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Abstract

In this paper we shall present rigorous path integral treatments of free motion on the Poincaré disc, the Poincaré upper half-plane and on the hyperbolic strip, three spaces which are analytically equivalent with each other. Whereas the path integral treatments on the disc and on the strip are new, we shall present two further path integral treatments for the upper half-plane to the existing one. All the calculations are mainly based on Fourier-expansions of the Feynman kernels which can be easily performed. The remaining path integrals on D, U and S can be reduced to the path integral problems on the pseudosphere Λ^2, Liouville quantum mechanics and the modified Pöschl-Teller potential problem, respectively, where the results are known. The corresponding normalized wave functions and the energy-spectrum are derived. The energy-spectra are the same in all three spaces and read $E = \frac{1}{2m}(p^2+\frac{1}{4})$ (p - momentum). We shall discuss the "zero point" energy $E_0 = \frac{1}{8m}$ which is due to the Heisenberg uncertainty relation. We shall also discuss the equivalences between the Feynman kernels on Λ^2, D, U and S.

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I. Introduction

In this paper we shall present complete path integral treatments for a particle moving freely on the **Poincaré disc** D

$$D := \{z = x_1 + ix_2 = re^{i\psi} | r < 1, \psi \in [0, 2\pi]\}, \tag{1}$$

and on the **hyperbolic strip** S

$$S := \{\eta = X + iY | X \in \mathbf{R}, Y \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})\}, \tag{2}$$

endowed with the hyperbolic geometry (see below). In addition we shall present an alternative treatment to our previous publication [14,16] for the path integral for a particle moving freely on the **Poincaré upper half-plane** U

$$U := \{\zeta = x + iy | x \in \mathbf{R}, y > 0\}, \tag{3}$$

Recently these models for a non-Euclidean geo- metry have become important in the theory of strings, in particular in the Polyakov approach for the bosonic string (see e.g.[12,29]), in the theory of quantum chaos (see [3,18,31], and for non-Euclidean harmonic analysis [34,35]. In the former two cases one considers bounded regions in the Poincaré upper half-plane, say, which are fundamental domains of discrete subgroups of $PSL(2,\mathbf{R})$ (see e.g.[19]). We shall not consider the motion in bounded domains; our paper will deal only with the free motion on the entire disc, upper half-plane and strip, respectively.[1]

All these three Riemannian spaces are analytically equivalent to each other.

1) In order to discuss this in some detail we start with the **pseudosphere** Λ^2 which is defined by:
$$\Lambda^2 := \{(y_1, y_2, y_3) | -y_1^2 + y_2^2 + y_3^2 = -R^2\} \tag{4}$$
(in the following we shall set $R = 1$). Λ^2 can be visualised as a hyperboloid embedded in $\mathbf{R}^3$. But be careful: Λ^2 has negative Gaussian curvature $K = -1$, as well as U, D and S, i.e. they are everywhere saddle-shaped. A more convenient description for Λ^2 reads in pseudospherical polar coordinates (τ, ϕ)[3,34,36]:
$$y_1 = \cosh\tau, \quad y_2 = \sinh\tau\cos\phi, \quad y_3 = \sinh\tau\sin\phi, \quad (t \geq 0, \phi \in [0, 2\pi]). \tag{5}$$
The metric g_{ab} associated with the line element $ds^2 = g_{ab}dq^a dq^b$ reads g_{ab} =diag $(1, \sinh^2\tau)$.
2) With the stereographic projection of Λ^2 onto the (x_1, x_2)-plane we get the Poincaré disc D:
$$z = x_1 + ix_2 = re^{i\psi} = \frac{y_2 + iy_3}{1 + y_1} = \tanh\frac{\tau}{2}(\sin\phi + i\cos\phi). \tag{6}$$
Here the metric reads $g_{ab} = [2/(1-r^2)]^2 \mathrm{diag}(1, r^2)$.
3) The Poincaré disc D can be mapped onto the Poincaré upper half-plane U by the Cayley-transformation:
$$\zeta = x + iy = \frac{-iz + i}{z + 1}, \qquad z = \frac{-\zeta + i}{\zeta + i}. \tag{7}$$
The metric reads $g_{ab} = 1/y^2 \cdot \delta_{ab}$.

[1] For an attempt to calculate energy-levels and wave-functions in bounded domains see Ref.[2].

4) With the help of the transformation

$$\eta = X + iY = -\ln(-i\zeta)\ (= 2\mathrm{artanh} z), \tag{8}$$

we can map the Poincaré upper half-plane (the Poincaré disc) onto the hyperbolic strip S. The metric reads $g_{ab} = 1/\cos^2 Y \cdot \delta_{ab}$.

The hyperbolic distance $r = d(p'', p')$ [p - any of the coordinates (τ, ϕ), (x_1, x_2), (x, y), (X, Y)] in these spaces is given by:

$$\left.\begin{aligned}
\cosh r &= \cosh\tau''\cosh\tau' - \sinh\tau''\sinh\tau'\cos(\phi''-\phi') && (\text{on } \Lambda^2)\\
&= 1 + \frac{2|z''-z'|^2}{(1-|z'|^2)(1-|z''|^2)} && (\text{on } D)\\
&= \frac{(x''-x')^2 + y''^2 + y'^2}{2y'y''} && (\text{on } U)\\
&= \frac{\cosh(X''-X')}{\cos Y'\cos Y''} - \tan Y'\tan Y'' && (\text{on } S).
\end{aligned}\right\} \tag{9}$$

In some recent papers [16,17] we have calculated the path integrals on U and Λ^{d-1} (the d-dimensional generalisation of Λ^2).[1] In formulating the path integral on U and on Λ^{d-1} our starting point has always been the Weyl-ordering prescription in the quantum Hamiltonian which leads to a Lagrangian path integral defined on midpoints (see [15,23,27,28]). But in this paper we shall use another prescription, which we call "product form"-definition. Let us summarize in short the most important facts of this prescription (for details see [13]).

We start with the generic case [i.e. the classical Lagrangian is given by $\mathcal{L}_{Cl}(q, \dot q) = \frac{m}{2} g_{ab}\dot q^a \dot q^b - V(q)$] and write the metric tensor g_{ab} in the form (which under reasonable assumptions is always possible, e.g. positive definite scalar product):

$$g_{ab}(q) = \sum_{c=1}^{d} h_{ac}(q) h_{bc}(q) \tag{10}$$

(d=dimension of the Riemannian manifold). The quantum Hamiltonian is constructed in the usual way by the Laplace-Beltrami operator Δ_{LB} (we set $\hbar = 1$; in the following sums over repeated indices are understood):

$$H = -\frac{1}{2m}\Delta_{LB} + V(q) = -\frac{1}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}\sqrt{g} g^{ab}\frac{\partial}{\partial q^b} + V(q) \tag{11}$$

(g=determinant of the metric tensor g_{ab}). We introduce momentum operators $p_a = -i(\partial_a + \Gamma_a/2)$ where $\Gamma_a = \partial_a \ln\sqrt{g}$. Rewriting the Hamiltonian (11) in terms of the momentum operators p_a we choose a **product ordering**-definition:

$$H = \frac{1}{2m} h^{ac}(q) p_a p_b h^{bc}(q) + V(q) + \Delta V(q) \tag{12}$$

[1] But see also [6,18] for a discussion of Λ^{d-1} and Λ^2,Λ^3, respectively, and [22] for a discussion for U.

with the well-defined quantum correction ΔV given by ($h := \det(h_{ab}) = \sqrt{g}$):

$$\begin{aligned}
\Delta V = \frac{1}{8m}\Bigg[4h^{ac}h^{bc}{}_{,ab} &+ 2h^{ac}h^{bc}\frac{h_{,ab}}{h}\\
&+ 2h^{ac}\left(h^{bc}{}_{,b}\frac{h_{,a}}{h} + h^{bc}{}_{,a}\frac{h_{,b}}{h}\right) - h^{ac}h^{bc}\frac{h_{,a}h_{,b}}{h^2}\Bigg].
\end{aligned} \tag{13}$$

There is an important special case of equation (13). Let us assume that g_{ab} is proportional to the unit tensor, i.e. $g_{ab} = f^2\delta_{ab}$. Then ΔV simplifies

$$\Delta V = \hbar^2 \frac{d-2}{8mf^4}\sum_{a=1}^{d}\left[(4-d)f_{,a}^2 + 2f\cdot f_{,aa}\right]. \tag{14}$$

This implies an important corollary:

Corollary: Assume that the metric has the special form $g_{ab} = f^2\delta_{ab}$ - or can be cast into this form by an orthogonal linear transformation. If the dimension of the space is $d = 2$, then the quantum correction ΔV vanishes.

For details we refer to [13]. Using the Trotter formula $e^{-it(A+B)} = s-\lim_{N\to\infty} \left(e^{-itA/N}e^{-itB/N}\right)^N$ (e.g. [30]) and the short-time approximation for the matrix element $< q''|e^{-i\epsilon H}|q' >$ one obtains in the usual manner the **Lagrangian path integral in the "product form"-definition** [$q^{(j)} = q(t^{(j)})$, $f^{(j)} = f(t^{(j)})$, $t^{(j)} = t' + j\epsilon$, $\epsilon = T/N = (t''-t')/N$, $N\to\infty$, $\Delta q^{(j)} = q^{(j)} - q^{(j-1)}$]

$$\begin{aligned}
K(q'', q'; T) &= \int \sqrt{g}\, Dq(t) \exp\left\{ i\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}h_{bc}\dot q^a \dot q^b - V(q) - \Delta V(q)\right]dt\right\}\\
&:= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{Nd}{2}}\prod_{j=1}^{N-1}\int\sqrt{g(q^{(j)})}\,dq^{(j)}\\
&\times \exp\left\{ i\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{ac}(q^{(j-1)})h_{bc}(q^{(j)})\Delta q^{a,(j)}\Delta q^{b,(j)} - \epsilon V(q^{(j)}) - \epsilon\Delta V(q^{(j)})\right]\right\}.
\end{aligned} \tag{15}$$

We shall always use in this paper the lattice definition of Eq.(15). The expression in square brackets is nothing but the classical Lagrangian with an additional quantum correction potential ΔV: $\mathcal{L}_{eff} = \mathcal{L}_{Cl} - \Delta V$. Clearly, one has to prove that with the short time kernel of this path integral the time-dependent Schrödinger equation

$$\left[-\frac{1}{2m}\Delta_{LB} + V(q)\right]\psi(q;t) = \frac{1}{i}\frac{\partial}{\partial t}\psi(q;t) \tag{16}$$

can be derived via the time evolution equation

$$\psi(q''; t'') = \int \sqrt{g(q')} K(q'', q'; T)\psi(q'; t')dq'. \tag{17}$$

This is in fact the case - see [13].

Our paper is organised as follows. In section II we shall present a complete path integral treatment on D. We have already noted in [17] a path integral solution on D, but this solution was derived in an indirect manner using the path integral solution on the pseudosphere Λ^2 followed by a change of variables. This indirect reasoning will be avoided here: we shall calculate the path integral on D by performing, first, a Fourier expansion and, second, a nonlinear transformation.

In section II we shall present a path integral calculation on the Poincaré upper half-plane U. In [16] we have already presented a path integral solution on U, but here we shall present some new aspects and two different alternative calculations. The first will follow an idea by Kubo [22], who in our opinion used a somewhat questionable lattice definition of the path integral. We shall discuss this aspect in some detail and present a rigorous calculation. The second approach will use a Fourier expansion and the path integral solution of Liouville quantum mechanics.

In section IV we shall present the path integral solution on the hyperbolic strip S. We shall calculate the path integral again by starting with a Fourier expansion; in the second step we shall use the path integral solution for the potential problem $V(x) = V_0/\cosh^2 x > 0$ ($x \in \mathbf{R}$). The path integral for this potential, which is a special form of a modified Pöschl-Teller potential, has been discussed by Böhm and Junker [5].

Section V will summarise our results.

In the appendix A we shall list some important properties of the Legendre functions P_ν^μ and Q_ν^μ. In appendix B the orthonormality and completeness of the eigenstates in S are shown. In appendix C we shall display some graphical plots of the wave-functions on S.

II. The Poincaré Disc

In order to formulate the path integral on the Poincaré disc D, let us start with the classical Lagrangian and Hamiltonian. They are given by [the metric reads $g_{ab} = [2/(1-r^2)]^2 \mathrm{diag}(1, r^2)$]:

$$\mathcal{L}_{Cl}(r, \dot r, \psi, \dot\psi) = 2m\frac{\dot r^2 + r^2\dot\psi^2}{(1-r^2)^2}, \qquad \mathcal{H}_{Cl}(r, p_r, \psi, p_\psi) = \frac{(1-r^2)^2}{8m}\left(p_r^2 + \frac{1}{r^2}p_\psi^2\right). \tag{1}$$

The quantum Hamiltonain reads:

$$H = -\frac{(1-r^2)^2}{8m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\psi^2}\right] \tag{2}$$

which can be easily deduced from the Laplace-Beltrami operator on D. The scalar product for functions $f_1, f_2 \in L^2(D)$ is given by

$$(f_1, f_2)_D = \int_0^1 \frac{4r\,dr}{(1-r^2)^2}\int_0^{2\pi} d\psi\, f_1(r,\psi) f_2^*(r,\psi). \tag{3}$$

For functions $\Psi \in D(H) \cap L^2(D)$ we have the boundary condition $\lim_{r\to 1}\Psi(r,\psi) = 0$ ($\psi \in [0, 2\pi]$). Because $h_{ab} = \sum_{c=1}^2 f_c\,\delta_{ac}\,\delta_{bc}$ with $f_1 = 2/(1-r^2)$, $f_2 = 2r/(1-r^2)$ we get, following our prescription outlined in the introduction, for the hermitian momenta

$$\left.\begin{aligned} \Gamma_r &= \frac{1}{r} + \frac{4r}{1-r^2}, \qquad & p_r &= \frac{1}{i}\left(\frac{\partial}{\partial r} + \frac{1}{2r} + \frac{2r}{1-r^2}\right), \\ \Gamma_\psi &= 0, & p_\psi &= \frac{1}{i}\frac{\partial}{\partial\psi}. \end{aligned}\right\} \tag{4}$$

