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I. Introduction

The investigation of the dynamics of holomorphic or anti-holomorphic λ-differentials on Riemann surfaces was initiated by the observation, that these differentials enter string theory via the Faddeev-Popov-procedure, albeit as local sections of Grassmann valued line bundles. Furthermore, it is well known that dual line bundles play an important role in the description of modular deformations of complex structures.
As usual in physics, the dynamics of conformal fields was analyzed mainly in terms of local sections living in a certain patch, implying a local operator formalism. Several useful formalisms have been set up to obtain global information via local operators, see e.g. [1,2].
Another powerful approach was initiated by the pioneering work of Krichever and Novikov [3,4], which permits the formulation of a global operator formalism on twice-punctured Riemann surfaces of arbitrary genus (see [5], and various recent issues of Phys. Lett. B), and leads to a considerable clarification of concepts.
However, the local approach is by no means obsolete, because the conceptual simplicity of the global approach has its price in form of certain technical difficulties. Very recently, the bases of holomorphic λ-differentials on twice-punctured surfaces introduced by Krichever and Novikov were generalized to the case of arbitrary many punctures [6,7]. This will be discussed, and further elaborated, in this talk.

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II. Holomorphic λ-Differentials on Punctured Riemann Surfaces

In this talk I will denote by $\Omega^{\lambda}_{-D}(X)$ the space of meromorphic λ-differentials ω on a compact Riemann surface X of genus g with the property $\mathrm{ord}_P(\omega) \geq D(P)$ for all points $P \in X$, where D is a divisor on X, and λ takes integer values.
The dimensions of the spaces of holomorphic λ-differentials displayed in Table 1 are well-known consequences of the Riemann-Roch theorem

$$\dim\Omega^{\lambda}_{-D}(X) = (2\lambda-1)(g-1) - \deg D + \dim\Omega^{1-\lambda}_{D}(X)$$

and the isomorphism $\Omega^{\lambda}(X) \to \Omega^{0}_{\lambda K}(X)$.

$$\dim\Omega^{\lambda}(X) = \begin{cases} (2\lambda-1)(g-1) & \text{if } (\lambda-1)(g-1) > 0 \\ g & \text{if } \lambda = 1 \\ 1 & \text{if } \lambda = 0 \text{ or } g = 1 \\ 0 & \text{if } \lambda(g-1) < 0 \end{cases}$$

Table 1: dimensions of spaces of holomorphic differentials

These results were exploited in different guises in string theory and conformal field theory. However, until recently such a table was lacking for meromorphic λ-differentials. The gap can be filled in the following fashion [7]:
A point $P \in X$ is called <u>λ-generic with respect to the divisor D</u> if the Wronskian of a basis of $\Omega^{\lambda}_{-D}(X)$ does not vanish in P or $\dim\Omega^{\lambda}_{-D}(X) = 0$.
Due to the compactness of X there are only finitely many non λ-generic points for a fixed divisor D.
Then it is easy to prove the following Lemma [6,7]:

Let P be a λ-generic point with respect to the divisor D. Then for any positive integer γ holds

$$\dim\Omega^{\lambda}_{-D-\gamma P}(X) = \begin{cases} \dim\Omega^{\lambda}_{-D}(X)-\gamma & \text{if } 0 \leq \gamma \leq \dim\Omega^{\lambda}_{-D}(X) \\ 0 & \text{if } \gamma \geq \dim\Omega^{\lambda}_{-D}(X) \end{cases}$$

For a vanishing divisor the non-generic points are of course the λ-Weierstrass-points. The proof is an immediate generalization of a classical proof of the Weierstrass gap theorem [8] and can be found in [7].
A divisor D will be called λ-generic if those finitely many points where the divisor does not vanish can be enumerated in such a way that

$$D = \sum_{j=1}^{N} D(P_j)P_j \quad \text{and for all } P_j:$$

If $D(P_j) > 0$ then P_j is λ-generic with respect to $\sum_{k=1}^{j-1} D(P_k)P_k$, and

if $D(P_j) < 0$ then P_j is $(1-\lambda)$-generic with respect to $\sum_{k=1}^{j-1} -D(P_k)P_k$.

Henceforth, the divisor D will be assumed to be λ-generic throughout, with the value of λ determined by the context.

