\Gamma(\bar{h}_1+h_\phi+\delta n)} \lim_{\bar{h}_2\to\infty} \frac{\Gamma(2\bar{h}_2)\Gamma(\bar{h}_1-h_\phi-\delta n)}{\Gamma(\bar{h}_2)\Gamma(\bar{h}_2+\bar{h}_1-h_\phi-\delta n)}. \tag{3.105}$$

The evaluation of the limits using Stirling's formula gives the explicit simple expression for the normalization of the case II lightcone blocks that we anticipated in eq. (3.102). We have thus completed all the goals we set ourselves for this section: computing all the lightcone blocks we need for our analysis of the crossing symmetry equation. We will not go there right away, however, but instead pause for a moment to comment on the behavior of vertex differential operators in the lightcone limit.

4 Interlude: lightcone limit of vertex operators

In the last few sections, we have analyzed second-order Casimir equations in the lightcone limit. But these are not the only differential equations that are needed to characterize conformal blocks. On the one hand, there are also higher-order Casimir operators. For the scalar four- and five-point functions we considered here, only the fourth-order Casimir equations are needed to characterize the blocks. It is clear from our discussion that in the lightcone limit, these can be expressed in terms of the second-order Casimir and Euler operators. So, there was no need to include them in our discussion. On the other hand, starting from $N = 5$ external scalars, Casimir operators no longer provide a complete set of differential equations. As was explained in [41, 44], additional vertex differential operators are needed in order to fully characterize multipoint conformal blocks. These have to do with the appearance of non-trivial tensor structures at the vertices. Just as the eigenvalues of the Casimir operators measure the weight and spin of exchanged fields in a given OPE channel, the eigenvalues of the vertex operators can be used to measure the choice of tensor structures. In the case of five-point blocks, a single such vertex operator needs to be taken into account.

As we have emphasized in the introduction already, vertex differential operators are relevant for the lightcone bootstrap in that they can be used to determine the scaling of

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tensor structures much in the same way as Casimir operators allow to extract the scaling of spin labels. The main obstacle to overcome in setting up a theory of vertex singular behavior is to understand how the eigenvalues of vertex operators are actually related to more conventional tensor structure labels. In general, very little is known. But in the lightcone limit, we can now add some significant new insight based on the results of the previous section. This new insight will suffice to determine the scaling of tensor structures in the next section.

Below we shall analyze the vertex differential operator for five-point functions, both for the case I and the case II lightcone regimes studied in the previous section. The resulting expressions for the vertex differential operator are remarkably simple. This simplicity will enable us to construct their eigenfunctions from a linear combination/integral transform of the lightcone blocks ψ_n we constructed in the previous section.

4.1 Vertex operator for case I

To begin with, let us first recall that in [44], the vertex operator was constructed out of generators $\mathcal{T}$ of the conformal Lie algebra as follows:

$$\mathcal{V} \equiv \mathcal{D}^{4,3}_{\rho,(12)3} = \frac{1}{2} str\left[(\mathcal{T}_1+\mathcal{T}_2)^3\,(\mathcal{T}_3)\right] . \tag{4.1}$$

All relevant notations can be found in [44] along with a Mathematica notebook that expresses $\mathcal{V}$ as a differential operator acting on the five cross-ratios of a five-point function. Starting from these expressions, we can now go to the lightcone limit LCL, see eq. (3.15). In doing so, it is straightforward to notice that the leading divergences of the operator $\mathcal{V}$ can be expressed in terms of the second-order Casimir operators $\mathcal{D}_{12}$ and $\mathcal{D}_{45}$. The precise relation is

$$\mathcal{V}^{\vec{\epsilon}} = -\epsilon_{15}^{-2}\epsilon_{23}^{-2}\left(\mathcal{D}_{12}^{(-1,0,-1)}\right)^2 + \epsilon_{15}^{-2}\epsilon_{23}^{-1}\epsilon_{34}^{-1}\mathcal{D}_{12}^{(-1,0,-1)}\mathcal{D}_{45}^{(-1,-1,0)} + \mathcal{O}\left(\epsilon_{15}^{-1}\right). \tag{4.2}$$

As in our discussion in section 3, we have suppressed the components $\epsilon_{12}, \epsilon_{45} = 0$ from the superscript on the right-hand side. In other words, the superscript only lists $(\epsilon_{15}, \epsilon_{34}, \epsilon_{23})$. To expose those terms in the vertex operator that are actually independent of the quadratic Casimir operators we have already studied, we propose to subtract $(\mathcal{D}_{12}\mathcal{D}_{45} - \mathcal{D}_{12}^2)/4$. Indeed, after this subtraction one finds

$$\left[\mathcal{V}+\mathcal{D}_{12}^2-\mathcal{D}_{12}\mathcal{D}_{45}\right]^{\vec{\epsilon}} = \epsilon_{15}^{-1}\left(\epsilon_{34}^{-1}\left(\epsilon_{23}^{-1}\,\mathcal{V}^{(-1,-1,-1)} + \mathcal{O}\left(\epsilon_{23}^{0}\right)\right)+\mathcal{O}\left(\epsilon_{34}^{0}\right)\right)+\mathcal{O}\left(\epsilon_{15}^{0}\right), \tag{4.3}$$

where

$$\mathcal{V}^{(-1,-1,-1)} = 2\left(\partial_{u_{s2}} - \frac{h_\phi}{u_{s2}}\right)\left(\partial_{u_{s3}} - \frac{h_\phi}{u_{s3}}\right)\partial_{u_{s5}}\left(\vartheta_{u_{s2}}+\vartheta_{u_{s3}}-\vartheta_{u_{s5}}-\vartheta_{u_1}-\vartheta_{u_4}+\frac{d-2}{2}\right). \tag{4.4}$$

In this form, the operator $\mathcal{V}^{(-1,-1,-1)}$ cannot be expressed entirely in terms of the second-order Casimir operators, i.e. it is an independent operator. On the other hand, it is also worth noticing that we can factor out the lightcone limit (3.66) or (3.68) of the second-order Casimir operator $\mathcal{D}_{12}$ or $\mathcal{D}_{45}$ from the lightcone limit of the fourth-order vertex operator. Hence, when applied to eigenfunctions of the lightcone Casimir operators, the lightcone vertex operator gives effectively a second order eigenvalue equation. This makes it rather easy to solve the combined system of Casimir and vertex differential equations in the lightcone limit.

In order to construct the eigenfunctions of the vertex operator in the lightcone limit explicitly, we note that it also commutes with the Euler operators $\vartheta_{u_{s1}}$ and $\vartheta_{u_{s4}}$, which implies we can diagonalize these and work in a basis of functions of the form

$$\psi^{\mathrm{CC}}_{(h_a,\bar{h}_a;\mathfrak{t})}(u_{si}) = u_{s1}^{h_1} u_{s4}^{h_2} \left(u_{s2}u_{s3}\right)^{h_\phi} g_{(h_a,\bar{h}_a;\mathfrak{t})}(u_{s2},u_{s3},u_{s5}) \,. \tag{4.5}$$

Here we have also extracted a prefactor $(u_{s2}u_{s3})^{h_\phi}$ as suggested by previous experience and the explicit form of $\mathcal{V}$. Note that we have also denoted the tensor structure label by $\mathfrak{t}$ in place of the label n we used above. This label $\mathfrak{t}$, that is to say the eigenvalue of the vertex operator, is now used to label eigenfunctions of the vertex operator. After conjugation with the prefactor in eq. (4.5) the relevant leading contribution of the subtracted vertex differential operator becomes

$$\widetilde{\mathcal{V}}^{(-1,-1,-1)} = 2\,\partial_{u_{s2}}\partial_{u_{s3}}\partial_{u_{s5}}\left(\vartheta_{u_{s2}}+\vartheta_{u_{s3}}-\vartheta_{u_{s5}}+2h_\phi-h_1-h_2+\frac{d-2}{2}\right). \tag{4.6}$$

By definition, the functions g in eq. (4.5) are eigenfunctions of $\tilde{\mathcal{V}}$ with an eigenvalue that is determined by $\mathfrak{t}$. To diagonalize $\tilde{\mathcal{V}}$, let us start from the basis of functions that diagonalize the leading Casimirs as well as $\partial_{u_{s5}}$, see section 3,

$$\psi^{\mathrm{CC}}_{(h_a,\bar{h}_a;\eta)}(u_{si}) \sim \mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;n)} u_{s1}^{h_1} u_{s4}^{h_2} \left(u_{s2}u_{s3}\right)^{h_\phi} e^{-\eta u_{s5}-\frac{\lambda_1 u_{s2}+\lambda_2 u_{s3}}{\eta}}. \tag{4.7}$$

Here $\sim$ implies that we display the leading term in the lightcone limit up to some numerical prefactor. As we outlined in the introductory paragraphs to this section, a joint eigenfunction of the Casimir operators and the vertex operator (4.6) is a superposition of the blocks (4.7), and can be thus written as

$$\psi^{\mathrm{CC}}_{(h_a,\bar{h}_a;\mathfrak{t})}(u_{si}) \sim u_{s1}^{h_1} u_{s4}^{h_2} \left(u_{s2}u_{s3}\right)^{h_\phi} \int_0^\infty f(\eta) e^{-\eta u_{s5}-\frac{\lambda_1 u_{s2}+\lambda_2 u_{s3}}{\eta}}\mathrm{d}\eta\,. \tag{4.8}$$

Here we assumed that only the positive part of the η spectrum contributes to the physical blocks and that said spectrum is continuous. Acting with the vertex differential operator $\mathcal{V}$ on the integral (4.8) and integrating by parts (assuming appropriate falloff at large η), we obtain a differential equation for the function $f(\eta)$,

$$-2\lambda_1\lambda_2\left[f'(\eta)+\frac{p+1}{\eta}f(\eta)\right] = \mathfrak{t}f(\eta), \qquad p := 2h_\phi-h_1-h_2+\frac{d-2}{2}. \tag{4.9}$$

This equation is easy to solve. As one can see immediately, the solution is determined up to a multiplicative constant f_0 by

$$f(\eta) = \frac{f_0}{2} e^{-\eta\frac{\mathfrak{t}}{2\lambda_1\lambda_2}}\eta^{-(1+p)}. \tag{4.10}$$

In writing this solution, we assumed that the eigenvalue $\mathfrak{t}$ of the vertex operator $\mathcal{V}$ scales with $\epsilon_{15}^{-1}\epsilon_{23}^{-1}\epsilon_{34}^{-1}$, just as the leading term of the vertex operator itself. Plugging this form of the function $f(\eta)$ into eq. (4.8) provides us with the blocks

$$\psi^{\mathrm{CC},I}_{(h_a,\bar{h}_a;\mathfrak{t})}(u_{si}) \overset{LCL_{\vec{\epsilon}}}{=} f_0\, u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi} \left(u_{s5}+\frac{\mathfrak{t}}{2\lambda_1\lambda_2}\right)^p \mathcal{K}_p\left[(u_{s2}\lambda_1+u_{s3}\lambda_2)\left(u_{s5}+\frac{\mathfrak{t}}{2\lambda_1\lambda_2}\right)\right], \tag{4.11}$$

where we used the integral representation of the modified Bessel-Clifford function $\mathcal{K}_p(x)$ in eq. (B.4). Thereby we have indeed obtained an explicit expression for the lightcone limit of the joint eigenfunctions of Casimir and vertex differential operators. The formula (4.10) for the coefficients of the eigenfunctions (4.7) in the η basis has a simple interpretation: in order to go from the η basis we used in section 3 to the $\mathfrak{t}$ basis in which the vertex differential operator is diagonal, we apply a Laplace transform, after multiplying the argument of the Laplace transform with η^{-1-p}. Recall from our discussion in section 3 that the variable η coincides with the usual tensor structure label n in the lightcone limit. Hence, in the limit, the tensor structure label $\mathfrak{t}$ that was introduced in [44] within the integrable systems approach to conformal blocks is essentially the Laplace transform of the standard basis labeled by n.

4.2 Vertex operator for case II

In our analysis of the eigenfunctions of Casimir operators, we have actually gone a little further and evaluated the eigenfunctions to all orders in u_{s2}. Quite remarkably, we can also include all these finite-order contributions in the analysis of the vertex operator. Once we consider all orders in ϵ_{23}, we obtain a more complicated expression for the vertex operator which reads

$$\begin{aligned}\widetilde{\mathcal{V}}^{(-1,-1,*)} = 2\Bigg[&(1-z_1)\,z_1^2\partial_{z_1}^2 + z_1\,(h_\phi+h_1-h_2)\,(\vartheta_{u_{s5}}-\vartheta_{u_{s3}})\\ &-\left(z_1\,(2h_\phi+h_1-h_2+1+\vartheta_{u_{s3}}-\vartheta_{u_{s5}})-2h_1+\frac{d-2}{2}\right)\vartheta_{z_1}\\ &-(h_\phi+h_1-h_2)\,h_\phi z_1+h_1^2+\frac{d^2}{12}-\frac{d}{4}\,(2h_1+1)\Bigg]\partial_{u_{s3}}\partial_{u_{s5}}\,,\end{aligned} \tag{4.12}$$

where we introduced $z_1 := 1-u_{s2}$. In our discussion of the leading order, we constructed eigenfunctions of the vertex operators as linear combinations of case I blocks in eq. (4.7). Now we need to work with the case II blocks (3.101) instead, i.e. we write the eigenfunctions $\psi_{\mathfrak{t}}$ of the vertex operators as a superposition of case II blocks,

$$\begin{aligned}g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;\mathfrak{t})}(u_{s2},u_{s3},u_{s5}) &= \sum_{m\geq 0} c^{\mathfrak{t}}_m g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;m)}(u_{s2},u_{s3},u_{s5})\\ &= \sum_{m\geq 0} c^{\mathfrak{t}}_m u_{s5}^{h_\phi+m} z_1^m \mathcal{K}_{h_\phi+m}(\bar{h}_1^2 u_{s5}+\bar{h}_2^2 u_{s3}u_{s5})\,.\end{aligned} \tag{4.13}$$

In writing this expansion, we have not explicitly displayed the dependence of the expansion coefficients $c^{\mathfrak{t}}_m$ on the quantum numbers $(h_a,\bar{h}_a)$ of the exchanged fields. The eigenvalue

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equation for the vertex operator acting on these blocks can be recast as a recurrence relation for the coefficients c,

$$\lambda_2 c^{\mathfrak{t}}_m \left(d^2 - 3d\,(2h_1 + 2m + 1) + 12\,(h_1 + m)^{\,2}\right) - 6\mathfrak{t}\, c^{\mathfrak{t}}_m \\ -12\lambda_1\lambda_2 c^{\mathfrak{t}}_{m-1}\,(h_\phi + h_1 - h_2 + m - 1) = 0\,,$$

where $\lambda_i = \mathcal{O}(\bar{h}_i^2)$, as usual. We can solve this case II lightcone recursion relation through the following explicit expression involving Pochhammer symbols $(\cdot)_{m-1}$,

$$c^{\mathfrak{t}}_m = \mathcal{N}\frac{\lambda_1^{m-1}\,(h_1 - h_2 + h_\phi + 1)_{m-1}}{\left(-\frac{d}{4} + h_1 - \frac{\sqrt{8t\lambda_2 - \frac{1}{3}d(d-12)\lambda_2^2}}{4\lambda_2} + 2\right)_{m-1}\left(-\frac{d}{4} + h_1 + \frac{\sqrt{8t\lambda_2 - \frac{1}{3}d(d-12)\lambda_2^2}}{4\lambda_2} + 2\right)_{m-1}}\,. \tag{4.14}$$

The square roots that appear in the solution suggest the following parametrization of the vertex operator eigenvalues

$$\mathfrak{t}_N = 2\lambda_2\left(\left(-\frac{d}{4} + h_1 + N\right)^2 - \frac{1}{4}d\left(1 - \frac{d}{12}\right)\right), \tag{4.15}$$

for some integer $N \in \mathbb{Z}_{\geq 0}$. Once this parametrization is adopted, we can simplify the expressions and make the coefficients c vanish for $m < N$, assuming N is a positive integer. By reabsorbing some constants in the normalization factor, we are then left with

$$c^{\mathfrak{t}}_m = \mathcal{N}'\frac{\lambda_1^m\,(h_1 - h_2 + h_\phi + N)_{m-N}}{(m-N)!\left(2h_1 - \frac{d-2}{2} + 2N\right)_{m-N}} \tag{4.16}$$

which, together with eq. (4.13), provide a representation for the vertex operator eigenfunctions at all orders in ϵ_{23}.

Alternative representations and consistency with previous results. There are two ways that we can check the consistency of the case II vertex operator spectrum (4.15) and eigenfunctions (4.13), (4.16) with other computations. First, we can reproduce the case I vertex operator eigenfunctions starting from the following representation of the case II eigenfunctions:

$$g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;\mathfrak{t}_N)}(u_{s2}, u_{s3}, u_{s5}) = \mathcal{N}'\int_0^\infty \frac{\mathrm{d}k}{k^{1+h_\phi}}\left(\frac{\lambda_1 z_1}{k}\right)^N {}_1F_1\begin{bmatrix} a+N \\ b+2N \end{bmatrix}\left(\frac{\lambda_1 z_1}{k}\right) e^{-\left(k u_{s5} + \frac{\lambda_1}{k} + \frac{\lambda_2 u_{s3}}{k}\right)}, \tag{4.17}$$

where $a := h_1 + h_\phi - h_2$ and $b := 2h_1 - (d-2)/2$. In the scaling limit

$$1 - z_1 = u_{s2} = \mathcal{O}(\epsilon_{23}), \quad \lambda_1 = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1}), \quad N^2 = \mathcal{O}(\epsilon_{23}^{-1}), \tag{4.18}$$

we can approximate the confluent hypergeometric function (see [56, eq. (13.7.1)]) by

$${}_1F_1\begin{bmatrix} a+N \\ b+2N \end{bmatrix}\left(\frac{\lambda_1 z_1}{k}\right) = \left(\frac{\lambda_1 z_1}{k}\right)^{a-b-N} e^{\frac{\lambda_1}{k}(1-u_{s2}) - \frac{kN^2}{\lambda_1}}\left(1 + \mathcal{O}(\epsilon_{23}^{1/2})\right). \tag{4.19}$$

In this case, the limiting form of the case II eigenfunctions is

$$g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;\mathfrak{t}_N)}(u_{s2},u_{s3},u_{s5}) \overset{\epsilon_{23}\to 0}{\sim} \frac{\mathcal{N}'}{\lambda_1^{b-a}}\left(u_{s5}+\frac{N^2}{\lambda_1}\right)^{h_\phi+a-b} \mathcal{K}_{h_\phi+a-b}\left([\lambda_1 u_{s2}+\lambda_2 u_{s3}]\left(u_{s5}+\frac{N^2}{\lambda_1}\right)\right). \tag{4.20}$$

This expression coincides with eq. (4.11) as $p = h_\phi + a - b$ and $\mathfrak{t} = \mathfrak{t}_N = 2\lambda_2 N^2$. The second check is the case of scalar exchange $\lambda_1 = 0+\mathcal{O}(1)$, for which there is only one single tensor structure which we expect to be at $N = 0$. The corresponding vertex operator eigenfunction must agree with case II blocks in the n-basis when $J_1 = \delta n = 0$. We can check this from the following Euler integral representation of the case II vertex operator eigenfunctions:

$$g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;\mathfrak{t}_N)}(u_{s2},u_{s3},u_{s5}) = \mathcal{N}'(\lambda_1 u_{s5} z_1)^N u_{s5}^{h_\phi}\int_0^1 \frac{\mathrm{d}t}{t(1-t)}\frac{t^{a+N}(1-t)^{b-a+N}}{\mathrm{B}_{a+N,b-a+N}}\mathcal{K}_{h_\phi+N}\left(\lambda_1 u_{s5}(1-z_1 t)+\lambda_2 u_{s3}u_{s5}\right). \tag{4.21}$$

For scalar exchange, where $\lambda_1 u_{s5} \to 0$, the integral over t factorizes. We are thus left with

$$g^{\mathrm{CC,II}}_{(h_a,\bar{h}_a;\mathfrak{t}_N)}(u_{s2},u_{s3},u_{s5}) \overset{\lambda_1 u_{s5}\to 0}{\sim} \mathcal{N}'(\lambda_1 u_{s5} z_1)^N u_{s5}^{h_\phi}\mathcal{K}_{h_\phi+N}\left(\lambda_2 u_{s3}u_{s5}\right). \tag{4.22}$$

We indeed retrieve the asymptotics of blocks with scalar exchange for $N = 0$, see appendix A.4 for a direct computation.

This concludes our discussion of case I and II solutions of the vertex differential equations. In this work, we shall stick to the more conventional basis of tensor structures that is labeled by the integer n, also for comparison of our bootstrap analysis in the next section with previous work. But it is very promising that the vertex operator basis can be calculated explicitly in the lightcone limit and that, at least in the leading order case I lightcone blocks, is related to the n basis through a simple integral transform.

5 Five-point lightcone bootstrap

In this section, we will apply the results on five-point lightcone blocks we derived in the previous section to the analysis of the crossing symmetry equation depicted in figure 1. In the first subsection, we shall address the leading contributions in the direct channel. The remaining subsections are then devoted to the crossed channel. There we shall explain how to reproduce the various direct channel terms from the crossed channel block expansion. Along the way we will determine various OPE coefficients involving two double-twist operators, see eqs. (5.12), (5.30) and (5.56). The first two of these are known from the work [26], so our derivation of these results simply represents a confirmation of their validity via a more rigorous treatment of the order of limits. Our formula (5.56), instead, is new. It applies in particular to theories for which the leading-twist field $\mathcal{O}_\star$ that appears in the

OPE of ϕ with itself has twist $h_\star > h_\phi$. The derivation of our new formula requires going beyond the leading terms in the lightcone limit. This is explained in the section 5.3, where we derive the large spin expansion of OPE coefficients at discrete tensor structures that reproduce all leading-twist exchanges in the direct channel. Finally, in section 5.4, we go over the applications of our results to specific models and check their validity against an independent computation of tree-level OPE coefficients in ϕ^3 theory.

5.1 Direct channel contributions to lightcone limit

We will expand both channels of the crossing equation (3.14) near two different sequences of lightcone limits: first the null polygon limit $X_{i(i+1)} = 0$, with the specific hierarchy stated in eq. (3.15), and then the limit $LCL_\epsilon^{(4)}$ in eq. (3.57), obtained by relaxing the $X_{23} = 0$ limit. Starting with the direct channel, we observe that the hierarchy of lightcone limits ensures an expansion whose first contributions are the leading-twist fields in the two operator products of the direct channel. More precisely, taking both $X_{15} \ll 1$ and $X_{34} \ll 1$, we obtain

$$\sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\rm DC}_{\mathcal{O}_1\mathcal{O}_2;n} = C_{\phi\phi\phi}\left((u_{s1}u_{s3})^{\frac{\Delta_\phi}{2}} + (u_{s5}u_{s2})^{\frac{\Delta_\phi}{2}}\right) + \\ + \sum_{n=0}^{J_\star} P^{(n)}_{\mathcal{O}_\star\mathcal{O}_\star}\,\psi_{\mathcal{O}_\star\mathcal{O}_\star;n}(u_{s1},u_{s2},u_{s4}) + \mathcal{O}(X_{15}^{h>h_\star}). \tag{5.1}$$

Here, we have used the conventions that were stated in eq. (3.10), i.e. we multiplied the five-point correlation function by a factor

$$\Omega_{\rm DC}(X_i) = \left(X_{15}X_{34}\sqrt{\frac{X_{12}X_{23}}{X_{13}}}\right)^{\Delta_\phi}.$$

The three terms that appear on the right-hand side correspond to intermediate exchange of $[\mathcal{O}_1|\mathcal{O}_2] = [\mathbf{1}|\phi], [\phi|\mathbf{1}]$ and $[\mathcal{O}_1|\mathcal{O}_2] = [\mathcal{O}_\star|\mathcal{O}_\star]$ at the top and bottom line respectively. Note that there cannot be identity exchange in both intermediate channels simultaneously. Furthermore, a single identity exchange forces the second intermediate exchange to coincide with the external operators, i.e. it forces ϕ exchange in the other operator product. The two terms with a single identity exchange are the first two terms on the right-hand side of the previous equation. The third term, which may involve a sum over tensor structures n, is associated with the leading-twist field $\mathcal{O}_\star$ in the operator product of ϕ with itself, under the assumption that this field is unique. We parameterize its weight and spin by $h_\star$ and $J_\star$, and we require $h_\star < 2h_\phi$ in order to avoid the infinite summation over the double-twist fields $[\phi\phi]_{n,J}$. For this term, there are a few mutually exclusive cases to distinguish. The leading-twist field $\mathcal{O}_\star$ may coincide with the external field ϕ itself, i.e. $[\mathcal{O}_1|\mathcal{O}_2] = [\mathcal{O}_\star|\mathcal{O}_\star] = [\phi|\phi]$. But this is not always realized, especially when the appearance of ϕ in the operator product of ϕ with itself is excluded by some selection rule. It turns out that $\mathcal{O}_\star \neq \phi$ falls again into two subcases, depending on whether $h_\star < h_\phi$ or $2h_\phi > h_\star > h_\phi$. Depending on which of the three scenarios is realized, the blocks $g_{\mathcal{O}_\star\mathcal{O}_\star;n}(u_{s1},u_{s2},u_{s4})$ possess the following asymptotic behavior in the fivefold lightcone limit (3.15), see section 3.2.