The Hamiltonian in the "product ordering" reads

$$H = \frac{1}{8m}\left[(1-r^2)p_r^2(1-r^2) + \frac{1-r^2}{r}p_\psi^2\frac{1-r^2}{r}\right] + \Delta V \tag{5}$$

with the quantum correction ΔV given by

$$\Delta V = -\frac{(1-r^2)^2}{32mr^2}. \tag{6}$$

With this "product form"-definition we get for the path integral on D:

$$\begin{aligned} K^D(r'', r', \psi'', \psi'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_0^{2\pi} d\psi^{(j)}\int_0^1 \frac{4r^{(j)}dr^{(j)}}{(1-r^{(j)2})^2} \\ \times\exp\left\{i\sum_{j=1}^N\left[\frac{2m}{\epsilon}\frac{\Delta^2 r^{(j)} + r^{(j)}r^{(j-1)}\Delta^2\psi^{(j)}}{(1-r^{(j)2})(1-r^{(j-1)2})} + \epsilon\frac{(1-r^{(j)2})^2}{32mr^{(j)2}}\right]\right\}. \end{aligned} \tag{7}$$

In order to calculate the path integral (7) we start by performing a Fourier expansion:

$$\left.\begin{aligned} K^D(r'', r', \psi'', \psi'; T) &= \sum_{l=-\infty}^{\infty} K_l^D(r'', r'; T)\, e^{il(\psi''-\psi')} \\ K_l^D(r'', r'; T) &= \frac{1}{2\pi}\int_0^{2\pi} K^D(r'', r', \psi'', \psi'; T) e^{-il(\psi''-\psi')} d\psi''. \end{aligned}\right\} \tag{8}$$

Equation (7) inserted into (8) gives for $K_l^D(T)$:

$$\begin{aligned} K_l^D(r'', r'; T) &= \frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}}\prod_{j=1}^{N-1}\int_0^{2\pi} d\psi^{(j)}\int_0^1\frac{4r^{(j)}dr^{(j)}}{(1-r^{(j)2})^2}\int_0^{2\pi} d\psi'' \\ &\quad\times\exp\left\{i\sum_{j=1}^N\left[\frac{2m}{\epsilon}\frac{\Delta^2 r^{(j)} + r^{(j)}r^{(j-1)}\Delta^2\psi^{(j)}}{(1-r^{(j)2})(1-r^{(j-1)2})} + \epsilon\frac{(1-r^{(j)2})^2}{32mr^{(j)2}} - l\Delta\psi^{(j)}\right]\right\} \\ &= \frac{1}{2\pi}\left[\frac{(1-r''^2)(1-r'^2)}{4r'r''}\right]^{\frac12}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N\prod_{j=1}^{N-1}\int_0^1\frac{2dr^{(j)}}{1-r^{(j)2}} \\ &\quad\times\exp\left\{i\sum_{j=1}^N\left[\frac{2m}{\epsilon}\frac{\Delta^2 r^{(j)}}{(1-r^{(j)2})(1-r^{(j-1)2})} - \frac{\epsilon(l^2-\frac14)}{8mr^{(j)2}}(1-r^{(j)2})^2\right]\right\} \\ &= \frac{1}{2\pi}\left[\frac{(1-r''^2)(1-r'^2)}{4r'r''}\right]^{\frac12}\int\frac{2}{1-r^2}Dr(t) \\ &\quad\times\exp\left\{i\int_{t'}^{t''}\left[2m\frac{\dot r^2}{(1-r^2)^2} - \frac{l^2-\frac14}{8mr^2}(1-r^2)^2\right]dt\right\}. \end{aligned} \tag{9}$$

We now look for a transformation $z = z(r)$ such that

$$\frac{4\dot r^2}{(1-r^2)^2} = \dot z^2. \tag{10}$$

With the ansatz $z(r)=\ln f(r)$ we find a differential equation for $f(r)$:

$$f'(r)-\frac{2}{1-r^2}f(r)=0 \qquad (11)$$

with solution $f(r)=\exp\left[2\int^r du/(1-u^2)\right]=(1+r)/(1-r)$. Therefore we get:

$$z(r)=\ln\frac{1+r}{1-r}. \qquad (12)$$

$z(r)$ has the property $[0,1)\mapsto[0,\infty)$. Note that $z(r)$ is essentially the hyperbolic distance of an arbitrary point in the disc from the origin [see Eq.(I.9)]. The inverse transformation reads $r(z)=\tanh\frac{z}{2}$ and maps $[0,\infty)\mapsto[0,1)$. For the various terms in the path integral (8) we have:

1) $(1-r^{(j)\,2})^2/r^{(j)\,2}=4/\sinh^2 z^{(j)}$;
2) $2dr^{(j)}/(1-r^{(j)\,2})=dz^{(j)}$;
3) For the term $\Delta^2 r^{(j)}/[(1-r^{(j)\,2})(1-r^{(j-1)\,2})]$ we have to perform a Taylor expansion up to fourth order in $\Delta z^{(j)}$. We get

$$\frac{4(r^{(j)}-r^{(j-1)})^2}{(1-r^{(j)\,2})(1-r^{(j-1)\,2})}\simeq(z^{(j)}-z^{(j-1)})^2+\frac{(z^{(j)}-z^{(j-1)})^4}{12}; \qquad (13)$$

4) Inserting equation (12) into the exponetial in (9) yields together with the identity[1] $\Delta^4 z^{(j)}\doteq 3\left(\frac{i\epsilon}{m}\right)^2$ (see [8,10,26]):

$$\exp\left[\frac{im}{2\epsilon}\frac{4\Delta^2 r^{(j)}}{(1-r^{(j)\,2})(1-r^{(j-1)\,2})}\right]\doteq\exp\left[\frac{im}{2\epsilon}\Delta^2 z^{(j)}-\frac{i\epsilon}{8m}\right]. \qquad (14)$$

Therefore we have for the coordinate transformed path integral (9):

$$\tilde K_l^D(z'',z';T)=\frac{1}{2\pi}(\sinh z'\sinh z'')^{-\frac12}e^{-\frac{iT}{8m}}$$
$$\times\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac N2}\prod_{j=1}^{N-1}\int_0^\infty dz^{(j)}\exp\left\{i\sum_{j=1}^N\left[\frac{m}{2\epsilon}\Delta^2 z^{(j)}-\frac{\epsilon(l^2-\frac14)}{2m\sinh^2 z^{(j)}}\right]\right\}$$
$$\equiv\frac{1}{2\pi}(\sinh z'\sinh z'')^{-\frac12}e^{-\frac{iT}{8m}}\int Dz(t)\exp\left\{i\int_{t'}^{t''}\left[\frac m2\dot z^2-\frac{l^2-\frac14}{2m\sinh^2 z}\right]dt\right\}. \qquad (15)$$

The path integral (15) describes scattering in $\mathbf{R}^+$ under the influence of the potential $V(z)=(l^2-\frac14)/2m\sinh^2 z$, a special case of the modified Pöschl-Teller potential (see section IV for details). But equation (15) is also nothing but the Fourier expanded path integral on the pseudosphere Λ^2. In order to see this we first consider the Feynman kernel for Λ^2 (see [6,17]). It reads:

$$K^{\Lambda^2}(\tau'',\tau',\phi'',\phi';T)=\frac{1}{2\pi^2}\int_0^\infty dp\sum_{l=-\infty}^\infty p\sinh\pi p$$
$$\times e^{-\frac{iT}{2m}(p^2+\frac14)}|\Gamma(\tfrac12+ip+l)|^2\mathcal{P}^{-l}_{ip-\frac12}(\cosh\tau')\mathcal{P}^{-l}_{ip-\frac12}(\cosh\tau'')e^{il(\phi''-\phi')}. \qquad (16)$$

[1] We use the symbol $\doteq$ (following DeWitt [7]) to denote "equivalence as far as use in the path integral is concerned".

A partial wave expansion yields:

$$K^{\Lambda^2}(\tau'',\tau',\phi'',\phi';T)=\sum_{l=-\infty}^\infty K_l^{\Lambda^2}(\tau'',\tau';T)e^{il(\phi''-\phi')} \qquad (17)$$

where the radial kernel $K_l^{\Lambda^2}(T)$ is given by [17]:

$$K_l^{\Lambda^2}(\tau'',\tau';T)=\frac{1}{2\pi^2}\int_0^\infty dp\,p\sinh\pi p$$
$$\times\exp\left[-\frac{iT}{2m}(p^2+\frac14)\right]|\Gamma(\tfrac12+ip+l)|^2\mathcal{P}^{-l}_{ip-\frac12}(\cosh\tau')\mathcal{P}^{-l}_{ip-\frac12}(\cosh\tau''). \qquad (18)$$

On the other hand we get for the path integral (see equation (9b)) on Λ^2 defined in the "product form"-definition [see the introduction, equations (I.5,13,15)]:

$$K_l^{\Lambda^2}(\tau'',\tau';T)=\frac{1}{2\pi}\int_0^{2\pi}K^{\Lambda^2}(\tau'',\tau',\phi'',\phi';T)e^{-il(\phi''-\phi')}d\phi''$$
$$=\frac{1}{2\pi}e^{-\frac{iT}{8m}}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N\left[\prod_{j=1}^{N-1}\int_0^\infty\sinh\tau^{(j)}d\tau^{(j)}\int_0^{2\pi}d\phi^{(j)}\right]\int_0^{2\pi}d\phi^{(N)}$$
$$\times\exp\left\{i\sum_{j=1}^N\left[\frac{m}{2\epsilon}(\Delta^2\tau^{(j)}+\sinh\tau^{(j)}\sinh\tau^{(j-1)}\Delta^2\phi^{(j)})-l\Delta\phi^{(j)}+\frac{\epsilon}{8m\sinh^2\tau^{(j)}}\right]\right\}$$
$$=\frac{1}{2\pi}(\sinh\tau'\sinh\tau'')^{-\frac12}e^{-\frac{iT}{8m}}$$
$$\times\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac N2}\prod_{j=1}^{N-1}\int_0^\infty d\tau^{(j)}\exp\left\{i\sum_{j=1}^N\left[\frac{m}{2\epsilon}\Delta^2\tau^{(j)}-\frac{\epsilon(l^2-\frac14)}{2m\sinh^2\tau^{(j)}}\right]\right\}. \qquad (19)$$

A direct comparision between equations (15), (18) and (19) yields the **Feynman kernel on the Poincaré disc** D:

$$K^D(r'',r',\psi'',\psi';T)=\frac{1}{2\pi^2}\int_0^\infty dp\sum_{l=-\infty}^\infty p\sinh\pi p\exp\left[-\frac{iT}{2m}\left(p^2+\frac14\right)\right]$$
$$\times|\Gamma(\tfrac12+ip+l)|^2\mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r'^2}{1-r'^2}\right)\mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r''^2}{1-r''^2}\right)e^{il(\psi''-\psi')} \qquad (20)$$

where we have inserted $z=\ln\frac{1+r}{1-r}$ to get the solution in terms of the variables of the disc D. Therefore the wave-functions and the energy-spectrum on the Poincaré disc D are given by:

$$\left.\begin{aligned}\Psi^D_{p,l}(r,\psi)&=\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma(\tfrac12+ip+l)\mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r^2}{1-r^2}\right)e^{il\psi}\\&=\sqrt{\frac{p\tanh\pi p}{2\pi}}\,\Phi_{p,l}(r)e^{il\psi}\\E_p&=\frac{1}{2m}\left(p^2+\frac14\right),\end{aligned}\right\} \qquad (21)$$

($p > 0, l \in \mathbf{Z}$) satisfying the orthonormality relation:

$$\int_0^1 dr \int_0^{2\pi} d\psi \frac{4r}{(1-r^2)^2} \Psi^D_{p',l'}(r,\psi)\Psi^{D*}_{p,l}(r,\psi) = \delta_{l,l'}\delta(p-p') \tag{22}$$

and forming a complete set:

$$\sum_{l=-\infty}^{\infty} \int_0^\infty dp \Psi^D_{p,l}(r',\psi')\Psi^{D*}_{p,l}(r,\psi) = \left[\frac{(1-r'^2)(1-r^2)}{4r'r}\right]^{\frac{1}{2}} \delta(r'-r)\delta(\psi'-\psi). \tag{23}$$

Note the metric expression $[g(q')g(q'')]^{-\frac{1}{4}}$ in the completeness relation (for a discussion see e.g. [28]). With the help of equation (A.8) we see that $\lim_{r\to 1} \Psi^D_{p,k}(r,\psi) = const. \cdot \lim_{r\to 1} \sqrt{1-r} = 0$ ($\phi \in [0,2\pi]$) and the boundary conditions are fulfilled. The $\Phi_{p,l}$ are given by (following Helgason [20]):

$$\Phi_{p,l}(r) = (1-r^2)^{\frac{1}{2}+ip} r^{|l|} \frac{\Gamma(|l|+\frac{1}{2}+ip)}{|l|!\Gamma(\frac{1}{2}+ip)} {}_2F_1(\tfrac{1}{2}+ip, |l|+\tfrac{1}{2}+ip; |l|+1; r^2), \tag{24}$$

and where we have used the representation ([11], p.1010, $|\frac{z-1}{z+1}| < 1$):

$$\mathcal{P}^\mu_\nu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^\nu {}_2F_1\left(-\nu, -\nu-\mu; 1-\mu; \frac{z-1}{z+1}\right). \tag{25}$$

A state with $p = 0$ does not exist because $\Psi_{0,k} \equiv 0$. It is also interesting that the "ground state" energy is different from zero: $E_0 = \frac{1}{8m}$. This is a pure quantum phenomenon - see also section III. The result coincides, of course, with our previous work [17]. With the transformations (I.6) and (11) it is obvious that the representations (15) and (19) can be transformed into each other, i.e. they are equivalent and thus we have $K^{\Lambda^2}(T) = K^D(T)$.

For $r \to 1$ we find with the help of Eq.(A9)

$$\Psi^D_{p,l}(r,\psi) \simeq (2\pi)^{-1} e^{il\psi} e^{ipd} / \sqrt{\cosh d} \tag{26}$$

where d denotes the hyperbolic distance of an arbitrary $z \in D$ from the origin (see Eq.(I.9)). Thus we get in the limit $r \to 1$ "cicular waves" in D.

