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From classical periodic orbits to the quantization of chaos [1]

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Abstract

We present a quantitative analysis of Selberg's trace formula viewed as an exact version of Gutzwiller's periodic–orbit theory for the quantization of classically chaotic systems. Two main applications of the trace formula are discussed in detail: i) The periodic–orbit sum rules giving a smoothing of the quantal energy–level density, ii) The Selberg zeta function as a prototype of a dynamical zeta function defined as an Euler product over the classical periodic orbits and its analytic continuation across the entropy barrier by means of a Dirichlet series. It is shown how the long periodic orbits can be effectively taken into account by a universal remainder term which is explicitly given as an integral over an "orbit–selection function". Numerical results are presented for two Riemann surfaces which demonstrate clearly the crucial role played by the long periodic orbits. A general rule for quantizing chaos is given for such systems where the Dirichlet series representing the Selberg zeta function converges on the critical line. Explicit formulas are given for the computation of the abscissas of absolute and conditional convergence, respectively, of these dynamical Dirichlet series. For the two Riemann surfaces considered, it turns out that one can cross the entropy barrier, but that the critical line cannot be reached by a convergent Dirichlet series. This seems to be the main reason why the recently conjectured Riemann–Siegel lookalike formula does not work in the case of these strongly chaotic systems.

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I Introduction

In this paper we shall discuss two different applications of Gutzwiller's periodic–orbit theory [1,2] considered as a semiclassical method to quantize classically chaotic systems for which quantization conditions like the WKB method fail. (Here only chaotic systems with a discrete energy spectrum are considered.)

On the one hand, periodic–orbit sum rules [3,4,5,6] are considered which allow in principle the determination of the energy levels of a given quantum mechanical system in terms of the lengths of the periodic orbits of the classical counterpart. These sum rules are smoothed versions of Gutzwiller's original trace formula [1] for the trace of the resolvent which possesses bad convergence properties [5]. The smoothing replaces the poles of the resolvent at the energy levels by regular maxima, a process which improves the convergence properties. The positions of the maxima yield then approximations to the energy levels.

On the other hand, functions like the Selberg zeta function [7,8,9] are considered allowing the determination of the quantal energies by searching for the non–trivial zeros. The periodic–orbit theory can be used to express such functions as Euler products over the classical periodic orbits. However, as in the case of the periodic–orbit sum rules, these expressions are in general not absolutely convergent in the region where the non–trivial zeros lie, and a careful discussion is needed. Berry and Keating [10] (see also [11]) have suggested a semiclassical quantization condition based on a Riemann–Siegel lookalike formula in analogy with the Riemann–Siegel formula for the zeros of the Riemann zeta function. A different Riemann–Siegel formula has been proposed by Bogomolny [12]. Recently, the Riemann–Siegel lookalike formula [10,11] has been tested [13] in the case of the hyperbola billiard [6]. The results for the energy levels are in good agreement with the energies obtained by a direct numerical solution of the Schrödinger equation. A detailed investigation [13] of the convergence properties shows, however, that the relevant series converges for the hyperbola billiard on the critical line and thus any reasonable truncation taking into account only a finite number of periodic orbits will yield good results. A similar result has been obtained [14] for Artin's billiard where the quantization condition is even exact since it has been derived from a Selberg trace formula.

In this paper the two ways of employing the periodic–orbit theory as a quantization condition for chaotic systems are discussed in detail in the example of the free motion of a point particle on a compact Riemann surface with constant negative curvature. This system is conservative and strongly chaotic (K–system). (An introduction to this field can be found in [2,15] and in our previous papers [3,4,17,18].) Here we deal with the simplest possible realization of compact Riemann surfaces with constant negative curvature having genus $g = 2$. Such surfaces correspond topologically to spheres with two handles, i.e. double tori. For this class of systems the periodic–orbit theory is exact as Gutzwiller [16] was the first to realize, because it can be expressed by Selberg's trace formula [7] which reads

$$\sum_{\{p_n\}} h(p_n) \;=\; \frac{\text{Area}(\mathcal{F})}{4\pi} \int_{-\infty}^{\infty} dp\, p\, \tanh(\pi p)\, h(p) \;+\; \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{l_n}{2 \sinh \frac{k\, l_n}{2}}\, g(k\, l_n) \quad . \tag{1}$$

Here the l. h. s. is purely quantum mechanical. It is a sum over the energy spectrum $\{E_n\}$ expressed by the momenta p_n via $E_n = \frac{1}{4} + p_n^2$, and $h(p)$ is an even function which is holomorphic in the strip $|\, \text{Im}\, p\, | \le \frac{1}{2} + \varepsilon, \varepsilon > 0$ and vanishes asymptotically for $|p| \to \infty$ faster than $\frac{1}{p^2}$. The r. h. s. is of classical nature consisting of the so–called zero length term proportional to $\text{Area}(\mathcal{F}) = 4\pi(g-1)$, and the periodic–orbit sum over the length spectrum $\{l_n\}$. $g(x)$ denotes the Fourier transform of $h(p)$. The exact trace formula (1) lies at the foundation of our discussion of the periodic–orbit theory.

II An exact expression for the remainder term

In general it is impossible to carry out the summation over the length spectrum analytically. Therefore, one is forced to a numerical evaluation of the trace formula. In the case of the regular hyperbolic octagon [17] we were able to compute the complete length spectrum [21] for the first 1500 different lengths which enclose approximately 4 million primitive periodic orbits because of a high degeneracy g_n due to the symmetry of this special system [4,17]. The shortest length is $l_1 = 3.057\ldots$, the length of the 1500th length is $l_{1500} = 18.092\ldots$. Even those first 4 million primitive periodic orbits are not sufficient to determine the energy spectrum by a simple evaluation of the trace formula because it leads to a sharp cut–off at l_{1500} in the summation over the length spectrum. This cut–off causes strong energy–dependent oscillations which are much more pronounced than the peaks corresponding to the energies of the system. To determine the energies, it is thus necessary to smooth the sharp cut–off by taking into account a remainder term which has to be computed analytically as follows.

If the length spectrum of the primitive periodic orbits is cut off at $l_n = L$, we obtain from (1) $(g = 2)$

$$\sum_{\{p_n\}} h(p_n) = \int_{-\infty}^{\infty} dp\, p\, \tanh(\pi p)\, h(p) + \sum_{\{l_n\}} \sum_{\substack{k=1 \\ k\, l_n \le L}}^{\infty} \frac{l_n}{2\sinh\frac{k\, l_n}{2}}\, g(k\, l_n) + R(L) \quad , \tag{2}$$

where the remainder term is given exactly by

$$R(L) := \sum_{\{l_n\}} \sum_{\substack{k=1 \\ k\, l_n > L}}^{\infty} \frac{l_n}{2\sinh\frac{k\, l_n}{2}}\, g(k\, l_n) = \sum_{k=1}^{\infty} \frac{1}{k} \int_{L-}^{\infty} \frac{l\, g(l)}{2\sinh\frac{l}{2}}\, dN\left(\frac{l}{k}\right) \quad . \tag{3}$$

Here the last integral is to be understood as a Riemann–Stieltjes integral, while $N(l)$ denotes the exact staircase function which counts the number of primitive periodic orbits whose length l_n is shorter than or equal to l $\;(N(l) \equiv 0$ for $l < l_1)$. Eq.(3) can be rewritten

$$R(L) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} \int_{L-}^{\infty} l\, e^{-ml}\, r(l)\, dN\left(\frac{l}{k}\right) \quad , \tag{4}$$

where $r(l)$ is called the "orbit–selection function" and is defined by

$$r(l) := e^{l/2}\, g(l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\, e^{(\frac{1}{2}+ip)l}\, h(p) \quad . \tag{5}$$

If there are no small eigenvalues $(0 < E_n \le \frac{1}{4})$, the following asymptotic behaviour has been shown to hold for any Fuchsian group of the first kind [19]

$$dN(l) = \frac{e^l}{l}\, dl + O\left(\frac{e^{\frac{3}{4}l}}{l}\right) dl \quad , \quad l \to \infty \quad . \tag{6}$$

Here the leading term is known as Huber's law [20] and describes the average proliferation of the number of primitive periodic orbits very well as can be seen from figs.2 and 3 of ref.[21]. Such an exponential law is characteristic for chaotic systems in general and leads to the notorious problems in the computation of the periodic–orbit sums for this class of systems. Since eq.(6) is valid for every compact Riemann surface of constant negative curvature $K = -1$ with genus 2 (having no small eigenvalues), this implies that the following approximation based on (6) contains no detailed information about the energy spectrum of the considered regular hyperbolic octagon, keeping in mind that all Riemann surfaces possess distinct energy spectra.