(−) in case $\mathcal{O}_\star \neq \phi$ and $h_\star < h_\phi$, then for all $0 \leq n < h_\phi - h_\star$ the blocks possess the following power law behavior in the lightcone limit,

$$\psi^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;n}(u_{si}) = \mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;n}\,(u_{s1}u_{s2}u_{s3}u_{s5})^{h_\star}\,(u_{s1}u_{s2})^n\,(1 + \mathcal{O}(u_{s2}))\,. \tag{5.2}$$

Note that even when $h_\phi - h_\star > 1$, such that the direct channel sum includes $n > 0$ contributions, the latter will be subleading of relative order $(X_{12}X_{23})^n$ compared to the $n = 0$ block.

(0) In case $\mathcal{O}_\star = \phi$ the tensor structure is trivial, i.e. $n = 0$, and the blocks possess a logarithmic divergence of the form

$$\psi^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;0}(u_{si}) = \mathcal{N}^{\mathrm{DC}\,(0)}_{\phi\phi;0}\,\,(u_{s1}u_{s2}u_{s3}u_{s5})^{h_\phi}\log u_{s1}\log u_{s2} + \mathcal{O}(u_{s2}\log u_{s1}). \tag{5.3}$$

(+) In case $\mathcal{O}_\star \neq \phi$ and $2h_\phi > h_\star > h_\phi$, then for all $n = 0, 1, \ldots, J_\star$, the blocks possess the following power law behavior in the lightcone limit,

$$\psi^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;n}(u_{si}) = \mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;n}\,(u_{s1}u_{s2}u_{s3}u_{s5})^{h_\star}\,(u_{s1}u_{s2})^{h_\phi - h_\star}\,(1 + \mathcal{O}(u_{s2}))\,. \tag{5.4}$$

Contrary to the case (−), the $n > 0$ blocks are not subleading.

In the fourfold lightcone limit (3.57), the blocks present the same asymptotics in the first four cross-ratios but acquire a non-trivial dependence in the remaining finite cross-ratio $u_{s2} := 1 - z$, see eq. (3.59).

5.2 Reproducing direct channel terms from crossed channel

We will now proceed to the term-by-term analysis of the direct channel (5.1) expanded up to order $(\epsilon_{15}\epsilon_{34})^{h_\star}$, reproducing it term by term in the crossed channel. Similarly to the standard works [4, 5], in all of the following computations we assume that the direct-channel contributions are reproduced uniquely by fields whose twist at large spin matches the leading power of the left-hand side. This has been proven in the context of four-point functions in [14], and is thus a natural assumption to make in a higher-point setting. The third term that does not involve identity exchange in the direct channel will require a separate discussion for each of the three alternative cases $(-, 0, +)$ listed above. We shall indeed find that all direct channel terms can be reproduced based on what is known about the behavior of anomalous dimensions and operator product coefficients from previous work on four- and five-point lightcone bootstrap, see in particular [4] and [26]. But there is one notable exception. In order to reproduce the terms without identity exchange for case (+), the operator product coefficients involving two double-twist families must satisfy a novel sum rule that involves a summation/averaging over tensor structures. We will actually be able to resolve this sum rule and derive the full dependence of these operator product coefficients on the tensor structure in the next subsection, but this requires leaving the strict lightcone limit (3.15).

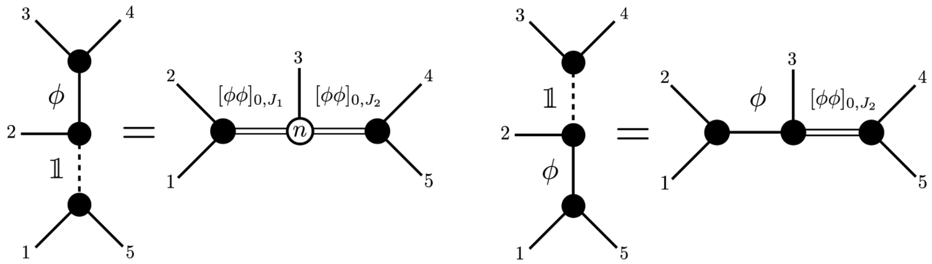

Figure 3. Lightcone bootstrap of five-point correlator (with an identity exchange).

5.2.1 Single internal identity exchange in direct channel

Let us begin with the two direct channel terms that involve a single identity exchange, either in the operator product (15) or in the operator product (34). Let us stress again that double identity exchange is excluded so that the terms we consider here represent the leading contributions to the lightcone limit of the direct channel. A graphical representation of the relevant crossing equations is given in figure 3.

$[\mathbf{1}|\phi]$-exchange in the direct channel. As we explained above, the first term on the left-hand side of the direct channel decomposition (5.1) is associated with the exchange of $[\mathcal{O}_1|\mathcal{O}_2] = [\mathcal{O}_{15}|\mathcal{O}_{34}] = [\mathbf{1}|\phi]$ in the direct channel. Such a term can contribute non-trivially to the direct channel whenever the field ϕ appears in the operator product of ϕ with itself. If this is the case, the leading term in the direct channel is the first term in eq. (5.1). Taking into account the prefactor on the right-hand side of the crossing symmetry equation (3.14) we deduce that

$$C_{\phi\phi\phi}\left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi} + \ldots = \sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\mathrm{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{5.5}$$

Here, $\ldots$ denotes other (subleading) terms in the direct channel. In our lightcone regime (3.15), we first take $u_{s5} \ll u_{s3} \ll u_{s2} \ll 1$. Then, in order to match the behavior of the leading direct channel term on the remaining cross ratios u_{s1} and u_{s4}, we conclude that the limiting crossed channel blocks must obey

$$\psi^{\mathrm{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}(u_{si}) \sim (u_{s1}u_{s4})^{2h_\phi}\widetilde{\psi}^{\mathrm{CC}}_{h_1h_2}(u_{s2},u_{s3},u_{s5}), \tag{5.6}$$

This implies double-twist exchange in both operator products of the crossed channel, i.e. we conclude that the direct channel terms must be reproduced from terms in the crossed channel that involve $\mathcal{O}_1,\mathcal{O}_2 = [\phi\phi]_{0,J}$.

Next, we observe that the leading contribution $u_{s5}^{-2h_\phi}$ is Casimir singular with respect to the relevant terms of both crossed-channel Casimir operators. Indeed, using the explicit expressions we spelled out in eqs. (3.66) and (3.68) we obtain

$$[\mathcal{D}^{(-1,0,-1)}_{12}]^n u_{s5}^{-2h_\phi} = (2h_\phi)_n(h_\phi)_n u_{s5}^{-2h_\phi-n}u_{s2}^{-n}\,, \tag{5.7}$$

$$[\mathcal{D}^{(-1,-1,0)}_{45}]^n u_{s5}^{-2h_\phi} = (2h_\phi)_n(h_\phi)_n u_{s5}^{-2h_\phi-n}u_{s3}^{-n}\,. \tag{5.8}$$

More specifically, the actions of the leading Casimir operators $\mathcal{D}_{12}$ and $\mathcal{D}_{45}$ increase the singularities by an order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$ and $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1})$ respectively. In direct analogy to the four-point discussion above eq. (2.50), this implies that the direct channel has infinite support when expanding in a basis of eigenvectors of the crossed-channel differential operators. This may be understood by acting with one of the leading Casimirs (say $\mathcal{D}_{12}^{(-1,0,-1)}$) on the crossing equation (5.5): the order of $\epsilon_{15}^{-1}\epsilon_{23}^{-1}$ only matches on both sides when the crossed channel blocks have $\lambda_1 = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$. Similarly, acting with $\mathcal{D}_{45}^{(-1,-1,0)}$ yields a match only for $\lambda_2 = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$. We can analogously understand the scaling of the tensor structure label by acting with the vertex operator we discussed in section 4. Acting with the leading vertex operator (4.4) on the left-hand side of eq. (5.5), we get:

$$\mathcal{V}^{(-1,-1,-1)}\left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi} = \frac{h_\phi^3}{u_{s2}u_{s3}u_{s5}}\left(2h_\phi - \frac{d-2}{2}\right)\left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi}. \tag{5.9}$$

Unless the field ϕ saturates the unitarity bound $2h_\phi = \frac{d-2}{2}$, which corresponds to it being a free scalar and having a vanishing five-point function, we see that the action of $\mathcal{V}^{(-1,-1,-1)}$ increases the singularities by an order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1}\epsilon_{34}^{-1})$, which requires its eigenvalue to scale as in eq. (4.11). For the superposition (4.8) with coefficients (4.10) to make sense, we see that the tensor structure label η needs to scale as $\mathcal{O}(\epsilon_{15}^{-1})$. We can thus conclude that, to reproduce the correct behavior in the crossed channel, we must sum over the case I blocks of eq. (3.69).

Having identified the correct regime, we can now address our goal to reproduce the direct channel from a sum over crossed channel lightcone blocks in the case I regime. Using the limits of blocks computed in the previous section (see eq. (3.81)), we can indeed reproduce the correct asymptotics from a large spin and tensor structure integral

$$C_{\phi\phi\phi}u_{s5}^{-2h_\phi} = (u_{s2}u_{s3})^{h_\phi}\int_{[\mathcal{O}(1),\infty)^3} \mathrm{d}\eta \frac{\mathrm{d}\lambda_1}{4\sqrt{\lambda_1}}\frac{\mathrm{d}\lambda_2}{4\sqrt{\lambda_2}}\mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;\eta)}P^{(\eta)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}}\, e^{-\eta u_{s5}-\frac{\lambda_1 u_{s2}+\lambda_2 u_{s3}}{\eta}}, \tag{5.10}$$

where $\lambda_a = \bar{h}_a^2 + \mathcal{O}(\bar{h}_a)$ and $\bar{h}_a = J_a + \mathcal{O}(1)$ at leading order. In writing the crossing equation we have divided both sides by $(u_{s1}u_{s4})^{2h_\phi}$. Through explicit computation of the integrals it is not difficult to verify that the crossing equation is satisfied if the coefficients P in the crossed channel obey

$$\frac{\mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;\eta)}}{16\sqrt{\lambda_1\lambda_2}}P^{(\eta)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} = C_{\phi\phi\phi}\frac{(\lambda_1\lambda_2)^{\Delta_\phi/2-1}}{\eta\,\Gamma^2(\Delta_\phi/2)\Gamma(\Delta_\phi)}. \tag{5.11}$$

Using our formula (3.83) for the normalization $\mathcal{N}^{\mathrm{CC,I}}$ of the case I lightcone blocks we obtain

$$P^{(\eta)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} = C_{\phi\phi\phi}4^{-J_1-J_2}\frac{32\times 16^{-\Delta_\phi}}{\Gamma^2(\Delta_\phi/2)\Gamma(\Delta_\phi)}\eta^{-2\eta-\Delta_\phi}e^{2\eta}(J_1J_2)^{\eta+\frac{3}{2}(\Delta_\phi-1)}\,. \tag{5.12}$$

Here we have used that $h_a = \Delta_\phi = 2h_\phi, \bar{h}_a = h_a + J_a$ and $\lambda_a \sim J_a^2$. Let us note that $\bar{h}_a$ can be replaced by $\bar{h}_a \to J_a$ except if $\bar{h}_a$ appears in the exponent, i.e. in the factor $4^{\bar{h}_a}$ of the normalization (3.83). Our result coincides with the findings of [26]. In order to compare the two expressions one should observe that our normalization conventions for blocks differ slightly from those used by Antunes et al., see [26, footnote 3]. This difference of conventions gives rise to an additional factor $2^{J_1+J_2}$.

$[\phi|\mathbb{1}]$-exchange in the direct channel. Let us now look at the second term $(u_{s2}u_{s5})^{h_\phi}$ in the direct channel that arises from the exchange of an identity field in the (34) OPE. Compared to the leading term in eq. (5.1), the second term is subleading in X_{15}, but higher order in X_{12}. In the presence of this second term, the crossing equation (5.5) is modified to

$$C_{\phi\phi\phi}\left[\left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi}+\left(\frac{u_{s1}u_{s2}u_{s4}^2}{u_{s3}u_{s5}}\right)^{h_\phi}\right]+\ldots=\sum_{\mathcal{O}_1,\mathcal{O}_2,n}P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\text{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{5.13}$$

Once again, ... denote other (subleading) terms in the direct channel. Since we have already reproduced the first term on the right-hand side, we would now like to see how the second term emerges from the crossed-channel sum on the right-hand side.

As a first step, we observe that the subleading direct channel term contains $u_{s1}^{h_\phi}$ rather than the $u_{s1}^{2h_\phi}$ that appears in the first term. Consequently, we expect to reproduce the subleading term from an exchange of the field ϕ itself in the (12) OPE of the crossed channel. For the (45) OPE the situation is unchanged with respect to the leading term, i.e. the crossed channel must involve the exchange of double-twist operators in the (45) OPE.

At the same time, we now find that the direct channel term under consideration is in the kernel of the crossed channel Casimir operator $\mathcal{D}_{12}$ at both leading $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$ and subleading $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{0})$ order — these two properties are easily verified using the explicit formulas (3.66) and (3.85) spelled out above. In fact, outside of any lightcone limit, the Casimir operator $\mathcal{D}_{12}$ is exactly diagonalized by the direct channel power law with eigenvalue $\frac{1}{2}\Delta_\phi(\Delta_\phi-d)$. We conclude that the second term of the direct channel is associated with ϕ exchange in the (12) OPE of the crossed channel. On the other hand, the action of the second crossed-channel Casimir operator $\mathcal{D}_{45}$ is just as singular as in eq. (5.8). Hence, we continue to have an infinite support of twist $h_2=2h_\phi$ operators in the (45) OPE with $\lambda_2=\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1})$.

According to these remarks, the lightcone blocks we need on the right-hand side satisfy the same differential equations as case II of our analysis in the previous section, see eq. (3.101). But in contrast to what we discussed above, the parameters J_1 is now fixed to $J_1=0$ (which also implies $n=0$) and hence the normalization prescription must be modified. In appendix A.4, we determine the correct normalization for the case at hand and obtain

$$\psi^{\text{CC,II}^*}_{\phi,[\phi\phi]_{0,J_2};0}(u_{si})\sim\frac{\Gamma(2h_\phi)}{\Gamma(h_\phi)}4^{\bar{h}_2}\sqrt{\frac{\bar{h}_2}{\pi}}\,(u_{s1}u_{s2}u_{s3}u_{s4}^2u_{s5})^{h_\phi}\mathcal{K}_{h_\phi}(u_{s3}u_{s5}J_2^2). \tag{5.14}$$

where the notation II* reminds us that $J_1=0$ and hence one should not copy the normalization factor from the expression (3.102). Note that the asymptotics in eq. (5.14) can also be determined from the vertex differential operator of section 4: these blocks are in fact eigenfunctions of the case II vertex operator with eigenvalue (4.15) for $N=0$, in the limit where $\lambda_1 u_{s5}\to 0$.

We now insert these results into the crossing symmetry equation. After dividing both sides of the equation by $C_{\phi\phi\phi}(u_{s1}u_{s2}u_{s4}^2)^{h_\phi}$, we conclude

$$(u_{s3}u_{s5})^{-h_\phi}=\frac{\Gamma(2h_\phi)}{\Gamma(h_\phi)}\int\frac{\mathrm{d}J_2}{2}C^2_{\phi\phi[\phi\phi]_{0,J_2}}4^{2h_\phi+J_2}\sqrt{\frac{J_2}{\pi}}\,(u_{s3}u_{s5})^{h_\phi}\mathcal{K}_{h_\phi}(u_{s3}u_{s5}J_2^2). \tag{5.15}$$

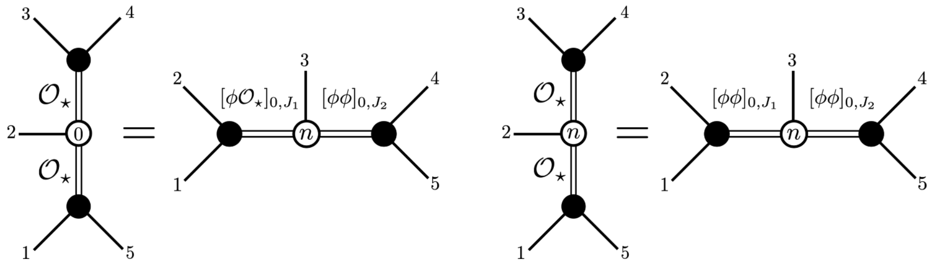

Figure 4. Lightcone bootstrap of five-point correlator with non-identity leading-twist exchange in the direct channel. Left: lighter exchange $h_\star < h_\phi$. Right: heavier exchange $h_\star > h_\phi$.

Here, we have also factorized the coefficients in the crossed channel as $P^{(n)}_{\mathcal{O}_1\mathcal{O}_2} = C_{\phi\phi\phi}\, C^2_{\phi\phi[\phi\phi]_{0,J_2}}$. This integral equation is easily verified using the following expression for the square of the OPE coefficient:

$$C^2_{\phi\phi[\phi\phi]_{0,J}} = \frac{8}{\Gamma(2h_\phi)^2} \frac{\sqrt{\pi}}{4^{2h_\phi+J}} J^{4h_\phi-3/2}, \tag{5.16}$$

which can be found e.g. in [4, eq. (12)], with $\ell \leftrightarrow J$ and $\Delta_\phi \leftrightarrow 2h_\phi$. In conclusion, we have determined how the second term in the direct channel can be reproduced from the crossed channel. In this case, however, the analysis does not provide access to any new dynamical data that goes beyond the four-point lightcone bootstrap.

5.2.2 No internal identity exchange in the direct channel

In this subsection, we address the direct channel contributions that appear in the second line of the expansion in eq. (5.1), in which there is no internal identity operator exchanged. As we explained in section 5.1, there are three alternatives to consider which we denoted by $(-), (0)$ and $(+)$, respectively. Here we shall address case (0) first before discussing $(-)$ and $(+)$.

$[\phi|\phi]$-exchange in the direct channel. Let us first consider the case in which the field ϕ is exchanged in both the (15) and (34) OPE. In particular, we shall assume that the field ϕ does appear in the operator product of ϕ with itself. The associated direct channel contribution is given by eq. (5.1). The crossing symmetry equation is

$$C^3_{\phi\phi\phi}\mathcal{N}^{\mathrm{DC}\,(0)}_{\phi\phi;0}(u^2_{s1}u_{s2}u^2_{s4}u^{-1}_{s5})^{h_\phi}\log u_{s1}\log u_{s2} + \ldots = \sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\mathrm{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{5.17}$$

Here, we have not shown explicitly the leading terms involving identities, which were accounted for earlier, i.e. the $\ldots$ denote these leading terms as well as all the subleading ones. With all the experience we have from the previous subsections, we can now skip the arguments that are mere repetitions of previous examples.

Let us first note that the u_{s1}, u_{s4} asymptotics impose double-twist exchange in both intermediate exchanges of the crossed channel, i.e. $\mathcal{O}_a = [\phi\phi]_{0,J_a}$ by the same argument as in section (5.6). Unlike the claim made in [26], however, the asymptotics of this direct channel contribution *cannot* be reproduced in every single block: the direct channel term that results from $[\phi|\phi]$-exchange is still Casimir singular for the three variables u_{s2}, u_{s3}, u_{s5},

$$[\mathcal{D}_{12}^{(-1,0,-1)}]^n u_{s2}^{h_\phi} u_{s5}^{-h_\phi} \log u_{s2} = (n-1)!(h_\phi)_n u_{s2}^{h_\phi - n} u_{s5}^{-h_\phi - n} \tag{5.18}$$

$$[\mathcal{D}_{45}^{(-1,-1,0)}]^n u_{s3}^0 u_{s5}^{-h_\phi} = (h_\phi)_n^2 u_{s3}^{-n} u_{s5}^{-h_\phi - n} \,. \tag{5.19}$$

Therefore, as argued in section 5.2.1, the crossed channel blocks must have the eigenvalues λ_1 of order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$ and λ_2 of order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1})$, respectively. Similarly, one can use again the vertex operator argument of the same subsection to argue the scaling of the tensor structure label. Acting with the operator (4.4) on the direct channel contribution, one gets

$$\begin{aligned} &\mathcal{V}^{(-1,-1,-1)}\Big[(u_{s1}^2 u_{s2} u_{s4}^2 u_{s5}^{-1})^{h_\phi} \log u_{s1} \log u_{s2}\Big] \\ &\quad = -\frac{2(u_{s1}^2 u_{s2} u_{s4}^2 u_{s5}^{-1})^{h_\phi}}{u_{s2}u_{s3}u_{s5}} h_\phi^2 \left(1 + \left(2h_\phi - \frac{d-2}{2}\right)\log u_{s1}\right), \end{aligned} \tag{5.20}$$

which implies $\mathfrak{t} = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1}\epsilon_{34}^{-1})$ and $\eta = \mathcal{O}(\epsilon_{15}^{-1})$. Consequently, the relevant crossing equation is similar to eq. (5.10) studied in the previous subsection, and we can use the case I blocks of eq. (3.81) for the crossed-channel conformal block decomposition. The main new feature is of course the logarithmic factors in u_{s1} and u_{s2}. The case of the $\log u_{s1}$ factor is well known from the four-point lightcone bootstrap: it can be reproduced in the crossed channel via an anomalous correction to the weight of double-twist families that takes the form

$$h_1 = h_{[\phi\phi]_{0,J_1}} = 2h_\phi + \frac{\gamma}{2J_1^{2h_\phi}} + \ldots = 2h_\phi + \frac{\gamma}{2\lambda_1^{h_\phi}} + \ldots \,. \tag{5.21}$$

Indeed, from eq. (3.81), we see that expanding the exponent h_1 of the factor $u_{s1}^{h_1}$ around $2h_\phi$ gives the desired $\log u_{s1}$ at leading order.

In order to understand the factor $\log u_{s2}$, we now study the sum (now approximated by an integral) on the right-hand side of the crossing symmetry equation (5.17). After inserting the relevant lightcone blocks (3.81), we obtain

$$\sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2} \psi^{\text{CC}}_{\mathcal{O}_1\mathcal{O}_2;n} = \tag{5.22}$$

$$\begin{aligned} &= (u_{s1}^2 u_{s2} u_{s3} u_{s4}^2)^{h_\phi} \int \mathrm{d}\eta \frac{\mathrm{d}\lambda_1}{4\sqrt{\lambda_1}} \frac{\mathrm{d}\lambda_2}{4\sqrt{\lambda_2}} \frac{\gamma \log u_{s1}}{2\lambda_1^{h_\phi}} \mathcal{N}^{\text{CC,I}} P^{(\eta)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} e^{-\eta u_{s5} - \frac{\lambda_1 u_{s2} + \lambda_2 u_{s3}}{\eta}} \\ &= -\frac{\gamma}{2} \frac{C_{\phi\phi\phi}}{\Gamma(\Delta_\phi)} \left(u_{s1}^2 u_{s2} u_{s4}^2 u_{s5}^{-1}\right)^{h_\phi} \log u_{s1} \log u_{s2}. \end{aligned} \tag{5.23}$$

In order to reach the third line, we inserted the formula (5.11) for the product $\mathcal{N}P$ and performed the integration over λ_1 using

$$\int_{L_1^2}^\infty \frac{\mathrm{d}\lambda_1}{\lambda_1} e^{-\lambda_1 u_{s2} \eta^{-1}} = -\log u_{s2} + \mathcal{O}(1), \quad \forall L_1 = \mathcal{O}(1). \tag{5.24}$$

Here, L_1 denotes a lower limit of spins above which the large spin approximation of the blocks is valid. This same kind of large spin integral with logarithmic asymptotics appears in the four-point bootstrap at subleading orders — see the discussion below [21, eq. (3.5)]. The rest of the crossing equation is similar to eq. (5.10), i.e. the integrals over large λ_2 and η reproduce the correct powers law behavior on the various cross-ratios. Comparison with the right-hand side of the crossing symmetry equation yields

$$\frac{\Gamma(\Delta_\phi)^2}{\Gamma(\Delta_\phi/2)^4} C^3_{\phi\phi\phi} = -\frac{\gamma}{2} \frac{C_{\phi\phi\phi}}{\Gamma(\Delta_\phi)} \,. \tag{5.25}$$

The result for the factor γ in the anomalous dimension of double-twist families is the same as for ϕ exchange in the direct channel of the four-point crossing equation, see e.g. [4, eq. (41)] after setting the quantum numbers involved to $\tau_m = \Delta_\phi$ and $\ell_m = 0$. In addition, one identifies the operator product coefficients of the four-point expansion as $P_m = C^2_{\phi\phi[\phi\phi]_{0,J_1}}$.