The prove of the orthonormality- and completeness relations (22) and (23), respectively, can be reduced to the orthonormality and completeness of the wave-functions $\Psi^{\Lambda^2}_{p,l}(r,\tau) = \sqrt{p \sinh \pi p / 2\pi^2}\, \Gamma(\frac{1}{2}+ip+l)\mathcal{P}^{-l}_{ip-\frac{1}{2}}(\cosh\tau)e^{il\phi}$ on the pseudo-sphere Λ^2. The orthonormality and completeness of these functions is well-known from Mehler-transformation theory (see [4,17,24]).

III. The Poincaré Upper Half-Plane

In recent publications [14,16] we have already calculated the path integral on the Poincaré upper half-plane U. There is also some work of Kubo [22], but his calculation seems questionable to us. Here we want to present two alternative calculations as in [16], adopting the idea of Kubo, but respecting all the time the crucial interrelation between the lattice definition of a path integral and the appropriate quantum correction ΔV on the one hand, and additional terms of $O(\epsilon)$ due to a Taylor expansion of the kinetic energy term for a non-linear transformation on the other.

In U the metric is given by $g_{ab} = \delta_{ab}/y^2$ ($x \in \mathbf{R}, y > 0$). The classical Lagrangian and the Hamiltonian read, respectively:

$$\mathcal{L}_{Cl}(x,\dot{x},y,\dot{y}) = \frac{m}{2y^2}(\dot{x}^2+\dot{y}^2), \qquad \mathcal{H}_{Cl}(x,p_x,y,p_y) = \frac{y^2}{2m}(p_x^2+p_y^2), \tag{1}$$

and the quantum Hamiltonian is given by

$$H = \frac{y^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{2}$$

The scalar product for functions $f_1, f_2 \in L^2(U)$ reads,

$$(f_1,f_2)_U = \int_{-\infty}^\infty dx \int_0^\infty \frac{dy}{y^2} f_1(x,y) f_2^*(x,y). \tag{3}$$

States $\Psi \in D(H)\cap L^2(U)$ must satify the boundary condition $\lim_{y\to 0}\Psi(x,y) = 0$ ($x \in \mathbf{R}$) as well as $\lim_{y\to\infty}\Psi(x,y) = 0$ ($x \in \mathbf{R}$). Following our prescription given in the introduction, we get for the hermitian momenta:

$$\left.\begin{aligned} \Gamma_x &= 0, & p_x &= \frac{1}{i}\frac{\partial}{\partial x} \\ \Gamma_y &= -\frac{2}{y}, & p_y &= \frac{1}{i}\left(\frac{\partial}{\partial y} - \frac{2}{y}\right). \end{aligned}\right\} \tag{4}$$

The Hamiltonian rewritten in the product ordering yields:

$$H = \frac{1}{2m} y(p_x^2 + p_y^2)y, \tag{5}$$

where we have applied the corollary in the introduction which yields for U: $\Delta V = 0$. Thus we can infer that the path integral on the Poincaré upper half-plane in the "product form"-definition reads

$$\begin{aligned} &K^U(x'',x',y'',y';T) \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_{-\infty}^\infty dx^{(j)} \int_0^\infty \frac{dy^{(j)}}{y^{(j)2}} \cdot \exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N \frac{\Delta^2 x^{(j)} + \Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right]. \end{aligned} \tag{6}$$

We want to emphasize again that in the "product form"-definition for the path integral on U, there is no additional quantum potential or curvature term. This fact has been overlooked in [22]. Via the time evolution equation (I.17) it is an easy computation to show that equation (6) is indeed the right path integral on U (see [16] for details).

We shall now present two alternative ways to calculate the path integral (6).

1) In the first approach in calculating the path integral (6) - following the idea of Kubo [22] - we start by integrating $K(T)$ over $x'' = x^{(N)}$. This gives:

$$\begin{aligned} \int_{-\infty}^\infty K(x'',x',y'',y';T)dx'' &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_0^\infty \frac{dy^{(j)}}{y^{(j)2}} \\ &\quad \times \exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N \frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right] \cdot \prod_{j=1}^N \int_{-\infty}^\infty dx^{(j)} \exp\left(-\frac{m}{2i\epsilon}\frac{\Delta^2 x^{(j)}}{y^{(j)}y^{(j-1)}}\right) \\ &= \sqrt{y'y''}\lim_{n\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1}\int_0^\infty \frac{dy^{(j)}}{y^{(j)}} \exp\left[\frac{im}{2\epsilon}\sum_{j=1}^N \frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right] \equiv \bar{K}(y'',y';T). \end{aligned} \tag{7}$$

We now look for a transformation $z = z(y)$ such that

$$\dot{z}^2 = \frac{\dot{y}^2}{y^2}. \tag{8}$$

A simple calculation similar as in section II yields $z = \ln y$, which is essentially the hyperbolic distance on the y-axis from $\zeta = i$ [see Eq.(I.9)], respectively the inverse transformation $y = e^z$. $z(y)$ has the property $(0,\infty) \mapsto \mathbf{R}$. We have $dy/y = dz$, and the kinetic term in the exponential in the path integral (7) gives in a Taylor expansion:

$$\frac{[y^{(j)} - y^{(j-1)}]^2}{y^{(j)}y^{(j-1)}} = e^{\Delta z^{(j)} - 2z^{(j)}}[e^{z^{(j)}} - e^{z^{(j)} - \Delta z^{(j)}}]^2 \simeq \Delta^2 z^{(j)} + \frac{\Delta^4 z^{(j)}}{12}. \tag{9}$$

Exploiting the path intergral identity $\Delta^4 z \doteq 3\left(\frac{im}{\epsilon}\right)^2$ we get:[1]

$$\exp\left(\frac{im}{2\epsilon}\frac{[y^{(j)} - y^{(j-1)}]^2}{y^{(j)}y^{(j-1)}}\right) \doteq \exp\left(\frac{im}{2\epsilon}\Delta^2 z^{(j)} - \frac{i\epsilon}{8m}\right). \tag{10}$$

Therefore the path integral (7) gives essentially a path integral of a free particle in $\mathbf{R}$ in the z-coordinate. We get with $\hat{K}(z'', z'; T) \equiv \tilde{K}(y'', y'; T)$:

$$\begin{aligned}\hat{K}(z'', z'; T) &= \exp\left[\frac{z' + z''}{2} - \frac{iT}{8m}\right] \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} \int_{-\infty}^{\infty} dz^{(j)} \exp\left(\frac{im}{2\epsilon}\sum_{j=1}^{N}\Delta^2 z^{(j)}\right)\\
&= \exp\left[\frac{z' + z''}{2} - \frac{iT}{8m}\right]\left(\frac{m}{2\pi iT}\right)^{\frac{1}{2}} \exp\left[\frac{im}{2T}(z'' - z')^2\right]\\
&= e^{-iT/8m}\sqrt{y'y''}\left(\frac{m}{2\pi iT}\right)^{\frac{1}{2}} \exp\left[-\frac{m}{2iT}\ln^2\left(\frac{y''}{y'}\right)\right] = \tilde{K}(y'', y'; T).\end{aligned} \tag{11}$$

We note that $l = \ln(y''/y')$ is nothing but the hyperbolic distance r in U if $(x'' - x') = 0$ where r is given by (I.9c). Introducing now

$$k := \frac{\xi^2 + y'^2 + y''^2}{2y'y''} \tag{12}$$

we obtain

$$\tilde{K}(y'', y'; T) = \int_l^{\infty} K(k, y'', y'; T)\frac{dk}{\sqrt{k - l}} = \left(\frac{m}{2\pi iT}\right)^{\frac{1}{2}} \exp\left(-\frac{m}{2iT}\operatorname{arcosh}^2 l - \frac{iT}{8m}\right). \tag{13}$$

This integral equation can be solved exactly and the result reads (see [22]):

$$K^U(r; T) = \sqrt{2}\left(\frac{m}{2\pi iT}\right)^{\frac{3}{2}} \int_r^{\infty} \frac{u\,du}{\sqrt{\cosh u - \cosh r}} \exp\left[-\frac{m}{2iT}u^2 - \frac{iT}{8m}\right] \tag{14}$$

This is the well-known result (see e.g. [18,25]). Note that $K(T)$ is only a function of the hyperbolic distance r.

[1] In Ref. [22] this fourth order term was overlooked, but compensated by an additional curvature term in Eq.(6), which is, however, wrong. Therefore the correct result is obtained by chance.

We introduce the Green's function $G(E) = \int_0^\infty e^{iTE} K(T)dT$.[1] This gives for (14):

$$\begin{aligned}G^U(r; E) &= \sqrt{2}\left(\frac{m}{2\pi i}\right)^{\frac{3}{2}} \int_r^{\infty} \frac{u\,du}{\sqrt{\cosh u - \cosh r}} du\\
&\qquad \times \int_0^{\infty} T^{-\frac{3}{2}} \exp\left[-\frac{m}{2iT}u^2 - \left(\frac{i}{8m} - iE\right)T\right] dT\\
&= \frac{m}{\pi\sqrt{2}} \int_r^{\infty} \frac{\exp\left(-iu\sqrt{2mE - \frac{1}{4}}\right)}{\sqrt{\cosh u - \cosh r}} du = \frac{m}{\pi} \mathcal{Q}_{-\frac{1}{2} - i\sqrt{2mE - \frac{1}{4}}}(\cosh r),\end{aligned} \tag{15}$$

which is the correct result (see [14,16,18]) and gives a closed expression for $G^U(E)$. In the T-integration use has been made of the integral ([11], p.340):

$$\int_0^{\infty} x^{\nu - 1} e^{-\beta/x - \gamma x} dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}) \tag{16}$$

(K_ν - modified Bessel function) and the expression $K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$. In the last step we used a well-known integral representation for the Legendre function of the second kind $\mathcal{Q}_\nu^\mu$ ([11], p.1002). Using the integrals (see [24], pp.819,732)

$$\left.\begin{aligned}\mathcal{Q}_{\nu - \frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2ab}\right) &= \int_0^{\infty} dp' \frac{p' \tanh \pi p'}{\nu^2 + p'^2} \mathcal{P}_{ip' - \frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2ab}\right)\\
\mathcal{P}_{\nu - \frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2ab}\right) &= \frac{4\sqrt{ab}}{\pi^2} \cos \nu\pi \int_0^{\infty} dk\, K_\nu(ak) K_\nu(bk) \cos ck,\end{aligned}\right\} \tag{17}$$

equation (15) can be rewritten as

$$\begin{aligned}&G^U(x'', y'', x', y'; E)\\
&= \frac{1}{\pi^3}\int_{-\infty}^{\infty} dk \int_0^{\infty} dp \frac{p \sinh \pi p}{\frac{1}{2m}(p^2 + \frac{1}{4}) - E} \sqrt{y'y''} K_{ip}(|k|y') K_{ip}(|k|y'') e^{ik(x'' - x')}.\end{aligned} \tag{18}$$

The representation (21) shows clearly that $G^U(E)$ has a cut on the positive real axis in the complex energy plane with a branch point at $E = 1/8m$. The energy-spectrum and the normalized wave-functions are therefore given by:

$$\left.\begin{aligned}\Psi_{p,k}^U(x, y) &= \sqrt{\frac{p \sinh \pi p}{\pi^3}} \sqrt{y} K_{ip}(|k|y) e^{ikx}\\
E_p &= \frac{1}{2m}\left(p^2 + \frac{1}{4}\right)\end{aligned}\right\} \tag{19}$$

($p > 0$, $k \in \mathbf{R} \setminus \{0\}$). They satisfy the orthonormality relation

$$\int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{dy}{y^2} \Psi_{p,k}^U(x, y) \Psi_{p',k'}^{U*}(x, y) = \delta(p' - p)\delta(k' - k) \tag{20}$$

[1] In order to work with well-defined mathematical formulas we shall assume that E has a small positive imaginary part $i\epsilon$, and write $E + i\epsilon$ (with real E) instead of E whenever necessary. Also, square roots will be positive. See e.g. [14,18] for details.

and form a complete set

$$\int_{-\infty}^{\infty} dk \int_0^{\infty} dp \Psi^U_{p,k}(x',y') \Psi^{U*}_{p,k}(x,y) = y'y''\delta(x'-x)\delta(y'-y). \tag{21}$$

Equations (20) and (21) have been proved in [16]. A state with $p=0$ does not exist because $\Psi^U_{0,k} \equiv 0$ The energy-spectrum coincides with the energy-spectrum for the Poincaré disc as it should be. With the help of the asymptotic expansion for the modified K_ν - Bessel functions: $K_\nu(z) \simeq \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}$ $(z \to 0)$ ([1], p.119), we see that $\lim_{y\to 0} |\Psi^U_{p,k}| = \lim_{y\to 0} \sqrt{y}/2\pi = 0$ $(x \in \mathbf{R})$. Similar $(x \in \mathbf{R})$: $\lim_{y\to\infty} |\Psi^U_{p,k}(x,y)| = const.\lim_{y\to\infty} e^{-|k|y} = 0$ $(K_\nu(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x}$ - see [1], p.122), and both boundary conditions are fulfilled. In the x-variable we have everywhere plane waves. The functions $\Psi^U_{p,k}$ allow an interpretation as "plane waves" in the limit $y \to 0$ where we have $\Psi^U_{p,k} \propto \sqrt{y}\, e^{\pm ipd}$. Here d denotes the hyperbolic distance of $(0,y)$ from $(0,1)$. These "plane waves" have, of course, a distortion proportional to $\sqrt{y}$. The "plane wave" interpretation is much easier for the functions $\Psi_p(y) := y^{\frac{1}{2}\pm ip}$ which are solutions of the eigenvalue problem $H\Psi = E\Psi$ with the Hamiltonian (2) with eigenvalue E_p. However, these functions are not normalizable in U; they are only normalizable in bounded domains.

Let us discuss in short the "zero-momentum" energy shift $E_0 = \frac{1}{8m}$ [32]. We consider the classical Hamiltonian (1) and insert (introducing $\hbar$) the Heisenberg uncertainty relations $xp_x \geq \hbar/2$ and $yp_y \geq \hbar/2$. This gives for the energy of quantum motion on U the lower bound:

$$E_U \geq \frac{\hbar^2}{8m}\left(1 + \frac{y^2}{x^2}\right) > \frac{\hbar^2}{8m}. \tag{22}$$

The value $E_0 = \inf_U E_U = \frac{\hbar^2}{8m}$ can never be taken on because $\{z | y = 0\} \notin U$. E_0 is the largest lower bound on U.