It follows from eqs.(4) and (6) that the leading behaviour of the remainder $R(L)$ is given exactly by

$$R(L) \;=\; R_1(L)\,\left[1+O(e^{-L/4})\right] \quad , \tag{7}$$

where

$$R_1(L) := \int_L^\infty dl\; r(l) \quad . \tag{8}$$

The simple universal expression (8) for the rest term depends only on the orbit–selection function (5) which measures the average contribution to the periodic–orbit sum rule (2) as a function of the continuous length–parameter l. For a large enough cut–off length L, the length spectrum is for $l_n > L$ so dense that the continuous approximation (8) should yield an excellent approximation to the remainder. $R_1(L)$ corresponds to an effective resummation of orbits with long periods. It follows from eq.(5) that

$$\lim_{L\to\infty} R(L) \;=\; \lim_{L\to\infty} R_1(L) \;=\; 0 \quad , \tag{9}$$

if the function $h(p)$ in (2) satisfies the conditions stated after eq.(1), which is consistent with the fact that the integral and all series in eq.(1) converge absolutely under these conditions. In this case the long periodic orbits are exponentially suppressed. An example of such an absolutely convergent periodic–orbit sum rule will be discussed in the next section.

III An absolutely convergent periodic–orbit sum rule

In ref.[3] we have introduced the Gaussian smoothing

$$h(p') = e^{-\frac{(p-p')^2}{\epsilon^2}} + e^{-\frac{(p+p')^2}{\epsilon^2}} \tag{10}$$

which fulfils obviously all the conditions stated after eq.(1). The Selberg trace formula (1) reads in this case

$$\begin{aligned}\sum_{n=0}^{\infty}\left[e^{-\frac{(p-p_n)^2}{\epsilon^2}} + e^{-\frac{(p+p_n)^2}{\epsilon^2}}\right] \;&=\; 2\int_0^\infty dp'\,p'\,\tanh(\pi p')\left[e^{-\frac{(p-p')^2}{\epsilon^2}} + e^{-\frac{(p+p')^2}{\epsilon^2}}\right]\\ &+\; \frac{\epsilon}{2\sqrt{\pi}}\sum_{\{l_n\}}\sum_{\substack{k=1\\ kl_n\le L}}^{\infty}\frac{l_n}{\sinh\frac{kl_n}{2}}\cos(pkl_n)e^{-\frac{\epsilon^2}{4}(kl_n)^2} \;+\; R_1(L,p)+\ldots\end{aligned} \tag{11}$$

This sum rule is absolutely convergent for any $\epsilon > 0$ and permits for $L\to\infty$ an arbitrarily accurate determination of the energies of the quantum mechanical system from the lengths of the periodic orbits of the classical system. Every periodic orbit contributes to the sum with a "cosine–wave", and the greater the length of the orbit the shorter the "wavelength". This shows clearly that the long periodic orbits determine the fine structure of the energy spectrum, whereas the short orbits describe only the coarse structure. Therefore, an evaluation of the trace formula with a finite number of orbits cannot yield the exact energies, and the cut–off length L determines the energy resolution.

The remainder term reads in this case (the dependence on the momentum p is explicitly denoted)

$$\begin{aligned}R_1(L,p) \;&:=\; \int_L^\infty dl\;r(l) \;=\; \frac{\varepsilon}{\sqrt{\pi}}\int_L^\infty dl\;\cos(p\,l)\,e^{-\frac{\varepsilon^2}{4}l^2+\frac{1}{2}l}\\ &=\; e^{-\frac{\varepsilon^2}{4}L^2+\frac{1}{2}L}\;\mathrm{Re}\left[\mathrm{erfc}(\rho)\,e^{\rho^2}e^{-ipL}\right] \quad \text{with} \quad \rho := \frac{\varepsilon L}{2}-\frac{1}{2\varepsilon}+i\frac{p}{\varepsilon} \quad .\end{aligned} \tag{12}$$

The real part of the energy–dependent term in brackets gives large oscillations which are needed to cancel the corresponding ones coming from the truncated periodic–orbit sum.

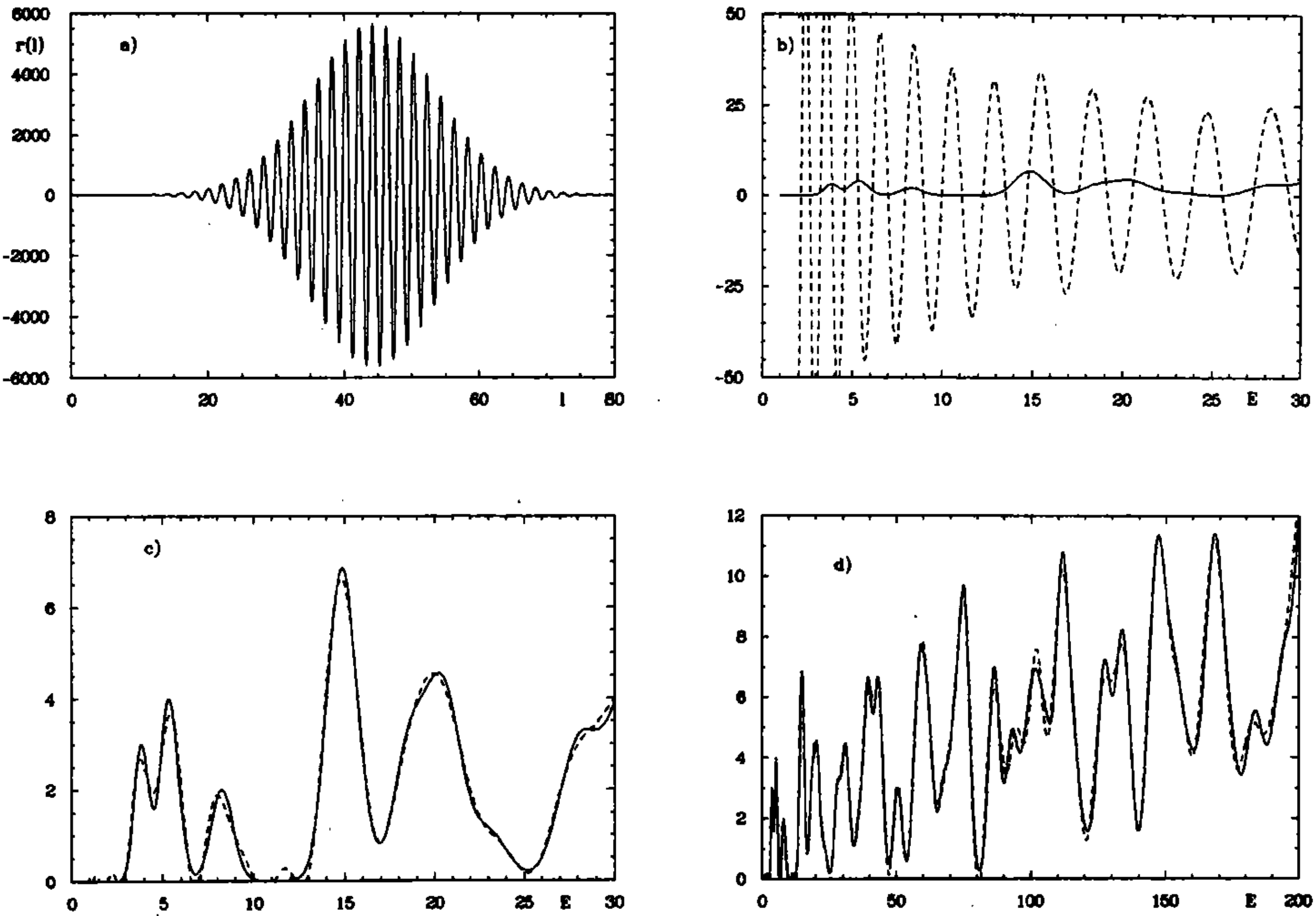

Figure 1: a) The orbit–selection function $r(l)$ for Gaussian smoothing is shown for $\epsilon = 0.15$ and $E = 10$. b) The evaluation of the Gaussian sum rule with a sharp cut–off at $L = 18.092$ shows strong oscillations (dashed curve) in disagreement with the curve computed directly from the energy levels (solid curve). c),d) If the remainder term $R_1(L,p)$ is added, the large oscillations caused by the sharp cut–off are canceled, leaving nice peaks at the correct quantal energies.