$[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange in the direct channel, $h_\star < h_\phi$ We will now consider the case where the two exchanged operators in the direct channel are identical and possess leading twist $h_\star < h_\phi$. The direct channel blocks for this case $(-)$ were given in eq. (5.2), and we had already stressed that the leading contribution comes solely from one block, namely the block with $n = 0$. Hence, the relevant term we need to reproduce from the crossed channel sum is

$$\cdots + P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star} \mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;0} u_{s1}^{h_\star+h_\phi} u_{s2}^{h_\star} u_{s3}^{h_\star-h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star-2h_\phi} + \cdots = \sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2} \psi^{\mathrm{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{5.26}$$

In this case, the power law $u_{s1}^{h_\phi+h_\star} u_{s4}^{2h_\phi}$ implies that operators $\mathcal{O}_1 = [\phi\mathcal{O}_\star]_{0,J_1}$ and $\mathcal{O}_2 = [\phi\phi]_{0,J_2}$ are exchanged in the two intermediate channels of the crossed channel. Furthermore, just as for case (0), the power law prefactor is Casimir singular with respect to the two Casimir operators $\mathcal{D}_{12}^{(-1,0,-1)}$ and $\mathcal{D}_{45}^{(-1,-1,0)}$,

$$\begin{aligned}
\big[\mathcal{D}_{12}^{(-1,0,-1)}\big]^n u_{s2}^{h_\star} u_{s5}^{h_\star-2h_\phi} &= (h_\phi - h_\star)_n (2h_\phi - h_\star)_n \, u_{s2}^{h_\star} u_{s5}^{h_\star-2h_\phi}, \\
\big[\mathcal{D}_{45}^{(-1,-1,0)}\big]^n u_{s3}^{h_\star-h_\phi} u_{s5}^{h_\star-2h_\phi} &= (2h_\phi - h_\star)_n^2 \, u_{s3}^{h_\star-h_\phi} u_{s5}^{h_\star-2h_\phi}.
\end{aligned}$$

Finally, the direct channel power law is singular with respect to the vertex operator $\mathcal{V}^{(-1,-1,-1)}$ given in eq. (4.4):

$$\frac{\mathcal{V}^{(-1,-1,-1)}\left(u_{s1}^{h_\star+h_\phi} u_{s2}^{h_\star} u_{s3}^{h_\star-h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star-2h_\phi}\right)}{u_{s1}^{h_\star+h_\phi} u_{s2}^{h_\star} u_{s3}^{h_\star-h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star-2h_\phi}} = -2\frac{(h_\star - h_\phi)(h_\star - 2h_\phi)^2 \left(2h_\phi - \frac{d-2}{2}\right)}{u_{s2}u_{s3}u_{s5}} \,. \tag{5.27}$$

Our standard reasoning from previous subsections implies that, in order to reproduce the direct channel term, we need to sum over crossed channel lightcone blocks with eigenvalue λ_1 of order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1})$, eigenvalue λ_2 of order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1})$, and tensor structure label η of order $\mathcal{O}(\epsilon_{15}^{-1})$. Consequently, we use the case I lightcone blocks that were given in eq. (3.81), in which case the crossed channel sum reduces to

$$\sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\mathrm{CC}}_{\mathcal{O}_1\mathcal{O}_2;n} = \\ u_{s1}^{h_\star+h_\phi} u_{s4}^{2h_\phi} u_{s2}^{h_\star} u_{s3}^{2h_\phi-h_\star} \int_{\mathbb{R}^3_+} \frac{\mathrm{d}\eta\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2}{16\sqrt{\lambda_1\lambda_2}} \mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;\eta)} P^{(\eta)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}} e^{-\eta u_{s5}-\frac{\lambda_1 u_{s2}+\lambda_2 u_{s3}}{\eta}} . \tag{5.28}$$

In order to reproduce the left hand side of the crossing equation (5.26), we must impose the following asymptotic behavior of the product

$$\frac{\mathcal{N}^{\mathrm{CC,I}}_{(h_a,\bar{h}_a;\eta)}}{16\sqrt{\lambda_1\lambda_2}} P^{(\eta)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}} = P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star}\mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;0} \frac{(\eta^{-1}\lambda_1)^{h_\phi-h_\star-1}}{\Gamma(h_\phi-h_\star)} \frac{(\eta^{-1}\lambda_2)^{2h_\phi-h_\star-1}}{\Gamma(2h_\phi-h_\star)} \frac{\eta^{2h_\phi-h_\star-3}}{\Gamma(2h_\phi-h_\star)} . \tag{5.29}$$

at large quantum numbers $\eta, \lambda_1, \lambda_2$. After plugging in our formulas (3.83) and (3.64) for the relevant normalizations $\mathcal{N}$, we deduce

$$P^{(\eta)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}} \\ = P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star}\mathcal{N}^{\mathrm{DC}}_{\mathcal{O}_\star\mathcal{O}_\star;0} \frac{32\times 4^{3h_\phi-h_\star}4^{-J_1-J_2}e^{2\eta}}{\Gamma(h_\phi-h_\star)\Gamma^2(2h_\phi-h_\star)} \eta^{-2\eta-2h_\phi} J_1^{\eta+3h_\phi-2h_\star-\frac{3}{2}} J_2^{\eta+4h_\phi-h_\star-\frac{3}{2}} , \tag{5.30}$$

with

$$\mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star\mathcal{O}_\star;0} = \frac{\Gamma^2(2h_\star)\Gamma^2(h_\phi-h_\star)}{\Gamma^2(h_\star)\Gamma^2(h_\phi)} . \tag{5.31}$$

To evaluate the crossed channel quantities we have used that $h_1 \to h_\phi + h_\star$, $h_2 \to 2h_\phi$ and $\bar{h}_a = h_a + J_a \to \sqrt{\lambda_a}$. In the direct channel, on the other hand, we have $h_a = \bar{h}_a = h_\star$. Our result coincides with the findings of [26], see eq. (48) therein. In order to compare the two expressions, one should observe that our normalization conventions for blocks differ slightly from those used by Antunes et al., see [26, footnote 3]. This difference of conventions gives rise to an additional factor $2^{J_1+J_2}$. In addition, the labels J_1 and J_2 are exchanged.

With this result, we have reproduced all computations of [26] pertaining to the scalar five-point function. In this context, our derivation explicitly shows that these OPE coefficients lie in a case I regime where the tensor structure label n scales to infinity faster than the spins J_1, J_2. In contrast, the derivation of [26] was based on a permutation of the order of limits in the crossed channel from $(\epsilon_{15}, \epsilon_{34}, \epsilon_{12}, \epsilon_{45}, \epsilon_{23})$ to $(\epsilon_{12}, \epsilon_{45}, \epsilon_{23}, \epsilon_{34}, \epsilon_{15})$, in which case $J_1, J_2 \to \infty$ before $n \to \infty$. In general, it is not clear that these results will continue to predict the leading asymptotics of OPE coefficients in this permuted order of limits. Nonetheless, in the explicit example [26, section 4.2] of ϕ^3 theory in $d = 6 - \varepsilon$ dimensions, which we will review in section 5.4 in more detail, the authors show that the case I asymptotics predicted from the lightcone bootstrap indeed coincide with the leading $J_1, J_2 \gg n \gg 1$ behavior of the OPE coefficients. To avoid the permutation of limits, it would be interesting to clarify the physical interpretation of this case I regime and, more generally, to better understand the analyticity properties of OPE coefficients in the tensor structure label.

$[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange in the direct channel, $2h_\phi > h_\star > h_\phi$ Let us now address the last case in which the leading-twist contribution $\mathcal{O}_\star$ to the internal exchange in the crossed channel has $2h_\phi > h_\star > h_\phi$. As we have pointed out in the first subsection, an infinite number of crossed channel blocks can contribute in this case at the same order, since the leading asymptotics do not depend on the quantum number n. Using our formula (5.4) for crossed channel lightcone blocks, the crossing equation becomes

$$\cdots + u_{s1}^{2h_\phi} u_{s2}^{h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi} \sum_n P^{(n)}_{\mathcal{O}_\star \mathcal{O}_\star} \mathcal{N}^{\mathrm{DC}\,(0)}_{\mathcal{O}_\star \mathcal{O}_\star; n} + \cdots = \sum_{\mathcal{O}_1, \mathcal{O}_2, n} P^{(n)}_{\mathcal{O}_1 \mathcal{O}_2} \psi^{\mathrm{CC}}_{\mathcal{O}_1 \mathcal{O}_2; n}(u_{si}). \tag{5.32}$$

The familiar power law $(u_{s1}u_{s4})^{2h_\phi}$ for the dependence on the cross-ratios u_{s1} and u_{s4} implies double-twist exchange of $[\phi\phi]_{0,J_i}$ in the (12) and (45) OPE of the crossed channel.

The first interesting new feature of the case under consideration arises from the kernel condition. In fact, using our explicit formula (3.66) for the leading term of the crossed channel Casimir operator $\mathcal{D}_{12}$, we find

$$\mathcal{D}_{12}^{(-1,0,-1)} u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi} = 0. \tag{5.33}$$

On the other hand, when we apply the next-to-leading term (3.85) in the Casimir operator D_{12} to the same power law on the left-hand side of the crossing equation (5.32), we find

$$[\mathcal{D}_{12}^{(-1,0,0)}]^n u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi} = (2h_\phi - h_\star)_n (h_\phi)_n u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi - n},$$

i.e. the direct channel is $\mathcal{D}_{12}^2$-singular at order $\mathcal{O}(\epsilon_{15}^{-1})$ but not at leading order. At the same time, the action of the leading term (3.68) in the second crossed channel Casimir operator $\mathcal{D}_{45}$ gives

$$[\mathcal{D}_{45}^{(-1,0,-1)}]^n u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi} = ((2h_\phi - h_\star)_n)^2 u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi - n}. \tag{5.34}$$

In comparison with the discussion in section 5.2.1, the crossed channel is now dominated by lightcone blocks with eigenvalue λ_1 of order $\mathcal{O}(\epsilon_{15}^{-1})$ and eigenvalue λ_2 of order $\mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1})$. To see the way the tensor structure label scales, we can use the case II vertex operator of section 4. Its action on the direct channel contribution reads

$$\frac{\mathcal{V}^{(-1,-1,0)} \left(u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi} \right)}{u_{s2}^{h_\phi} u_{s1}^{2h_\phi} u_{s3}^{h_\star - h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star - 2h_\phi}} = \frac{(h_\star - 2h_\phi)^2 \left(36h_\phi^2 - 12dh_\phi + d^2 - 3d\right)}{6u_{s3}u_{s5}}, \tag{5.35}$$

which shows that its eigenvalue scales like λ_2, that is, precisely like in eq. (4.15) with N finite. This regime corresponds precisely to the case II blocks (3.101) we studied at the end of the previous section. Consequently, the crossed-channel sum on the right-hand side of equation (5.32) becomes

$$\sum_{\mathcal{O}_1, \mathcal{O}_2, n} P^{(n)}_{\mathcal{O}_1 \mathcal{O}_2} \psi^{\mathrm{CC}}_{\mathcal{O}_1 \mathcal{O}_2; n}(u_{si}) = \left(u_{s1}^2 u_{s4}^2 u_{s2} u_{s3} u_{s5}\right)^{h_\phi} \int_{\mathbb{R}_+^2} \frac{\mathrm{d}\lambda_1 \mathrm{d}\lambda_2}{16\sqrt{\lambda_1\lambda_2}} \tag{5.36}$$

$$\sum_{\delta n=0}^{\infty} \mathcal{N}^{\mathrm{CC,II}}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}; J_1 - \delta n} P^{(J_1 - \delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} \mathcal{K}_{h_\phi + \delta n}(\lambda_1 u_{s5} + \lambda_2 u_{s3} u_{s5}).$$

In order to match this crossed-channel integral with the direct-channel term on the left-hand side of the crossing equation (5.32), we propose the following Ansatz:

$$\frac{\mathcal{N}^{\text{CC,II}}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2};J_1-\delta n}}{16\sqrt{\lambda_1\lambda_2}} P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} = b_{\delta n}\frac{2\lambda_1^{h_\phi+\delta n-1}}{\Gamma(h_\phi+\delta n)\Gamma(2h_\phi-h_\star)}\frac{\lambda_2^{2h_\phi-h_\star-1}}{\Gamma(2h_\phi-h_\star)}, \tag{5.37}$$

where $b_{\delta n}$ is a set of constants that depend on the choice of tensor structure. As we shall see, these constants cannot be determined by our equation (5.32). After plugging the Ansatz (5.37) into the right-hand side of eq. (5.36), we evaluate the two integrations over λ_1 and λ_2 with the help of the following integral formula:

$$\int_{\mathbb{R}_+^2}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}x^\alpha y^\beta \mathcal{K}_\alpha(x+y) = \frac{1}{2}\Gamma(\alpha)\Gamma(\beta)^2, \tag{5.38}$$

for $\alpha = h_\phi + \delta n$ and $\beta = 2h_\phi - h_\star$. This identity is proven in appendix B. Once the dust settles, we see that the crossing equation (5.32) is indeed satisfied, provided that the undetermined constants $b_{\delta n}$ satisfy the following sum rule:

$$\sum_{n=0}^{J_\star} P^{(n)}_{\mathcal{O}_\star\mathcal{O}_\star}\mathcal{N}^{\text{DC}}_{\mathcal{O}_\star\mathcal{O}_\star;n} = \sum_{\delta n=0}^{\infty} b_{\delta n} \tag{5.39}$$

In other words, while the crossing symmetry equation (5.32) suffices to determine the dependence of the operator product coefficients in eq. (5.37) on the spins J_1 and J_2 of the intermediate fields, it only constrains the 'average' over the choice δn of tensor structure. It is clear that we could not have done better here because the lightcone blocks from case II depend nontrivially on only two of the five cross-ratios, namely on u_{s3} and u_{s5}. The dependence of the other three cross-ratios is through a simple power law that matches directly the dependence of the lightcone blocks in the direct channel and moreover does not depend on δn. Hence, at the order we have considered here, crossing symmetry is unable to resolve the δn dependence of the operator product coefficients beyond the sum rule (5.39). In the next subsection, we will explain how to determine the coefficients $b_{\delta n}$ in eq. (5.37) by solving the crossing equation at subleading orders in u_{s2}.

5.3 Solving OPE coefficients for discrete tensor structures

In the last subsection, we determined the operator product coefficients involving two double-twist families in the limit of large tensor structure label η, see eqs. (5.12) and (5.30). Through the analysis of the five-point crossing equation (5.32) in theories for which the leading-twist field $\mathcal{O}_\star$ that appears in the operator product of ϕ with itself has weight $h_\star > h_\phi$, we were able to at least constrain the operator product coefficients with finite tensor structure label δn through the sum rule (5.39). Our goal in this subsection is to do better and to fully determine the operator product coefficients for finite tensor structure label $\delta n = J_1 - n$. Our comments in the final paragraph of the previous subsection suggest that our goal can be reached if we manage to analyze corrections to the crossing symmetry constraint (5.32) that are subleading in the cross-ratio u_{s2}, i.e. we should look at the crossing equation in the regime

$$LCL^{(4)}_{\vec{\epsilon}} : X_{15} \ll X_{34} \ll X_{12} \ll X_{45} \ll 1, \tag{5.40}$$

which imposes no condition on X_{23}. Fortunately, we have already determined both direct channel and case II crossed channel blocks in this regime. For the direct channel, we constructed the full u_{s2} dependence of the relevant blocks in eq. (3.59), on the way from the partial lightcone limit $LCL^{(2)}$ to the full lightcone limit. By relaxing the last lightcone limit and summing over crossed channel blocks in a different scaling regime of spins, we can also bootstrap the leading contributions with $h_\star \leq h_\phi$ to derive OPE coefficients in the less stringent scaling limit $J_2 \gg J_1, n \gg 1$ with $J_1 - n = \delta n = 0, 1, 2, \ldots, \infty$.

5.3.1 Single internal identity exchange in the direct channel

$[1|\phi]$-exchange in the direct channel. Relaxing the $u_{s2} \to 0$ limit does not change the leading contribution:

$$C_{\phi\phi\phi}\left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi} + \ldots = \sum_{\mathcal{O}_1,\mathcal{O}_2,n} P^{(n)}_{\mathcal{O}_1\mathcal{O}_2}\psi^{\text{CC}}_{\mathcal{O}_1\mathcal{O}_2;n}. \tag{5.41}$$

We have constructed the full u_{s2} dependence of the case II crossed channel blocks, see eq. (3.101) in section 3.3:

$$\begin{aligned}&\psi^{(12),(45)}_{([\phi\phi]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)}(u_{si}) \\ &\overset{LCL^{(4)}}{\sim} \mathcal{N}^{\text{LS,II}}_{([\phi\phi]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)}(u_{s1}^2u_{s2}u_{s3}u_{s4}^2u_{s5})^{h_\phi}(u_{s5}(1-u_{s2}))^{\delta n}\,\mathcal{K}_{h_\phi+\delta n}(u_{s3}u_{s5}J_2^2).\end{aligned} \tag{5.42}$$

The normalization $\mathcal{N}^{\text{CC,II}}_{([\phi\phi]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)} := \mathcal{N}^{\text{CC,II}}_{(2h_\phi,2h_\phi,2h_\phi+J_1,2h_\phi+J_2;J_1-\delta n)}$ is given in eq. (3.102). Hence, there is nothing that prevents us from evaluating crossing symmetry in the regime $LCL^{(4)}$, without sending u_{s2} to zero. In this case, the crossing equation becomes

$$\begin{aligned}C_{\phi\phi\phi}u_{s5}^{-2h_\phi} &= u_{s3}^{h_\phi}(1-z)^{h_\phi} \\ \times \sum_{\delta n=0}^{\infty} z^{\delta n}&\int \frac{\mathrm{d}^2\lambda}{16\sqrt{\lambda_1\lambda_2}}\mathcal{N}^{\text{CC,II}}_{([\phi\phi]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)}P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}}u_{s5}^{h_\phi+\delta n}\mathcal{K}_{h_\phi+\delta n}(\lambda_1u_{s5}+\lambda_2u_{s3}u_{s5}).\end{aligned}$$

Homogeneity of the equation in $\epsilon_{15}, \epsilon_{34}$ imposes $\mathcal{N}P \propto \lambda_1^{2h_\phi+\delta n-1/2}\lambda_2^{h_\phi-1/2}$, so without loss of generality we can write

$$\frac{\mathcal{N}^{\text{CC,II}}_{([\phi\phi]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)}P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}}}{16\sqrt{\lambda_1\lambda_2}} = C_{\phi\phi\phi}\frac{2\lambda_1^{2h_\phi+\delta n-1}\lambda_2^{h_\phi-1}}{\Gamma(2h_\phi+\delta n)\Gamma(h_\phi)\Gamma(2h_\phi)}a_{\delta n}, \tag{5.43}$$

for some series of coefficients $a_{\delta n}$. Plugging this back into the crossing equation and using the double integral formula of appendix B yields a sum rule for the a-coefficients:

$$1 = (1-z)^{h_\phi}\sum_{\delta n=0}^{\infty} a_{\delta n}z^{\delta n}. \tag{5.44}$$

The sum rule is easily solved by imposing $a_{\delta n} = (h_\phi)_{\delta n}/\delta n!$, such that the full solution is

$$\frac{\mathcal{N}^{\text{CC,II}}_{(2h_\phi,2h_\phi,2h_\phi+J_1,2h_\phi+J_2;J_1-\delta n)}P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}}}{16J_1J_2} \sim C_{\phi\phi\phi}\frac{2J_1^{4h_\phi+2\delta n-2}J_2^{2h_\phi-2}}{\Gamma(h_\phi)\Gamma(2h_\phi)^2}\frac{(h_\phi)_{\delta n}}{\delta n!(2h_\phi)_{\delta n}}. \tag{5.45}$$

The quantity P on the left-hand side is given by the product

$$P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} = C_{\phi\phi[\phi\phi]_{0,J_1}} C^{(J_1-\delta n)}_{[\phi\phi]_{0,J_2}\phi[\phi\phi]_{0,J_1}} C_{\phi\phi[\phi\phi]_{0,J_2}} . \tag{5.46}$$

Since the operator product coefficients $C_{\phi\phi[\phi\phi]_{0,J}}$ for large J are known (see eq. (5.16)), our new formula (5.45) allows us to compute the coefficients in the center of the expression (5.46) for large J_a but any finite integer value of the label δn. We will discuss this formula further in the section 5.4 and test it through a non-trivial example. But before that, we will first go through the other direct channel exchanges.

$[\phi|\mathbf{1}]$-exchange in the direct channel. As we saw previously, the identity exchange in (34) forces the exchange of ϕ in the (12) OPE of the crossed channel. As a result, the crossed channel block decomposition is a specific example of case II scaling where $\lambda_1 = 0 + \mathcal{O}(1)$ is fixed. The corresponding solution to the crossing equation was already computed in the second paragraph of section 5.2.1. Hence, the $[\phi|\mathbf{1}]$ does not provide any new insight into OPE coefficients.

5.3.2 No internal identity exchange in the direct channel

Even in the relaxed lightcone limit $LCL^{(4)}_{\vec{\epsilon}}$, the exchange of $\mathcal{O}_\star$ in the (15) OPE of the direct channel will yield differing asymptotics in the $\epsilon_{12} \to 0$ limit depending on whether $h_\star < h_\phi$, $h_\star = h_\phi$ or $h_\star > h_\phi$. As we determined in the previous section, these X_{12} asymptotics affect the twist of operators in the (12) OPE of the crossed channel: $[\phi\mathcal{O}_\star]$ is exchanged when $h_\star \leq h_\phi$, $[\phi\phi]$ is exchanged when $h_\star \geq h_\phi$, and an anomalous dimension appears when $h_\star = h_\phi$. Each contribution will have a non-trivial dependence on the finite cross-ratio u_{s2} corresponding to the lightcone blocks for $\mathcal{O}_\star$ exchange — the latter were computed in full detail in section 3.2.

$[\mathcal{O}_\star|\phi]$-exchange in the direct channel. If there is ϕ exchange in the (34) OPE of the direct channel, then for any $\mathcal{O}_\star$ exchange in the (12) OPE, the left hand side of the crossing equation will take the form

$$\mathrm{DC} = \dots + C_{\phi\phi\phi} \left(\frac{u_{s1}u_{s4}}{u_{s5}}\right)^{2h_\phi} u_{s5}^{h_\star} \frac{C^2_{\phi\phi\mathcal{O}_\star}}{\mathrm{B}_{\bar{h}_\star}} \log u_{s1} u_{s2}^{h_\phi}\, {}_2F_1\left[\begin{matrix} h_\star, 2h_\phi - h_\star \\ 2h_\star \end{matrix}\right](z) + \dots \tag{5.47}$$

If we define $\gamma_0 := 2\mathrm{B}^{-1}_{\bar{h}_\star} C^2_{\phi\phi\mathcal{O}_\star} \Gamma(2h_\phi - h_\star)^2 \Gamma(2h_\phi)^{-2}$, then it is easy to verify that the direct channel is reproduced by the perturbation $\frac{1}{2}\gamma_0 \frac{\partial}{\partial h_1}(P\psi) = \gamma_0 \log u_{s1}(P\psi) + \dots$ of the direct channel sum computed in eq. (5.45) for $[\mathbf{1}|\phi]$ exchange, i.e.

$$\mathrm{DC} = \dots + \int \frac{\mathrm{d}J_1 \mathrm{d}J_2}{4} \left(\log u_{s1} \frac{\gamma_0}{2J_1^{2h_\star}}\right) P^{(J-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} \psi^{\mathrm{CC}}_{(2h_\phi, 2h_\phi + J_a; J_1 - \delta n)} + \dots \tag{5.48}$$

Hence, the direct channel contribution from $[\mathcal{O}_\star|\phi]$-exchange is nicely reproduced by crossed channel terms, but it does not provide any new information on OPE coefficients.