2) In our second approach in calculating the path integral (5) we start by performing a Fourier expansion of $K^U(T)$:

$$\left.\begin{aligned} K^U(x'',x',y'',y';T) &= \int_{-\infty}^{\infty} K^U_k(y'',y';T)\, e^{ik(x''-x')} dk \\ K^U_k(y'',y';T) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K^U(x'',x',y'',y';T) e^{-ik(x''-x')} dx''. \end{aligned}\right\} \tag{23}$$

This gives if equation (6) inserted in (23) for $K^U_k(T)$:

$$\begin{aligned} K^U_k(y'',y';T) &= \frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_0^{\infty}\frac{dy^{(j)}}{y^{(j)2}} \exp\left[\frac{im}{2\epsilon}\sum_{j=1}^{N}\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}}\right] \\ &\qquad \times \prod_{j=1}^{N}\int_{-\infty}^{\infty} dx^{(j)} \exp\left[-\frac{m}{2i\epsilon}\frac{\Delta^2 x^{(j)}}{y^{(j)}y^{(j-1)}} - ik\Delta x^{(j)}\right] \\ &= \frac{\sqrt{y'y''}}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1}\int_0^{\infty}\frac{dy^{(j)}}{y^{(j)}} \exp\left\{i\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\frac{\Delta^2 y^{(j)}}{y^{(j)}y^{(j-1)}} - \epsilon\frac{k^2 y^{(j)}y^{(j-1)}}{2m}\right]\right\} \\ &\equiv \frac{\sqrt{y'y''}}{2\pi}\int\frac{Dy(t)}{y}\exp\left[i\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot{y}^2}{y^2} - \frac{k^2y^2}{2m}\right)dt\right]. \end{aligned} \tag{24}$$

Performing the transformation $z = \ln y$, $y = e^z$ in (24) and repeating the same procedure as in equation (9) we get with $\hat{K}_k(z'',z';T) \equiv K^U_k(y'',y';T)$:

$$\hat{K}_k(z'',z';T) = \frac{1}{2\pi}e^{(z'+z'')/2 - iT/8m}\int Dz(t)\exp\left[i\int_{t'}^{t''}\left(\dot{z}^2 - \frac{k^2}{2m}e^{2z}\right)dt\right]. \tag{25}$$

This path integral is nothing but a path integral for the potential of Liouville quantum mechanics with the potential $V(z) = \frac{k^2}{2m}e^{2z}$. This path integral was calculated in [16] and the result for $\hat{K}_k$ therefore reads,

$$\begin{aligned} &\hat{K}_k(z'',z';T) \\ &= \frac{1}{\pi^3}e^{(z''+z')/2}\int_0^{\infty} dp\, p\sinh\pi p\, \exp\left[-\frac{iT}{2m}\left(p^2 + \frac{1}{4}\right)\right] K_{ip}(|k|e^{z'})K_{ip}(|k|e^{z''}). \end{aligned} \tag{26}$$

Inserting $y = e^z$ we get finally for the Feynman kernel on U:

$$\begin{aligned} K^U(x'',x',y'',y';T) &= \frac{1}{\pi^3}\int_{-\infty}^{\infty} dk \int_0^{\infty} dp\, p\sinh\pi p \\ &\quad \times \exp\left[-\frac{iT}{2m}\left(p^2 + \frac{1}{4}\right)\right]\sqrt{y'y''}K_{ip}(|k|y')K_{ip}(|k|y'')e^{ik(x''-x')} \end{aligned} \tag{27}$$

with the correct energy-spectrum and wave-functions as in (18) - see [16,34,35]. Equation (25) is, of course, the Fourier transformed of equation (18).

The spectral representations (II.15) and (28) can be transformed into each other. In order to achieve this we use the integral ([11], p.732):

$$\int_0^{\infty} K_\nu(ax)K_\nu(bx)\cos cx\, dx = \frac{\pi^2}{4\sqrt{ab}\cos\nu\pi}\mathcal{P}_{-\frac{1}{2}+\nu}\left(\frac{a^2+b^2+c^2}{2ab}\right), \tag{28}$$

and the addition theorem for the associated Legendre functions: ([11], p.1014)

$$\mathcal{P}_\nu(zz' - \sqrt{z^2-1}\sqrt{z'^2-1}\cos\phi) = \sum_{l=-\infty}^{\infty}(-1)^l e^{il\phi}\frac{\Gamma(\nu-l+1)}{\Gamma(\nu+l+1)}\mathcal{P}^l_\nu(z)\mathcal{P}^l_\nu(z'). \tag{29}$$

This enables us to derive the identity:

$$\begin{aligned} &\frac{1}{\pi^3}\int_{-\infty}^{\infty} dk \int_0^{\infty} dp\, p\sinh\pi p\, e^{-\frac{iT}{2m}(p^2+\frac{1}{4})}\sqrt{y'y''}K_{ip}(|k|y')\,K_{ip}(|k|y'')\,e^{ik(x''-x')} \\ &= \frac{1}{2\pi^2}\sum_{l=-\infty}^{\infty}\int_0^{\infty} dp\, p\sinh\pi p\, e^{-\frac{iT}{2m}(p^2+\frac{1}{4})}|\Gamma(\tfrac{1}{2}+ip-l)|^2 \\ &\qquad \times e^{il(\phi''-\phi')}\mathcal{P}^l_{-\frac{1}{2}+ip}(\cosh\tau')\mathcal{P}^l_{-\frac{1}{2}+ip}(\cosh\tau''), \end{aligned} \tag{30}$$

where use has been made of equation (I.9). Thus equation (30) shows the equivalence between the Feynman kernels on U and Λ^2, i.e. we have $K^U(T) = K^{\Lambda^2}(T)$.

IV. The Hyperbolic Strip

In order to formulate the path integral on the hyperbolic strip we start with the classical Lagrangian and Hamiltonian. They are given by (the metric reads $g_{ab} = \delta_{ab}/\cos^2 Y$):

$$\mathcal{L}_{Cl}(X,\dot{X},Y,\dot{Y}) = \frac{m}{2}\frac{\dot{X}^2+\dot{Y}^2}{\cos^2 Y}, \qquad \mathcal{H}_{Cl}(X,p_X,Y,p_Y) = \frac{\cos^2 Y}{2m}(p_X^2+p_Y^2). \tag{1}$$

The quantum Hamiltonian reads:

$$H = -\frac{\cos^2 Y}{2m}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) \tag{2}$$

which can be constructed in the usual way from the Laplace-Beltrami operator on S. The scalar-product for functions $f_1, f_2 \in L^2(S)$ is given by

$$(f_1, f_2)_S = \int_{-\infty}^{\infty} dX \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dY\, \frac{f_1(X,Y) f_2^*(X,Y)}{\cos^2 Y}. \tag{3}$$

States $\Psi \in D(H) \cap L^2(S)$ must satisfy the boundary condition $\lim_{Y\to\pm\frac{\pi}{2}} \Psi(X,Y) = 0$ $(X \in \mathbf{R})$. We have $h_{ab} = \delta_{ab}/\cos Y$, and the momentum operators $p_a = -i(\partial_a + \Gamma_a/2)$ read therefore,

$$\left.\begin{aligned} \Gamma_X &= 0 & p_X &= \frac{1}{i}\frac{\partial}{\partial X} \\ \Gamma_Y &= 2\tan Y & p_Y &= \frac{1}{i}\left(\frac{\partial}{\partial Y} + \tan Y\right). \end{aligned}\right\} \tag{4}$$

Again we can apply the corollary of the introduction and deduce that the quantum correction ΔV is given by: $\Delta V = 0$. Thus the Hamiltonian H expressed in terms of the momentum operators (4) in the product ordering reads,

$$H = \frac{1}{2m}\cos Y (p_X^2 + p_Y^2)\cos Y. \tag{5}$$

With the "product form"-definition we get for the path integral on S:

$$K^S(X'', X', Y'', Y'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_{-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dX^{(j)}dY^{(j)}}{\cos^2 Y^{(j)}} \cdot \exp\left(\frac{im}{2\epsilon}\sum_{j=1}^{N}\frac{\Delta^2 X^{(j)} + \Delta^2 Y^{(j)}}{\cos Y^{(j)}\cos Y^{(j-1)}}\right). \tag{6}$$

In order to calculate the path integral (6) we perform a Fourier expansion:

$$\left.\begin{aligned} K^S(X'', X', Y'', Y'; T) &= \int_{-\infty}^{\infty} K_k^S(Y'', Y'; T) e^{ik(X''-X')} dk \\ K_k^S(Y'', Y'; T) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} K^S(X'', X', Y'', Y'; T) e^{-ik(X''-X')} dX''. \end{aligned}\right\} \tag{7}$$

The path integral (6) inserted into (7b) gives for $K_k^S(T)$:

$$\begin{aligned} K_k^S(Y'', Y'; T) &= \frac{1}{2\pi}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dY^{(j)}}{\cos^2 Y^{(j)}} \\ &\quad\times \prod_{j=1}^{N}\int_{-\infty}^{\infty} dX^{(j)} \exp\left(-\frac{m}{2i\epsilon}\frac{\Delta^2 X^{(j)} + \Delta^2 Y^{(j)}}{\cos Y^{(j)}\cos Y^{(j-1)}} - ik\Delta X^{(j)}\right) \\ &= \frac{1}{2\pi}\sqrt{\cos Y' \cos Y''}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dY^{(j)}}{\cos Y^{(j)}} \\ &\quad\times \exp\left[i\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}\frac{\Delta^2 Y^{(j)}}{\cos Y^{(j)}\cos Y^{(j-1)}} - \frac{\epsilon k^2}{2m}\cos Y^{(j)}\cos Y^{(j-1)}\right)\right]. \end{aligned} \tag{8}$$

We now look for a transformation $z = z(Y)$ such that $\dot{z}^2 = \dot{Y}^2/\cos^2 Y$. $z(Y)$ is essentially the hyperbolic distance on the Y-axis from the origin [see Eq.(I.9)]. With the ansatz $z(Y) = \ln f(Y)$ we find a differential equation for f: $f'(Y) - f(Y)/\cos Y = 0$. The solution reads,

$$f(Y) = \exp\left(\int^Y \frac{du}{\cos u}\right) = \tan\left(\frac{Y}{2} - \frac{\pi}{4}\right) = \sqrt{\frac{1+\sin Y}{1-\sin Y}}. \tag{9}$$

Therefore we get for $z(Y)$:

$$z(Y) = \frac{1}{2}\ln\left(\frac{1+\sin Y}{1-\sin Y}\right). \tag{10}$$

$z(Y)$ is essentially the hyperbolic distance on the Y-axis from the origin. The transformation $z = z(Y)$ maps $(-\frac{\pi}{2}, \frac{\pi}{2}) \mapsto \mathbf{R}$. The inverse transformation $Y = Y(z)$ reads $Y(z) =$arcsin[tanh z] and maps $\mathbf{R} \mapsto (-\frac{\pi}{2}, \frac{\pi}{2})$. For the various terms in the path integral (8) we have:

1) The potential term: $k^2\cos^2 Y^{(j)}/2m = k^2/(2m\cosh^2 z^{(j)})$.
2) The measure: $dY^{(j)}/\cos Y^{(j)} = dz^{(j)}$.
3) In the kinetic term in the exponential we have to perform a Taylor expansion up to fourth order in $\Delta z^{(j)}$. We get

$$\frac{\Delta^2 Y^{(j)}}{\cos Y^{(j)}\cos Y^{(j)}} \simeq \Delta^2 z^{(j)} + \frac{\Delta^4 z^{(j)}}{12}\left(1 + \frac{1}{\cosh^2 z^{(j)}}\right). \tag{11}$$

4) Again we need the identity $\Delta^4 z^{(j)} \doteq 3(i\epsilon/m)^2$ to get for the exponential in the path integral (8):

$$\begin{aligned} &\exp\left[\frac{im}{2\epsilon}\frac{\Delta^2 Y^{(j)}}{\cos Y^{(j)}\cos Y^{(j)}} - \frac{i\epsilon k^2}{2m}\cos^2 Y^{(j)}\right] \\ &\qquad \doteq \exp\left[\frac{im}{2\epsilon}\Delta^2 z^{(j)} - \frac{i\epsilon(k^2+\frac{1}{4})}{2m\cosh^2 z^{(j)}} - \frac{i\epsilon}{8m}\right]. \end{aligned} \tag{12}$$

We define $\hat{K}(T)$ by $K_k^S(Y'', Y'; T) =: \frac{1}{2\pi}\sqrt{\cos Y' \cos Y''}e^{-\frac{iT}{8m}}\hat{K}(z'', z'; T)$. Therefore we have to study the path integral:

$$\hat{K}(z'', z'; T) = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1}\int_{-\infty}^{\infty} dz^{(j)} \exp\left\{i\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\Delta^2 z^{(j)} - i\epsilon\frac{k^2+\frac{1}{4}}{2m\cosh^2 z^{(j)}}\right]\right\}. \tag{13}$$

This Feynman kernel describes scattering under the influence of the potential

$$V(z) = \frac{k^2+\frac{1}{4}}{2m\cosh^2 z} \qquad (z \in \mathbf{R}). \tag{14}$$

The potential (14) is a special case of the general modified Pöschl-Teller potential which is defined by (c, s - constants):

$$V^{PT}(r) = \frac{c/m}{\cosh^2 r} + \frac{s/m}{\sinh^2 r} \qquad (r > 0). \tag{15}$$

A general discussion for this potential problem for the bound and scattering states is due to Frank and Wolf [9]. A treatment by the factoriztion method was given by Infeld and Hull [21]. A simple discussion for $s = 0$ and bound states can be found in 'ter Haar [33]. A path integral treatment is due to Böhm and Junker [5,6]. Adopting the notation of Frank and Wolf, the path integral solution for the potential problem (15) for the continuous states is given by:

$$K^{PT}(r'', r'; T) = \int_0^\infty dp\, \Psi_\kappa^{(k_1,k_2)*}(r')\Psi_\kappa^{(k_1,k_2)}(r'')e^{-iE_\kappa T} \tag{16}$$

where the normalized wave-functions read:

$$\begin{aligned}\Psi_\kappa^{(k_1,k_2)}(r) = N_\kappa^{(k_1,k_2)}&(\cosh r)^{2k_1-\frac{3}{2}}(\sinh r)^{2k_2-\frac{1}{2}} \\ &\times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r)\end{aligned} \tag{17}$$

with energy $E_\kappa = -(1/2m)(2\kappa-1)^2$ where $\kappa = \frac{1}{2}(1+ip)$ ($p > 0$), and the normalisation $N_\kappa^{(k_1,k_2)}$ is given by:

$$\begin{aligned}N_\kappa^{(k_1,k_2)} &= \frac{1}{\pi\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2}} \\ &\times \Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{\frac{1}{2}}.\end{aligned} \tag{18}$$