Now the trace formula can be evaluated by computing the periodic–orbit sum truncated at the cut–off length L using the known length spectrum $\{l_n\}$. Then the contribution of the omitted periodic orbits has to be taken into account by (12). This approach may fail because (12) contains no information about the fine structure of the energy spectrum.

We have studied the question whether this approach yields reasonable results in the case of the regular hyperbolic octagon where the length spectrum $\{l_n\}$ is completely known [21] until $L = l_{1500} = 18.092\ldots$. This L–value determines the shortest wavelength and thereby the resolution. Let us assume that the resolution is limited by the distance $\Delta p = \frac{\pi}{L}$ between two zeros of the cosine–wave. Demanding that the Gaussian curve of a given energy level is as small as $e^{-2} = 0.135\ldots$ at the location of the neighbouring level, one obtains $\Delta p = \sqrt{2}\,\epsilon$ and therefore $\epsilon = \frac{\pi}{\sqrt{2}\,L} \simeq 0.12$. In the following computation we choose $\epsilon = 0.15$ which allows the resolution of the energy levels with $E_n < 10$.

Figure 1a shows the orbit–selection function $r(l)$ for $\epsilon = 0.15$ at the energy $E = 10$. It is frightening to see that the periodic orbits of length $l = \frac{1}{\epsilon^2} \simeq 44$ yield the largest contribution to the periodic–orbit sum. If one would try to sum all terms up to $L = 60$ one has to deal with roughly 2×10^{24} periodic orbits according Huber's law. Apart from the overwhelming storage problem, the computer time is utopian. Assuming 100 floating point operations for a single summand, a supercomputer with 1 GFlops would need 6×10^9 years for the evaluation of the periodic–orbit sum at a single energy value! Furthermore, fig. 1a shows that the periodic orbits up to $L = 18.092$ contribute scarcely to the periodic–orbit sum. In figure 1b the evaluation of the periodic–orbit sum truncated at $L = 18.092$ is shown (dashed curve). One recognizes large oscillations caused by the cut–off to be compared with the

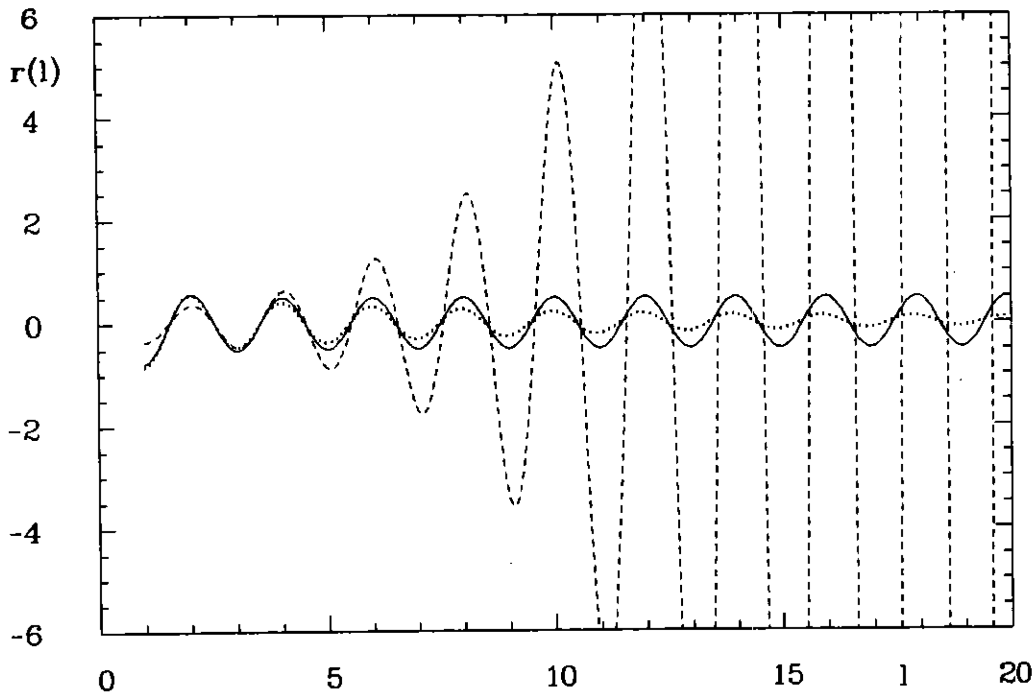

Figure 2: The orbit–selection function $r(l)$ for the Breit–Wigner smoothing is shown for the three cases $\alpha = 1.2$ (dotted curve), $\alpha = 1$ (solid curve) and $\alpha = 0.3$ (dashed curve).

peaks of the energy levels represented by the solid curve computed directly form the energies (l. h. s. of (11)). Of course, this bad result comes at no surprise after the above discussion. More astonishing is, however, the remarkably good result, which is obtained if the remainder term (12) is taken into account, as figures 1c,d show. Here the dashed curves are computed by evaluating the periodic–orbit sum up to $L = 18.092$ and adding the remainder term (12), whereas the solid curves are computed from the energies directly. All energy levels with $E_n < 10$ are resolved with the correct degeneracy d_n [4] ($E_1 = 3.838(d_1 = 3), E_2 = 5.353(d_2 = 4)$ and $E_3 = 8.249(d_3 = 2)$), and even the coarse structure up to $E = 200$ is correctly reproduced. This demonstrates that the remainder term, which contains no information about the fine structure of the energy levels, describes the collective behaviour of the very long periodic orbits correctly.

IV A divergent periodic–orbit sum rule and its analytical continuation across the entropy barrier

In contrast to the last section, where the remainder term was well defined, we want to discuss in this section a periodic–orbit sum rule which is not absolutely convergent. Such sum rules were used very often in the past without having control over the remainder. Therefore, the following discussion is to a large extent historically motivated, because absolutely convergent periodic–orbit sum rules like the Gaussian smoothing are now available. Instead of considering the trace of the Green's function, we want to study the Breit–Wigner smoothing which we have already discussed in [4]. With the choice

$$h(p') = \frac{\alpha^2 \, E}{(E - E')^2 + \alpha^2 \, E} \quad , \quad E = p^2 + \frac{1}{4} > \frac{1}{4} \quad , \tag{13}$$

Selberg's trace formula (1) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\alpha^2 E}{(E - E_n)^2 + \alpha^2 E} \quad &= \quad -2\alpha\sqrt{E} \; \mathrm{Im} \; \Psi(\frac{1}{2} + A_- - iA_+) \;\; + \\ & \frac{\alpha\sqrt{E}}{4(A_+^2 + A_-^2)} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{l_n e^{-A_- k l_n}}{\sinh \frac{k l_n}{2}} \left[A_+ \cos(A_+ k l_n) + A_- \sin(A_+ k l_n)\right] \end{aligned} \tag{14}$$

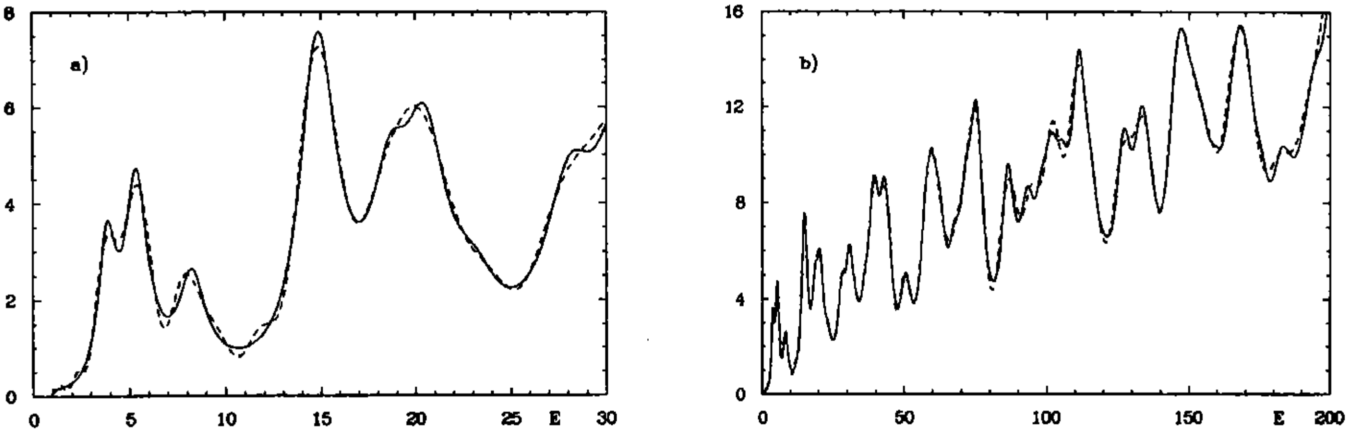

Figure 3: The Breit–Wigner smoothing is shown for $\alpha = 0.3$ computed directly from the energy spectrum (solid curves) and computed from the periodic–orbit sum including the remainder term (15) (dashed curves).