$[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange in the direct channel, $h_\star < h_\phi$. We can now look at the direct channel contribution given by

$$\mathrm{DC} = \cdots + P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star} u_{s1}^{h_\star+h_\phi} u_{s3}^{h_\star-h_\phi} u_{s4}^{2h_\phi} u_{s5}^{h_\star-2h_\phi} (1-z)^{h_\star} z^{J_\star} {}_2F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star + h_\star - h_\phi \\ 2\bar{h}_\star \end{bmatrix} (z) + \dots \tag{5.49}$$

For this contribution, the crossing equation reduces to

$$(u_3u_5)^{h_\star-2h_\phi}(1-z)^{h_\star-h_\phi} z^{J_\star} {}_2F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star + h_\star - h_\phi \\ 2\bar{h}_\star \end{bmatrix} (z) =$$
$$\sum_{\delta n=0}^{\infty} \int \frac{\mathrm{d}\lambda_1 \mathrm{d}\lambda_2}{16\sqrt{\lambda_1\lambda_2}} \mathcal{N}^{\mathrm{CC,II}}_{([\phi\mathcal{O}_\star]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)} P^{(J_1-\delta n)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}} z^{\delta n} u_5^{h_\phi+\delta n} \mathcal{K}_{h_\phi+\delta n}(\lambda_1 u_5 + \lambda_2 u_3 u_5),$$

where the normalization $\mathcal{N}^{\mathrm{CC,II}}_{([\phi\mathcal{O}_\star]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)} := \mathcal{N}^{\mathrm{CC,II}}_{(h_\phi+h_\star,2h_\phi,h_\phi+h_\star+J_1,2h_\phi+J_2;J_1-\delta n)}$ is again to be found in eq. (3.102). Homogeneity of the crossing equation in $\epsilon_{15}, \epsilon_{34}$ fixes the dependence of the OPE coefficients in λ_1, λ_2 up to a series in δn, which we express as

$$\frac{\mathcal{N}^{\mathrm{CC,II}}_{([\phi\mathcal{O}_\star]_{0,J_1},[\phi\phi]_{0,J_2};J_1-\delta n)} P^{(J_1-\delta n)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}}}{16\sqrt{\lambda_1\lambda_2}} = P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{2\lambda_1^{h_\phi+\delta n-1}\lambda_2^{2h_\phi-h_\star-1}}{\Gamma(h_\phi+\delta n)\Gamma(2h_\phi-h_\star)^2} b^-_{\delta n}. \tag{5.50}$$

Plugging this form back into the crossing equation yields the sum rule

$$(1-z)^{h_\star} z^{J_\star} {}_2F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star + h_\star - h_\phi \\ 2\bar{h}_\star \end{bmatrix} (z) = \sum_{\delta n=0}^{\infty} b^-_{\delta n} z^{\delta n}. \tag{5.51}$$

The solution to this sum rule is immediate: the coefficients $b^-_{\delta n}$ on the right are coefficients of the function on the left in a power series expansion around $z=0$. By combining the previous two equations we obtain the following expression for the OPE coefficients

$$\frac{\mathcal{N}^{\mathrm{CC,II}}_{(h_\phi+h_\star,2h_\phi,h_\phi+h_\star+J_1,2h_\phi+J_2;J_1-\delta n)} P^{(J_1-\delta n)}_{[\phi\mathcal{O}_\star]_{0,J_1}[\phi\phi]_{0,J_2}}}{16\sqrt{\lambda_1\lambda_2}} = \tag{5.52}$$
$$= P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{2\lambda_1^{h_\phi+\delta n-1}\lambda_2^{2h_\phi-h_\star-1}}{\Gamma(h_\phi+\delta n)\Gamma(2h_\phi-h_\star)^2\delta n!} \frac{\mathrm{d}^{\delta n}}{\mathrm{d}z^{\delta n}} (1-z)^{h_\star} z^{J_\star} {}_2F_1 \begin{bmatrix} \bar{h}_\star, \bar{h}_\star + h_\star - h_\phi \\ 2\bar{h}_\star \end{bmatrix} (z)\Big|_{z=0}.$$

This is a genuinely new result on OPE coefficients involving the double twist families $[\phi\mathcal{O}_\star]$ and $[\phi\phi]$. We shall discuss its applications in the next subsection. But before we do so, let us analyze the final case in which $[\mathcal{O}_\star|\mathcal{O}_\star]$ is exchanged in the direct channel.

$[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange in the direct channel, $h_\star > h_\phi$. When $h_\star > h_\phi$, the relevant behavior of direct channel blocks is

$$\psi^{(15),(34)}_{\mathcal{O}_\star\mathcal{O}_\star;n}(u_{si}) \overset{LCL^{(4)}}{\sim} (u_{s1}u_{s2})^{h_\phi}(u_{s3}u_{s5})^{h_\star} g^{\mathrm{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;n}(z), \quad z := 1-u_{s2}. \tag{5.53}$$

An integral representation for the function g^{fin} was given in equation (3.60). It is also possible to work out the following power series expansion, see eq. (3.61) in section 3.2. For

$\mathcal{O}_1 = \mathcal{O}_\star = \mathcal{O}_2$, it reads

$$g^{\text{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;n}(z) = \frac{\Gamma(2\bar{h}_\star)\Gamma(h_{\star\phi}+n)}{\Gamma(\bar{h}_\star)\Gamma(\bar{h}_\star+h_{\star\phi}+n)} \sum_{k=0}^{\infty} \frac{(2\bar{h}_\star-\bar{h}_\phi)_k}{k!} \frac{(\bar{h}_\star)_k}{(2\bar{h}_\star)_k} \frac{(h_{\star\phi}+n)_k}{(h_{\star\phi}+n+\bar{h}_\star)_k} \\ {}_2F_1\left[\begin{matrix}\bar{h}_\star-2h_\phi-n, \bar{h}_\star \\ 2\bar{h}_\star+k\end{matrix}\right](z)\, {}_2F_1\left[\begin{matrix}n-\bar{h}_\star-h_\star, \bar{h}_\star \\ \bar{h}_\star+h_{\star\phi}+n+k\end{matrix}\right](1-z). \tag{5.54}$$

If we plug our Ansatz (5.37) for the operator product coefficients into the crossing equation, we obtain the following constraint:

$$\sum_{n=0}^{J_\star} P^{(n)}_{\mathcal{O}_\star\mathcal{O}_\star} g^{\text{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;n}(z) = \sum_{\delta n=0}^{\infty} b_{\delta n} z^{\delta n}. \tag{5.55}$$

This constraint now replaces the z-independent sum rule (5.39) found in the previous subsection, the latter of which is recovered for $z = 1-u_{s2} = 1$. Since the new equation (5.55) contains a power series in z on both sides, it suffices to completely determine the coefficients $b_{\delta n}$. In conclusion, the operator product coefficients for two double-twist exchanges at subleading order are given by

$$\frac{\mathcal{N}^{\text{CC,II}}_{(2h_\phi,2h_\phi,2h_\phi+J_1,2h_\phi+J_2;J_1-\delta n)}}{16\sqrt{\lambda_1\lambda_2}} P^{(J_1-\delta n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} \sim \\ \sim \frac{2\lambda_1^{h_\phi+\delta n-1}\lambda_2^{2h_\phi-h_\star-1}}{\Gamma(h_\phi+\delta n)\Gamma(2h_\phi-h_\star)^2} \sum_{m=0}^{J_\star} P^{(m)}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{\mathrm{d}^{\delta n} g^{\text{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;m}}{\delta n!\,\mathrm{d}z^{\delta n}}\Big|_{z=0}. \tag{5.56}$$

It is not difficult to evaluate arbitrary derivatives of the function $g^{\text{fin}}(z)$ at $z = 0$ using the series expansion formula (5.54). Recall that OPE coefficients on the right hand side are given by $P^{(m)}_{\mathcal{O}_\star\mathcal{O}_\star} = C^2_{\phi\phi\mathcal{O}_\star} C^{(m)}_{\mathcal{O}_\star\phi\mathcal{O}_\star}$ with a label m that enumerates non-trivial tensor structures in case $\mathcal{O}_\star$ carries spin $J_\star \neq 0$. Assuming this data is known, one can use our formula to compute the quantity P on the left-hand side of the equation via the expression (3.102) for the normalization of case II lightcone blocks.

The last formula can be made even more explicit if the intermediate field $\mathcal{O}_\star$ is scalar. In this case, the finite sum over m consists of a single term with $m = 0$ and the function $g^{\text{fin}}(z)$ reduces to a Gauss hypergeometric series,

$$g^{\text{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;0}(z) = \frac{\mathrm{B}_{h_\star,h_\phi}\mathrm{B}_{h_\star,h_\star-h_\phi}}{\mathrm{B}^2_{h_\star}} {}_2F_1\left[\begin{matrix}h_\phi, h_\phi \\ h_\star+h_\phi\end{matrix}\right](z). \tag{5.57}$$

From this expression, it is straightforward to evaluate the derivatives of g^{fin} that appear in eq. (5.56). The net effect is encoded in the replacement

$$\sum_{m=0}^{J_\star} P^{(m)}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{\mathrm{d}^{\delta n} g^{\text{fin}}_{\mathcal{O}_\star\mathcal{O}_\star;m}}{\delta n!\,\mathrm{d}z^{\delta n}}\Big|_{z=0} \mapsto P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{\mathrm{B}_{h_\star,h_\phi}\mathrm{B}_{h_\star,h_\star-h_\phi}}{\mathrm{B}^2_{h_\star}} \frac{(h_\phi)^2_{\delta n}}{\delta n!\,(h_\star+h_\phi)_{\delta n}}. \tag{5.58}$$

With this substitution rule, our new result (5.56) on the contributions to double twist OPE coefficients that stem from $[\mathcal{O}_\star\mathcal{O}_\star]$-exchange in the direct channel applies to the case of scalar $\mathcal{O}_\star$.

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Summary. We have now bootstrapped OPE coefficients of large-spin double-twist operators in the crossed channel as a function of direct-channel leading-twist exchanges $\mathcal{O}_\star$ and their CFT data $P^{(n)}_{\mathcal{O}_\star\mathcal{O}_\star}$. We were able to obtain asymptotics of double-twist OPE coefficients in two regimes: either $n \gg J_1, J_2 \gg 1$ (I) or $J_2 \gg J_1, n \gg 1$ with $J_1 - n = \delta n = 0, 1, 2, \ldots, \infty$ (II). The results in regime (I), derived in section 5.2, only apply to direct-channel exchanges with $h_\star \leq h_\phi$. The OPE coefficients in this regime were computed in [26] using different methods, and we find exact agreement between their results and ours. On the other hand, the results in regime (II), derived in section 5.3, apply for any $h_\star < 2h_\phi$ exchanges in the direct channel. It is actually the first time that a discrete tensor structure dependence for the OPE coefficients $C^{(n)}_{[\phi\phi]\phi[\phi\phi]}$, $C^{(n)}_{[\phi\mathcal{O}_\star]\phi[\phi\phi]}$ has been resolved. This progress came about because we were able to control the lightcone blocks on both sides of the crossing symmetry equation with only four, rather than the usual five lightcone limits taken. While in principle, the relevant information on lightcone blocks resides in the integral formula and hence in the usual lightcone OPE, the lightcone Casimir operators played a crucial role in extracting this data.

5.4 Applications to specific models

To explore the predictions that follow from formulas (5.45), (5.52) and (5.56), we would like to discuss possible applications, at least briefly. In the first subsection, we shall go through each of the new formulas and list a few cases to which they apply, mostly within the context of models in $d = 3$. We shall then employ one of the concrete realizations in the second subsection to check our result (5.45) against leading order perturbative computations in scalar ϕ^3 theory.

5.4.1 Possible leading-twist exchanges

Our results on OPE coefficients were obtained by solving the crossing equation for specific leading-twist exchanges in the direct channel. Which of these leading-twist exchanges actually appear depends on the specific model under investigation and the choice of the field ϕ. We will review this case-by-case:

- $[\mathbf{1}|\phi]$-exchange only exists if $C_{\phi\phi\phi} \neq 0$ and entails $P_{[\phi\phi][\phi\phi]} \propto C_{\phi\phi\phi}$ in the crossed channel, see eq. (5.45). Of course, in all models with $\mathbb{Z}_2$ symmetry, the OPE coefficients $C_{\phi\phi\phi}$ vanish if ϕ is $\mathbb{Z}_2$-charged. Hence, there are two cases where this contribution to the OPE coefficients does not vanish. On the one hand, we can consider $\mathbb{Z}_2$-charged operators in models that break the $\mathbb{Z}_2$ symmetry, like ϕ^3 theory. Alternatively, our formula (5.45) applies to any $\mathbb{Z}_2$-singlet operator, even in models with $\mathbb{Z}_2$ symmetry, such as the field $\phi \equiv \epsilon$ in the $d = 3$ Ising model or $\phi \equiv \varphi^2$ at the Wilson-Fisher fixed-point. The first realization within scalar ϕ^3 theory provides a nice opportunity to check formula (5.45) against an independent perturbative calculation of the OPE coefficients at tree level. The latter are derived in appendix D and they are compared with our bootstrap result in section 5.4.2 below.

- $[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange with $0 < h_\star < h_\phi$ entails $P_{[\phi\mathcal{O}_\star][\phi\phi]} \propto P^{(0)}_{\mathcal{O}_\star\mathcal{O}_\star}$ in the crossed channel, see eq. (5.52). These exchanges can of course only appear when ϕ is *not* the lowest dimension scalar of the theory. A typical example would be $(\phi, \mathcal{O}_\star) = (\epsilon, T_{\mu\nu})$ in the Ising model, or more generally $\mathcal{O}_\star = T_{\mu\nu}$ whenever $\Delta_\phi > d-2$. Since these cases involve external composite fields, performing explicit perturbative calculations of our formula (5.52) would be significantly more difficult than for the case we mentioned at the end of the previous paragraph. This makes eq. (5.52) an interesting prediction of the multipoint bootstrap.

- $[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange with $h_\star > h_\phi$ entails $P_{[\phi\phi][\phi\phi]} \propto P^{(n)}_{\mathcal{O}_\star\mathcal{O}_\star}$, as detailed in our formula (5.56). When $[\mathbf{1}|\phi]$-exchange contributes to the direct channel — see our discussion in the first item — the contributions from $[\mathcal{O}_\star|\mathcal{O}_\star]$-exchange correspond to subleading terms in the large spin expansion of the double-twist OPE coefficients, such as $\mathcal{O}_\star = T_{\mu\nu}$ in ϕ^3 theory or $(\phi, \mathcal{O}_\star) = (\epsilon, \epsilon')$ in the Ising model. On the other hand, one can also find models in which the exchange of $[\mathcal{O}_\star|\mathcal{O}_\star]$ with $h_\star > h_\phi$ is leading, like the Gross-Neveu-Yukawa models with $\phi \equiv \sigma$ the pseudoscalar and $\mathcal{O}_\star \equiv T_{\mu\nu}$. In all of these cases, $\mathcal{O}_\star$ must be the unique field with twist $h_\star$ in the spectrum. Such a twist gap is characteristic of strongly interacting CFTs, making formula (5.56) a non-trivial prediction of the five-point lightcone bootstrap that would be difficult to reproduce with perturbative methods.

Before we conclude this short list of applications, we would like to briefly comment on an aspect that specifically applies to five-point functions in $d = 3$-dimensional models. It is well known (see e.g. [55, section 4.2.3]) that three-point tensor structures of two STTs and one scalar can only be parity-odd in $d = 3$, whereas the results we have stated above only apply to parity-even tensor structures. Fortunately, it is not difficult to derive analogous expressions for parity-odd tensor structures. As demonstrated in appendix E, the parity-odd five-point blocks in either channel are obtained by a simple shift of parameters $(h_\phi, n) \to (h_\phi + 1/2, n+1)$.[8] We can therefore perform the same analysis and bootstrap parity-odd OPE coefficients of double-twist operators. There are only two qualitative differences: first, odd parity excludes scalar exchange in either channel. Second, the direct channel asymptotics now depend on the sign of $h_\star - h_\phi + 1/2$, such that $[\phi\mathcal{O}_\star]$ only appears if $\frac{d-2}{2} \leq h_\star < h_\phi - 1/2$, which in turn requires $\Delta_\phi > d-1$. In the case where formula (5.56) applies, i.e. $h_\star > h_\phi - 1/2$, the shift of parameters leads to the following explicit result:

$$\frac{\mathcal{N}^{\mathrm{CC,II}}_{([\sigma\sigma]_{0,J_1}[\sigma\sigma]_{0,J_2};J_1-\delta n-1/2)}}{16 J_1 J_2} P^{(J_1-\delta n,\mathrm{odd})}_{[\sigma\sigma]_{0,J_1}[\sigma\sigma]_{0,J_2}} \sim \frac{2 J_1^{2(h_\sigma+\delta n-3/2)} J_2^{2(2h_\phi-h_\star-1)}}{\Gamma(h_\phi+\delta n-1/2)\Gamma(2h_\phi-h_\star)^2} \sum_{m=0}^{J_\star-1} P^{(m,\mathrm{odd})}_{\mathcal{O}_\star\mathcal{O}_\star} \frac{\mathrm{d}^{\delta n} g^{\mathrm{fin},h_\phi\to h_\phi+1/2}_{\mathcal{O}_\star\mathcal{O}_\star;m+1}}{\delta n!\,\mathrm{d}z^{\delta n}}\Big|_{z=0}. \tag{5.59}$$

For pseudoscalar five-point functions, the comments on applications that we listed in the final item of our list will therefore apply after replacing the parity-even result (5.56) with the parity-odd result (5.59) above.

[8] Here, h_ϕ is to be understood as the dependence of the blocks on the conformal dimension of the external field when the quantum numbers $h_a, \bar{h}_a, n$ are kept arbitrary.

5.4.2 Explicit checks in ϕ^3 theory at tree level

In this subsection, we perform explicit checks for the predicted leading large spin asymptotics of double-twist OPE coefficients that reproduce the identity exchange in the (15) OPE of the direct channel. This corresponds to eq. (5.11) for case I scaling and eq. (5.45) for case II scaling. We check that these expressions can be obtained from the exact OPE coefficients of ϕ^3 theory at tree-level. At this first order in perturbation theory, the five-point correlator is disconnected and takes the form

$$\langle\phi(X_1)\dots\phi(X_5)\rangle = \langle\phi(X_1)\phi(X_5)\rangle\langle\phi(X_2)\phi(X_3)\phi(X_4)\rangle + \text{perms} + \mathcal{O}(g^2). \tag{5.60}$$

This applies to either the ϵ-expansion around the six-dimensional conformal field theory, or the holographic description of a perturbative ϕ^3 theory in AdS$_d$. In appendix D, we compute the double-twist OPE coefficients in the OPE decomposition of this correlator and obtain

$$P^{(n)}_{[\phi\phi]_{0,J_1}[\phi\phi]_{0,J_2}} = C_{\phi\phi\phi}\frac{1}{n!}\prod_{i=1}^{2} P_{[\phi\phi]_{0,J_i}}\frac{J_i!}{(J_i-n)!}(\Delta_\phi/2)_{J_i-n}\,\mathcal{F}^{(n)}_{(\Delta_\phi;J_1,J_2)} + \mathcal{O}(g^2), \tag{5.61}$$

where $P_{[\phi\phi]_{0,J}} = C^2_{\phi\phi[\phi\phi]_{0,J}}$ are the OPE coefficients of the GFF four-point function, given. in e.g. [4, eq. (11)], and

$$\mathcal{F}^{(n)}_{(\Delta_\phi;J_1,J_2)} = \sum_{j=0}^{n}\binom{n}{j}\frac{(\Delta_\phi/2)_{n-j}}{(\Delta_\phi)_j(\Delta_\phi)_{J_1-j}(\Delta_\phi)_{J_2-j}} \tag{5.62}$$

$$= \frac{(\Delta_\phi/2)_n}{(\Delta_\phi)_{J_1}(\Delta_\phi)_{J_2}}\,{}_3F_2\left[\begin{matrix}-n, & 1-\Delta_\phi-J_1,\ 1-\Delta_\phi-J_2\\ 1-\Delta_\phi/2-n, & \Delta_\phi\end{matrix}\right](1). \tag{5.63}$$

Comparison with case I asymptotics. In the case I regime $(J_1^2, J_2^2, n) = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{23}^{-1}, \epsilon_{15}^{-1}\epsilon_{34}^{-1}, \epsilon_{15}^{-1})$ with tensor structures larger than spin, we cannot directly compare the asymptotics of eq. (5.12) with the tree-level OPE coefficients in eq. (5.61) — the derivation in the latter case requires $[\phi\phi]_{0,J}$ to be local operators with integer spin, in which case $n \leq J_i$. Nevertheless, we can meaningfully compare the two results by inverting the order of limits from $(\epsilon_{15}, \epsilon_{34}, \epsilon_{23})$ to $(\epsilon_{34}, \epsilon_{23}, \epsilon_{15})$ in eq. (5.61) to put the spins and tensor structures back into a physical regime. Indeed, given the explicit expression for the tree-level correlator in eq. (5.60), it is easy to compare the expansion in the two different orders of limits. While the permutation makes the identity exchange in (15) subleading (compared to the identity in (34)), it will instead appear at next-to-leading order in the crossing equation of the tree-level correlator. In the limit $(J_1^2, J_2^2, n) = \mathcal{O}(\epsilon_{23}^{-1}, \epsilon_{34}^{-1}, 1)$ we find

$$\mathcal{F}^{(n)}_{(\Delta_\phi;J_1,J_2)} = \frac{1}{(\Delta_\phi)_{J_1}(\Delta_\phi)_{J_2}}\frac{(J_1J_2)^n}{n!(\Delta_\phi)_n}\left[1 + \mathcal{O}(J_1^{n-1}J_2^n, J_1^nJ_2^{n-1})\right]. \tag{5.64}$$

This result coincides with the large J_1, J_2 asymptotics predicted in [26]. As a result, isolating this term and taking the limit $J_1, J_2 \gg n \gg 1$ in the ratio of Γ functions indeed reproduces the lightcone bootstrap prediction of eq. (5.12) in case I asymptotics.

Retrieving case II asymptotics. In the case II regime $(J_1, J_2, n) = \mathcal{O}(\epsilon_{15}^{-1}, \epsilon_{15}^{-1}\epsilon_{34}^{-1}, \epsilon_{15}^{-1})$ with $J_1 - n = \delta n = 0, 1, \ldots, \infty$, the spins and tensor structure label take physical values that allow for a direct check of eq. (5.45) against eq. (5.61). In this regime, we find

$$\mathcal{F}^{(J_1-\delta n)}_{(\Delta_\phi;J_1,J_2)} = \frac{1 + \mathcal{O}((J_2 - J_1)^{-1})}{(\Delta_\phi)_{\delta n}(\Delta_\phi)_{J_2-J_1+\delta n}(\Delta_\phi)_{J_1-\delta n}}. \tag{5.65}$$

After this approximation, we can again take the large J_1, J_2 limit in eq. (5.61) and retrieve the solution to the crossing equation in eq. (5.45).

6 Outlook: six-point comb channel lightcone bootstrap

The goal of this final section is to provide a detailed outlook on the second part of this work, in which we will apply the new methodology we have developed here to study triple-twist operators. While [26] explored crossing symmetry between two OPE channels with snowflake topology, our upcoming paper addresses the duality between a direct snowflake channel and a crossed channel of comb topology, see figure 5. The treatment of the crossed comb channel is made possible by the new technology we have — to a large extent — developed above. And it is the comb channel that will give us access to triple-twist data.

6.1 Crossing a snowflake into a comb

In our forthcoming work [52], we will further extend the methods and results of this paper to six points. Indeed, the setup of figure 1 naturally generalizes to a six-point setup with the comb-channel expansion (12)3(4(56)) as the crossed channel, and a direct channel containing the (16) OPE. In particular, the snowflake channel (16)(23)(45) leads to the planar crossing equation depicted in figure 5. In the snowflake channel, all intermediate operators are STT with twist denoted by $h_a, a = 1, 2, 3$ and spin $J_a, a = 1, 2, 3$. The tensor structures at the central vertex are parameterized by three integers ℓ_1, ℓ_2, ℓ_3. In the comb channel, the nine degrees of freedom in the sum over conformal blocks are divided among twists h_1, h_2, h_3, spins J_1, J_2, κ_2, J_3, and tensor structures n_1, n_2, where the new symbol κ_2 represents the length of the second row of the Young tableau associated with the mixed-symmetry tensor exchanged at the middle leg. Note that in $d = 3$, where the scalar six-point function has one degree of freedom less, there is an extra relation between the tensor structure labels ℓ_1, ℓ_2, ℓ_3 in the snowflake channel, while $\kappa_2 = 0$ in the comb channel. As in our treatment of five-point crossing, we are using the same symbols for the twists and STT spins of the intermediate fields in the direct and crossed channel. We trust that it will be clear from the context which set we are referring to.