The numbers k_1, k_2 are defined by c and s and are explicitly given by:

$$k_1 = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{4} - 2c}\right), \qquad k_2 = \frac{1}{2}\left(1 \pm \sqrt{\frac{1}{4} + 2s}\right). \tag{19}$$

From equation (15) we read off $s = 0$ and $c = (k^2 + \frac{1}{4})/2$, therefore $k_1 = \frac{1}{2}(1+ik)$ ($k \in \mathbf{R}$) and $k_2 = \frac{1}{4}, \frac{3}{4}$. This means that we get two wave-functions corresponding to $k_2 = \frac{1}{4}$ and $k_2' = \frac{3}{4}$ which have even and odd parity, respectively. They are given by (define $\Psi_{p,k}^{(1)} \equiv \Psi_\kappa^{(k_1,\frac{1}{4})}$, $\Psi_{p,k}^{(2)} \equiv \Psi_\kappa^{(k_1,\frac{3}{4})}$):

$$\left.\begin{aligned}\Psi_{p,k}^{(1)}(r) &= \sqrt{\frac{p\sinh\pi p}{2\pi^3}}|\Gamma[\tfrac{1}{4}+\tfrac{i}{2}(k+p)]\Gamma[\tfrac{1}{4}+\tfrac{i}{2}(k-p)]| \\ &\quad\times(\cosh r)^{\frac{1}{2}+ik}\, {}_2F_1[\tfrac{1}{4}+\tfrac{i}{2}(k-p), \tfrac{1}{4}+\tfrac{i}{2}(k+p); \tfrac{1}{2}; -\sinh^2 r] \\ \Psi_{p,k}^{(2)}(r) &= \sqrt{\frac{p\sinh\pi p}{\pi^3}}|\Gamma[\tfrac{3}{4}+\tfrac{i}{2}(k+p)]\Gamma[\tfrac{3}{4}+\tfrac{i}{2}(k-p)]| \\ &\quad\times(\cosh r)^{\frac{1}{2}+ik}\sinh r\, {}_2F_1[\tfrac{3}{4}+\tfrac{i}{2}(k-p), \tfrac{3}{4}+\tfrac{i}{2}(k+p); \tfrac{3}{2}; -\sinh^2 r].\end{aligned}\right\} \tag{20}$$

We emphasize that $\Psi_{p,k}^{(1,2)}$ are defined in the halfspace $\mathbf{R}^+$. In order to get the normalized wave-function in $\mathbf{R}$ we have to construct a linear combination of $\Psi_{p,k}^{(1)}$ and

$\Psi_{p,k}^{(2)}$ including an additional normalisation factor $2^{-\frac{1}{2}}$ because $\int_0^\infty dr|\Psi_{p,k}^{(1,2)}(r)|^2 = \frac{1}{2}\int_{-\infty}^\infty dr|\Psi_{p,k}^{(1,2)}(r)|^2$. This is possible because V^{PT} can be extended for $s = 0$ to the entire $\mathbf{R}$. We make the ansatz ($z \in \mathbf{R}$):

$$\Psi_{p,k}(z) = \sigma_1 2^{-\frac{1}{2}}\Psi_{p,k}^{(1)}(z) + \sigma_2 2^{-\frac{1}{2}}\Psi_{p,k}^{(2)}(z) \tag{21}$$

with the normalisation condition $|\sigma_1|^2 + |\sigma_2|^2 = 1$. We have two possibilities in choosing $\sigma_{1,2}$:

1) We set

$$\left.\begin{aligned}\sigma_1 &= \sqrt{\frac{2\pi^3}{\cosh^2\pi k + \sinh^2\pi p}}\frac{2^{ip}\Gamma[\frac{1}{4}+\frac{i}{2}(k+p)]\cos[\frac{i}{2}\pi(k+p)-\frac{\pi}{4}]}{\sqrt{\pi}\Gamma[\frac{3}{4}+\frac{i}{2}(k-p)]|\Gamma[\frac{1}{4}+\frac{i}{2}(k-p)]\Gamma[\frac{1}{4}+\frac{i}{2}(k+p)]|} \\ \sigma_2 &= \sqrt{\frac{2\pi^3}{\cosh^2\pi k + \sinh^2\pi p}}\frac{2^{ip}\Gamma[\frac{3}{4}+\frac{i}{2}(k+p)]\sin[\frac{i}{2}\pi(k+p)-\frac{\pi}{4}]}{\sqrt{\pi}\Gamma[\frac{1}{4}+\frac{i}{2}(k-p)]|\Gamma[\frac{3}{4}+\frac{i}{2}(k-p)]\Gamma[\frac{3}{4}+\frac{i}{2}(k+p)]|}.\end{aligned}\right\} \tag{22}$$

This gives for the normalized wave-function with equation (A.6) and the property ${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1[a, c-b; c; z/(z-1)]$ of the hypergeometric function:

$$\Psi_{p,k}(z) = \sqrt{\frac{p\sinh\pi p}{2(\cosh^2\pi k + \sinh^2\pi p)}}\, P_{ik-\frac{1}{2}}^{ip}(\tanh z). \tag{23}$$

Note the symmetries $\Psi_{p,-k} = \Psi_{p,k}$ and $\Psi_{-p,k} = \Psi_{p,k}^*$. Equation (26) and the appendix B will show that the $\Psi_{p,k}$ have indeed the correct normalisation.

2) The second possibilty gives for $\sigma_{1,2}$:

$$\left.\begin{aligned}\sigma_1 &= -\sqrt{\frac{8\pi}{\cosh^2\pi k + \sinh^2\pi p}}\frac{\sqrt{\pi}2^{ip}\Gamma[\frac{1}{4}+\frac{i}{2}(k+p)]\sin[\frac{i}{2}\pi(k+p)-\frac{\pi}{4}]}{2\Gamma[\frac{3}{4}+\frac{i}{2}(k-p)]|\Gamma[\frac{1}{4}+\frac{i}{2}(k-p)]\Gamma[\frac{1}{4}+\frac{i}{2}(k+p)]|} \\ \sigma_2 &= \sqrt{\frac{8\pi}{\cosh^2\pi k + \sinh^2\pi p}}\frac{\sqrt{\pi}2^{ip}\Gamma[\frac{3}{4}+\frac{i}{2}(k+p)]\cos[\frac{i}{2}\pi(k+p)-\frac{\pi}{4}]}{2\Gamma[\frac{1}{4}+\frac{i}{2}(k-p)]|\Gamma[\frac{3}{4}+\frac{i}{2}(k-p)]\Gamma[\frac{3}{4}+\frac{i}{2}(k+p)]|}.\end{aligned}\right\} \tag{24}$$

This gives the wave-function (see equation (A.7) similar as for the solution (23), note $\tilde{\Psi}_{p,-k} \neq \tilde{\Psi}_{p,k}$):

$$\tilde{\Psi}_{p,k}(z) = \sqrt{\frac{2p\sinh\pi p}{\pi^2(\cosh^2\pi k + \sinh^2\pi p)}}\, Q_{ik-\frac{1}{2}}^{ip}(\tanh z). \tag{25}$$

With the asymptotic expansions for P_ν^μ and Q_ν^μ for $x = \tanh z \to \pm 1$ (see appendix A) we get for $p >> |k|$ (α, β, γ - phase factors, common phase factors neglected):

$$
\left.\begin{aligned}
\Psi_{p,k}(z) &\simeq \sqrt{\frac{1}{2\pi}}\left(1+\frac{\cosh^2\pi k}{\sinh^2\pi p}\right)^{-\frac{1}{2}} e^{ipz} && (z\to+\infty)\\
\Psi_{p,k}(z) &\simeq \sqrt{\frac{1}{2\pi}}\left(e^{ipz}+\frac{\cosh\pi k}{\sinh\pi p}e^{-ipz+i\alpha}\right) && (z\to-\infty)\\
\tilde\Psi_{p,k}(z) &\simeq \sqrt{\frac{1}{2\pi}}\left(1+\frac{\cosh^2\pi k}{\sinh^2\pi p}\right)^{-\frac{1}{2}}\cdot\left(e^{ipz}+\frac{1}{\cosh\pi p}e^{-ipz+i\beta}\right) && (z\to+\infty)\\
\tilde\Psi_{p,k}(z) &\simeq \sqrt{\frac{1}{2\pi}}\left(e^{ipz}+\frac{\cosh\pi k}{\sinh\pi p}e^{-ipz+i\gamma}\right) && (z\to-\infty).
\end{aligned}\right\} \tag{26}
$$

From these representations we see that a consistent scattering interpretation, i.e. incoming normalized wave-function $\simeq e^{ipz}/\sqrt{2\pi}$ - for $p >> |k|$, small reflected part of $O(\cosh\pi k/\sinh\pi p)$ and transmitted wave-function $\simeq e^{ipz}/\sqrt{2\pi}$ is only possible for $\Psi_{p,k}$, but not for $\tilde\Psi_{p,k}$. In order to construct $\hat K(T)$ from $K^{PT}(T)$ we have to sum up the even and odd contributions, as well $p>0$ and $p<0$ (incoming from the left and right, respectively), i.e. (e - even, o - odd):

$$
\hat K(T)=|\sigma_1|^2K^{PT}_{ee}(T)+\sigma_1\sigma_2^*K^{PT}_{eo}(T)+\sigma_2\sigma_1^*K^{PT}_{oe}(T)+|\sigma_2|^2K^{PT}_{oo}(T)+(p\leftrightarrow -p). \tag{27}
$$

Thus we can write down the path integral solution of equation (13) yielding,

$$
\begin{aligned}
\hat K(z'',z';T) &= \frac{1}{2}\int_0^\infty dp\frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\\
&\times\left[P^{ip}_{ik-\frac12}(\tanh z'')P^{-ip}_{ik-\frac12}(\tanh z')+P^{-ip}_{ik-\frac12}(\tanh z'')P^{ip}_{ik-\frac12}(\tanh z')\right]e^{-\frac{iT}{2m}p^2}\\
&=\frac{1}{2}\int_{-\infty}^\infty dp\frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}P^{ip}_{ik-\frac12}(\tanh z'')P^{-ip}_{ik-\frac12}(\tanh z')e^{-\frac{iT}{2m}p^2}.
\end{aligned} \tag{28}
$$

Therefore we get finally the solution of the path integral (6):

$$
\begin{aligned}
K^S(X'',X',Y'',Y';T) &= \frac{1}{4\pi}\int_{-\infty}^\infty dk\int_{-\infty}^\infty dp\frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\\
&\times P^{ip}_{ik-\frac12}(\sin Y'')P^{-ip}_{ik-\frac12}(\sin Y')e^{ik(X''-X')}e^{-\frac{iT}{2m}(p^2+\frac14)}.
\end{aligned} \tag{29}
$$

Wave-functions and energy-spectrum are given by

$$
\left.\begin{aligned}
\Psi^S_{p,k}(X,Y) &= \sqrt{\frac{p\sinh\pi p}{4\pi(\cosh^2\pi k+\sinh^2\pi p)}}\sqrt{\cos Y}P^{ip}_{ik-\frac12}(\sin Y)e^{ikX}\\
E_p &= \frac{1}{2m}\left(p^2+\frac14\right)
\end{aligned}\right\} \tag{30}
$$

($p,k\in\mathbf{R}$). They are orthonormal

$$
\int_{-\infty}^\infty dX\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dY}{\cos^2 Y}\Psi^S_{p,k}(X,Y)\Psi^{S*}_{p',k'}(X,Y)=\delta(p'-p)\delta(k'-k), \tag{31}
$$

and form a complete set:

$$
\int_{-\infty}^\infty dk\int_{-\infty}^\infty dp\,\Psi^S_{p,k}(X',Y')\Psi^{S*}_{p,k}(X,Y)=\cos Y'\cos Y\delta(X'-X)\delta(Y'-Y). \tag{32}
$$

Relations (31) and (32) are proved in appendix B. A state with $p=0$ does not exist due to $\Psi^S_{0,k}\equiv 0$. The energy-spectrum coincides, of course, with (II.21) and (III.19). With equation (A.8) we see that $\lim_{Y\to\pm\frac{\pi}{2}}\Psi^S_{p,k}(X,Y)=const.\lim_{Y\to\pm\frac{\pi}{2}}\sqrt{\cos Y}=0$ ($X\in\mathbf{R}$) as it should be.

This solution, of course, coincides with the operator approach in considering the eigenvalue problem

$$
-\frac{\cos^2 Y}{2m}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\Psi(X,Y)=E\Psi(X,Y). \tag{33}
$$

With the ansatz $\Psi(X,Y)=e^{ikX}\sqrt{\cos Y}P(\sin Y)$ one gets the differential equation

$$
(1-t^2)P''(t)-2tP'(t)+\left[-(k^2+\tfrac14)-\frac{\frac14-2mE}{1-t^2}\right]P(t)=0, \tag{34}
$$

where $t=\sin Y$. Equation (34) is a well-known differential equation for the Legendre functions P^μ_ν and Q^μ_ν. With $\nu=ik-\frac12$ and $\mu=i\sqrt{2mE-\frac14}$ and respecting the correct boundary conditions the result (30) is reproduced (including additional normalisation).