with $A_\pm = 2^{-1/2}\{\sqrt{(E-\frac{1}{4})^2+\alpha^2 E} \pm E \mp \frac{1}{4}\}^{1/2}$ and $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$. The l. h. s. is a sum over Breit–Wigner resonances with width $\Gamma = 2\alpha\sqrt{E}$. In the limit $\alpha \to 0$ the resonances would become ever sharper and an arbitrarily accurate energy resolution would be possible. However, the periodic–orbit sum on the r. h. s. is absolutely convergent only for $\alpha > 1$ which is too large to resolve even the first excited state. Therefore, this sum rule does not seem to be suitable for a determination of the energy levels from the length spectrum $\{l_n\}$.

To illustrate the bad behaviour of the Breit–Wigner smoothed periodic–orbit sum for $\alpha \leq 1$, we present in figure 2 the orbit–selection function $r(l)$ for the three cases $\alpha = 1.2$, $\alpha = 1$ and $\alpha = 0.3$, respectively. Only for $\alpha > 1$ a sufficiently large cut–off length L would give a reasonable result. In the desired case of $\alpha < 1$, an increasing cut–off length L leads to exponentially increasing oscillations of the periodic–orbit sum as figure 2 suggests, and no reasonable result is expected.

Let us now compute the remainder term assuming $\alpha > 1$, where the following integral exists because $A_- > \frac{1}{2}$ for $\alpha > 1$:

$$\begin{aligned} R_1(L,E) &= \int_L^\infty dl\, r(l) = \frac{\alpha\sqrt{E}}{2(A_+^2+A_-^2)} \int_L^\infty dl\, e^{-(A_- - \frac{1}{2})l}\, [\, A_+\cos(A_+ l) + A_-\sin(A_+ l)\,] \\ &= -\frac{\alpha\sqrt{E}}{2(A_+^2+A_-^2)}\, \frac{e^{-(A_- - \frac{1}{2})L}}{A_+^2 + (A_- - \frac{1}{2})^2} \;\times \\ &\quad \left[\left(\frac{A_+}{2} - \alpha\sqrt{E}\right)\cos(A_+ L) + \left(\frac{A_-}{2} + E - \frac{1}{4}\right)\sin(A_+ L)\right] \quad \text{for} \quad \alpha > 1\,. \end{aligned} \tag{15}$$

Adding this remainder term to the truncated periodic–orbit sum, gives for $\alpha > 1$ a good description as in the case of the Gaussian smoothing. This was to be expected, since $R_1(L,E) \to 0$ for $\alpha > 1$ and $L \to \infty$. However, while the integral in (15) would diverge for $\alpha < 1$, the resulting function obtained by integration keeping $\alpha > 1$, is well defined for $0 < \alpha < 1$. Thus we are led to define the remainder term for $\alpha < 1$ by the analytic continuation of the result given in (15). Note, however, that $R_1(L,E) \to \infty$ for $L \to \infty$ since $A_- < \frac{1}{2}$ for $\alpha < 1$. But on the other hand $R_1(L,E) \to 0$ for fixed cut–off L in the semiclassical limit $E \to \infty$. The numerical evaluation of this approximation (dashed curve) is in good agreement with the "true" curve computed directly from the energy levels (solid curve) even for values of α as small as 0.3, see figure 3. This result indicates, that this approximation, which was originally derived only for $\alpha > 1$, yields indeed a meaningful analytical continuation of the periodic–orbit sum across the critical point $\alpha = 1$ ("entropy barrier").

In [4] we have already presented an evaluation of the Breit–Wigner sum rule using a length spectrum which was derived by a code (see [17]). The Breit–Wigner sum rule was shown for $\alpha = 0.3$ and the remarkably good result was somewhat miraculous. However, the length spectrum used in [4] was not complete for larger lengths and showed a smooth decrease in the number of periodic orbits with increasing length. It seems to be a general feature that an algorithm based on a code produces for a fixed code length periodic orbits whose lengths are Gaussian distributed (see e. g. [6]). The evaluation of the periodic–orbit sum with such a length spectrum has no trouble with a sharp cut–off. In the case of our length spectrum which is complete up to l_{1500}, one can mimic such a behaviour with a wrong "Huber's law" like

$$\widehat{N}(l) \;=\; \frac{\gamma}{l} e^{-\frac{(l-\mathcal{L})^2}{2\sigma^2}} \tag{16}$$

where γ and $\mathcal{L}$ are fit parameters. These can be determined by the requirements $N_H(L) = \widehat{N}(L)$ and $\frac{d}{dl}N_H(l)|_L = \frac{d}{dl}\widehat{N}(l)|_L$ where L is the cut–off length and $N_H(l)$ is the correct Huber's law. The width σ in (16) should be chosen as large as possible. For (16) it is again possible to compute a remainder term $R_1(L,E)$ which is valid for $\alpha > 0$. We have checked that this ansatz yields a comparably good result as (15) which shows that it is indeed not necessary to deal with the correct Huber's law. The result obtained is nearly identical to the one shown in figure 3 except for small energies $E < 2$, because the value $\sigma^2 = 50$ which was used is not able to cancel the strong oscillations in that range. Notice that the wavelength of the oscillations of $r(l)$ is the longer the smaller the energy and therefore the limit $\sigma \to \infty$ is required for $E \to \frac{1}{4}$ which causes numerical troubles. Nevertheless, this demonstrates that an evaluation of a not absolutely convergent trace formula can lead to reasonable results if a length spectrum is used which is computed by a code yielding a "soft" cut–off.

V The Selberg zeta function and a rule for quantizing chaos

The Selberg zeta function is defined [7] as the Euler product ($s = \frac{1}{2} - ip$, $E = s(1-s)$)

$$Z(s) \;=\; \prod_{\{l_n\}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)l_n}\right) \quad , \quad \mathrm{Re}\, s > 1 \quad . \tag{17}$$

It can be shown with the aid of the Selberg trace formula (1) (see e.g. [4,9]) that the logarithmic derivative of $Z(s)$ has after analytical continuation non–trivial poles located at $s_n = \frac{1}{2} \pm ip_n$ and hence $Z(s)$ has its non–trivial zeros exactly at the quantal energies given by $E_n = \frac{1}{4} + p_n^2$.

Unfortunately, the Euler product (17) cannot be used directly for the determination of the energy levels because it converges only for $\mathrm{Re}\, s > 1$. In general, a periodic–orbit sum converges absolutely if [5] $\mathrm{Im}\, p > \tau - \frac{\bar\lambda}{2}$ ($\hbar \equiv 1$), where τ denotes the topological entropy and $\bar\lambda$ the metric entropy. This result carries over to the convergence of a zeta–function like (17). In the case of the Riemann surfaces considered here one has $\tau = \bar\lambda = 1$ and thus $\mathrm{Im}\, p > \frac{1}{2}$ to ensure the absolute convergence of the Euler product according to the condition $\mathrm{Re}\, s > 1$. This "entropy barrier" is therefore at a distance of $\frac{1}{2}$ from the critical line $\mathrm{Re}\, s = \frac{1}{2}$ which is very large in comparison with the mean distance $\frac{1}{2p}$ between the zeros on the critical line. Already at the very low energy $p = 1$, i. e. $E = 1.25$, the mean distance between the zeros is equal to the distance from the critical line. There are other systems where this distance is much smaller which facilitates the calculation of the zeros of the zeta function.