The sequence of lightcone limits in which we propose to analyze the crossing symmetry relations shown in figure 5 is given by

$$LCL_{\vec{\epsilon}} : X_{16} \ll X_{23}, X_{45} \ll X_{12}, X_{56} \ll X_{46} \ll X_{34} \ll 1. \tag{6.1}$$

The first three limits we perform clearly expose leading-twist contributions in the direct snowflake channel. The remaining three limits favor leading-twist terms in the crossed

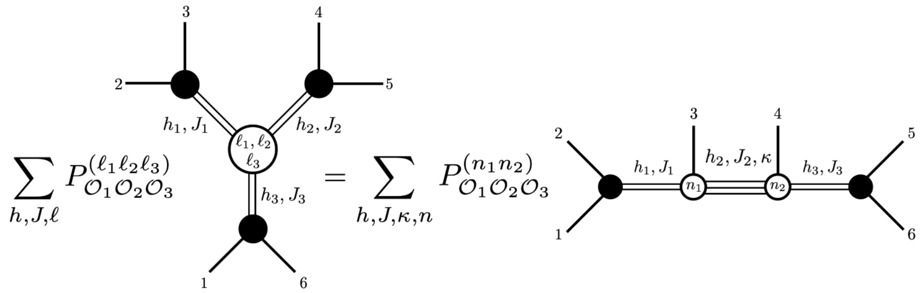

Figure 5. Graphical representation of a planar six-point crossing symmetry equation from the snowflake channel (61)(23)(45) to the comb (12)3(4(56)). Internal legs are labeled by half-twists h and spin labels J, κ (the lengths of the two rows of a mixed-symmetry tensor Young tableau), while tensor structure labels are ℓ_1, ℓ_2, ℓ_3 at the central spinning vertex of the snowflake and n_1, n_2 at the two innermost vertices of the comb.

channel. Note in particular that the limit $X_{45}, X_{56} \ll X_{46} \to 0$ projects to leading twist in the (4(56)) OPE of the comb channel. Through the final limit in which X_{34} is sent to zero, we reach the kinematics of a null polygon in which $X_{i(i+1)} \ll 1$ for all $i = 1, \ldots, 6$, with the additional lightcone constraint $X_{46} = 0$ imposed before we complete the null polygon.

A convenient choice of cross-ratios that generalizes the snowflake cross-ratios $u_{s1}, \ldots, u_{s5}$ we used in the context of five-point functions and that additionally preserves the $\mathbb{Z}_2$ reflection symmetry of the comb channel is given by

$$
\begin{aligned}
u_1 &= \frac{u_1}{v_2} = \frac{X_{12}X_{35}}{X_{13}X_{25}}\,, \quad & u_2 &= \frac{u_2}{v_1 v_3} = \frac{X_{13}X_{46}}{X_{14}X_{36}}\,, \quad & u_3 &= \frac{u_3}{v_2} = \frac{X_{24}X_{56}}{X_{25}X_{46}}\,, \\
v_1 &= \frac{X_{14}X_{23}}{X_{13}X_{24}}\,, & v_2 &= \frac{X_{25}X_{34}}{X_{24}X_{35}}\,, & v_3 &= \frac{X_{36}X_{45}}{X_{35}X_{46}}\,, \\
\mathcal{U}_1 &= \frac{U_1^{(5)}}{v_1 v_2} = \frac{X_{15}X_{24}}{X_{14}X_{25}}\,, & \mathcal{U}_2 &= \frac{U_2^{(5)}}{v_2 v_3} = \frac{X_{26}X_{35}}{X_{25}X_{36}}\,, & \mathcal{U}^6 &= \frac{U^{(6)}}{v_1 v_2 v_3} = \frac{X_{16}X_{24}X_{35}}{X_{14}X_{25}X_{36}}\,.
\end{aligned}
\tag{6.2}
$$

In terms of these cross-ratios, the lightcone regime $LCL_{\vec{\epsilon}}$ we specified in eq. (6.1) corresponds to

$$
LCL_{\vec{\epsilon}} : \mathcal{U}^6 \ll v_1, v_3 \ll u_1, u_3 \ll u_2 \ll v_2 \ll 1. \tag{6.3}
$$

Note that two cross-ratios $\mathcal{U}_1, \mathcal{U}_2$ do not contain any of the X_{ij} that are sent to zero and hence they are left unconstrained in our limit. There is one less independent cross ratio in $d = 3$ kinematics, which is in encoded in the relation $\mathcal{U}_1 + \mathcal{U}_2 = 1$ at leading order in the lightcone limit (6.3). Finally, to fix our conventions for snowflake and comb channel blocks, we write down the corresponding conformal block decompositions below:

$$
\begin{aligned}
\langle \phi(X_1) \ldots \phi(X_6) \rangle &= (X_{16}X_{23}X_{45})^{-\Delta_\phi} \sum_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3,\ell_1,\ell_1,\ell_3} P^{(\ell_1\ell_2\ell_3)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\, \psi^{(16)(23)(45)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;\ell_1\ell_2\ell_3}(u_a, v_a, \mathcal{U}_s, \mathcal{U}^6) \\
&= \left(\frac{X_{12}X_{34}X_{56}}{\sqrt{u_2 v_1 v_3}} \right)^{-\Delta_\phi} \sum_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3,n_1,n_2} P^{(n_1n_2)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\, \psi^{(12)3(4(56))}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;n_1n_2}(u_a, v_a, \mathcal{U}_s, \mathcal{U}^6).
\end{aligned}
$$

In order to analyze the crossing symmetry constraints we start with an explicit expansion of the direct channel. In the lightcone regime $LCL^{(6)}_{\vec{\epsilon}}$ where six of the seven limits in eq. (6.1) are performed, i.e. the cross-ratio v_2 is left unconstrained, the direct channel expansion reads

$$\sum_{\mathcal{O},\ell} P^{(\ell_1\ell_2\ell_3)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\psi^{\rm DC}_{\mathcal{O};\ell} \sim \Big[1 + C^2_{\phi\phi\mathcal{O}_\star}\,(u_2v_1v_3)^{h_\star}{}_2F_1\begin{bmatrix}\bar h_\star, \bar h_\star\\ 2\bar h_\star\end{bmatrix}(1-v_2) \tag{6.4}$$
$$+ C^2_{\phi\phi\mathcal{O}_\star}\,(\frac{v_1\mathcal{U}^6}{\mathcal{U}_2})^{h_\star}{}_2F_1\begin{bmatrix}\bar h_\star, \bar h_\star\\ 2\bar h_\star\end{bmatrix}(1-u_1/\mathcal{U}_2)$$
$$+ C^2_{\phi\phi\mathcal{O}_\star}\,(\frac{v_3\mathcal{U}^6}{\mathcal{U}_1})^{h_\star}{}_2F_1\begin{bmatrix}\bar h_\star, \bar h_\star\\ 2\bar h_\star\end{bmatrix}(1-u_3/\mathcal{U}_1)$$
$$+ \Big(u_2v_1v_3\,\mathcal{U}^6\Big)^{h_\star}\sum_{\ell} P^{(\ell_1\ell_2\ell_3)}_{\mathcal{O}_\star\mathcal{O}_\star\mathcal{O}_\star}\left(g_{\mathcal{O}_\star\mathcal{O}_\star\mathcal{O}_\star;\ell}\,(0, v_2\mathcal{U}_1, 0, u_2, \mathcal{U}_1, \mathcal{U}_2) + \mathcal{O}(X_{12,56})\right)$$
$$+ \mathcal{O}(X_{16}^{h>h_\star})\Big].$$

In writing the direct channel expansion we have dropped the prefactor $(X_{16}X_{23}X_{45})^{-\Delta_\phi}$ so that we obtain functions of the cross ratios only. On the fourth line, we used the same conventions as in [26] for six-point OPE coefficients and lightcone blocks, except that we replaced their label k by $\mathcal{O}$. The snowflake lightcone blocks $g_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3;\ell_1\ell_2\ell_3}(u_1, u_3, u_5, U_1, U_2, U_3)$ are given by the integral formula [26, eq. (22)]. The latter is a direct consequence of the lightcone OPE. Therefore, the direct channel expansions are under good control.

In [26], the leading terms of the direct channel expansion were reproduced from a crossed channel with snowflake topology. Here we address the same problem, but for a crossed channel of comb topology, i.e. the crossing equation we analyze takes the form

$$\Big(\frac{u_1\sqrt{u_2}u_3v_2}{\mathcal{U}^6\sqrt{v_1v_3}}\Big)^{\Delta_\phi}\sum_{\mathcal{O}_a;\ell_a} P^{(\ell_1,\ell_2,\ell_3)}_{\mathcal{O}_1\mathcal{O}_2,\mathcal{O}_3}\psi^{\rm DC}_{\mathcal{O}_a,\ell_a} = \sum_{\mathcal{O}_a;n_\rho} P^{(n_1,n_2)}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\;\psi^{\rm CC}_{\mathcal{O}_a,n_\rho}(u_i, v_i, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}^6). \tag{6.5}$$

The prefactor on the left-hand side is $\Omega_{\rm DC}\Omega^{-1}_{\rm CC}$. The twists $(h_a)^3_{a=1}$ of the three intermediate operators that are exchanged at leading order in the crossed channel may be read off from the exponents of the cross-ratios $(u_a)^3_{a=1}$ on the left-hand side of the crossing equation (6.5). The scaling of the spin labels (J_1, J_2, J_3, κ) in the crossed channel, on the other hand, can be determined from the degree of singularity of the three second-order comb-channel Casimir operators $\mathcal{D}^2_{12}, \mathcal{D}^2_{456}, \mathcal{D}^2_{56}$ along with the fourth-order operator $\mathcal{D}^4_{456}$ for the middle leg.

The leading contribution in the direct channel expansion (6.4), corresponding to an exchange of three identities, does not probe any new data in the crossed channel because it is reproduced by a sum over contributions of $(\mathcal{O}_1;\mathcal{O}_2;\mathcal{O}_3) = ([\phi\phi]_{0,J_1};\phi;[\phi\phi]_{0,J_3})$ at large J_1, J_3. In particular, the exchange of ϕ at the middle leg follows from the exact diagonalization of $\mathcal{D}^2_{456}$ by the direct channel contribution with eigenvalue $\frac{1}{2}\Delta_\phi(\Delta_\phi - d)$.

All of the remaining contributions of the direct channel expansion (6.4) are reproduced by exchanges of three double-twist operators $[\phi\phi]_{0,J_1}$, $[\phi\mathcal{O}_\star]_{0,J_2,\kappa}$, $[\phi\phi]_{0,J_3}$ in the crossed channel. Therein, the leading logarithmic singularity coming from the ${}_2F_1(1-v_2)$ is reproduced by what we call "case I" scaling, namely

$$\mathrm{LS_I}:\quad J_1^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{23}^{-1}),\quad J_2^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{34}^{-1}),\quad J_3^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{45}^{-1}). \tag{6.6}$$

The remaining contributions are what we call "case II", in that J_2 scales only with X_{16}:

$$\mathrm{LS}_{\mathrm{II}}: \quad J_1^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{23}^{-1}), \quad J_2^2 = \mathcal{O}(\epsilon_{16}^{-1}), \quad J_3^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{45}^{-1}). \tag{6.7}$$

Following our analysis of case II scaling in the five-point crossing equation, we expect it will be necessary to solve the crossing equation at all orders in X_{34} to completely fix the double-twist data that appears in the crossed channel.

Explicit solution in case I. After describing the general setup, we now want to anticipate a few of the new results. In particular, we state the lightcone limit of the comb channel blocks in the regime $LCL_{\vec{\epsilon}}$, see eq. (6.1), with scaling LS_{I} as specified in eq. (6.6), and in $d = 3$, where the two unconstrained cross-ratios satisfy the extra relation $\mathcal{U}_1 + \mathcal{U}_2 = 1$. These lightcone blocks take the form

$$\frac{\psi_{\mathcal{O};n}(u_a, v_a, \mathcal{U}_1, \mathcal{U}^6)}{\prod_{a=1}^3 u_a^{h_a}(v_1 v_2^2 v_3)^{h_\phi}} \underset{\mathrm{LS}_{\mathrm{I}}}{\overset{LCL_{\vec{\epsilon}}}{\sim}} \frac{\mathcal{N}_{\mathcal{O};n}^{\mathrm{I},3d}\,\delta(n_2\mathcal{U}_1 - n_1(1-\mathcal{U}_1))}{\exp\Big((n_1+n_2)\mathcal{U}^6 + \frac{\lambda_1 v_1}{n_1} + \frac{\lambda_2 v_2}{n_1+n_2} + \frac{\lambda_3 v_3}{n_3}\Big)}, \tag{6.8}$$

as we shall prove in [52]. Here, the scaling $n_1, n_2 = \mathcal{O}(\epsilon_{16}^{-1})$ of the tensor structure labels is further corroborated by the asymptotics of the vertex operators, and the normalization is given by

$$\mathcal{N}_{\mathcal{O};n}^{\mathrm{I},3d} = \frac{1}{\sqrt{\pi}} 4^{\bar{h}_1+\bar{h}_2+\bar{h}_3-1/2} \frac{\left(\frac{n_1}{e}\right)^{n_1}\left(\frac{n_2}{e}\right)^{n_2}(n_1 n_2)^{h_\star}}{J_1^{n_1+h_\star} J_2^{n_1+n_2+2h_\phi-1/2} J_3^{n_2+h_\star}} (n_1+n_2)^{2h_\phi-1}. \tag{6.9}$$

Plugging these blocks back into the crossing equation, we can reproduce the $\log v_2^{-1}$ term in the first line of the direct channel expansion eq. (6.4) if the OPE coefficients are given by

$$P^{(n_1 n_2)}_{[\phi\phi]_{0,J_1}[\phi\mathcal{O}_\star]_{0,J_2}[\phi\phi]_{0,J_3}} \sim \frac{C^2_{\phi\phi\mathcal{O}_\star}}{\mathrm{B}_{\bar{h}_\star}} \frac{4^{\frac{5}{2}-\sum_{a=1}^3 \bar{h}_a}}{\sqrt{\pi}} \frac{J_1^{2\Delta_\phi-h_\star-n_1-1} J_2^{\Delta_\phi-n_1-n_2-\frac{1}{2}} J_3^{2\Delta_\phi-h_\star-n_2-1}}{\Gamma(\Delta_\phi)\Gamma(\Delta_\phi-h_\star)^2\, e^{n_1+n_2} n_1^{n_1+\Delta_\phi} n_2^{n_2+\Delta_\phi}}. \tag{6.10}$$

Setting $J_3 = J_1$, $n_1 = n_2 = n$ and dividing both sides by $C^2_{\phi\phi[\phi\phi]_{0,J_1}}$ in eq. (5.16), we obtain a new result for OPE coefficients of two double-twist operators,

$$C^{(n)}_{[\phi\phi]_{0,J_1}\phi[\phi\mathcal{O}_\star]_{0,J_2}} \sim \frac{C^2_{\phi\phi\mathcal{O}_\star}}{\mathrm{B}_{\bar{h}_\star}} \frac{4^{1-\bar{h}_1-\bar{h}_2}}{\sqrt{\pi}} \frac{\Gamma(\Delta_\phi)}{\Gamma(\Delta_\phi-h_\star)^2} \frac{J_1^{2\Delta_\phi-2h_\star-2n-\frac{1}{2}} J_2^{\Delta_\phi-2n-\frac{1}{2}}}{e^{2n} n^{2n+\Delta_\phi}}. \tag{6.11}$$

The result holds at leading order for large spins J_a and large tensor structure label n, but it does not assume any additional relations between these large quantum numbers.

Higher order terms and case II. On the first line of the direct channel expansion in eq. (6.4), the remaining non-divergent terms in the expansion of the four-point lightcone block ${}_2F_1(1-v_2)$ around $v_2 = 0$, should be reproduced in the crossed channel by a large spin sum over double-twist OPE coefficients with the case II scaling defined by eq. (6.7). Expanding both sides of the crossing symmetry equation (6.5) in the same basis of functions of the cross-ratio v_2, we expect to solve the crossed channel OPE coefficients for the

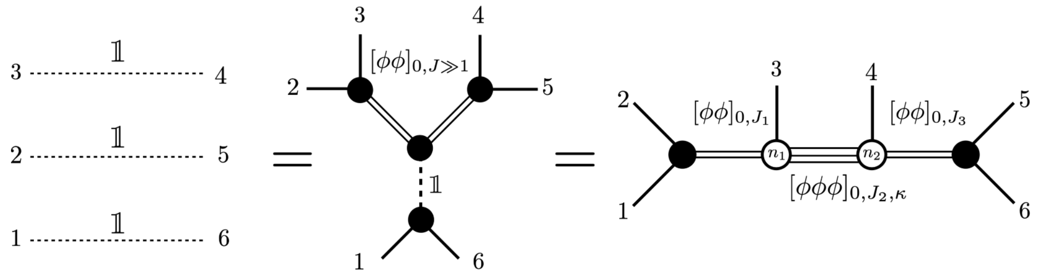

Figure 6. Leading contribution to the six-point crossing equation with triple-twist exchange in the crossed channel.

direct channel data similarly to how we solved for the OPE coefficients $P^{(J-\delta n)}_{[\phi\phi][\phi\phi]}$ with discrete dependence on the tensor structure δn in section 5. Next, we expect the four-point contributions from the second and third line to be reproduced by the OPE coefficients and anomalous dimensions of the double-twist operators $[\phi\phi]_{n_{1,3},J_{1,3}}$ at the left and right legs of the comb channel. Moving on to the fourth line of the direct channel expansion (6.4), we will analyze the asymptotics of the lightcone block $g_{\mathcal{O}_\star\mathcal{O}_\star\mathcal{O}_\star}(0, v_2\mathcal{U}_1^5, 0, u_2, \mathcal{U}_1^5, \mathcal{U}_1^5)$ in the limit where the cross-ratio $u_2 \ll 1$. In comparison to the setup of [24–26], this amounts to taking one of the three origin limits ($U_1 \to 0$ in their notation) before taking the last null polygon limit ($u_3 \to 0$ in their notation). Following a differential-operator-based analysis of these asymptotics in the spirit of our analysis above, we will show that $g_{\mathcal{O}_\star\mathcal{O}_\star\mathcal{O}_\star} = \mathcal{O}(\log u_2)$ at leading order in this limit. This leading contribution on the second line can only be reproduced from a $\mathcal{O}(J_2^{-2h_\star})$ correction to the anomalous dimension of the double-twist operators $[\phi\mathcal{O}_\star]_{0,J_2,\kappa}$ in the crossed channel.

6.2 Triple-twist data and non-planar crossing equation

In the crossing equation of figure 5, triple-twist exchanges appear in the crossed channel only if double-twist exchanges appear in the direct channel. At the same time, for identity exchange in the (16) OPE, a four-point crossing between (23)(45) and (25)(34) relates double-twist exchange in the (16)(23)(45) channel to three identity exchanges in the (16)(25)(34) channel, see figure 6. This observation motivates us to study a novel crossing equation where (16)(25)(34) is the direct channel. In this case, the relevant sequence of lightcone limits is given by

$$LCL_{\vec{\epsilon}'} : X_{16} \ll X_{34} \ll X_{12}, X_{56} \ll X_{23}, X_{45}, X_{46} \ll 1. \tag{6.12}$$

If we only expand the direct channel explicitly in the first two limits $X_{16} \ll X_{34} \ll 1$, then the non-planar crossing equation takes the form

$$\begin{aligned}&\left(\frac{u_1 u_2^{\frac{3}{2}} u_3\sqrt{v_1 v_3}}{\mathcal{U}^6}\right)^{\Delta_\phi}\left(1 + v_2^{h_\star}{}_2F_1\begin{bmatrix}\bar{h}_\star, \bar{h}_\star\\ 2\bar{h}_\star\end{bmatrix}(1 - v_1 v_3 u_2) + \mathcal{O}(X_{16}^{h>h_\star})\right)\\ &\quad= \sum_{\Xi} P_\Xi \psi_\Xi^{\mathrm{CC}}(u_a, v_a, \mathcal{U}_s, \mathcal{U}^6),\end{aligned} \tag{6.13}$$

where the crossed-channel sum is specified by

$$\Xi = ([\phi\phi]_{0,J_1}[\phi\phi\phi]_{0,J_2,\kappa}[\phi\phi]_{0,J_3}; n_1 n_2)\,. \tag{6.14}$$

Acting with the comb channel Casimirs at leading order in the lightcone limit (6.12), we then find that the crossed channel sum is dominated by the large spin regime

$$\mathrm{LS}_{\mathrm{II}'} : J_1^2 = \mathcal{O}(\epsilon_{16}^{-1}), \quad J_2^2 = \mathcal{O}(\epsilon_{16}^{-1}\epsilon_{34}^{-1}), \quad J_3^2 = \mathcal{O}(\epsilon_{16}^{-1}). \tag{6.15}$$

This "case II′" scaling regime is the closest analog to the five-point case II blocks computed in section 3.3. From this point of view, one may expect that a solution to the crossing equation at all orders in X_{23}, X_{45} would further constrain the triple-twist OPE coefficients in the crossed channel, as we saw in section 5 for double-twist OPE coefficients $P^{(J-\delta n)}_{[\phi\phi][\phi\phi]}$. However, the direct channel contribution on the left-hand side of eq. (6.13) is exactly the same for $X_{23}, X_{45} \ll 1$ or $X_{23}, X_{45} = \mathcal{O}(1)$, which suggests that we may not obtain any new information by relaxing the latter limits. We will address this question in our upcoming work [52].

The next to leading $\mathcal{O}(X_{34}^{h_\star})$ contribution to the non-planar crossing equation (6.13) in the direct channel is

$$v_2^{h_\star}{}_2F_1\begin{bmatrix}\bar{h}_\star, \bar{h}_\star \\ 2\bar{h}_\star\end{bmatrix}(1 - v_1 v_3 u_2) = v_2^{h_\star}\left(\frac{\log u_2^{-1}}{\mathrm{B}_{\bar{h}_\star}} + \mathcal{O}(u_2^0)\right). \tag{6.16}$$

In the crossed channel, we expect this logarithm to come from a correction to the anomalous dimension of $[\phi\phi\phi]_{0,J_2,\kappa}$.

6.3 Concluding remarks

This work can be seen as a proof of concept for the multipoint lightcone bootstrap. In particular, the notions of Casimir and vertex singularity of the direct channel provide an efficient criterion for the presence of universal data in the crossed channel. The variation in the degree of Casimir/vertex singularity, which first appears non-trivially at $N > 4$ points, translates into different scaling regimes in the spin and tensor structures of the multipoint data. While subleading scaling limits raise the difficulty in computing the blocks (cf. case I and case II blocks in this paper), the payoff is a higher resolution in the conformal field theory data.

As we briefly discussed in this section, the triple-twist data of the six-point comb channel appears only in subleading regimes, where the spins and tensor structures do not all scale independently. Solving in the full lightcone limit, we expect the crossing equation to fix only certain averages of this data over a subset of spins and tensor structures. It would be interesting to better understand how far these averages can be resolved by relaxing certain lightcone limits and, if not, to find a natural physical interpretation.

We believe that our integrability-based approach to lightcone blocks can be generalized to a wide class of OPE channels and lightcone limits. In this approach, the blocks in the relevant regime are computed by interpolating the OPE limit and the lightcone limit with the differential equations of the integrable system. From this point of view, the computation is analogous to an integrable scattering problem for a basis of many-body quantum mechanical wave functions. While its technical implementation may seem daunting, we are motivated by the simplicity of the differential equations in lightcone limits. It would be interesting to investigate how the standard integrability machinery may adapt to and simplify in such limits, particularly in determining the spectra of the vertex differential operators and relations between different bases of tensor structures.