Unfortunately we are not able to prove $K^S(T)=K^{\Lambda^2}(T)$, say, directly by manipulations in the representation (29). However, this equivalence has to hold. Because we know that $K(T)$ and $G(E)$, respectively, are only functions of the hyperbolic distance r we can state with the help of equations (I.9) and (III.15) the following identity: The Green's function $G^S(E)$ on S is given by:

$$
\begin{aligned}
G^S(X'',X',Y'',Y';E) &= \frac{m}{\pi}Q_{-\frac12-i\sqrt{2mE-\frac14}}\left[\frac{\cosh(X''-X')}{\cos Y'\cos Y''}-\tan Y'\tan Y''\right]\\
&=\frac{1}{4\pi}\int_{-\infty}^\infty dk\int_{-\infty}^\infty dp\frac{1}{\frac{1}{2m}(p^2+\frac14)-E}\cdot\frac{p\sinh\pi p}{\cosh^2\pi k+\sinh^2\pi p}\\
&\times P^{ip}_{ik-\frac12}(\sin Y'')P^{-ip}_{ik-\frac12}(\sin Y')e^{ik(X''-X')}
\end{aligned} \tag{35}
$$

V. Summary

In this paper we have given complete path integral treatments for

1) the Poincaré disc D,
2) the Poincaré upper half-plane U and
3) the hyperbolic strip S.

In detail: 1) Starting with the "product form"-definition for path integrals we first formulated the path integrals on D yielding:

$$
\begin{aligned}
K^D(r'', r, \psi'', \psi'; T) \\
= \int \frac{4r}{(1-r^2)^2} Dr(t) \int D\psi(t) \exp\left[i \int_{t''}^{t''} \left(2m \frac{\dot r^2 + r^2 \dot\psi^2}{(1-r^2)^2} + \frac{(1-r^2)^2}{32m} \right) dt \right], \quad (1)
\end{aligned}
$$

and the quantum correction ΔV was given by $\Delta V = -\frac{(1-r^2)^2}{32mr^2}$. We performed a Fourier expansion which separated the ψ and r path integrations. For the remaining path integral over r we found with the transformation $z(r) = \ln\frac{1+r}{1-r}$ that it could be transformed into a path integral on the pseudosphere Λ^2. Care has to be taken in the Taylor expansion in the kinetic term in the exponential in order to get all terms of $O(\epsilon)$. For the path integral on Λ^2 we used the solution calculated in a previous publication which gives finally the Feynman kernel on the Poincaré disc D:

$$
\begin{aligned}
K^D(r'', r', \psi'', \psi, ; T) = \frac{1}{2\pi^2} \int_0^\infty dp \sum_{l=-\infty}^{\infty} p \sinh \pi p \\
\times e^{-\frac{iT}{2m}(p^2+\frac14)} |\Gamma(\tfrac12 + ip + l)|^2 \mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r'^2}{1-r'^2}\right) \mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r''^2}{1-r''^2}\right) e^{il(\psi''-\psi')}. \quad (2)
\end{aligned}
$$

Wave-functions and the energy-spectrum are given by:

$$
\left.
\begin{aligned}
\Psi^D_{p,l}(r,\psi) &= \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \Gamma(\tfrac12 + ip + l) \mathcal{P}^{-l}_{ip-\frac12}\left(\frac{1+r^2}{1-r^2}\right) e^{il\psi} \\
E_p &= \frac{1}{2m}\left(p^2 + \frac14\right),
\end{aligned}
\right\} \quad (3)
$$

($p > 0, l \in \mathbf{Z}$). It was shown $K^D(T) = K^{\Lambda^2}(T)$.

2) The path integral on the Poincaré upper half-plane was already calculated in a previous work but motivated by the paper of Kubo and some ambiguities in his calculation we presented two further possibilities to calculate the path integral on D. With the "product form"-definition we found that the quantum correction ΔV vanishes and the path integral reads:

$$
K^U(x'', x', y'', y'; T) = \int Dx(t) \int \frac{Dy(t)}{y^2} \exp\left(\frac{im}{2} \int_{t'}^{t''} \frac{\dot x^2 + \dot y^2}{y^2} dt \right). \quad (4)
$$

The first approach to calculate this path integral follows the idea of Kubo. After integrating out the x-dependence we transfomed the remaining path integral with the help of the transformation $z = \ln y$ into a path integral of a free particle in $\mathbf{R}$. Again care has to be taken to consider all terms of $O(\epsilon)$ in the Taylor expansion of the kinetic term. We derived an integral equation which could be solved yielding

$$
K(r;T) = \sqrt{2}\left(\frac{m}{2\pi i T}\right)^{\frac32} \int_r^\infty \frac{u\,du}{\sqrt{\cosh u - \cosh r}} \exp\left[-iu\sqrt{2mE - \frac14} \right], \quad (5)
$$

where r is the hyperboloc distance in U. Introducing the Green's function $G(E) = \int e^{iTE} K(T) dT$ we get explicitly:

$$
G(r;E) = \frac{m}{\pi} \mathcal{Q}_{-\frac12 - i\sqrt{2mE-\frac14}}(\cosh r). \quad (6)
$$

Note that $G(E)$ and $K(T)$ are only functions of the hyperbolic distance r and thus **equations (5) and (6) give the solutions in all the four spaces Λ^2, D, U and S.**

The second approach started with a Fourier expansion similar to that for the Poincaré disc D. The x and y path integration were separated and with the same steps as in the first approach (transformation $z = \ln y$, Taylor expansion) we get the path integral for Liouville quantum mechanics. This problem was solved in [16] and insertion yields for the path integral on U:

$$
\begin{aligned}
K^U(x'', x', y'', y'; T) = \frac{1}{\pi^3} \int_{-\infty}^\infty dk \int_0^\infty dp p \sinh \pi p \\
\times e^{-\frac{iT}{2m}(p^2+\frac14)} \sqrt{y'y''} K_{ip}(|k|y') K_{ip}(|k|y'') e^{ik(x''-x')}. \quad (7)
\end{aligned}
$$

Energy-spectrum and the normalised wave-functions are given by:

$$
\left.
\begin{aligned}
\Psi^U_{p,k}(x,y) &= \sqrt{\frac{p \sinh \pi p}{\pi^3}} \sqrt{y} K_{ip}(|k|y) e^{ikx} \\
E_p &= \frac{1}{2m}\left(p^2 + \frac14\right)
\end{aligned}
\right\} \quad (8)
$$

($p > 0,\ k \in \mathbf{R} \setminus \{0\}$). Finally it was shown that $K^U(T) = K^{\Lambda^2}(T)$.

3) In order to calculate the path integral on the hyperbolic strip again we applied the "product form"-definition. As for the path integral on the Poincaré upper half-plane the quantum correction turned out to vanish and the path integral on S was given by:

$$
K^S(X'', X', Y'', Y'; T) = \int DX(t) \int \frac{DY(t)}{\cos^2 Y} \exp\left(\frac{im}{2} \int_{t'}^{t''} \frac{\dot X^2 + \dot Y^2}{\cos^2 Y} dt \right). \quad (9)
$$

We performed a Fourier expansion which separated the X and Y path integrations. With the transformation $z = \frac12 \ln\frac{1+\sin Y}{1-\sin Y}$ we get by careful Taylor expansion up to fourth order in $\Delta z^{(j)}$ a path integral in $\mathbf{R}$ with the scattering potential $V(z) = (k^2 + \frac14)/\cosh^2 z$. This potential is a special case of the modified Pöschl-Teller potential $V^{PT}(r) = (c/m)/\cosh^2 r + (s/m)/\sinh^2 r$ ($r > 0$), which has been algebraically [9,21] and by path integrals [5,6] studied in the literature. The known (path integral-) solution in the half-space $\mathbf{R}^+$ has to be extended to the entire $\mathbf{R}$ in order to get the solution for our problem. A linear combination of even and odd wave-functions has to be done which satisfies the appropriate boundary conditions. Discussing these problems in some extent we get the path integral solution for the $1/\cosh^2 z$-potential in $\mathbf{R}$ and thus finally the Feynman kernel on S:

$$
\begin{aligned}
K^S(X'', X', Y'', Y'; T) = \frac{1}{4\pi} \int_{-\infty}^\infty dk \int_{-\infty}^\infty dp \frac{p \sinh \pi p}{\cosh^2 \pi k + \sinh^2 \pi p} \\
\times P^{ip}_{ik-\frac12}(\tanh Y'') P^{-ip}_{ik-\frac12}(\tanh Y') e^{ik(X''-X')} e^{-\frac{iT}{2m}(p^2+\frac14)}. \quad (10)
\end{aligned}
$$

Wave-functions and energy-spectrum are given by ($X \in \mathbf{R},\ |Y| < \frac{\pi}{2}$):

$$
\left.
\begin{aligned}
\Psi^S_{p,k}(X,Y) &= \sqrt{\frac{p \sinh \pi p}{4\pi(\cosh^2 \pi k + \sinh^2 \pi p)}} \sqrt{\cos Y} P^{ip}_{ik-\frac12}(\sin Y) e^{ikX} \\
E_p &= \frac{1}{2m}\left(p^2 + \frac14\right).
\end{aligned}
\right\} \quad (11)
$$

Therefore we have completed rigorous path integral treatments on the four Riemannian spaces D, U, S and Λ^2, which are all equivalent to each other and have negative Gaussian curvature $K = -1$. The path integral on Λ^2 is nothing but a special case of the d-dimensional pseudosphere Λ^{d-1}. The path integral on Λ^{d-1} has been calculated in [17] and the path integral on U with a magnetic field in [14].

It is a quite interesting feature that the path integrals on U, D and S (free motion on spaces with constant negative curvature) are connected to potential problems in flat space. This connection is due to the symmetery properties of these spaces. The symmetry in D, say, reads $SU(1,1)$ (pseudo-unitary group) [3,5], which is e.g. isomorphic to $PSL(2,\mathbf{R})$ (projective special linear group) in U [19], or to the orthochronous Lorentz transformations in Λ^2 [3]; these symmetries are "hidden" in the potential problems.

We have shown that that all solutions on D, U and S are satisfying the boundary conditions and orthogonality and completeness relations. We have also shown that the Feynman kernels on Λ^2, D and U are equivalent to each other, which is, of course, due to the fact that all the spaces Λ^2, D, U and S are equivalent to each other. A direct proof of the equivalence of $K^S(T)$ with the other Feynman kernels could not be given. However, deducing from the equivalence of all these spaces we could state that the Green's function $G^S(E)$ on S is given by

$$G^S(X'',X',Y'',Y';E) = \frac{m}{\pi} Q_{-\frac{1}{2}-i\sqrt{2mE-\frac{1}{4}}}\left[\frac{\cosh(X''-X')}{\cos Y'\cos Y''} - \tan Y' \tan Y''\right]. \quad (12)$$

Once more the essential connection between ordering prescriptions in the quantum Hamiltonian, the lattice definition in the path integral and the appropriate quantum corrections has become clear. Without these quantum corrections path integrals on curved manifolds cannot be understood. Otherwise "correct" results are obtained just by chance. It has also become clear that the Taylor expansion up to fourth order in $\Delta^4 q^{(j)}/\epsilon$ for the various transformations we have done give important and indispensable contributions. Otherwise the "zero" momentum energy E_0 would be incorrect, namely $E_0 = 0$ instead of $E_0 = \frac{1}{8m}$. This feature has been discussed in section III. This zero point energy can be understood as a pure quantum phenomenon. E_0 is the largest lower bound for the energy a state on $U(D,S,\Lambda^2)$ can have due to the Heisenberg uncertainty relation.

We thus have added three further examples to the short list of exactly solvable path integrals. The examples demonstrate once more the consistency as well the universal utility and feasibility of our general method developed in [13,15].

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Appendix A: The Associated Legendre Functions $\mathcal{P}_\nu^\mu$ and $\mathcal{Q}_\nu^\mu$

The functions $\mathcal{P}_\nu^\mu$ and $\mathcal{Q}_\nu^\mu$ are linearly independent solutions of the differential equation

$$(1-z^2)\frac{d^2u(z)}{dz^2} - 2z\frac{du(z)}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]u(z) = 0, \quad (1)$$

and are defined by means of the hypergeometric function ${}_2F_1$:

$$\mathcal{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} {}_2F_1(-\nu,\nu+1;1-\mu;\tfrac{1-z}{2}) \quad (2)$$

$$\mathcal{Q}_\nu^\mu(z) = \frac{e^{\mu\pi i}\Gamma(\nu+\mu+1)\sqrt{\pi}}{2^{\nu+1}\Gamma(\nu+3/2)}(z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1}$$
$$\times\ {}_2F_1\left(\tfrac{\nu+\mu+2}{2},\tfrac{\nu+\mu+1}{2};\nu+\tfrac{3}{2};\tfrac{1}{z^2}\right). \quad (3)$$

They are called associated Legendre functions (or spherical functions) of the first and second kind, respectively[1]. They are uniquely defined in the intervalls $|1-z| < 2$ and $|z| > 1$, respectively. They can be extended to the entire z-plane where a cut along the real axis from $-\infty$ to $+1$ has to be made.

The so-called conical functions $\mathcal{P}^\mu_{-\frac{1}{2}+ip}$ have the special property

$$\mathcal{P}^\mu_{-\frac{1}{2}+ip}(z) = \mathcal{P}^\mu_{-\frac{1}{2}-ip}(z), \quad (4)$$

which is due to the general property $\mathcal{P}_\nu^\mu = \mathcal{P}^\mu_{-\nu-1}$. If $\mu = m \in \mathbf{Z}$

$$\mathcal{P}_\nu^m(z) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\mathcal{P}_\nu^{-m}(z). \quad (5)$$

Important expansions for $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ ($|x| < 1$) read ([11], p.1011):

$$P_\nu^\mu(x) = \frac{2^\mu \cos\pi\frac{\nu+\mu}{2}\Gamma(\frac{\mu+\mu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu-\mu}{2}+1)}(1-x^2)^{\frac{\mu}{2}}{}_2F_1\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right)$$
$$+2\frac{2^\mu \sin\pi\frac{\nu+\mu}{2}\Gamma(\frac{\mu+\mu}{2}+1)}{\sqrt{\pi}\Gamma(\frac{\nu-\mu+1}{2})}x(1-x^2)^{\frac{\mu}{2}}{}_2F_1\left(\frac{\nu+\mu}{2}+1,\frac{\mu-\nu+1}{2};\frac{3}{2};x^2\right) \quad (6)$$

$$Q_\nu^\mu(x) = -\frac{\sqrt{\pi}2^\mu \sin\pi\frac{\nu+\mu}{2}\Gamma(\frac{\mu+\mu+1}{2})}{2\Gamma(\frac{\nu-\mu}{2}+1)}(1-x^2)^{\frac{\mu}{2}}{}_2F_1\left(\frac{\nu+\mu+1}{2},\frac{\mu-\nu}{2};\frac{1}{2};x^2\right)$$
$$+\frac{\sqrt{\pi}2^\mu \cos\pi\frac{\nu+\mu}{2}\Gamma(\frac{\mu+\mu}{2}+1)}{\Gamma(\frac{\nu-\mu+1}{2})}x(1-x^2)^{\frac{\mu}{2}}{}_2F_1\left(\frac{\nu+\mu}{2}+1,\frac{\mu-\nu+1}{2};\frac{3}{2};x^2\right). \quad (7)$$

[1] We use $\mathcal{P}_\nu^\mu(z)$, $\mathcal{Q}_\nu^\mu(z)$ for $z \in \mathbf{C}\setminus[-1,1]$ and $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$ for $x \in (-1,1)$ for the Legendre functions of the first and second kind, respectively.