The well–known functional equation for $Z(s)$ gives on the critical line, $s = \frac{1}{2} + ip$, the exact relation ($E > \frac{1}{4}$)

$$Z(\frac{1}{2} + ip) \;=\; e^{-2\pi i \bar{\mathcal{N}}(E)} Z(\frac{1}{2} - ip) \quad \text{with} \quad \bar{\mathcal{N}}(E) = \int_{1/4}^{E} dE' \tanh\left(\pi\sqrt{E' - \frac{1}{4}}\right) \quad , \tag{18}$$

which implies that both sides of the following equation are real

$$Z(\frac{1}{2} + ip)\, e^{i\pi\bar{\mathcal{N}}(E)} \;=\; Z(\frac{1}{2} - ip)\, e^{-i\pi\bar{\mathcal{N}}(E)} \quad . \tag{19}$$

We are thus led to define for real values of p the real function [13,14]

$$\xi(p) \;:=\; \text{Re}\;\left\{ Z(\frac{1}{2}+ip)\, e^{i\pi\bar{\mathcal{N}}(E)} \right\} \tag{20}$$

whose only zeros are located exactly at the quantal momenta $p = p_n = \sqrt{E_n - \frac{1}{4}}$. The condition

$$\xi(p) \;=\; 0 \tag{21}$$

constitutes an exact rule for quantizing the chaos on the compact Riemann surfaces considered in this paper.

In order to make use of the quantization condition (21), we need an analytic continuation of $Z(s)$ to be inserted in eq.(20). To this end we transform the Euler product (17) into a Dirichlet series. Following [8,10] we rewrite the product over k in (17) with the help of Euler's identity [22]

$$\prod_{k=0}^{\infty} \left(1 - y\,x^k\right) \;=\; 1 \;+\; \sum_{m=1}^{\infty} \frac{(-1)^m\, y^m\, x^{\frac{1}{2}m(m-1)}}{\prod_{l=1}^{m}(1-x^l)} \quad , \quad |x|<1 \quad , \quad y \in \mathbf{C} \quad , \tag{22}$$

into a sum which reads

$$Z(s) \;=\; \prod_{\{l_n\}} \left(1 \;+\; \sum_{m=1}^{\infty} \frac{a_{mn}}{N_n^s}\right) \quad \text{with} \quad a_{mn} := \frac{(-1)^m\, e^{-\frac{1}{2}m(m-1)l_n}}{\prod_{k=1}^{m}(1-e^{-k\,l_n})} \quad \text{and} \quad N_n := e^{l_n} \quad . \tag{23}$$

Expanding the product over the periodic orbits, we arrive with the definitions

$$A_\alpha := \prod_i a_{m_i n_i} \quad , \quad N_\alpha := \prod_i N_{n_i}^{m_i} \tag{24}$$

at a Dirichlet series over "pseudo orbits" with lengths $L_\alpha := \sum_i m_i l_{n_i}$

$$Z(s) \;=\; 1 \;+\; \sum_\alpha \frac{A_\alpha}{N_\alpha^s} \quad , \quad \text{Re}\; s > 1 \quad . \tag{25}$$

Let us assume for a moment that the Dirichlet series (25) converges on the critical line $\text{Re}\, s = \frac{1}{2}$. (A detailed discussion of the convergence properties will be given in section VI.) It is then justified to insert the series (25) into (20), and one obtains the explicit series expansion [13,14]

$$\xi(p) \;=\; \cos\{\pi\bar{\mathcal{N}}(E)\} \;+\; \sum_\alpha \frac{A_\alpha}{\sqrt{N_\alpha}}\, \cos\{\pi\bar{\mathcal{N}}(E) - p\ln N_\alpha\} \quad . \tag{26}$$

Here the first term generates already zeros according to Weyl's law, i. e. leads to approximations to the energy levels with the correct mean level density, while the sum over the pseudo orbits describes the fluctuation properties of the energy levels. In case of convergence one expects to obtain a good approximation to the function $\xi(p)$, if the series (26) is truncated at a sufficiently long pseudo–length L. In ref.[13,14] it has been shown that such a truncation yields indeed good approximations to the energy levels for the hyperbola billiard and Artin's billiard, respectively. It is natural to assume that the optimal truncation is obtained by choosing for the cut–off $L = L^*$, where L^* is determined by the condition that the last term taken into account in the series (26) is locally stationary as a function of the energy E, i. e. is given by

$$\frac{d}{dE}\,\{\pi\bar{\mathcal{N}}(E) \;-\; pL_\alpha\} \;=\; 0 \quad . \tag{27}$$

(This condition has already been imposed by Berry and Keating [10] in their discussion of the Riemann–Siegel lookalike formula which will be discussed in section VII.) Using (18) we are thus

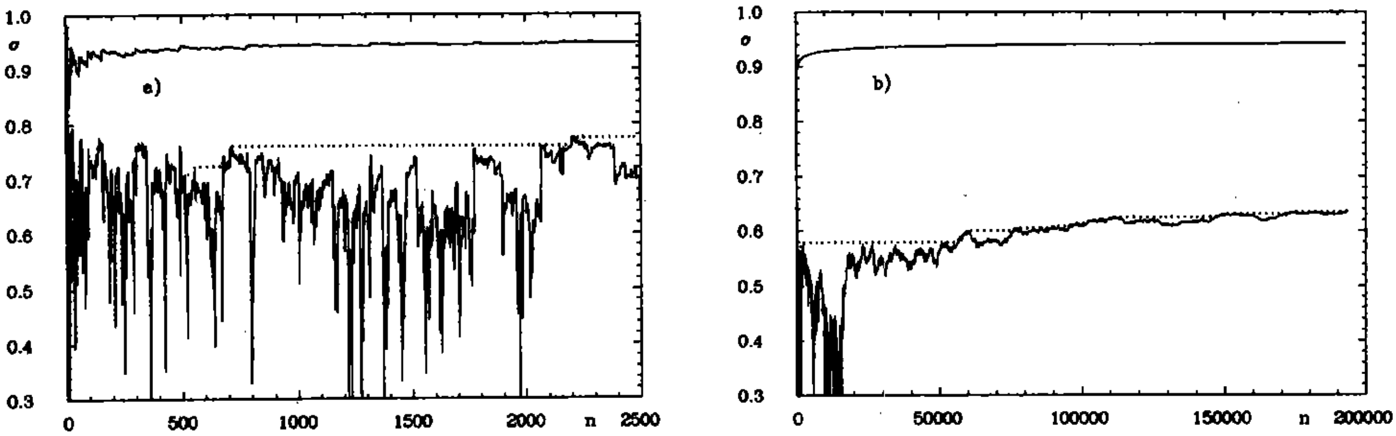

Figure 4: The abscissas of convergence of the Dirichlet series (25) are shown for the regular octagon a) and the asymmetric one b).

led to the energy–dependent cut–off pseudo–length $L^* = 2\pi p \tanh(\pi p)$ and finally to the following rule for quantizing chaos

$$\cos\{\pi \bar{\mathcal{N}}(E)\} + \sum_{L_\alpha \leq L^*} A_\alpha \, e^{-L_\alpha/2} \cos\{\pi \bar{\mathcal{N}}(E) - p L_\alpha\} = 0 \quad . \tag{28}$$

We expect that the quantization condition (28) generates good semiclassical approximations to the energy levels for those compact Riemann surfaces (and chaotic systems, in general) for which the Dirichlet series (25) converges conditionally on the critical line. A satisfactory test of (28) has already been carried out for the hyperbola billiard in ref.[13]. It is worthwhile to notice that (28) is – apart from an overall normalization factor of 2 – identical to the Riemann–Siegel lookalike formula of [10], see eq.(34) below. But in contrast to the arguments given in [10], we have not required any conjecture concerning the resummation of the long orbits. The derivation of the quantization rule (28) is mainly based on the functional equation and on the convergence of the Dirichlet series on the critical line.

In the next section we shall show for two compact Riemann surfaces that the corresponding Dirichlet series (25) do <u>not</u> converge on the critical line. In this case the quantization condition (28) looses its justification. Indeed, the numerical results to be presented in section VII seem to indicate that the condition (28) fails for these systems.