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A Five point lightcone blocks

A.1 Integral representation from lightcone OPE

The lightcone OPE for two scalars can be written as

$$\phi_1(X_1)\phi_2(X_2) \overset{X_{12}\to 0}{\sim} \sum_{\mathcal{O}_3} \frac{X_{12}^{h_3-h_1-h_2} C_{\phi_1\phi_2\mathcal{O}_3}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}_3+h_{12},\bar{h}_3+h_{21}}} \int_{\mathbb{R}^2_+} \frac{\mathrm{d}s_1}{s_1} s_1^{\bar{h}_3+h_{21}} \frac{\mathrm{d}s_2}{s_2} s_2^{\bar{h}_3+h_{12}} \mathcal{O}_3(X,Z), \quad \text{(A.1)}$$

with $X := s_1X_1 + s_2X_2$, $X \wedge Z := X_1 \wedge X_2$. We write the Euler Beta function as $\mathrm{B}_{a,b} := \Gamma(a+b)^{-1}\Gamma(a)\Gamma(b)$ and use the shorthand notation "$\mathbb{R}^\times := \int_0^\infty r^{-1}\mathrm{d}r$" to denote the volume of the dilation group, which acts on the integration variables as $(s_1,s_2) \to (rs_1, rs_2)$. Since the integrand is invariant under any such dilation, we may factorize $\mathbb{R}^\times$ via a change of variables or by using the Faddeev-Popov method. In particular, the more familiar formula of Ferrara et al. is obtained by the change of variables $(s_1,s_2) = (rt, r(1-t))$, for $(r,t) \in \mathbb{R}_{>0} \times [0,1]$,

$$\phi_1(X_1)\phi_2(X_2) \overset{X_{12}\to 0}{\sim} \sum_{\mathcal{O}_3} \frac{X_{12}^{h_3-h_1-h_2} C_{\phi_1\phi_2\mathcal{O}_3}}{\mathrm{B}_{\bar{h}_3+h_{12},\bar{h}_3+h_{21}}} \int_0^1 \frac{\mathrm{d}t}{(t(1-t))^{1-\bar{h}_3}} (1/t-1)^{h_{12}} \mathcal{O}_3(X_2 - t(X_2-X_1), X_2 - X_1).$$

Applying this OPE twice in a five-point function of identical scalars yields an integral formula for lightcone blocks,

$$\psi^{(12),(45)}_{\mathcal{O}_1\mathcal{O}_2;n}(u_{si}) \overset{u_{s1,4}\to 0}{\sim} X_{12}^{h_1} X_{45}^{h_2} (X_{15}X_{24} - X_{14}X_{25})^n \int_{\mathbb{R}^4_+} \frac{\mathrm{d}^4 \log s\, (s_1s_2)^{\bar{h}_1}(s_4s_5)^{\bar{h}_2}}{(\mathbb{R}^\times)^2\, \mathrm{B}_{\bar{h}_1} \mathrm{B}_{\bar{h}_2}} \quad \text{(A.2)}$$

$$X_{a3}^{h_2-\bar{h}_1-h_\phi+n} X_{b3}^{h_1-\bar{h}_2-h_\phi+n} X_{ab}^{-\bar{h}_{12;\phi}} J_{a,3b}^{J_1-n} J_{b,3a}^{J_2-n}, \quad \text{(A.3)}$$

where $X_a = s_1 X_1 + s_2 X_2$, $X_b = s_4 X_4 + s_5 X_5$ and $X_a \wedge Z_a = X_1 \wedge X_2$, $X_b \wedge Z_b = X_5 \wedge X_4$, such that e.g. $X_{a3} = s_1 X_{13} + s_2 X_{23}$. To efficiently reduce this formula to cross-ratio space, we fix the gauge $X_i = X_i^\star$, with

$$X_2^\star = (1, 0, \mathbf{0}), \quad X_3^\star = (0, 1, \mathbf{0}), \quad X_4 = (1, 1, \mathbf{n}), \tag{A.4}$$

in lightcone coordinates $\mathrm{d}X^2 = -\mathrm{d}X^+ \mathrm{d}X^- + \mathrm{d}\mathbf{X}^2$ and for some unit vector $\mathbf{n} \in S^{d-1}$, such that

$$X_{15}^\star = u_{s2} u_{s3} u_{s5}, \qquad X_{14}^\star = u_{s2}, \qquad X_{13}^\star = 1, \qquad X_{12}^\star = u_{s1} u_{s3} \tag{A.5}$$

$$X_{25}^\star = u_{s3}, \qquad X_{24}^\star = 1, \qquad X_{23}^\star = 1, \tag{A.6}$$

$$X_{35}^\star = 1, \qquad X_{34}^\star = 1, \tag{A.7}$$

$$X_{45}^\star = u_{s2} u_{s4}. \tag{A.8}$$

We then evaluate the integrand at $X_{ij}^\star = X_i^\star \cdot X_j^\star$ to obtain a function of the cross-ratios.

A.1.1 Equivalence with solution to Casimir equations

In section 3.2, we derived an Euler integral representation of five point blocks directly from the second-order Casimir equations of blocks with OPE limit boundary conditions. This representation can also be retrieved directly from the above lightcone integral after expressing the $X_{ij}^\star(u_{si})$ in terms of OPE cross-ratios $\bar{z}_a, z_a, w$, where

$$(u_{si})_{i=1}^5 = \left(\frac{z_1}{1-z_2}\bar{z}_1, 1-z_1, 1-z_2, \frac{z_2}{1-z_1}\bar{z}_2, 1-\frac{(1-w)z_1 z_2}{(1-z_1)(1-z_2)}\right) + \mathcal{O}(\bar{z}_a). \tag{A.9}$$

The inverse map is

$$(\bar{z}_1, \bar{z}_2, z_1, z_2, w) = \left(\frac{u_{s1} u_{s3}}{(1-u_{s2})}, \frac{u_{s2} u_{s4}}{(1-u_{s3})}, 1-u_{s2}, 1-u_{s3}, 1-\frac{u_{s2} u_{s3}(1-u_{s5})}{(1-u_{s2})(1-u_{s3})}\right) + \mathcal{O}(u_{s1,4}). \tag{A.10}$$

We then obtain

$$X_{a3}^\star = s_1 + s_2, \qquad X_{3b}^\star = s_4 + s_5, \tag{A.11}$$

$$X_{ab}^\star = (s_1 + s_2)(s_4 + s_5)(1 - z_1 S_1 - z_2 S_5 + w z_1 z_2 S_1 S_5), \tag{A.12}$$

$$J_{a,b3}^\star = z_1(s_4 + s_5)(1 - w z_2 S_5), \qquad J_{b,a3}^\star = z_2(s_1 + s_2)(1 - w z_1 S_1), \tag{A.13}$$

where $S_1 := s_1/(s_1 + s_2)$ and $S_5 := s_5/(s_4 + s_5)$. If we plug this formula into eq. (A.3) and change variables to $(s_1, s_2; s_4, s_5) = (r_1 t_1, r_1(1-t_1); r_2(1-t_2), r_2 t_2)$, $r_1, r_2 \in \mathbb{R}_{>0}$, $t_1, t_2 \in [0, 1]$, then we obtain

$$\psi_{\mathcal{O}_1 \mathcal{O}_2; n}(u_{si}(\bar{z}_a, z_a, w)) \overset{\bar{z}_a \to 0}{\sim} \prod_{a=1}^{2} \bar{z}_a^{h_a} z_a^{\bar{h}_a} (1-w)^n \tilde{F}_{(h_a, \bar{h}_a; n)}(z_1, z_2, w), \tag{A.14}$$

where $\tilde{F}$ corresponds precisely to the integral formula (3.25) derived from the Casimir equations.

A.1.2 Euler transformation and alternative representation

When $h_a > h_\phi$, the Euler integral formula (3.25) does not converge in the limit $X_{(a+1)(a+2)} \to 0$, i.e. when $u_{s(a+1)} \to 0$ at $u_{s(b+1)}, u_{s5}$ fixed. In this case, it will be convenient to analyze the integral in the cross-ratios (u_a, v_a, x), where

$$(u_{si})_{i=1}^5 = \left(\frac{u_1}{v_2}, v_1, v_2, \frac{u_2}{v_1}, 1-x\right). \tag{A.15}$$

The inverse map is

$$(u_1, u_2, v_1, v_2, x) = (u_{s1}u_{s3}, u_{s2}u_{s4}, u_{s2}, u_{s3}, 1-u_{s5}). \tag{A.16}$$

After the change of variables $\tilde{s}_1 = s_1 v_1$, $\tilde{s}_5 = s_5 v_2$, we may rewrite the tensor structures in the integrand as

$$\begin{aligned} X^\star_{ab} &= (\tilde{s}_1 + s_2)(\tilde{s}_5 + s_4)\left(1 - x\tilde{S}_1\tilde{S}_5\right), \\ J^\star_{a,3b} &= (1-v_1)(\tilde{s}_5 + s_4)\left(1 + \frac{v_1 x}{1-v_1}\tilde{S}_5\right), \qquad J^\star_{b,a3} = (1-v_2)(\tilde{s}_1 + s_2)\left(1 + \frac{v_2 x}{1-v_2}\tilde{S}_1\right), \end{aligned}$$

where $\tilde{S}_1 = \tilde{s}_1/(\tilde{s}_1 + s_2)$ and $\tilde{S}_5 = \tilde{s}_5/(s_4 + \tilde{s}_5)$. Plugging this into eq. (A.3) then yields the result

$$\psi_{\mathcal{O}_1\mathcal{O}_2;n}(u_{si}(u_a, v_a, x)) \overset{u_a \to 0}{\sim} \prod_{a=1}^{2} u_a^{h_a}(1-v_a)^{J_a - n} v_a^{-h_{b\phi}} x^n F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, x), \tag{A.17}$$

where $F_{\mathcal{O}_1\mathcal{O}_2;n}$ is the integral defined in eq. (3.44). The two integral representations are related by a generalized Euler transformation,

$$\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(1-v_1, 1-v_2, w(v_1, v_2, x)) = \prod_{a \neq b} v_a^{-(h_{b\phi}+n)} F_{\mathcal{O}_1\mathcal{O}_2;n}(v_1, v_2, x). \tag{A.18}$$

where the change of variables $w \leftrightarrow x$ is given by

$$x(z_a, w) = \frac{z_1 z_2 (1-w)}{(1-z_1)(1-z_2)}, \quad w(v_a, x) = 1 - \frac{v_1 v_2 x}{(1-v_1)(1-v_2)}. \tag{A.19}$$

For $a \neq b$ and $h_{b\phi} + n \neq 0$, this relation ensures that F converges at $v_a = 0$ when f diverges and vice versa.

A.2 Expansion of lightcone blocks around decoupling limit

The goal of this section is to expand blocks as a power series in x around the decoupling limit $x = 0$. With applications to crossed channel blocks in mind, we will assume $h_1, h_2 > h_\phi$ such that $F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, x)$ converges at $v_a = 0$. Setting $\tilde{s}_{1,5} \equiv s_{1,5}$ for simplicity, such that $X_a = \frac{s_1}{v_2}X_1 + s_2 X_2$ and $X_b = s_4 X_4 + \frac{s_5}{v_2}X_5$, we can express the integral as

$$\begin{aligned} F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, x) = &\int_{\mathbb{R}_+^2} \frac{\mathrm{d}^2 \log(s_1, s_2)\,(S_1 S_2)^{\bar{h}_1}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}_1}(1-(1-v_1)S_1)^{\bar{h}_1 - h_{2\phi} - n}} \int_{\mathbb{R}_+^2} \frac{\mathrm{d}^2 \log(s_4, s_5)\,(S_4 S_5)^{\bar{h}_2}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}_2}(1-(1-v_2)S_4)^{\bar{h}_2 - h_{1\phi} - n}} \\ &\left(1 + \frac{v_1 x}{1-v_1}S_5\right)^{J_1 - n}\left(1 + \frac{v_2 x}{1-v_2}S_1\right)^{J_2 - n}(1 - xS_1S_5)^{-\bar{h}_{12;\phi}}. \end{aligned} \tag{A.20}$$

To obtain a power series in x, we first expand the integrand and then express each coefficient as a differential operator acting on $F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, 0)$.

A.2.1 Factorization in the decoupling limit

Evaluating eq. (A.20) at $x = 0$, we find a factorization into a product of two integrals,

$$\begin{aligned} &F_{\mathcal{O}_1\mathcal{O}_2;n}(v_1, v_2, 0) \\ &= \int_{\mathbb{R}_+^2} \frac{\mathrm{d}^2 \log(s_1, s_2)\,(S_1S_2)^{\bar{h}_1}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}_1}(1-(1-v_1)S_1)^{\bar{h}_1-h_{2\phi}-n}} \int_{\mathbb{R}_+^2} \frac{\mathrm{d}^2 \log(s_4, s_5)\,(S_4S_5)^{\bar{h}_2}}{\mathbb{R}^\times \mathrm{B}_{\bar{h}_2}(1-(1-v_2)S_4)^{\bar{h}_2-h_{1\phi}-n}}. \end{aligned} \tag{A.21}$$

Each integral is equal to one of the Gauss hypergeometric functions in eq. (3.74). More generally, the three-parameter Gauss hypergeometric function admits an integral representation

$${}_2F_1\begin{bmatrix} a, b \\ c \end{bmatrix}(1-v) = \int_{\mathbb{R}_+^2} \frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}s_2}{s_2}\frac{1}{\mathbb{R}^\times}\,{}_2f_1\begin{bmatrix} a, b \\ c \end{bmatrix}(1-v; s_1, s_2), \tag{A.22}$$

$${}_2f_1\begin{bmatrix} a, b \\ c \end{bmatrix}(1-v; s_1, s_2) = \frac{s_1^{c-b}s_2^b}{\mathrm{B}_{b,c-b}}\frac{(s_1+s_2)^{a-c}}{(s_1+s_2v)^a}, \tag{A.23}$$

which reduces to the Euler Beta integral representation after the change of variables $(s_1, s_2) = (r(1-t), rt)$, $(r, t) \in \mathbb{R}_{>0} \times [0, 1]$. In this homogeneous integral representation, the Euler transformation of the Gauss hypergeometric follows from

$${}_2f_1\begin{bmatrix} a, b \\ c \end{bmatrix}(1-v; s_1v, s_2) = v^{c-a-b}\,{}_2f_1\begin{bmatrix} c-a, c-b \\ c \end{bmatrix}(1-v; s_2, s_1). \tag{A.24}$$

A.2.2 Expansion around the decoupling limit

We can now rewrite the integral in eq. (A.20) as

$$\begin{aligned} F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, x) = \int_{\mathbb{R}_+^4} \frac{\mathrm{d}^4 \log s}{(\mathbb{R}^\times)^2}\,&{}_2f_1\begin{bmatrix} \bar{h}_1 - h_{2\phi} - n, \bar{h}_1 \\ 2\bar{h}_1 \end{bmatrix}(1-v; s_1, s_2) \\ &{}_2f_1\begin{bmatrix} \bar{h}_2 - h_{1\phi} - n, \bar{h}_2 \\ 2\bar{h}_2 \end{bmatrix}(1-v; s_5, s_4) \\ &\left(1 + \frac{v_1x}{1-v_1}S_5\right)^{J_1-n}\left(1 + \frac{v_2x}{1-v_2}S_1\right)^{J_2-n}(1-xS_1S_5)^{-\bar{h}_{12;\phi}} \end{aligned} \tag{A.25}$$

Since $0 \leq S_1, S_5 \leq 1$ for $s_i \in \mathbb{R}_+$ we can expand each factor into a binomial series when $0 \leq x \leq 1$ and $0 \leq (1-v_a)^{-1}v_ax \leq 1$. In this case, we observe that all higher-order corrections to the integrand in x are proportional to

$$\begin{aligned} x^{k+m_1+m_2}S_1^{k+m_2}\,&{}_2f_1\begin{bmatrix} \bar{h}_1 - h_{2\phi} - n, \bar{h}_1 \\ 2\bar{h}_1 \end{bmatrix}(1-v_1; s_1, s_2) \\ S_5^{k+m_2}\,&{}_2f_1\begin{bmatrix} \bar{h}_2 - h_{1\phi} - n, \bar{h}_2 \\ 2\bar{h}_2 \end{bmatrix}(1-v_2; s_5, s_4), \end{aligned}$$

for some triplets of positive integers k, m_1, m_2. At the same time, these integrand shifts S_i^ν are equivalent to

$$S_1^\nu\,{}_2f_1\begin{bmatrix} a, b \\ c \end{bmatrix}(1-v; s_1, s_2) = \frac{(c-b)_\nu}{(c)_\nu}\,{}_2f_1\begin{bmatrix} a, b \\ c+k \end{bmatrix}(1-v; s_1, s_2) \tag{A.26}$$

The right-hand side lifts to the integral itself, and to express it in a simple way we define an operator $\mathcal{S}_1^\nu$ that realizes this integrand transformation,

$$\mathcal{S}_1^\nu \cdot {}_2F_1\begin{bmatrix} a,b \\ c \end{bmatrix}(1-v) := \frac{(c-b)_\nu}{(c)_\nu}{}_2F_1\begin{bmatrix} a,b \\ c+\nu \end{bmatrix}(1-v). \tag{A.27}$$

The operator $\mathcal{S}_1^\nu$ obviously depends on the parameters a, b, c and the variable v, but we will not need to make this dependence explicit in future uses. Using the explicit form of the integrand ${}_2f_1(z; s_1, s_2)$, it is not difficult to write this transformation as a differential operator,

$$\begin{aligned}\frac{(c-b)_\nu}{(c)_\nu}{}_2F_1\begin{bmatrix} a,b \\ c+\nu \end{bmatrix}(1-v) &= v^{c+\nu-a-b}\frac{(c-b)_\nu}{(c)_\nu}{}_2F_1\begin{bmatrix} c-a+\nu, c-b+\nu \\ c+\nu \end{bmatrix}(1-v)\\ &= v^{c+\nu-a-b}\frac{(-\partial_v)^\nu}{(c-a)_\nu}{}_2F_1\begin{bmatrix} c-a, c-b \\ c \end{bmatrix}(1-v)\\ &= v^{c-a-b}\frac{v^\nu(-\partial_v)^\nu}{(c-a)_\nu}v^{a+b-c}{}_2F_1\begin{bmatrix} a,b \\ c \end{bmatrix}(1-v).\end{aligned}$$

Since the differential operator acting on the ${}_2F_1$ is homogeneous of degree zero in v, it can be written as a function of $\vartheta_v := v\partial_v$. By inspecting its action on the eigenbasis of ϑ_v, we find

$$v^\nu(-\partial_v)^\nu \cdot v^n = (-n)(-n+1)\dots(-n+\nu)v^n \Rightarrow v^\nu(-\partial_v)^\nu = (-\vartheta_v)_\nu.$$

Altogether, we obtain two useful representations of $\mathcal{S}_1^\nu$, defined in eq. (A.27), as a differential operator.

$$\text{Acting on } {}_2F_1\begin{bmatrix} a,b \\ c \end{bmatrix}(1-v), \quad \mathcal{S}_1^\nu = v^{c-a-b}\frac{v^\nu(-\partial_v)^\nu}{(c-a)_\nu}v^{a+b-c} = \frac{(c-a-b-\vartheta_v)_\nu}{(c-a)_\nu}. \tag{A.28}$$

Here, for the all orders expansion of lightcone blocks in x, we will use two such shift operators,

$$\mathcal{S}_a^\nu \cdot {}_2F_1\begin{bmatrix} \bar{h}_a - h_{b\phi} - n, \bar{h}_a \\ 2\bar{h}_a \end{bmatrix}(1-v_a) := \frac{(\bar{h}_a)_\nu}{(2\bar{h}_a)_\nu}{}_2F_1\begin{bmatrix} \bar{h}_a - h_{b\phi} - n, \bar{h}_a \\ 2\bar{h}_a+\nu \end{bmatrix}(1-v_a), \quad a \neq b = 1,2. \tag{A.29}$$

The resulting formulas for $\mathcal{S}_1^\nu, \mathcal{S}_2^\nu$ as differential operators can then be read from eq. (A.28) by inserting the corresponding values of (a, b, c) in the ${}_2F_1$'s. We will also adopt the notation of eq. (3.74) for the product of hypergeometrics at $x = 0$, that is

$$f_a^{\text{dec}}(v_a) := {}_2F_1\begin{bmatrix} \bar{h}_a - h_{b\phi} - n, \bar{h}_a \\ 2\bar{h}_a \end{bmatrix}(1-v_a), \quad a \neq b = 1,2. \tag{A.30}$$

Finally, with all of the necessary conventions listed in eq. (A.28)—(A.30), the all-orders expansion of five-point lightcone blocks around the decoupling limit $x = 0$ can be expressed as

$$F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a, x) = (1 - x\,\mathcal{S}_1\mathcal{S}_2)^{-\bar{h}_{12;\phi}}\left(1 + x\frac{v_1}{z_1}\mathcal{S}_2\right)^{J_1-n}\left(1 + x\,\mathcal{S}_1\frac{v_2}{z_2}\right)^{J_2-n} f_1^{\text{dec}}(v_1)f_2^{\text{dec}}(v_2). \tag{A.31}$$

In this formula, one should understand the three factors on the right-hand side as a binomial power series of the form $(1-x\lambda)^{-\Delta} = \sum_{k=0}^{\infty} \frac{x^k}{k!}(\Delta)_k \lambda^k$. This culminates in a triple-sum power series expansion in x,

$$F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a,x) = \sum_{k=0}^{\infty} \frac{(\bar{h}_{12;\phi})_k}{k!} \prod_{b\neq a=1}^{2} \sum_{m_a=0}^{J_a-n} \binom{J_a-n}{m_a} \left(\frac{v_a}{z_a}\right)^{m_a} \mathcal{S}_a^{k+m_b} f_a^{\text{dec}}(v_a)\, x^{k+m_1+m_2}. \tag{A.32}$$

A.2.3 Dual expansion

While less useful for our purposes, note that an analogous expansion around the decoupling limit $\mathcal{X} := 1-w = 0$ can be defined from the dual integral formula $\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_a,w)$ of eq. (3.44), which can be rewritten as

$$\begin{aligned}
&\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_a,1-\mathcal{X}) \\
&= \int_{\mathbb{R}_+^4} \frac{\mathrm{d}^4 \log s}{(\mathbb{R}^\times)^2}\, {}_2f_1\left[\begin{matrix}\bar{h}_1+h_{2\phi}+n, \bar{h}_1 \\ 2\bar{h}_1\end{matrix}\right](z_1;s_2,s_1)\, {}_2f_1\left[\begin{matrix}\bar{h}_2+h_{1\phi}+n, \bar{h}_2 \\ 2\bar{h}_2\end{matrix}\right](z_2;s_4,s_5) \\
&\quad \left(1+\frac{\mathcal{X}z_1S_5}{1-z_1S_5}\right)^{J_1-n} \left(1+\frac{\mathcal{X}z_2S_1}{1-z_2S_1}\right)^{J_2-n} \left(1-\frac{\mathcal{X}z_1z_2S_1S_5}{(1-z_1S_5)(1-z_2S_1)}\right)^{-\bar{h}_{12;\phi}}.
\end{aligned} \tag{A.33}$$

If we now define

$$\tilde{f}_a^{\text{dec}}(z_a) := (1-z_a)^{-(h_{b\phi}+n)} f_a^{\text{dec}}(1-z_a) = {}_2F_1\left[\begin{matrix}\bar{h}_a+h_{b\phi}+n, \bar{h}_a \\ 2\bar{h}_a\end{matrix}\right](z_a), \tag{A.34}$$

then all higher $\mathcal{X}$ corrections can be expressed in terms of the differential operators

$$\tilde{\mathcal{S}}_a^\nu \cdot \tilde{f}_a^{\text{dec}}(z_a) := \frac{z_a^\nu \partial_{z_a}^\nu \tilde{f}_a^{\text{dec}}(z_a)}{(\bar{h}_a+h_{b\phi}+n)_\nu}, \quad a\neq b=1,2. \tag{A.35}$$

In this case, we may express the expansion of $\tilde{F}$ around the decoupling limit as

$$\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_a,1-\mathcal{X}) = \left(1-\mathcal{X}\tilde{\mathcal{S}}_1\tilde{\mathcal{S}}_2\right)^{-\bar{h}_{12;\phi}} \left(1+\mathcal{X}\tilde{\mathcal{S}}_2\right)^{J_1-n} \left(1+\mathcal{X}\tilde{\mathcal{S}}_1\right)^{J_2-n} \tilde{f}_1^{\text{dec}}(z_1)\tilde{f}_2^{\text{dec}}(z_2). \tag{A.36}$$

A.3 Application to direct channel blocks

In our analysis of the direct channel, we will be interested in lightcone limits of the form

$$(z_a,w) = (1,1)+\mathcal{O}(\epsilon_a) \iff (v_a,x) = (\mathcal{O}(\epsilon_a),\mathcal{O}(1)), \quad \epsilon_a \to 0. \tag{A.37}$$

The asymptotics of blocks in this limit depend on the sign of $h_{b\phi}+n$: for $h_{b\phi}+n \geq 0$, these are easiest to derive from the expansion of $F_{\mathcal{O}_1\mathcal{O}_2;n}(v_a,x)$ in eq. (A.32), while for $h_{b\phi}+n \leq 0$, these are easiest to derive from the expansion of $\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_a,w)$ in eq. (A.36).