The asymptotic behaviour for the Legendrefunctions ([24], pp.196/197) for $t \to \pm 1$ are given by ($\mu \notin \mathbf{N}_0$):

$$\left.\begin{aligned}
P_\nu^\mu(x) &\simeq \frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1-x}\right)^{\frac{\mu}{2}} && (x \to +1)\\
P_\nu^\mu(x) &\simeq \frac{\Gamma(-\mu)}{\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)}\left(\frac{1+x}{2}\right)^{\frac{\mu}{2}}\\
&\quad - \frac{\Gamma(\mu)}{\pi}\sin\pi\nu\left(\frac{1+x}{2}\right)^{-\frac{\mu}{2}} && (x \to -1)\\
Q_\nu^\mu(x) &\simeq \frac{1}{2}\Gamma(\mu)\cos\pi\mu\left(\frac{1-x}{2}\right)^{-\frac{\mu}{2}}\\
&\quad + \frac{\Gamma(-\mu)\Gamma(1+\mu+\nu)}{2\Gamma(1+\nu-\mu)}\left(\frac{1-x}{2}\right)^{\frac{\mu}{2}} && (x \to +1)\\
Q_\nu^\mu(x) &\simeq -\frac{1}{2}\Gamma(\mu)\cos\nu\pi\left(\frac{1+x}{2}\right)^{-\frac{\mu}{2}}\\
&\quad - \frac{\cos\pi(\nu+\mu)\Gamma(-\mu)\Gamma(1+\mu+\nu)}{2\Gamma(1+\nu-\mu)}\left(\frac{1+x}{2}\right)^{\frac{\mu}{2}} && (x \to -1).
\end{aligned}\right\} \quad (8)$$

For $z \to \infty$ we have for $\mathcal{P}_\nu^\mu$ ([24], p.197):

$$\mathcal{P}_\nu^\mu(z) \simeq \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(1+\nu-\mu)}\frac{(2z)^\nu}{\sqrt{\pi}} + \frac{\Gamma(-\frac{1}{2}-\nu)}{\Gamma(-\mu-\nu)}\frac{(2z)^{-1-\nu}}{\sqrt{\pi}}. \quad (9)$$

Appendix B: Proof of the Equations (IV.31) and (IV.32)

Let us consider the orthonormality relation for wave-functions $\Psi_{p,k}^S, \Psi_{q,k'}^S$ ($p, q \neq 0$) where $\Psi_{p,k}^S(X,Y) = N\sqrt{\cos Y}\, P_{ik-\frac{1}{2}}^{ip}(\sin Y)e^{ikX}$ to calculate the normalisation N:

$$\begin{aligned}
(\Psi_{p,k}^S, \Psi_{q,k'}^S) &= \int_{-\infty}^{\infty} dX \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dY}{\cos^2 Y}\Psi_{p,k}^S(X,Y)\,\Psi_{q,k'}^{S*}(X,Y)\\
&= N^2 \int_{-\infty}^{\infty} dX \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dY}{\cos Y} P_{ik-\frac{1}{2}}^{ip}(\sin Y)P_{ik'-\frac{1}{2}}^{-iq}(\sin Y)\, e^{i(k-k')X}\\
&= 2\pi N^2\delta(k-k')\underbrace{\int_{-1}^{1}\frac{P_{ik-\frac{1}{2}}^{ip}(t)\,P_{ik-\frac{1}{2}}^{-iq}(t)}{1-t^2}dt}_{=:I(p,q,k)}. \quad (1)
\end{aligned}$$

Note the identity

$$I(p,k,q) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{P_{ik-\frac{1}{2}}^{ip}(\sin Y)P_{ik-\frac{1}{2}}^{-iq}(\sin Y)}{\cos Y}dY = \int_{-\infty}^{\infty} P_{ik-\frac{1}{2}}^{ip}(\tanh z)P_{ik-\frac{1}{2}}^{-iq}(\tanh z)dz \quad (2)$$

where the transformation $z = \frac{1}{2}\ln\left(\frac{1+\sin Y}{1-\sin Y}\right)$ has been made. Thus we see that the normalisation of the $\Psi_{p,k}^S$ give simultaneously the normalisation of the $\Psi_{p,k}$ of equation (IV.23). In order to calculate $I(p,q,k)$ we use the integral theorem (see [24], p.191):

$$\begin{aligned}
&\int_a^b \left[(\nu-\sigma)(\nu+\sigma+1) + \frac{\rho^2-\mu^2}{1-t^2}\right]P_\nu^\mu(t)P_\sigma^\rho(t)dt\\
&= \left[(1-t^2)\left(P_\nu^\mu(t)\frac{dP_\sigma^\rho(t)}{dt} - P_\sigma^\rho(t)\frac{dP_\nu^\mu(t)}{dt}\right)\right]\Bigg|_a^b\\
&= \left\{\sqrt{1-t^2}\left[P_\nu^{1+\mu}(t)\,P_\sigma^\rho(t) - P_\nu^\mu(t)\,P_\sigma^{1+\rho}(t)\right] + (\mu-\rho)tP_\nu^\mu(t)\,P_\sigma^\rho(t)\right\}\Bigg|_a^b \quad (3)
\end{aligned}$$

where use has been made of the relation ([24], p.171)

$$(\nu-\mu)tP_\nu^\mu(t) - (\nu+\mu)P_{\nu-1}^\mu(t) = \sqrt{1-t^2}P_\nu^{\mu+1}(t). \quad (4)$$

We set $a = \epsilon - 1$, $b = 1 - \epsilon$, $\nu = \sigma = ik - \frac{1}{2}$, $\mu = ip$ and $\rho = -iq$, therefore

$$\begin{aligned}
I(p,q,k) &= \lim_{\epsilon\to 0}\int_{\epsilon-1}^{1-\epsilon}\frac{P_{ik-\frac{1}{2}}^{ip}(t)\,P_{ik-\frac{1}{2}}^{-iq}(t)}{1-t^2}dt\\
&= \frac{1}{p^2-q^2}\lim_{\epsilon\to 0}\Bigg\{\sqrt{1-t^2}\left[P_{ik-\frac{1}{2}}^{1+ip}(t)P_{ik-\frac{1}{2}}^{-iq}(t) - P_{ik-\frac{1}{2}}^{ip}(t)P_{ik-\frac{1}{2}}^{1-iq}(t)\right]\\
&\qquad + i(p+q)P_{ik-\frac{1}{2}}^{ip}(t)\,P_{ik-\frac{1}{2}}^{-iq}(t)\Bigg\}\Bigg|_{\epsilon-1}^{1+\epsilon}. \quad (5)
\end{aligned}$$

We need the distributional relations:

$$\left.\begin{aligned}
\lim_{N\to\infty}\frac{\sin Nx}{x} &= \pi\delta(x)\\
\lim_{N\to\infty}\frac{\cos Nx}{x} &= 0.
\end{aligned}\right\} \quad (6)$$

These relations can be deduced if one considers for test functions ψ:

$$\begin{aligned}
\lim_{N\to\infty}\int_{-\infty}^{\infty}\psi(x)\frac{\sin Nx}{x}dx &= \lim_{N\to\infty}\int_{-\infty}^{\infty}\psi(y/N)\frac{\sin y}{y}dy = \pi\psi(0)\\
\lim_{N\to\infty}\int_{-\infty}^{\infty}\psi(x)\frac{\cos Nx}{x}dx &= \lim_{N\to\infty}\int_{-\infty}^{\infty}\psi(y/N)\frac{\cos y}{y}dy = \psi(0)P\int_{-\infty}^{\infty}\frac{\cos y}{y}dy = 0.
\end{aligned}$$

We apply equation (6) to the expression

$$\lim_{\epsilon\to 0}\frac{(2/\epsilon)^{i\frac{x}{2}}}{x} = \frac{1}{2}\lim_{\epsilon\to 0}\frac{\cos[\frac{x}{2}\ln(2/\epsilon)] + i\sin[\frac{x}{2}\ln(2/\epsilon)]}{\frac{x}{2}} = i\pi\delta(x). \quad (7)$$

With equation (7) we get in equation (5) for the first term at the upper limit:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\sqrt{1-t^2}}{p^2-q^2}\left[P_{ik-\frac{1}{2}}^{1+ip}(t)P_{ik-\frac{1}{2}}^{-iq}(t) - P_{ik-\frac{1}{2}}^{ip}(t)P_{ik-\frac{1}{2}}^{1-iq}(t)\right]\Bigg|_{t=1-\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{2}{p^2-q^2}\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}(p-q)}\left[\frac{1}{\Gamma(-ip)\Gamma(1+iq)} - \frac{1}{\Gamma(1-ip)\Gamma(iq)}\right]\\
&= \frac{2\sinh\pi p}{p}\delta(p-q). \quad (8)
\end{aligned}$$

For the second term at the upper limit

$$\lim_{\epsilon\to 0}\frac{i(p-q)}{p^2-q^2}tP^{ip}_{ik-\frac{1}{2}}(t)P^{-iq}_{ik-\frac{1}{2}}(t)\Big|_{t=1-\epsilon}$$
$$=\lim_{\epsilon\to 0}\frac{i}{p-q}\frac{(2/\epsilon)^{\frac{i}{2}(p-q)}}{\Gamma(1-ip)\Gamma(1+iq)}=-\frac{\sinh\pi p}{p}\delta(p-q). \quad (9)$$

At the lower limit we get for the first term:

$$\lim_{\epsilon\to 0}\frac{\sqrt{1-t^2}}{p^2-q^2}\left[P^{1+ip}_{ik-\frac{1}{2}}(t)P^{-iq}_{ik-\frac{1}{2}}(t)-P^{ip}_{ik-\frac{1}{2}}(t)P^{1-iq}_{ik-\frac{1}{2}}(t)\right]\Big|_{t=\epsilon-1}$$
$$=\lim_{\epsilon\to 0}\frac{\sqrt{2\epsilon}}{p^2-q^2}\Bigg\{-\frac{\Gamma(1+ip)}{\pi}\sin\pi(ik-\tfrac{1}{2})\cdot\left(\frac{2}{\epsilon}\right)^{\frac{1+ip}{2}}$$
$$\times\left[\frac{\Gamma(iq)}{\Gamma(\frac{1}{2}+ik+iq)\Gamma(\frac{1}{2}-ik+iq)}\left(\frac{2}{\epsilon}\right)^{-\frac{i}{2}q}-\frac{\Gamma(-iq)}{\pi}\sin\pi(ik-\tfrac{1}{2})\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}q}\right]$$
$$+\left[\frac{\Gamma(-ip)}{\Gamma(\frac{1}{2}+ik-ip)\Gamma(\frac{1}{2}-ik-ip)}\left(\frac{2}{\epsilon}\right)^{-\frac{i}{2}p}-\frac{\Gamma(-iq)}{\pi}\sin\pi(ik-\tfrac{1}{2})\left(\frac{2}{\epsilon}\right)^{-\frac{i}{2}q}\right]$$
$$\times\frac{\Gamma(1-iq)}{\pi}\sin\pi(ik-\tfrac{1}{2})\left(\frac{2}{\epsilon}\right)^{\frac{1-iq}{2}}\Bigg\}$$
$$=\lim_{\epsilon\to 0}\frac{2\cosh^2\pi k}{\pi(p^2-q^2)}\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}(p-q)}\left[\Gamma(1+ip)\Gamma(-iq)-\Gamma(ip)\Gamma(1-iq)\right]$$
$$=-2\frac{\cosh^2\pi k}{p\sinh\pi p}\delta(p-q), \quad (10)$$

where we have used the identity

$$\lim_{\epsilon\to 0}\frac{1}{p-q}\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}(p+q)}=\frac{\pi}{i}\frac{p+q}{p-q}\delta(p+q)=0.$$

For the second term at the lower limit we get finally:

$$\lim_{\epsilon\to 0}\frac{i(p+q)}{p^2-q^2}tP^{ip}_{ik-\frac{1}{2}}(t)P^{-iq}_{ik-\frac{1}{2}}(t)\Big|_{t=\epsilon-1}$$
$$=\lim_{\epsilon\to 0}\frac{i(\epsilon-1)}{p+q}\left[\frac{\Gamma(-ip)}{\Gamma(\frac{1}{2}+ik-ip)\Gamma(\frac{1}{2}-ik-ip)}\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}p}-\frac{\Gamma(ip)}{\pi}\sin\pi(ik-\tfrac{1}{2})\left(\frac{2}{\epsilon}\right)^{-\frac{i}{2}p}\right]$$
$$\times\left[\frac{\Gamma(iq)}{\Gamma(\frac{1}{2}+ik+iq)\Gamma(\frac{1}{2}-ik+iq)}\left(\frac{2}{\epsilon}\right)^{-\frac{i}{2}q}-\frac{\Gamma(-iq)}{\pi}\sin\pi(ik-\tfrac{1}{2})\left(\frac{2}{\epsilon}\right)^{\frac{i}{2}q}\right]$$
$$=-\pi\left|\frac{\Gamma(ip)}{\Gamma(\frac{1}{2}+ik+ip)\Gamma(\frac{1}{2}+ik-ip)}\right|^2\delta(p-q)+\frac{|\Gamma(ip)|^2}{\pi}\cosh^2\pi k\delta(p-q)$$
$$=\frac{-\cosh\pi(p-k)\cosh\pi(p+k)+\cosh^2\pi k}{p\sinh\pi p}\delta(p-q)=-\frac{\sinh\pi p}{p}\delta(p-q), \quad (11)$$

where we have used $\cosh(x-y)\cosh(x+y)=\cosh^2 x+\sinh^2 y$. Therefore we get for $I(p,q,k)$:

$$I(p,q,k)=2\frac{\cosh^2\pi k+\sinh^2\pi p}{p\sinh\pi p}\delta(p-q) \quad (12)$$