VI Convergence of the Dirichlet series

The discussion in the preceding section has shown that the question of convergence of the Dirichlet series (25) is of crucial importance for the application of the quantization condition (28). For Dirichlet series of the type (25) one has the following formulas for the abscissa of absolute convergence σ_a

$$\sigma_a = \limsup_{n\to\infty} \frac{1}{\ln N_n} \ln \sum_{\alpha=1}^{n} |A_\alpha| \tag{29}$$

and for the abscissa of (conditional) convergence $\sigma_c \leq \sigma_a$

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\ln N_n} \ln \left| \sum_{\alpha=1}^{n} A_\alpha \right| . \tag{30}$$

In ref.[23] we have shown that the number $N_p(L)$ of pseudo–orbits with length L_α smaller than or equal to L is asymptotically given by

$$N_p(L) = \frac{Z(2)}{Z'(1)} \, e^L + \ldots \quad , \quad L \to \infty \tag{31}$$

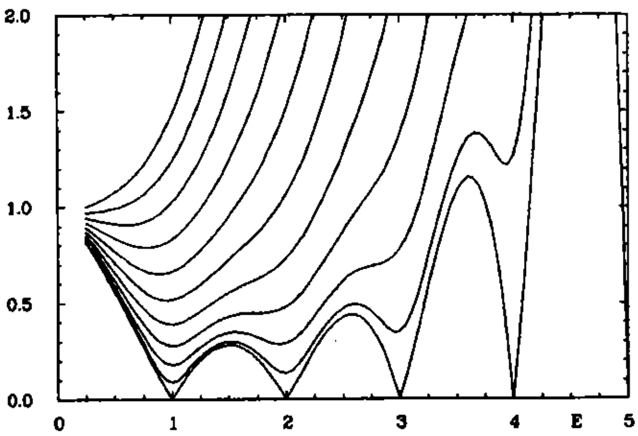

Figure 5: The product (32) is shown for different distances from the critical line $\sigma = \frac{1}{2}$. The lowest curve belongs to $\sigma = \frac{1}{2}$ revealing all zeros, while the remaining curves belong to $\sigma = 0.55, 0.6, \ldots, 1$ in ascending order.

where $Z'(1) > 0$ since the zeta function (17) has a simple zero at $s = 1$. For the regular Riemann surface discussed before one obtains [23] $Z(2)/Z'(1) = 0.3930$, while for the asymmetric Riemann surface discussed below one gets $Z(2)/Z'(1) = 0.4277$.

Fig.4a,b indicate that $\sigma_a = 1$, i. e. the Dirichlet series (25) is absolutely convergent for $\mathrm{Re}\, s > 1$, as expected. The crucial question is whether $\sigma_c < 1$, i. e. whether the entropy barrier at $\mathrm{Re}\, s = 1$ can be crossed by means of the Dirichlet series. Figure 4a,b show the sequences $\sigma_a(n)$ and $\sigma_c(n)$, respectively, whose limits for $n \to \infty$ determine the abscissas σ_a and σ_c according to eqs.(29) and (30). The upper curves belong to σ_a and the lower ones to σ_c. The dotted curves represent the upper limits of $\sigma_c(n)$. For the regular octagon (fig.4a) and for the asymmetric octagon (fig.4b) one obtains $\sigma_c \simeq 0.78$ and $\sigma_c \simeq 0.64$, respectively, at the largest n-value. One can speculate that σ_c is the smaller the lesser symmetries the system possesses and that a completely desymmetrized system has $\sigma_c = 0.5$ (or $\sigma_c = 0$ for systems whose critical line is at $\mathrm{Re}\, s = 0$) allowing the computation of the zeta-function on the critical line. As already mentioned, it has been shown recently that this is indeed the case for the desymmetrized hyperbola billiard [13] and for Artin's billiard [14].

For the systems considered in this paper, the important fact is that one has with the Dirichlet series (25) a representation in terms of classical orbits for the zeta-function which is valid beyond the entropy barrier at $\mathrm{Re}\; s = 1$. However, in our case the computation of the energy levels with the Dirichlet series is still impossible, because of the large distance from the critical line $\sigma = \frac{1}{2}$. This is illustrated in figure 5, where the expression

$$\left| e^{\gamma s(s-1)} \prod_{n=1}^{\infty} \left[\left(1 + \frac{s(s-1)}{E_n}\right) e^{-s(s-1)/E_n} \right] \right| \quad \text{with} \quad s = \sigma + ip\,,\ E = s(1-s) \quad \text{and} \quad E_n = n \tag{32}$$

is shown which is the modulus of a function whose zeros ar exactly at $s = \frac{1}{2} \pm i\sqrt{n - \frac{1}{4}}$. The curves belong to $\sigma = 0.5, 0.55, 0.60, \ldots, 1$ in ascending order. To resolve the first zero $E_1 = 1$ it suffices to use $\sigma = 0.6$; for the second at $E_2 = 2$, already $\sigma = 0.55$ is necessary. With increasing energy, the required value for σ has to approach the critical value $\sigma = 0.5$ ever faster. Thus the periodic-orbit sum rules seem to be more useful until a more effective representation of the zeta function in terms of the classical orbits is found. At present one can use the zeta function approach for the determination of energy levels only in the case of those systems, where the Dirichlet series (25) already allows the computation of the zeta function on the critical line.

VII Test of the Riemann–Siegel lookalike formula

Recently, Berry and Keating [10] have conjectured a quantization rule in analogy with the Riemann–Siegel formula for the Riemann zeta function. As already mentioned, this rule is identical to our rule (28), but it has been conjectured in [10] that this rule is generally valid, not only if the Dirichlet series is convergent on the critical line as has been assumed in our derivation of (28). We briefly repeat the arguments given in [10].

Without worrying about convergence problems, one obtains from (25)

$$Z(\frac{1}{2}+ip)\, e^{i\pi\bar{\mathcal{N}}(E)} \;=\; e^{i\pi\bar{\mathcal{N}}(E)} \;+\; \sum_{\alpha} \frac{A_\alpha}{\sqrt{N_\alpha}}\, e^{i(\pi\bar{\mathcal{N}}(E)-p\ln N_\alpha)} \quad . \tag{33}$$

Splitting this series over pseudo–orbits appropriately into two parts corresponding to $L_\alpha \leq L^*$ and $L_\alpha > L^*$, respectively, the authors conjecture that the second divergent series over the long pseudo–orbits is approximately equal to the complex conjugate of the first term on the r. h. s. of eq.(33) and of the first series over the short pseudo–orbits and they thus arrive at

$$Z(\frac{1}{2}+ip)\, e^{i\pi\bar{\mathcal{N}}(E)} \;\simeq\; 2\, \cos(\pi\bar{\mathcal{N}}(E)) \;+\; 2 \sum_{L_\alpha\leq L^*} A_\alpha\, e^{-L_\alpha/2}\, \cos\left(\pi\bar{\mathcal{N}}(E) - pL_\alpha\right) \quad . \tag{34}$$

For the cut–off L^* the condition (27) is imposed. Eq.(34) is called the Riemann–Siegel lookalike formula. Notice that in Berry and Keating's approach the first term in eq.(34) corresponding to Weyl's law requires already the conjecture concerning the resummation of long orbits and could not be derived.

We want to check the conjecture (34) in the case of two extreme examples of Riemann surfaces. The first is the regular Riemann surface which was used as a studying object in the preceding chapters. This surface is ill suited for a test of the conjecture (34) because the lengths possess extremely high degeneracies. In [17] we have shown that the mean degeneracy $\widehat{g}$ increases exponentially as $\widehat{g} \sim 8\sqrt{2}\frac{e^{l/2}}{l}$ which is a consequence of the high symmetry the system possesses. This causes serious problems because (34) is expected to give only the leading term, and as in the case of the classical Riemann–Siegel formula there should be correction terms which smooth the discontinuities caused by (27). These discontinuities are extremely large in the case of the regular Riemann surface, because the truncation condition (27) leads with increasing energy to large jumps proportional to $\widehat{g}$.