Asymptotics of blocks with $h_{b\phi}+n<0$. Consider first the lightcone limit $z_a = 1+\mathcal{O}(\epsilon_a)$, $\mathcal{X}=1-w=\mathcal{O}(\epsilon_a)$, in the case where $h_{b\phi}+n<0$. In this case, $\tilde{f}_a^{\mathrm{dec}}(z_a)$ converges at $z_a=1$ and

$$\mathcal{X}^m \tilde{\mathcal{S}}_a^m \cdot \tilde{f}_a^{\mathrm{dec}}(z_a) = \mathcal{O}\left(\epsilon_a^{\min(|h_{b\phi}+n|,m)}\right), \quad \mathcal{X}^m \tilde{f}_a^{\mathrm{dec}}(z_a) = \mathcal{O}\left(\epsilon_a^m\right), \quad \forall m\in\mathbb{Z}_{>0}. \tag{A.38}$$

As a result, the whole function $\tilde{F}$ converges in this double scaling limit, which coincides with the evaluation of $\tilde{F}$ at $z_a=1, w=1$. Taking $a=2$ without loss of generality, we may thus write

$$\lim_{\epsilon_2\to 0} \tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_1, 1+\mathcal{O}(\epsilon_2), 1+\mathcal{O}(\epsilon_2)) = \tilde{f}_1^{\mathrm{dec}}(z_1). \tag{A.39}$$

Asymptotics of blocks with $h_{b\phi}+n<0$. In this case, we know that $\tilde{f}_a^{\mathrm{dec}}(1-\epsilon_a) = f_a^{\mathrm{dec}}(\epsilon_a) = \mathrm{B}_{\bar{h}_2}^{-1}\log\epsilon_a^{-1} + \mathcal{O}(\epsilon_a^0)$ admits a logarithmic divergence. At the same time, it is easy to check from the differential operator expression for $\tilde{\mathcal{S}}_a^\nu$ in eq. (A.35) (respectively $\mathcal{S}_a^\nu$ in eq. (A.28)) that all higher powers in $1-w=\mathcal{X}$ (respectively x) in the expansion of $\tilde{F}$ (respectively F) around the decoupling limit are subleading, i.e. for any $m\in\mathbb{Z}_{>0}$

$$\mathcal{X}^m\tilde{\mathcal{S}}_a^m \tilde{f}_a^{\mathrm{dec}} = \mathcal{O}(\epsilon_a^0), \qquad \mathcal{X}^m \tilde{f}_a^{\mathrm{dec}} = \mathcal{O}(\epsilon_a^m\log\epsilon_a),$$
$$x^m\mathcal{S}_a^m f_a^{\mathrm{dec}} = \mathcal{O}(\epsilon_a^0), \qquad \left(\frac{x v_a}{1-v_a}\right)^m \tilde{f}_a^{\mathrm{dec}} = \mathcal{O}(\epsilon_a^m\log\epsilon_a).$$

It follows that the leading asymptotics in the double scaling limit coincides with the leading asymptotics when $z_a\to 1$ in the decoupling limit $\mathcal{X}=0$. In concrete terms, for $z_1=1-v_1$,

$$\begin{aligned}\lim_{v_2\to 0}\frac{\tilde{F}_{\mathcal{O}_1\mathcal{O}_2;n}(z_1,1-v_2,1+\mathcal{O}(v_2))}{\log v_2^{-1}} &= \tilde{f}_1^{\mathrm{dec}}(z_1)\lim_{v_2\to 0}\frac{\tilde{f}_2^{\mathrm{dec}}(1-v_2)}{\log v_2^{-1}} = \frac{\tilde{f}_1^{\mathrm{dec}}(z_1)}{\mathrm{B}_{\bar{h}_2}}\\ &= \lim_{v_2\to 0}\frac{F_{\mathcal{O}_1\mathcal{O}_2;n}(v_1,v_2,x)}{\log v_2^{-1}} = f_1^{\mathrm{dec}}(v_1)\lim_{v_2\to 0}\frac{\tilde{f}_2^{\mathrm{dec}}(v_2)}{\log v_2^{-1}} = \frac{f_1^{\mathrm{dec}}(v_1)}{\mathrm{B}_{\bar{h}_2}}.\end{aligned}$$

Asymptotics of blocks with $h_{b\phi}+n>0$. In this case, not only does $f_a^{\mathrm{dec}}(v_a)$ converge at $v_a=0$, but

$$\mathcal{S}_a^\nu\cdot f_a^{\mathrm{dec}}(v_a) = \frac{\mathrm{B}_{\bar{h}_a,h_{b\phi}+n}}{\mathrm{B}_{\bar{h}_a}}\frac{(2\bar{h}_a)_\nu(h_{b\phi}+n)_\nu}{(\bar{h}_a+h_{b\phi})_\nu(\bar{h}_a)_\nu} + \mathcal{O}(v_a). \tag{A.40}$$

We can thus write

$$\begin{aligned}&\lim_{v_a\to 0} F_{\mathcal{O}_1\mathcal{O}_2;n}(v_1,v_2,x) = \qquad\qquad (A.41)\\ &= \frac{\mathrm{B}_{\bar{h}_a,h_{b\phi}+n}}{\mathrm{B}_{\bar{h}_a}}\sum_{k=0}^{\infty}\frac{(\bar{h}_{12;\phi})_k}{k!}\sum_{m_b=0}^{J_b-n}\binom{J_b-n}{m_b}\left(\frac{v_b}{z_b}\right)^{m_a}\frac{(2\bar{h}_a)_{k+m_b}(h_{b\phi}+n)_{k+m_b}}{(\bar{h}_a+h_{b\phi})_{k+m_b}(\bar{h}_a)_{k+m_b}}\mathcal{S}_b^k f_b^{\mathrm{dec}}(v_b)x^{k+m_b}.\end{aligned}$$

A.4 Application to crossed channel blocks

In regimes relevant to the crossed channel OPE decomposition, the quantum numbers $\bar{h}_1,\bar{h}_2,n$ diverge by a specified ϵ-scaling with the limits $v_{1,2}=\mathcal{O}(\epsilon_{23,34})$ and $1-x=\mathcal{O}(\epsilon_{15})$. The spins $\bar{h}_1,\bar{h}_2$ are large numbers in all of these cases, and the tensor structure n may also scale with ϵ_{15}. Using the differential operator representation (A.28) of the $\mathcal{S}$-operators and their powers, we find

$$\mathcal{S}_a^k(\vartheta) = \bar{h}_a^{-k}(h_{b\phi} + n - \vartheta)_k \left(1 + \mathcal{O}(\bar{h}_a^{-1})\right). \tag{A.42}$$

Consequently, the two last factors in eq. (A.31) lead to corrections that scale non-trivially with J, v,

$$\left(1 + x\frac{v_a}{z_a}\mathcal{S}_b\right)^{J_a-n} = 1 + \mathcal{O}\left(\frac{(J_a-n)v_a}{\bar{h}_b}\right), \quad a \neq b = 1, 2. \tag{A.43}$$

These corrections will be subleading in any regime where

$$\bar{h}_2^{-2}, v_2 = \mathcal{O}(\epsilon_{34}), \quad \bar{h}_1 \ll \bar{h}_2. \tag{A.44}$$

This approximation applies consistently to all large spin limits considered in the crossed channel (12)3(45) that are relevant to the bootstrap analysis of section 5.

External scalar exchange in the crossed channel. Consider the special case

$$\mathcal{O}_1 = \phi, \quad n = 0, \quad \mathcal{O}_2 = [\phi\phi]_{0,J_2}, \quad J_2^2 = \mathcal{O}(\epsilon_{15}^{-1}\epsilon_{34}^{-1}). \tag{A.45}$$

In this case, blocks satisfy the same Casimir equations (3.89), (3.90) as case II blocks, but with $\lambda_1 u_{s5} \to 0$[9] and $\delta n = 0$. We can thus write their asymptotics as

$$F_{\phi[\phi\phi]_{0,J_2};0}(u_{s2}, u_{s3}, 1-u_{s5}) \sim \mathcal{N}^{\phi}_{\phi[\phi\phi]_{0,J_2}} u_{s5}^{h_\phi} \mathcal{K}_{h_\phi}(J_2^2 u_{s3} u_{s5}). \tag{A.46}$$

In this case, however, the normalization $\mathcal{N}^\phi$ cannot be computed in the same way as case II blocks because we now have $h_1 = h_\phi - n$ instead of $h_1 > h_\phi - n$, meaning that the limit of the block at $u_{s3}J_2^2 \ll u_{s5}^{-1}$ has different asymptotics than $(v, x) \to (0, 1)$ at finite J_2. More specifically, the hypergeometrics in the decoupling limit simplify to

$$f_1^{\text{dec}}(v_1) = 1, \quad f_2^{\text{dec}}(v_2) = {}_2F_1\begin{bmatrix}\bar{h}_2, \bar{h}_2 \\ 2\bar{h}_2\end{bmatrix}(1 - v_2). \tag{A.47}$$

We can recognize on the right-hand side a four-point lightcone block, with logarithmic divergence as $v_2 \to 0$. Instead, we will determine the normalization $\mathcal{N}^\phi$ by direct computation, taking the lightcone blocks at the $(0, v_2, x)$ node[10] along the two green arrows in figure 2 to the $(0, 0, 1)$ node where eq. (A.46) applies. In this specific case, the expansion around the decoupling limit takes the form

$$F_{\phi[\phi\phi]_{0,J_2};0}(v_a, x) = \mathrm{B}_{\bar{h}_2}^{-1} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{(h_\phi)_k}{(2h_\phi)_k} \frac{\Gamma(\bar{h}_2 + k)^2}{\Gamma(2\bar{h}_2 + k)}\, {}_2F_1\begin{bmatrix}\bar{h}_2, \bar{h}_2 \\ 2\bar{h}_2 + k\end{bmatrix}(1 - v_2). \tag{A.48}$$

We begin with the first limit

$$\mathrm{LS}_{34}: \quad J_2^{-2}, v_2 = \mathcal{O}(\epsilon_{34}), \quad \epsilon_{34} \to 0. \tag{A.49}$$

[9] This implies that the blocks are in the kernel of *both* $\mathcal{D}_{12}^{(-1,0,-1)}$ and $\mathcal{D}_{12}^{(-1,0,0)}$.

[10] Since $f_1^{\text{dec}}(v_1) = 1$, the function F is independent of v_1 in this case.

In this regime, the Gauss hypergeometric functions reduce to modified Bessel functions following the identity

$$\lim_{\bar{h}\to\infty} \frac{\Gamma(\bar{h}+k)^2}{\Gamma(2\bar{h}+k)}\, {}_2F_1\left[\begin{matrix}\bar{h},\bar{h}\\ 2\bar{h}+k\end{matrix}\right](1-\bar{h}^{-2}y) = 2\mathcal{K}_{-k}(y). \tag{A.50}$$

We thus obtain

$$F_{\phi[\phi\phi]_{0,J_2};0}(v_a,x) \overset{\text{LS}_{34}}{\sim} 2\text{B}_{\bar{h}_2}^{-1} \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{(h_\phi)_k}{(2h_\phi)_k}\mathcal{K}_{-k}(v_2 J_2^2). \tag{A.51}$$

At this stage, we refrained from applying the Stirling formula to the Beta function $\text{B}_{\bar{h}} = \Gamma(2\bar{h})^{-1}\Gamma(\bar{h})^2$ in order to keep the above expression more compact. Now, to reach the limiting form of eq. (A.46), we take the second large spin limit to the $(0,0,1)$ node,

$$\text{LS}_{34,15}: \quad v_2 J_2^2, 1-x = \mathcal{O}(\epsilon_{15}^{-1}), \quad \epsilon_{15}\to 0. \tag{A.52}$$

In this limit, the sum (A.51) is dominated by the regime $k = \mathcal{O}(\epsilon_{15}^{-1})$. Approximating the sum over $k = \mathcal{O}(\epsilon_{15}^{-1})$ by an integral and using the large k formulas

$$(\Delta)_k = \sqrt{\frac{\pi}{k}}\left(\frac{k}{e}\right)^{k+\Delta}\Gamma(\Delta)^{-1}\left(1+\mathcal{O}(k^{-1})\right), \quad x^k = e^{-k(1-x)}\left(1+\mathcal{O}(k^{-1})\right), \tag{A.53}$$

$$\mathcal{K}_k(kx) = \frac{1}{2}\sqrt{\frac{\pi}{k}}\left(\frac{k}{e}\right)^k e^{-x}\left(1+\mathcal{O}(k^{-\frac{1}{2}})\right), \tag{A.54}$$

we find

$$F^{(12),(45)}_{\phi[\phi\phi]_{0,J_2};0} \overset{\text{LS}_{34,51}}{\sim} \frac{2\Gamma(2h_\phi)}{\text{B}_{\bar{h}_2}\Gamma(h_\phi)}\frac{1}{2}\int_{\mathcal{O}(1)}^{\infty}\frac{\text{d}k}{k^{1+h_\phi}}e^{-k(1-x)-\frac{v_2 J_2^2}{k}}. \tag{A.55}$$

After a change of variables $t = k(1-x)$, we can identify the integral representation (B.4) of the Bessel-Clifford function, with the ratio of Gamma functions on the left providing the desired normalization of eq. (A.46):

$$\mathcal{N}^{\phi}_{\phi[\phi\phi]_{0,J_2}} = 2\frac{\Gamma(2h_\phi)}{\Gamma(h_\phi)}\lim_{\bar{h}_2\to\infty}\Gamma(2\bar{h}_2)\Gamma(\bar{h}_2)^{-2} = \frac{\Gamma(2h_\phi)}{\Gamma(h_\phi)}4^{\bar{h}_2}\sqrt{\frac{\bar{h}_2}{\pi}}. \tag{A.56}$$

B Double integral of a Bessel Function

B.1 Modified Bessel-Clifford function

For any $\alpha\in\mathbb{R}$, the modified Bessel-Clifford function $x\mapsto\mathcal{K}_\alpha(x)$ is defined as the solution to the differential equation

$$\partial_x(x\partial_x+\alpha)\mathcal{K}_\alpha(x) = \mathcal{K}_\alpha(x), \tag{B.1}$$

with $x\to 0$ asymptotics depending on the sign of α,

$$\mathcal{K}_\alpha(x) \overset{x\to 0}{\sim} \begin{cases} \frac{1}{2}\Gamma(-\alpha), & \alpha<0,\\ -\frac{1}{2}\log x, & \alpha=0\\ \frac{1}{2}\Gamma(\alpha)x^{-\alpha}, & \alpha>0.\end{cases} \tag{B.2}$$

It is related to the modified Bessel function $K_\alpha(y)$ by the relation

$$\mathcal{K}_\alpha(x) = x^{-\alpha/2} K_\alpha(2\sqrt{x}) \iff K_\alpha(y) = \left(\frac{y}{2}\right)^\alpha \mathcal{K}_\alpha\left(\frac{y^2}{4}\right). \tag{B.3}$$

This solution admits an integral representation,

$$\mathcal{K}_\alpha(x) = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} e^{-(t+t^{-1}x)}. \tag{B.4}$$

Applying the change of variables $t' = t^{-1}x$ in this formula then implies the Euler transformation

$$\mathcal{K}_\alpha(x) = x^{-\alpha}\mathcal{K}_{-\alpha}(x). \tag{B.5}$$

This Bessel-Clifford function controls the asymptotics of a class of Gauss hypergeometric functions by virtue of the identity

$$\lim_{\bar{h}\to\infty} \frac{\Gamma(\bar{h}+c-a)\Gamma(\bar{h}+c-b)}{2\,\Gamma(2\bar{h}+c)}\, {}_2F_1\left[\begin{matrix}\bar{h}+a, \bar{h}+b \\ 2\bar{h}+c\end{matrix}\right]\left(1-\bar{h}^{-2}x\right) = \mathcal{K}_{a+b-c}(x). \tag{B.6}$$

B.2 The double integral

In this appendix, we will prove the identity (5.38) by direct computation of a generalization of the integral on its left-hand side, namely

$$I(\alpha,\beta) := \int_{\mathbb{R}_+^2} \frac{\mathrm{d}x\mathrm{d}y}{xy} x^\alpha y^\beta \mathcal{K}_\gamma(x+y). \tag{B.7}$$

Since $x, y \geq 0$, it is natural to intuit the argument of the Bessel-Clifford function $\mathcal{K}_\gamma$ as the square radius of a circle in the plane. To substantiate this intuition, we make the change of variables

$$x = r^2\cos^2\theta, \quad y = r^2\sin^2\theta, \quad (r,\theta) \in \mathbb{R}_+ \times [0, \pi/2). \tag{B.8}$$

The variables (r,θ) can be understood as polar coordinates on the plane, and the domain of θ ensures a bijection with the upper right quadrant $(\sqrt{x}, \sqrt{y}) \in \mathbb{R}_+^2$. The measure transforms as

$$\frac{\mathrm{d}x\mathrm{d}y}{xy} = 4\frac{\mathrm{d}(\sqrt{x})\,\mathrm{d}(\sqrt{y})}{\sqrt{xy}} = 4\frac{r\mathrm{d}r\mathrm{d}\theta}{r^2\cos\theta\sin\theta} = 2\frac{\mathrm{d}(r^2)}{r^2}\frac{\mathrm{d}\theta}{\cos\theta\sin\theta}.$$

The double integral thereby factorizes in polar coordinates and takes the form

$$I(\alpha,\beta) = 2\int_0^\infty \frac{\mathrm{d}(r^2)}{r^2}(r^2)^{\alpha+\beta}\mathcal{K}_\gamma(r^2)\int_0^{\pi/2}\mathrm{d}\theta\,\cos^{2\alpha-1}\theta\sin^{2\beta-1}\theta. \tag{B.9}$$

Using $\mathcal{K}_\gamma(r^2) = r^{-\gamma}K_\gamma(2r)$, we retrieve two known integrals of special functions for $z = 2r$ and θ, namely

$$2\int_0^\infty \frac{\mathrm{d}z}{z}\left(\frac{z}{2}\right)^{2\alpha+2\beta-\gamma} K_\gamma(z) = \frac{1}{2}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-\gamma), \quad \mathrm{Re}(2\alpha+2\beta-\gamma) > |\mathrm{Re}(\gamma)|$$

$$\int_0^{\pi/2}\mathrm{d}\theta\,\cos^{2\alpha-1}\theta\sin^{2\beta-1}\theta = \frac{1}{2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

Putting everything together, this gives us

$$\int_{\mathbb{R}_+^2} \frac{\mathrm{d}x\mathrm{d}y}{xy} x^\alpha y^\beta \mathcal{K}_\gamma(x+y) = \frac{1}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\gamma). \tag{B.10}$$

When applied to the bootstrap analysis around eq. (5.38), we have $\alpha = \gamma = h_\phi + \delta n$ and $\beta = 2h_\phi - h_\star$, while in the case II bootstrap analysis of section 5.3.1 we have $\alpha = 2h_\phi + \delta n$, $\beta = h_\phi$, and $\gamma = h_\phi + \delta n$, with parameters always satisfying $0 < h_\phi < h_\star < 2h_\phi$ and $\delta n > 0$. We conclude that in both cases the integral converges, and our ansätze for the OPE coefficients in eqs. (5.37) and (5.45) are consistent.

C Comments on tensor structure larger than spin

In our construction of crossed channel five-point lightcone blocks we re-scaled the spins J_1 and J_2 of the exchanged fields as well as the tensor structure n. For case II blocks the scaling behavior was consistent with the usual condition $n \leq \min(J_1, J_2)$. But in the study of case I blocks, n was sent to ∞ faster than J_1, J_2. Here we want to explain why this can make sense when the spins become continuous. Let us consider the STT-STT-scalar three-point vertex operator we introduced in [43]. Using the same notations as in the main text, this operator is given by

$$\frac{1}{2} str\left[\mathcal{T}_1^3 \mathcal{T}_3\right], \tag{C.1}$$

where we are considering legs 1 and 2 to be associated with STTs with spin J_1, J_2, and leg 3 to be associated with a scalar field. We know that the eigenfunctions of this vertex operator are a basis of the space of tensor structures that can also be spanned by simple powers of the cross-ratio $\mathcal{X}$ defined in eq. (3.12). This means that we can recast the action of the vertex operator in terms of the basis $\mathcal{X}^n$, which acquires the following form:

$$\mathcal{V}\mathcal{X}^n = c_{-1}\mathcal{X}^{n-1} + c_0\mathcal{X}^n + c_1\mathcal{X}^{n+1} + c_2\mathcal{X}^{n+2}. \tag{C.2}$$

In particular, we have

$$c_{-1} = -n\left(n + \frac{d-4}{2}\right)(2n + \Delta_1 - \Delta_2 - \Delta_3 - J_1 - J_2)(2n - \Delta_1 + \Delta_2 - \Delta_3 - J_1 - J_2), \tag{C.3}$$

$$c_1 = 4(n-J_1)(n-J_2)\left[\frac{d}{2}(\Delta_1 + \Delta_2 + \Delta_3 - n + 1 - d) + n(2J_1 + 2J_2 - 3n + \Delta_3 - 2) - \Delta_1\Delta_2 - \Delta_3 + J_1 - J_1J_2 + J_2 - 1\right], \tag{C.4}$$

$$c_2 = 4(n-J_1)(n-J_1+1)(n-J_2)(n-J_2+1). \tag{C.5}$$

From here it is easy to understand the usual dimension of the space of tensor structures for integer spins. In fact, the space of powers $\mathcal{X}^n$ with $0 \leq n \leq \min(J_1, J_2)$ is closed under the action of the vertex operator: c_{-1} stops lowering the order of $\mathcal{X}^n$ at $n = 0$, while c_2 and c_1 will stop raising powers respectively at $n = \min(J_1, J_2) - 1$ and $n = \min(J_1, J_2)$.

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When taking large spin limits, however, we are often replacing the summation over spins in the OPE with integrals. In other words, a continuum limit is taken for the spins, which implies on the other hand that the space of tensor structures ceases to be finite. In fact, if we assume the tensor structure $\mathcal{X}^0$ to be part of the space of allowed tensor structures, repeated action of $\mathcal{V}$ on it will always produce higher powers of $\mathcal{X}$, as the zeros of the power raising coefficients, $n = J_i$, $n = J_i - 1$, will never be reached for non-integer spins. Conversely, if assume the tensor structure $\mathcal{X}^{\min(J_1,J_2)}$ to be allowed in our space, the vertex operator action shows us that the n spectrum is not bounded from below, allowing for the label n to scale faster (in absolute value) than the spins J_i. With this argument, we can conclude that as long as the continuum limit for the spins J_i can be taken, the tensor structure n is allowed to scale faster than the spin J_i.

D First-order computation of OPE coefficients in ϕ^3 theory

In this appendix, we will determine the OPE coefficients of ϕ^3 theory at first order in perturbation theory in the cubic coupling g, i.e. at tree level. This can be thought of as being either the $d = 6 - \epsilon$ expansion or the holographic description of a perturbative ϕ^3 theory in AdS$_d$. To obtain these OPE coefficients, we perform a direct computation of the following three-point function:

$$\langle [\phi\phi]_{0,J_1}(P_1, Z_1)\phi(X_3)[\phi\phi]_{0,J_2}(P_2, Z_2)\rangle = \Omega_{J_1 J_2} \sum_{n=0}^{\min(J_1,J_2)} C^{(n)}_{[\phi\phi]_{0,J_1}\phi[\phi\phi]_{0,J_2}} \mathcal{X}^n, \tag{D.1}$$

where we define

$$\Omega_{J_1 J_2} := X_{13}^{-h_\phi} X_{23}^{-h_\phi} X_{12}^{-3h_\phi} \frac{J_{1,23}^{J_1} J_{2,13}^{J_2}}{(X_{13}X_{23}X_{12})^{J_1+J_2}}, \quad \mathcal{X} = \frac{H_{12}X_{13}X_{23}}{J_{1,23}J_{2,13}}. \tag{D.2}$$

Since the double-twist operators $[\phi\phi]_{0,J}$ are not renormalized at leading order, we can make use of explicit formulas relating them to bilinear forms of the field ϕ. As a result, we can relate the above three-point function to the tree-level five-point function of the field ϕ, which takes the form

$$\langle \phi(X_1)\dots\phi(X_5)\rangle = C_{\phi\phi\phi}\, X_{15}^{-2h_\phi}\, (X_{23}X_{34}X_{24})^{-h_\phi} + \text{perms} + \mathcal{O}(g^2), \tag{D.3}$$

where $C_{\phi\phi\phi} = \mathcal{O}(g)$. Using an efficient expression for the bilinear forms derived in section D.1, we compute the three-point function and the resulting OPE coefficients of eq. (5.61) in section D.2.

D.1 Parameterization and normalization of GFF double-twist operators

The extensive literature on perturbative conformal field theories contains many different (but physically equivalent) expressions for double-twist operators $[\phi\phi]_{n,J}$ in generalized free field theory. Our approach, which applies to leading-twist operators $[\phi\phi]_{0,J}$, is similar to previous works of Derkachov and Manashov — see e.g. [57] for the original paper or [58] for a more recent review.