[note that $I(p,q,k)$ gives the normalistion of the $\Psi_{p,k}$ of equation (IV.23)] and the normalisation reads,

$$N=\left[\frac{p\sinh\pi p}{4\pi(\cosh^2\pi k+\sinh^2\pi p)}\right]^{\frac{1}{2}}. \quad (13)$$

The next question is, do the functions (IV.29) form a complete set? In order to prove this we have to study the integral

$$V=\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dpN^2(p,k)\sqrt{\cos y'\cos y''}P^{ip}_{ik-\frac{1}{2}}(\sin y')P^{-ip}_{ik-\frac{1}{2}}(\sin y'')e^{ik(x'-x'')} \quad (14)$$

Unfortunately we are not in the position to give a rigorous proof of the completeness property of the $\Psi^S_{p,k}$, and can offer only a sketch. However, we want to emphasize that the approximations (13) and (14) are quite good (see below), even for small values of $|p|>1$, because the relevant terms have exponential character. For the Legendre functions we apply the asymptotic expansion for $|p|\to\infty$ ([24], pp.169,195, leading term only):

$$P^{ip}_{ik-\frac{1}{2}}(x)\simeq\frac{1}{\Gamma(1-ip)}\left(\frac{1+x}{1-x}\right)^{\frac{ip}{2}} \quad (15)$$

and let (for a fixed $k\in\mathbf{R}$)

$$N^2(p,k)=\frac{\sinh^2\pi p}{\cosh^2\pi k+\sinh^2\pi p}\simeq 1 \quad (16)$$

in the same limit. The product of the two Legendre functions therefore yields:

$$P^{ip}_{ik-\frac{1}{2}}(\sin Y')P^{-ip}_{ik-\frac{1}{2}}(\sin Y'')\simeq\frac{\sinh\pi p}{\pi p}\left[\left(\frac{1+\sin Y'}{1-\sin Y'}\right)\left(\frac{1-\sin Y''}{1+\sin Y''}\right)\right]^{\frac{ip}{2}}. \quad (17)$$

Inserted into (12) this gives

$$V=\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dp|N(p,k)|^2\sqrt{\cos Y'\cos Y''}P^{ip}_{ik-\frac{1}{2}}(\sin Y')P^{-ip}_{ik-\frac{1}{2}}(\sin Y'')e^{ik(X''-X')}$$
$$\simeq\frac{\sqrt{\cos Y'\cos Y''}}{4\pi^2}\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dp\frac{\sinh^2\pi p}{\cosh^2\pi k+\sinh^2\pi p}$$
$$\times\left[\left(\frac{1+\sin Y'}{1-\sin Y'}\right)\left(\frac{1-\sin Y''}{1+\sin Y''}\right)\right]^{\frac{ip}{2}}e^{ik(X''-X')}$$
$$\simeq\frac{\sqrt{\cos Y'\cos Y''}}{4\pi^2}\int_{-\infty}^{\infty}dk\,e^{ik(X''-X')}$$
$$\times\int_{-\infty}^{\infty}dp\exp\left\{\frac{ip}{2}\log\left[\left(\frac{1+\sin Y'}{1-\sin Y'}\right)\left(\frac{1-\sin Y''}{1+\sin Y''}\right)\right]\right\}$$
$$=\sqrt{\cos Y'\cos Y''}\delta(X''-X')\delta\left\{\frac{1}{2}\log\left[\left(\frac{1+\sin Y'}{1-\sin Y'}\right)\left(\frac{1-\sin Y''}{1+\sin Y''}\right)\right]\right\}. \quad (18)$$

In the last step we have to use

$$\left.\begin{aligned} \delta[f(x)-f(x_0)] &= \frac{1}{|f'(x)|}\Bigg|_{x=x_0} \delta(x-x_0) \quad \text{where} \\ f(Y) &= \frac{1}{2}\log\frac{1+\sin Y}{1-\sin Y}, \quad f'(Y)=\frac{1}{\cos Y}. \end{aligned}\right\} \tag{19}$$

Therefore we get

$$V = \cos Y' \cos Y'' \delta(X''-X')\delta(Y''-Y') \tag{20}$$

which is the desired completeness relation.

Appendix C: Numerical Evaluation of the Wave-Functions on the Hyperbolic Strip

In this appendix we want to show a selection of plots of the wave-functions $\Psi^S_{p,k}$ on S for some values of the parameters p and k. Our motivation to do this stems from the fact that the Legendre functions P^μ_ν with pure imaginary upper index are quite uncommon. For the numerical evaluation the power expansion (A.2) for the P^μ_ν in terms of hypergeometric functions has been used, i.e.

$$\Psi^S_{p,k}(X,Y) = \left[\frac{p\sinh\pi p}{4\pi(\cosh^2\pi k+\sinh^2\pi p)}\right]^{\frac{1}{2}} \frac{1}{\Gamma(1-ip)}$$
$$\times\, e^{ikX}\sqrt{\cos Y}\left(\frac{1+\sin Y}{1-\sin Y}\right)^{\frac{ip}{2}} {}_2F_1(\tfrac{1}{2}-ik,\tfrac{1}{2}+ik;1-ip;(1-\sin Y)/2). \tag{C1}$$

The X-dependence is trivial; we furthermore omit the $\sqrt{\cos Y}$-factor to get essentially the functions $\Psi_{p,k}$ (IV.23):

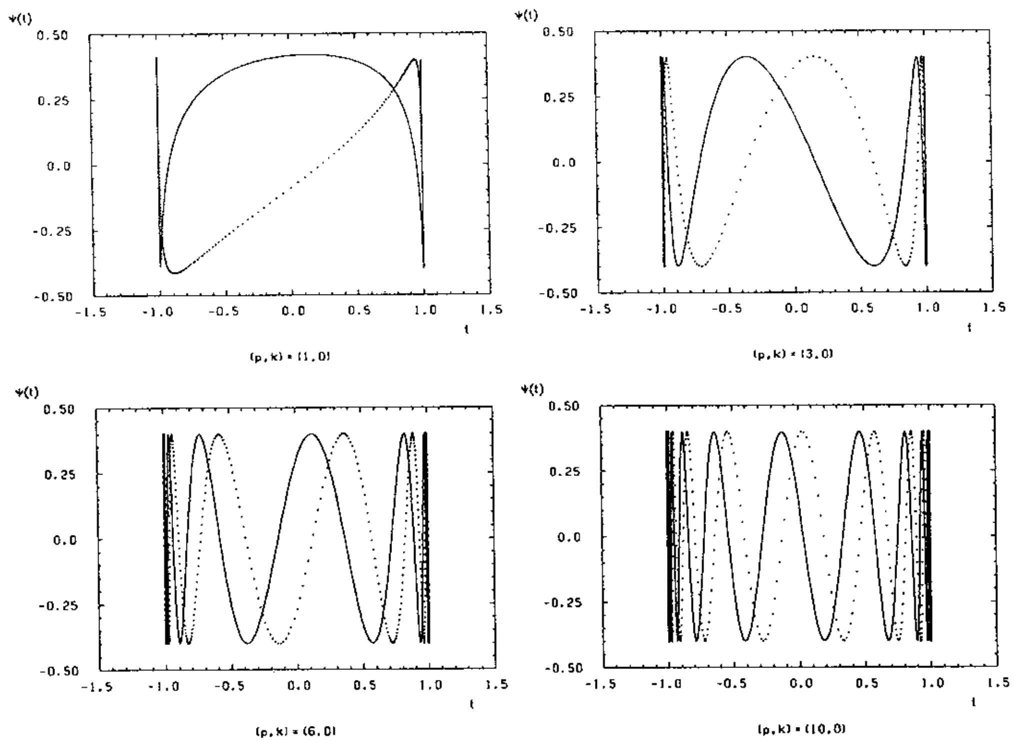
$$\Psi_{p,k}(t) = \left[\frac{p\sinh\pi p}{2(\cosh^2\pi k+\sinh^2\pi p)}\right]^{\frac{1}{2}} \frac{1}{\Gamma(1-ip)}$$
$$\times\left(\frac{1+t}{1-t}\right)^{\frac{ip}{2}} {}_2F_1(\tfrac{1}{2}-ik,\tfrac{1}{2}+ik;1-ip;(1-t)/2) \tag{C2}$$

where we have mapped $(-\frac{\pi}{2},\frac{\pi}{2}) \mapsto (-1,1)$ by $t=\sin Y$. In figure 1 we show $\Psi_{p,k}(t)$ for fixed $k=0$ and $p=1,3,6,10$, respectively. The solid and dotted lines represent the real- and imaginary parts of the wave-functions, respectively.

As t approaches $|t|\to 1$ the wave-functions start oscillating very rapidly and in fact in the interval $(-1,1)$ there are infinitely many oscillations for all values of the parameters p and k. By considering the asymptotic expansion of the $\Psi_{p,k}$ for $t\to 1$, say, we have

$$\Psi_{p,k}(t) \simeq (2\pi)^{-\frac{1}{2}}\exp\left[ip\ln\sqrt{\frac{1+t}{1-t}}\right]. \tag{C3}$$

Figure 1: Real- and imaginary parts of the wave-functions $\Psi_{1,0}$, $\Psi_{3,0}$, $\Psi_{6,0}$ and $\Psi_{10,0}$



Here $\ln\sqrt{\frac{1-t}{1-t}} = r$ denotes the hyperbolic distance from $(X,0)$ on S if r restricted to the Y-direction [set $X'=X''=X$ in Eq.(I.9)]. This feature allows one to say that the $\Psi_{p,k}$ represent plane waves in the interval $(-1,1)$ endowed with the hyperbolic geometry, distorted, of course, in S by the factor $(1-t^2)^{-\frac{1}{4}}$. Figure 2 shows this characteristic oscillation for the right part of the interval (-1,1) for $t\to 1$. From Eq.(IV.26) it is also legitimate to speak of the $\Psi_{p,k}$ as "plane waves" in $\mathbb{R}$ distorted by the potential well (IV.14).

On the left and right hand side of figure 3 we have displayed the effect of the variation of k for fixed p in $\mathrm{Re}\Psi_{p,k}$ and $\mathrm{Im}\Psi_{p,k}$ (here p=10), respectively. In comparison with the $k=0$ shape we have for increasing k an increasing phase shift and an increasing distortion of the wave functions. The solid, dotted and broken lines indicate the values $k=0,5,7$ respectively.

Figure 2: Real- and imaginary parts of the wave-function $\Psi_{10,0}$ as t approaches $t \to 1$

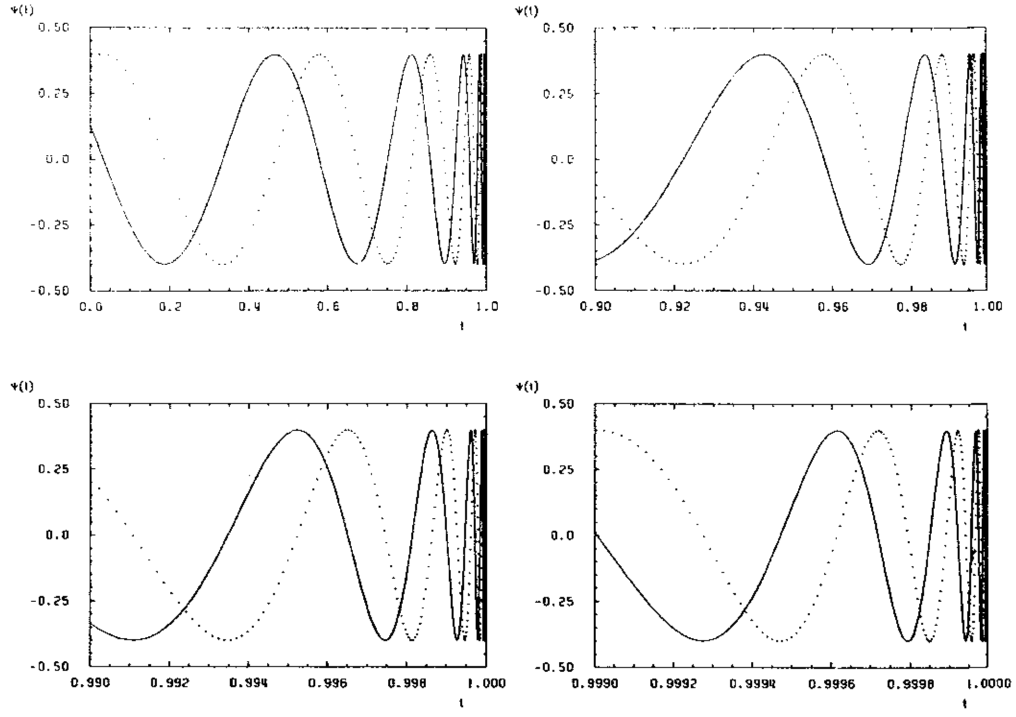
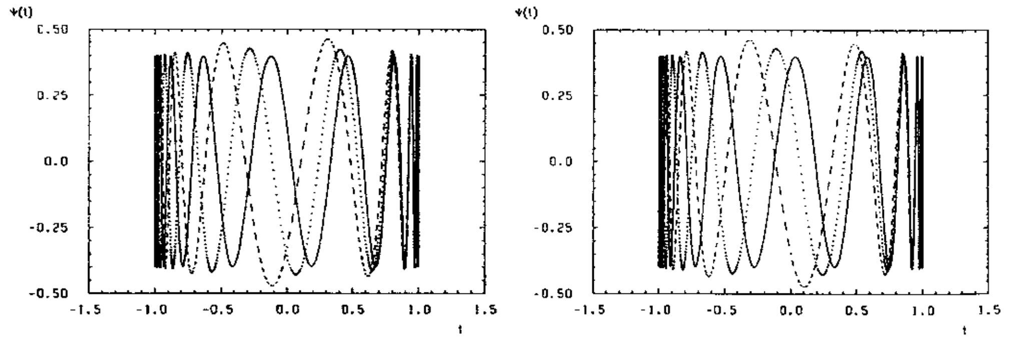


Figure 3: Real- and imaginary parts of the wave-functions $\Psi_{10,0}$, $\Psi_{10,5}$ and $\Psi_{10,7}$



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