The other case is an asymmetric Riemann surface which is nearly optimally suited for a test of (34) because the lengths are at most fourfold degenerated. A twofold degeneracy is due the time–reversal the system obeys, and a further twofold degeneracy is possible because of the parity symmetry which even the asymmetric Riemann surfaces possess. Therefore the unknown correction terms should be smaller in the asymmetric case. In our earlier study of energy–level statistics we have computed the lower part of the energy spectra for 40 different asymmetric Riemann surfaces [18]. Among these Riemann surfaces we now choose one which avoids near–degeneracy between the first 7 energy levels. The levels of the chosen surface obey $E_{n+1} - E_n > 0.6$ for $n < 7$, where the mean level spacing is unity. This means that the leading term $\cos\{\pi\bar{\mathcal{N}}(E)\}$ in (34) already yields good approximations for the spacings between the levels E_n, so that the truncated sum should only give relatively small corrections. Thus this Riemann surface is ideally suited for a test of (34).

In figure 6a we display the conjecture (34) for the regular octagon using the complete pseudo–length spectrum up to $L = 18.092$ allowing an evaluation up to $E = 8.54$. The dots represent the location of the energy levels as obtained from our finite–element computations [4]. These energy levels are degenerated. The curve shown in fig.6a is strongly fluctuating which prevents a determination of the desired zeros. This is a special effect due to the already discussed high symmetry of this system. The fluctuations are much more modest in the more generic case of the asymmetric octagon as shown in figure 6b. However, the zeros do not agree with the energy levels known from our finite–element computation. Furthermore, one recognizes in the case of the asymmetric octagon that the leading

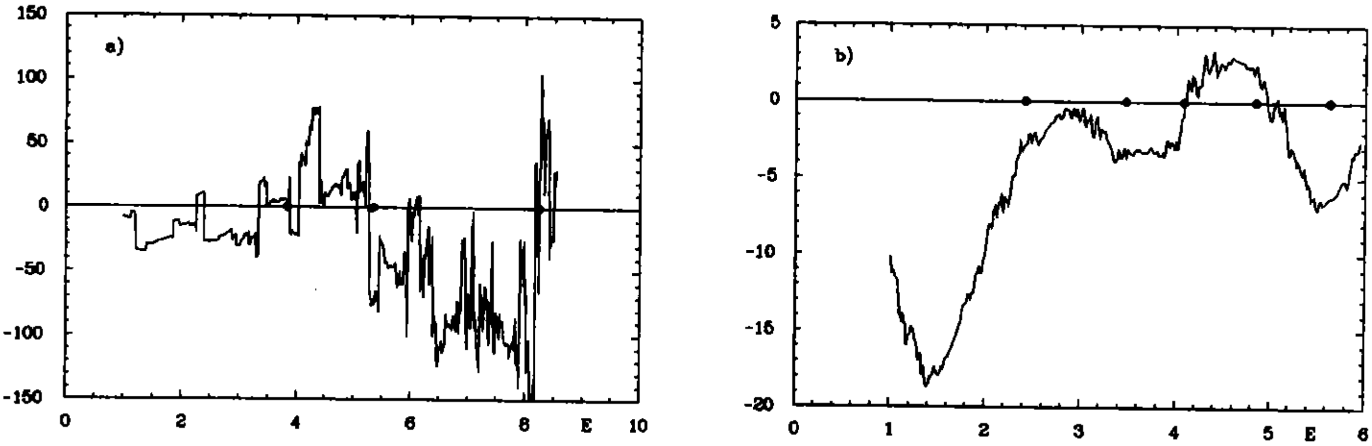

Figure 6: The Berry–Keating conjecture (34) is tested for the regular octagon a) and for an asymmetric octagon b).

term 2 $\cos(\pi\bar{N}(E))$ is completely surmounted by the orbit sum leading to amplitudes of order 20, i. e. ten times the contribution of the "leading" term. The asymmetric case allows the evaluation of the conjecture only up to $E = 5.94$ because the length spectrum is known only up to $L = 15$.

Thus the results presented in figure 6a,b do not support the conjecture. Rather they indicate that the good results obtained in refs.[13,14] are a consequence of the convergence of the corresponding Dirichlet series on the critical line, while in the present case there is no convergence on the critical line, and therefore, the conjecture breaks down.

VIII A smoothed version of the Riemann–Siegel lookalike formula

The large fluctuations seen in fig.6a in the case of the regular octagon are due to the exponentially growing degeneracies of the lengths of the pseudo–orbits, as already mentioned. Now we "derive" a smoothed version of the conjecture (34) based on a formal manipulation in analogy to [11].

The starting point is the functional equation (19) on the critical line, where we insert the Dirichlet series (25). The unity in (25) is taken into account by the choice $A_0 = N_0 = 1$. Then one obtains the formal functional equation

$$e^{-i\pi\bar{N}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}} e^{iS_\alpha} = e^{i\pi\bar{N}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}} e^{-iS_\alpha} \quad \text{with} \quad S_\alpha = p \ln N_\alpha \ . \tag{35}$$

Applying the operation

$$\int_{-\infty}^{\infty} dE'\, e^{-\frac{(E'-E)^2}{2\sigma^2}}\, e^{ixE'} \tag{36}$$

to this formal functional equation leads to

$$e^{-i\pi\bar{N}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}}\, e^{ip\ln N_\alpha} e^{-\frac{\sigma^2}{2}\left(x-\pi\bar{d}(E)+\frac{\ln N_\alpha}{2p}\right)^2} = \tag{37}$$

$$e^{i\pi\bar{N}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}}\, e^{-ip\ln N_\alpha} e^{-\frac{\sigma^2}{2}\left(x+\pi\bar{d}(E)-\frac{\ln N_\alpha}{2p}\right)^2}$$

where the expansions

$$\bar{N}(E') \simeq \bar{N}(E) + \bar{d}(E)(E'-E) \quad \text{and} \quad S_\alpha(E') \simeq p\ln N_\alpha + \frac{\ln N_\alpha}{2p}(E'-E)$$

have been used. As discussed in [11] this manipulations are only of a formal nature and therefore constitute no proof of the relation (37). The first point is that one starts with the functional equation

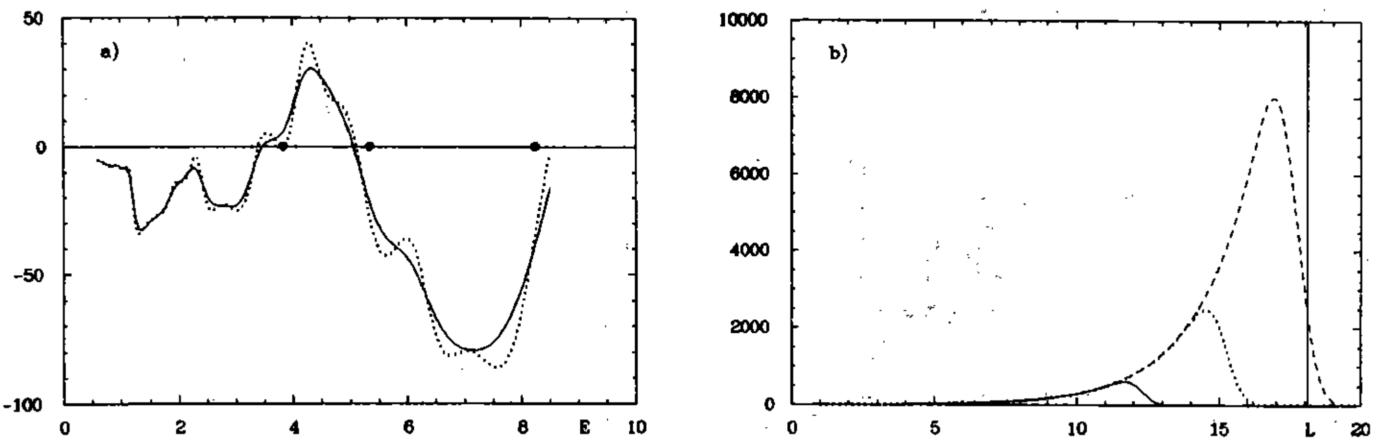

Figure 7: a) The conjectured approximation (39) is shown for $\sigma = 10$ (solid curve) and for $\sigma = 15$ (dotted curve). b) The orbit–selection function (40) is presented for $\sigma = 10$ at $E = 4$ (solid), $E = 6$ (dotted) and $E = 8$ (dashed). The vertical line at $L = 18.092$ marks the length up to which the pseudo–length spectrum is completely known.

of the zeta–function, where one has inserted a representation which does not converge on the critical line in general. The second flaw is the interchange of integration and summation in the "derivation" of (37) which is only valid if the sum is absolutely convergent.