Generating function of double-twist operators. Double-twist operators and their descendants can always be expressed as linear combinations of products of derivatives of ϕ. In the leading-twist sector $\Delta - J = 2\Delta_\phi$, these linear combinations can be expressed as

$$(Z\partial_P)^M[\phi\phi]_{0,J}(P,Z) = \psi_{J,M}(\partial_{\alpha_1},\partial_{\alpha_2})\phi(P+\alpha_1 Z)\phi(P+\alpha_2 Z)|_{\alpha_i=0}, \tag{D.4}$$

where $\psi_{J,M}(\alpha_1,\alpha_2)$ is a homogeneous polynomial of degree $J+M$. The right hand side of this formula is to be understood as the result of a limiting procedure. Indeed, the operator product $\phi(X_1)\phi(X_2)$ is divergent when $X_{12}\to 0$ due to the identity contribution in the GFF OPE:

$$\phi(X_1)\phi(X_2) = X_{12}^{-\Delta_\phi}\,\mathbf{1} + \mathcal{O}(X_{12}^0).$$

Instead, we define the operator product in eq. (D.4) as the lightcone limit of the OPE with the identity contribution subtracted. More specifically, if X_1, X_2 are generic and P_1, P_2 are two lightlike separated points, then we define the GFF operator product at lightlike separation by

$$\phi(P_1)\phi(P_2) := \lim_{X_1,X_2\to 0}\left(\phi(X_1+P_1)\phi(X_2+P_2) - X_{12}^{-\Delta_\phi}\,\mathbf{1}\right). \tag{D.5}$$

In this case, the derivative ∂_{α_i} acts as the operator $Z\partial_P$ on the i'th insertion of the field ϕ, which increases both the spin and the conformal dimension by one. Similarly, the action of $(Z\partial_P)^M$ on the primary $[\phi\phi]_{0,J}$ does not change the twist, so the corresponding descendants remain at leading twist.

Given the definition (D.5), the product in eq. (D.4) is finite and can therefore be expanded as a power series around $\alpha_1, \alpha_2 = 0$. We can organize this power series into leading-twist primaries and descendants:

$$\phi(P+\alpha_1 Z)\phi(P+\alpha_2 Z) = \sum_{J,M=0}^{\infty} \hat\psi_{J,M}(\alpha_1,\alpha_2)(Z\partial_P)^M[\phi\phi]_{0,J}(P,Z), \tag{D.6}$$

where $\hat\psi_{J,M}(\alpha_1,\alpha_2)$ is a homogeneous polynomial of degree $J+M$. The above expansion agrees with eq. (D.4) if

$$(\psi_{J,M},\hat\psi_{J',M'}) := \psi_{J,M}(\partial_{\alpha_1},\partial_{\alpha_2})\hat\psi_{J',M'}(\alpha_1,\alpha_2)|_{\alpha_i=0} = \delta_{JJ'}\delta_{MM'}, \tag{D.7}$$

which defines a duality relation between the two spaces of polynomials. The polynomials $\psi, \hat\psi$ are further constrained by the action of the conformal subalgebra that preserves leading-twist fields. The latter is isomorphic to $\mathfrak{sl}(2)$, and its representation on the space of fields spanned by $\{(Z\partial_P)^M[\phi\phi]_{0,J}\}_{M=0}^{\infty}$ is given by the generators $Z\partial_P, Z\partial_Z - P\partial_P, P\partial_Z$. By virtue of the relation (D.6), (D.7), the spaces spanned by $\{\psi_{J,M}\}_M$ and $\{\hat\psi_{J,M}\}_M$ also transform in a representation of $\mathfrak{sl}(2)$. These representations are all isomorphic to lowest or highest weight representations, and the explicit action of the raising/lowering operators on the polynomials $\psi, \hat\psi$ is

$$\sum_{i=1}^{2}(\alpha_i^2\partial_{\alpha_i} + 2\alpha_i\Delta_\phi)\psi_{J,0}(\alpha_1,\alpha_2) = 0, \quad \psi_{J,M+1}(\alpha_1,\alpha_2) = (\alpha_1+\alpha_2)\psi_{J,M}(\alpha_1,\alpha_2),$$

$$\sum_{i=1}^{2}\partial_{\alpha_i}\hat\psi_{J,0}(\alpha_1,\alpha_2) = 0, \quad \hat\psi_{J,M+1}(\alpha_1,\alpha_2) = \sum_{i=1}^{2}(\alpha_i^2\partial_{\alpha_i} + 2\alpha_i\Delta_\phi)\hat\psi_{J,M}(\alpha_1,\alpha_2).$$

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The same relations can be found in [58, section 5]. The second-order differential equation satisfied by $\psi_{J,0}$ has a unique polynomial solution, which specifies $[\phi\phi]_{0,J}$ up to a multiplicative constant (by plugging the solution into eq. (D.4)). However, the corresponding expression is tedious to manipulate when computing correlation functions of $[\phi\phi]_{0,J}$. Instead, we will exploit the simple functional form of $\hat{\psi}_{J,0}$:

$$\begin{aligned}&\hat{\psi}_{J,0}(\lambda\alpha_1,\lambda\alpha_2)=\lambda^J\hat{\psi}_{J,0}(\alpha_1,\alpha_2),\\&(\partial_{\alpha_1}+\partial_{\alpha_2})\hat{\psi}_{J,0}(\alpha_1,\alpha_2)=0\Longrightarrow\hat{\psi}_{J,0}(\alpha_1,\alpha_2)=C_J(\alpha_1-\alpha_2)^J.\end{aligned}\tag{D.8}$$

Next, we use an integral transform to relate ψ to $\hat{\psi}$:

$$\psi_{J,0}(\partial_{\alpha_1},\partial_{\alpha_2})=\prod_{i=1}^{2}\int_C\frac{\mathrm{d}t_i}{2\pi\mathrm{i}}\frac{\Gamma(\Delta_\phi)}{t_i^{\Delta_\phi}}e^{t_i}\hat{\psi}_{J,0}\left(t_1^{-1}\partial_{\alpha_1},t_2^{-1}\partial_{\alpha_2}\right).\tag{D.9}$$

Here, C is a Hankel contour going from $t_i=-\infty-\mathrm{i}\epsilon$ to $t_i=-\infty+\mathrm{i}\epsilon$ after winding once around $t_i=0$. The Hankel contour is commonly used to study the Gamma function away from $\mathrm{Re}(z)>0$. In fact, for any non-integer z (see [56, eq. (5.9.2)]):

$$\frac{1}{\Gamma(z)}=\int_C\frac{\mathrm{d}t}{2\pi\mathrm{i}}\frac{e^t}{t^z}.\tag{D.10}$$

Note that the inverse transform can be found in [57, eq. (2.14)]:

$$\hat{\psi}_{J,0}(\alpha_1,\alpha_2)=\prod_{i=1}^{2}\int_0^\infty\frac{\mathrm{d}t_i}{t_i}\frac{t_i^{\Delta_\phi}}{\Gamma(\Delta_\phi)}e^{-t_i}\psi_{J,0}(t_1\alpha_1,t_2\alpha_2).\tag{D.11}$$

Assuming Δ_ϕ non-integer, we can expand $\hat{\psi}_{J,0}$ in eq. (D.9) into monomials to obtain

$$\psi_{J,0}(\partial_{\alpha_1},\partial_{\alpha_2})=C_J\sum_{k=0}^{J}\binom{J}{k}(-1)^k\frac{\partial_{\alpha_1}^k}{(\Delta_\phi)_k}\frac{\partial_{\alpha_2}^{J-k}}{(\Delta_\phi)_{J-k}}.\tag{D.12}$$

This combination of derivatives indeed reproduces a well-known form of double-twist operators that is commonly used in the literature. In this appendix, with applications to three-point functions of double-twist operators in mind, we instead introduce a generating function

$$\sum_{J=0}^{\infty}\frac{s^J}{J!}\frac{\psi_{J,0}(\partial_{\alpha_1},\partial_{\alpha_2})}{C_J}=\prod_{i=1}^{2}\int_C\frac{\mathrm{d}t}{2\pi\mathrm{i}}\frac{\Gamma(\Delta_\phi)}{t_i^{\Delta_\phi}}e^{t_i}e^{s\left(\frac{\partial_{\alpha_1}}{t_1}-\frac{\partial_{\alpha_2}}{t_2}\right)}.\tag{D.13}$$

To equate the left and right hand sides, we of course needed to permute the sum over spins with the integration over t_1,t_2 — this applies as long as the sum over spins converges. Plugging the above generating function of $\psi_{J,0}$'s back into eq. (D.4), we obtain a generating function of $[\phi\phi]_{0,J}$'s:

$$\sum_{J=0}^{\infty}\frac{s^J}{J!}C_J^{-1}[\phi\phi]_{0,J}(P,Z)=\frac{\Gamma(\Delta_\phi)^2}{(2\pi\mathrm{i})^2}\int_{C\times C}\mathrm{d}t_1\mathrm{d}t_2\,e^{t_1+t_2}\phi(t_1P+sZ)\phi(t_2P-sZ).\tag{D.14}$$

In this expression, the invariance of the primaries under gauge transformations $Z\to Z+\beta P$ on the left hand side translates to the invariance of the integral under shifts $(t_1,t_2)\to(t_1+s\beta,t_2-s\beta)$ on the right hand side — this is the main benefit of introducing the generating function.

Fixing the normalization. To derive normalized three-point functions of $[\phi\phi]_{0,J}$ from the generating function in eq. (D.14), we first need to fix the undetermined constant C_J. We can derive this constant from the normalized three-point function

$$\langle [\phi\phi]_{0,J}(P_1,Z_1)\phi(X_3)\phi(X_4)\rangle = C_{\phi\phi[\phi\phi]_{0,J}}(X_{13}X_{14})^{-\Delta_\phi}\left(\frac{J_{1,34}}{X_{13}X_{14}}\right)^J, \tag{D.15}$$

where

$$C^2_{\phi\phi[\phi\phi]_{0,J}} = \frac{(1+(-)^J)}{J!}\frac{(\Delta_\phi)_J^2}{(2\Delta_\phi+J-1)_J} = \frac{(1+(-)^J)}{J!}\frac{\Gamma(\Delta_\phi+J)^2}{\Gamma(\Delta_\phi)^2}\frac{\Gamma(2\Delta_\phi-1+J)}{\Gamma(2\Delta_\phi-1+2J)}. \tag{D.16}$$

Inserting the generating function, we obtain

$$\begin{gathered}\sum_{J=0}^{\infty}\frac{s^J}{J!C_J}\langle[\phi\phi]_{0,J}(P_1,Z_1)\phi(X_3)\phi(X_4)\rangle = \\ \frac{\Gamma(\Delta_\phi)^2}{(2\pi \mathrm{i})^2}\int_C \mathrm{d}t_1\mathrm{d}t_2 e^{t_1+t_2}\langle\phi(t_1P_1+sZ_1)\phi(t_2P_1-sZ_1)\phi(X_3)\phi(X_4)\rangle.\end{gathered} \tag{D.17}$$

Now, given the definition of the GFF operator product in eq. (D.5), the four-point function on the right hand side will only involve two Wick contractions, namely

$$\begin{aligned}&\langle\phi(t_1P_1+sZ_1)\phi(t_2P_1-sZ_1)\phi(X_3)\phi(X_4)\rangle\\ &\quad=\sum_{3\le i\ne j\le 4}(t_1X_{1i}+sZ_1\cdot X_i)^{-\Delta_\phi}(t_2X_{1j}-sZ_1\cdot X_j)^{-\Delta_\phi}.\end{aligned}$$

After the gauge transformation/change of variables

$$(t_1',t_2') = \left(t_1+s\frac{Z_1\cdot X_3}{X_{13}}, t_2-s\frac{Z_1\cdot X_3}{X_{13}}\right),$$

we obtain

$$\begin{gathered}\langle\phi(t_1P_1+sZ_1)\phi(t_2P_1-sZ_1)\phi(X_3)\phi(X_4)\rangle = \\ (X_{13}X_{14})^{-\Delta_\phi}\left[t_1'^{-\Delta_\phi}\left(t_2'+s\frac{J_{1,34}}{X_{13}X_{14}}\right)^{-\Delta_\phi}+t_2'^{-\Delta_\phi}\left(t_1'-s\frac{J_{1,34}}{X_{13}X_{14}}\right)^{-\Delta_\phi}\right].\end{gathered}$$

The integrand can be expanded into a power series in s, and after integrating over t_1', t_2' we obtain

$$\langle[\phi\phi]_{0,J}(P_1,Z_1)\phi(X_3)\phi(X_4)\rangle = C_J(1+(-1)^J)(X_{13}X_{14})^{-\Delta_\phi}\left(\frac{J_{1,34}}{X_{13}X_{14}}\right)^J. \tag{D.18}$$

To retrieve the normalization of eq. (D.15), we must set $C_J = \frac{1}{2}C_{\phi\phi[\phi\phi]_{0,J}}$.

D.2 Application to OPE coefficients of two double-twist operators

The computation of $\langle[\phi\phi]_{0,J_1}(P_1,Z_1)\phi(X_3)[\phi\phi]_{0,J_2}(P_2,Z_2)\rangle$ can be done similarly to the computation of $\langle[\phi\phi]_{0,J_1}(P_1,Z_1)\phi(X_3)\phi(X_4)\rangle$ in section D.1. First of all, we have

$$\begin{gathered}\sum_{J_1,J_2=0}^{\infty}\frac{4s_1^{J_1}s_2^{J_2}}{J_1!J_2!}\frac{\langle[\phi\phi]_{0,J_1}(P_1,Z_1)\phi(X_3)[\phi\phi]_{0,J_2}(P_2,Z_2)\rangle}{C_{\phi\phi[\phi\phi]_{0,J_1}}C_{\phi\phi[\phi\phi]_{0,J_2}}} = \\ \prod_{i=1,2,4,5}\frac{\Gamma(\Delta_\phi)}{2\pi\mathrm{i}}\int_C \mathrm{d}t_i\, e^{t_i}\langle\phi(t_1P_1+s_1Z_1)\phi(t_2P_1-s_1Z_1)\phi(X_3)\phi(t_4P_2-s_2Z_2)\phi(t_5P_2+s_2Z_2)\rangle.\end{gathered}$$

We can now insert the tree-level form of the five-point function in eq. (D.3), with all Wick contractions of the first two and last two fields subtracted following eq. (D.5). The correlator simplifies after the gauge transformation/change of variables

$$(t'_1, t'_2, t'_4, t'_5) = \left(t_1 + s_1 \frac{Z_1 \cdot P_1}{X_{12}}, t_2 - s_1 \frac{Z_1 \cdot P_1}{X_{12}}, t_4 - s_2 \frac{Z_2 \cdot P_1}{X_{12}}, t_5 + s_2 \frac{Z_2 \cdot P_1}{X_{12}} \right),$$

in terms of which the five-point correlator takes the form

$$\begin{aligned} &\langle \phi(t_1 P_1 + s_1 Z_1) \phi(t_2 P_1 - s_1 Z_1) \phi(X_3) \phi(t_4 P_2 - s_2 Z_2) \phi(t_5 P_2 + s_2 Z_2) \rangle \\ &= C_{\phi\phi\phi} (X_{12}^3 X_{13} X_{23})^{-h_\phi} \left(t'_1 t'_5 + s_1 s_2 \frac{H_{12}}{X_{12}^2} \right)^{-2h_\phi} \left(t'_2 t'_4 + s_1 s_2 \frac{H_{12}}{X_{12}^2} \right)^{-h_\phi} \\ &\quad \left(t'_2 + s_1 \frac{J_{1,23}}{X_{12} X_{13}} \right)^{-\frac{1}{2}\Delta_\phi} \left(t'_4 + s_2 \frac{J_{2,13}}{X_{12} X_{23}} \right)^{-h_\phi} + \text{perms}, \end{aligned}$$

where $h_\phi := \Delta_\phi/2$ and

$$\text{perms} = [t'_1 \leftrightarrow t'_2, s_1 \to -s_1] + [t'_4 \leftrightarrow t'_5, s_2 \to -s_2] + [t'_1 \leftrightarrow t'_2, s_1 \to -s_1, t'_4 \leftrightarrow t'_5, s_2 \to -s_2]. \tag{D.19}$$

Since the integral is symmetric under the above permutations of t'_i, the four terms differ only by the signs of s_1, s_2. Therefore, in a power series expansion in s_1, s_2, the three permutations will be related to the first one by multiplicative factors of $(-1)^{J_1}$, $(-1)^{J_2}$ and $(-1)^{J_1+J_2}$ respectively, such that the total three-point function is the integral of the first term multiplied by $(1 + (-1)^{J_1})(1 + (-1)^{J_2})$.

Note that H_{12} always appears with a factor of $s_1 s_2$ in these expressions — this ensures that we are within the physical range $n \leq \min(J_1, J_2)$ when expanding a three-point function in the n-basis. The OPE coefficients $C^{(n)}_{[\phi\phi]_{0,J_1} \phi [\phi\phi]_{0,J_2}}$ in this basis can therefore be obtained from the coefficient of $s_1^{J_1-n} s_2^{J_2-n} (s_1 s_2 H_{12})^n$ in a series expansion around $s_1, s_2, H_{12} = 0$. After dividing out by the kinematical prefactor, we can write this coefficient for even J_1, J_2 as

$$\frac{\partial_{s_1}^{J_1-n} \partial_{s_2}^{J_2-n} \partial_{H_{12}}^n \langle \phi\phi\phi\phi\phi \rangle |_{s_1,s_2,H_{12}=0}}{4n! \Omega_{J_1 J_2} (\mathcal{X}/H_{12})^n} = \prod_{i=1}^{2} (h_\phi)_{J_i - n} \sum_{j=0}^{n} \frac{(2h_\phi)_j (h_\phi)_{n-j}}{j!(n-j)!} \frac{(t'_1 t'_5)^{-2h_\phi - j}}{t_2'^{2h_\phi + J_1 - j} t_4'^{2h_\phi + J_2 - j}}.$$

Each of the $j = 0, \ldots, n$ summands is proportional to a product of powers $t_i'^{-(2h_\phi + M)}$, where M is a non-negative integer, each of which integrates to $1/(2h_\phi)_M$. We therefore obtain

$$\prod_{i=1}^{2} \frac{(J_i - n)!}{J_i!} \frac{C^{(n)}_{[\phi\phi]_{0,J_1} \phi [\phi\phi]_{0,J_2}}}{C_{\phi\phi[\phi\phi]_{0,J_1}} C_{\phi\phi[\phi\phi]_{0,J_2}}} = \prod_{i=1}^{2} (h_\phi)_{J_i - n} \sum_{j=0}^{n} \frac{(h_\phi)_{n-j}}{j!(n-j)!} \frac{1}{(2h_\phi)_j (2h_\phi)_{J_1 - j} (2h_\phi)_{J_2 - j}}. \tag{D.20}$$

Multiplying the left hand side by $P_{[\phi\phi]_{0,J_1}} P_{[\phi\phi]_{0,J_2}}$ reproduces the product of OPE coefficients entering the five-point function, and we retrieve the formula in eq. (5.61).

E Lightcone blocks with parity-odd tensor structures

In section 3, our derivation of the lightcone blocks for the direct and crossed channels has been operated under the implicit assumption that the tensor structures at the central vertex of the OPE channel are even under parity transformations. While this is certainly true for CFTs in $d \geq 4$, in the special case of three-dimensional CFTs one can also have tensor structures that are odd under parity transformations. As reviewed in [43, section 3.2], these tensor structures arise because of the possibility of constructing three-point invariants using the totally-antisymmetric symbol ε. For a three-point function $\langle \mathcal{O}_1(X_1, Z_1)\mathcal{O}_2(X_2, Z_2)\phi(X_3)\rangle$ expressed in embedding space coordinates and polarizations, one can see that

$$\left(\varepsilon_{ABCDE} X_1^A X_2^B X_3^C Z_1^D Z_2^E\right)^2 = 2\,\mathcal{X}(1-\mathcal{X})\frac{J_{1,23}^2 J_{2,31}^2}{X_{12}X_{23}X_{13}} \tag{E.1}$$

which implies that the presence of the parity-odd tensor structures $\varepsilon_{ABCDE} X_1^A X_2^B X_3^C Z_1^D Z_2^E$ manifests in cross-ratio space as the presence of the factor $\sqrt{\mathcal{X}(1-\mathcal{X})}$.

With this in mind, we can aim to understand what changes does the presence of these tensor structures imply for the expressions of five-point lightcone blocks we derived in section 3. In the OPE limit, we have that $\sqrt{\mathcal{X}(1-\mathcal{X})} = \sqrt{w(1-w)}$, so the presence of parity-odd can be imposed as a correction to the OPE-limit asymptotics of the five-point blocks. For the direct channel, this tells us that the expression of blocks for odd tensor structures will change from eq. (3.24) to

$$\psi^{\mathrm{DC-odd}}_{(h_a,\bar{h}_a;n)}(u_{si}(\bar{z}_a, z_a, w)) \overset{LCL^{(2)}}{\sim} \prod_{a=1}^{2} \bar{z}_a^{h_a} z_a^{\bar{h}_a}(1-w)^n\sqrt{w(1-w)}\tilde{F}^{(\mathrm{odd})}_{(h_a,\bar{h}_a;n)}(z_1, z_2, w)\,, \tag{E.2}$$

where the yet to be specified function $\tilde{F}^{(\mathrm{odd})}_{(h_a,\bar{h}_a;n)}$ must asymptote to a constant in the OPE limit.

To compute the form of $\tilde{F}^{(\mathrm{odd})}_{(h_a,\bar{h}_a;n)}$, we can use the Casimir differential equations to express this object in terms of the parity-even solution $\tilde{F}$. In fact, denoting the differential operators (3.29) that act on $\tilde{F}$ as $\mathcal{D}_a(h_\phi; h_a, \bar{h}_a; n)$, one can directly check that the operators acting on $\tilde{F}^{(\mathrm{odd})}$ correspond to

$$\mathcal{D}_a^{\mathrm{odd}} = \mathcal{D}_a\left(h_\phi + \frac{1}{2}; h_a, \bar{h}_a; n+1\right) \tag{E.3}$$

which implies that their eigenfunctions must be equal once one performs the same shift in parameters. The same result applies to the crossed channel once a prefactor analogous to that in eq. (E.2) is extracted, so this constitutes for both cases a simple recipe that we can use to avoid computing from scratch the conformal blocks with parity-odd tensor structures.

For the direct channel, following this recipe means we just need to correct eq. (3.63) to

$$\psi^{\mathrm{DC-odd}\,(0)}_{(h_a,\bar{h}_a;n)}(u_{si}) \overset{LCL_{\bar{\epsilon}}}{\sim} \mathcal{N}^{\mathrm{DC}\,(0)}_{(h_\phi+\frac{1}{2};h_a,\bar{h}_a;n+1)}(u_{s1}u_{s2})^{h_\phi}u_{s5}^{h_1}u_{s3}^{h_2}\, f_{n+h_{2\phi}+\frac{1}{2}}(u_{s2}) f_{n+h_{1\phi}+\frac{1}{2}}(u_{s1}). \tag{E.4}$$

For the crossed channel, we can once more the distinguish the solutions for two types of scaling of eigenvalues: the case I blocks with scaling (3.69) are such that the correction to eq. (3.81)

$$\psi^{\mathrm{CC-odd,I}}_{(h_a,\bar h_a;n)}(u_{si}) \overset{LCL_{\vec\epsilon}}{\sim} \mathcal{N}^{\mathrm{CC,I}}_{(h_\phi+\frac{1}{2};h_a,\bar h_a;n+1)} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi} e^{-(n+1)u_{s5}-\frac{\bar h_1^2 u_{s2}+\bar h_2^2 u_{s3}}{n+1}} \tag{E.5}$$

is negligible due to $n+1 \sim n + \mathcal{O}(\epsilon_{15}^0)$, while the case II solution (3.101) becomes in the presence of odd tensor structures

$$\psi^{\mathrm{CC-odd,II}}_{(h_a,\bar h_a;J_1-\delta n)}(u_{si}) \overset{LCL^{(4)}_{\vec\epsilon}}{\sim} \mathcal{N}^{\mathrm{CC,II}}_{(h_\phi+\frac{1}{2};h_a,\bar h_a;n+1)} u_{s1}^{h_1} u_{s4}^{h_2} (u_{s2}u_{s3})^{h_\phi}$$
$$(1-u_{s2})^{\delta n-\frac{1}{2}} u_{s5}^{h_\phi+\delta n-\frac{1}{2}} \mathcal{K}_{h_\phi+\delta n-\frac{1}{2}}\left(\bar h_1^2 u_{s5} + \bar h_2^2 u_{s3} u_{s5}\right). \tag{E.6}$$

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