Now (37) is integrated over x from zero to infinity yielding

$$e^{-i\pi\bar{\mathcal{N}}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}} e^{ip\ln N_\alpha} \operatorname{erfc}\left(\frac{\sigma}{\sqrt{2}2p}(\ln N_\alpha - 2\pi p\bar{d}(E)\,)\right) = \tag{38}$$

$$e^{i\pi\bar{\mathcal{N}}(E)} \sum \frac{A_\alpha}{\sqrt{N_\alpha}} e^{-ip\ln N_\alpha} \operatorname{erfc}\left(-\frac{\sigma}{\sqrt{2}2p}(\ln N_\alpha - 2\pi p\bar{d}(E)\,)\right) \quad .$$

With $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$ one arrives at

$$e^{i\pi\bar{\mathcal{N}}(E)}\, Z(\frac{1}{2} + ip) \sim \sum \frac{A_\alpha}{\sqrt{N_\alpha}} \cos(\pi\bar{\mathcal{N}}(E) - p\ln N_\alpha) \operatorname{erfc}\left(\frac{\sigma}{\sqrt{2}2p}(\ln N_\alpha - 2\pi p\bar{d}(E)\,)\right) \tag{39}$$

This relation represents a smoothed version of the conjecture (34). In the limit $\sigma \to \infty$ the original conjecture is recovered because of $\lim_{x\to\infty} \operatorname{erfc}(x) = 0$ and $\lim_{x\to-\infty} \operatorname{erfc}(x) = 2$, i.e. pseudo–orbits with lengths $L_\alpha = \ln N_\alpha > 2\pi p\bar{d}(E)$ are suppressed whereas their contribution is taken into account by the pseudo–orbits of length $L_\alpha < 2\pi p\bar{d}(E)$. The limit $\sigma \to 0$ yields the representation which one would have obtained if the Dirichlet series would have been at least conditionally convergent on the critical line. The crucial point is that the intermediate range of σ corresponds to a soft cut–off in the conjecture resulting in a formula without any discontinuities. Following the philosophy in [10], one expects that the correction terms which should smooth this discontinuities should be much smaller.

In figure 7a an evaluation of (39) is shown in the case of the regular octagon for $\sigma = 10$ and $\sigma = 15$, respectively. The large fluctuations are now absent. Larger values of σ lead to increasing fluctuations tending towards the result shown in figure 6. Remarkably, the zeros seem to occur roughly at the right places. However, only the first zero at $E = 3.83$, which is threefold degenerated, is correct. The curve at $\sigma = 10$ shows a slight tendency towards a threefold zero and at $\sigma = 15$ three adjacent zeros occur. The next zeros are of the wrong order, because the zero at $E = 5.35$ should be fourfold and the one at $E = 8.25$ twofold, whereas the figure reveals an odd order of the zeros.

To be sure, that the smoothed cut–off does not demand pseudo–orbits of length $L_\alpha > 18.092$, which are not taken into account, one can compute an analogue of the orbit–selection function (5) for

(39). Because of $N^p(L) \sim e^L$ and $\frac{1}{\sqrt{N_\alpha}} = e^{-L_\alpha/2}$ one is led by replacing $A_\alpha \cos(\pi \bar{\mathcal{N}}(E) - p \ln N_\alpha)$ by 1 to

$$r(L) := e^{L/2} \text{ erfc } \left(\frac{\sigma}{\sqrt{2}2p}(L - 2\pi p d(E')) \right) \quad . \tag{40}$$

For $\sigma = 10$ function (40) is shown in figure 7b for the energies $E = 4, 6$ and 8. Only for $E \simeq 8$ pseudo-orbits of length $L > 18.092$ are necessary, so that missing pseudo-orbits cannot be the cause of the failure at least at the fourfold zero at $E = 5.35$.

Therefore, we conclude that there must be other correction terms in addition to the suggested ones which should smooth the discontinuities, if the conjecture is correct at all. One possibility is that (34) or (39) are only correct for systems having Dirichlet series representing the zeta-function which are conditionally convergent on the critical line. The other possibility is that the semiclassical limit is reached unusually late in this case. It is possible that the resummed tail possesses lengths and altered Dirichlet coefficients which are too distinct from the original ones to be simply replaced by the head. Nevertheless, it may be that the approximation is useful in the semiclassical limit. It is therefore too early to decide on the validity of the conjecture at this stage.

IX Summary and Discussion

In this paper we have addressed the question of how to obtain quantal energies from the classical periodic orbits for systems whose classical limit is strongly chaotic. The general framework has been Gutzwiller's periodic orbit theory. But in order to avoid the problems connected with possible corrections of higher order in $\hbar$, we have considered the motion of a particle sliding freely on a compact Riemann surface with genus two. In this case Gutzwiller's trace formula is exact, since it is identical to the Sélberg trace formula. Since the mathematical problems encountered, in particular the problems of convergence, are certainly not caused by the fact that the Selberg trace formula is exact, we can conclude that these problems will show up in the same manner in the treatment of more general systems for which the trace formula is only semiclassically valid. All attempts which have been discussed in this paper to determine the quantal energies had to fight against the entropy barrier as the main obstruction hiding the critical line. Two main roads for approaching the critical line have been discussed. In the first case one deals with smoothed versions of the trace formula leading to the so-called periodic-orbit sum rules [3,4]. In the second case one deals directly with the Selberg zeta function.

In the first part of this paper we have discussed in detail the evaluation of two periodic-orbit sum rules using a length spectrum which is completely known up to a certain cut-off length. The remarkable point is that the sum over the long periodic orbits can be well approximated by a universal remainder (8) which is determined by the orbit-selection function, eq.(5), and which takes into account the leading asymptotic proliferation of the length spectrum (Huber's law). Since Huber's law is responsible for the entropy barrier at $\text{Re } s = 1$, inclusion of the remainder (8) is already sufficient for crossing the entropy barrier. This becomes especially clear in the case of the Breit-Wigner smearing which is absolutely convergent only for $\alpha > 1$. But if the rest term (15), calculated for $\alpha < 1$ by analytic continuation, is added, the sum rule involves in the semiclassical limit the Selberg zeta function along the line $\text{Re } s = \frac{1+\alpha}{2}$, i.e. in the case $\alpha = 0.3$ shown in fig.3 along the line $\text{Re } s = 0.65$ which is well behind the entropy barrier. Nevertheless it appears that the Gaussian smoothing (11) is the most favourable choice since it is absolutely convergent for any finite smearing $\varepsilon > 0$.

Until recently the main problem with the Selberg zeta function was that the Euler product (17) is undefined for $\text{Re } s < 1$ and thus useless for a direct analytic continuation. The situation changes, however, drastically if instead of (17) the Dirichlet series (25) is used. With the help of (30) it is possible in principle to check for a given problem whether $\sigma_c < \sigma_a = 1$ and thus whether the entropy barrier can be crossed. As shown in fig.4a,b this is indeed the case for the two Riemann surfaces discussed in this paper. However, in both cases one obtains $\frac{1}{2} < \sigma_c < 1$, which implies that the critical line is still beyond reach. For systems where the Dirichlet series converge on the critical line, we have derived the quantum condition (28) which has been successfully tested in [13,14]. For the two Riemann surfaces we have shown that the quantization rule (28) or the Riemann–Siegel lookalike formula (34) do not reproduce the correct eigenvalues. This was expected in the case of (28) since its derivation breaks down for $\sigma_c > \frac{1}{2}$. Concerning the Riemann–Siegel lookalike formula (34) our results do not give support to the Berry–Keating conjecture [10,11]. In sect. VIII we have discussed a smoothed version of the Riemann–Siegel lookalike formula and have checked it in the case of the regular octagon. Although the large fluctuations of the original formula are now absent, the result does not yield the zeros with the correct degeneracies. We cannot exclude, however, that the conjecture becomes only true in the semiclassical limit, even though there are several independent results which indicate that semiclassical laws like Weyl's law are valid in the mean down to the ground state. At present we are inclined to believe that eq.(28) is the correct much sought–after rule for quantizing chaos for those chaotic systems whose Dirichlet series converge on the critical line, but that the Berry–Keating conjecture, to be applied in the general case, still needs a better foundation.

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