By an alternate application of the Riemann-Roch theorem and the Lemma it is possible to calculate Table 2 [7]:

A) $|(2\lambda-1)(g-1)| > g$ and D is λ-generic:

$$\dim\Omega^{\lambda}_{-D}(X) = \begin{cases} (2\lambda-1)(g-1)-\deg D & \text{if } \deg D \le (2\lambda-1)(g-1) \\ 0 & \text{if } \deg D \ge (2\lambda-1)(g-1) \end{cases}$$

B) $\lambda = 0$ or $g = 1$ and D is 0-generic:

$$\dim\Omega^{0}_{-D}(X) = \begin{cases} 0 & \text{if } \deg D \ge 1-g \text{ and } \exists P\colon D(P) > 0 \\ 1 & \text{if } \deg D \ge -g \text{ and } \forall P\colon D(P) \le 0 \\ 1-g-\deg D & \text{if } \deg D \le -g \end{cases}$$

C) $\lambda = 1$ and D is 1-generic:

$$\dim\Omega_{-D}(X) = \begin{cases} g-1-\deg D & \text{if } \deg D \le g-1 \text{ and } \exists P\colon D(P) < 0 \\ g-\deg D & \text{if } \deg D \le g \text{ and } \forall P\colon D(P) \ge 0 \\ 0 & \text{if } \deg D \ge g \end{cases}$$

Table 2: dimensions of spaces of meromorphic differentials

The proof proceeds by induction on the number N of punctures appearing in the divisor D [7].

$Y = X\backslash\{P_1,\dots P_N\}$ will denote the punctured surface associated to X and the divisor $D = \sum_{j=1}^{N} D(P_j)P_j$.

An immediate consequence of Table 2 is the Corollary:

A) $|(2\lambda-1)(g-1)| > g$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is λ-generic for every enumeration of points and $\deg D = 2\lambda(g-1)-g$, then there exists an up to multiplication by a constant unique λ-differential ω which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(\omega) = D(P_j)$.

B) $\lambda = 0$ or $g = 1$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is 0-generic for every enumeration of points and $\deg D = -g$, and $\exists P_j\colon D(P_j) > 0$, or if D vanishes, then there exists an up to multiplication by a constant unique function f which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(f) = D(P_j)$.

C) $\lambda = 1$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is 1-generic for every enumeration of points and

$\deg D = g-2$ and $\exists P_j\colon D(P_j) < -1$

or

$\deg D = g-2$ and $\exists\{P_j,P_k\}\colon j \ne k$ and $D(P_j) = D(P_k) = -1$

or

$\deg D = g-1$ and $\forall P_j\colon D(P_j) \ge 0$,

then there exists an up to multiplication by a constant unique differential μ which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(\mu) = D(P_j)$.

Once the correct pole orders are identified it is not hard to write down representations of the uniquely determined λ-differentials in terms of prime forms and theta functions. (For an introduction see e.g. [9]). This has been done for the case N = 2 already by Bonora et al. [5], and in [10]. Generalization to $N \ge 2$ requires only minor modifications:

If the picture of a divisor D of degree 0 under the usual map from the Picard group Pic(X) to the Jacobian torus Jac(X) of X is again denoted by D (by an obvious abuse of notation, because D now denotes both the divisor and its equivalence class modulo principal divisors), and if $\Delta = \Delta_B+(g-1)B$ is the Riemann divisor class, then a quasiperiodic $g/2$-differential σ with $\mathrm{div}(\sigma) = 0$ can be constructed [11]:

$$\sigma(P) = \vartheta\Big(P-\sum_{j=1}^{g} S_j+(g-1)B+\Delta_B\Big)\Big/\prod_{j=1}^{g} E(P,S_j)$$

provided the divisor $E = \sum_{j=1}^{g} S_j$ satisfies $\dim\Omega_{-E}(X) = 0$. B is of course the base point of the Jacobian map for divisors with nonvanishing degree. I usually omit the period matrix in the argument of the theta function.

For $g \ge 2$, the λ-differentials mentioned in the Corollary are in the generic case:

$$\omega_D(P) = \vartheta(P+D-(2\lambda-1)(g-1)B-(2\lambda-1)\Delta_B)\cdot\sigma(P)^{2\lambda-1}\cdot E(P,P_1)^{2\lambda(g-1)-g}\cdot \prod_{j=2}^{N}\left(\frac{E(P,P_j)}{E(P,P_1)}\right)^{D(P_j)}$$

and in the special cases

$\lambda = 0,\ D = 0\colon f_0(P) = 1$

$\lambda = 1,\ D \ge 0$:

$$\mu_D(P) = \vartheta(P+D-g\cdot B-\Delta_B)\cdot\frac{\sigma(P)}{E(P,B)}\cdot E(P,P_1)^{g-1}\cdot\prod_{j=2}^{N}\left(\frac{E(P,P_j)}{E(P,P_1)}\right)^{D(P_j)}$$

The exponents in the above formulas are determined by the required pole orders and the correct transformation behaviour, while the arguments of the theta functions are fixed by the periodicity properties.

In the generic case, the missing g zeros of $\omega_D(P)$ are determined by the Riemann theorem [12]:

$$\sum_{j=1}^{g} Q_j+D = 2\lambda[\Delta_B+(g-1)B]$$

while in the case $\lambda = 1$, $D \ge 0$ the Riemann theorem yields for the missing g-1 zeros

$$\sum_{j=1}^{g-1} Q_j+D = 2[\Delta_B+(g-1)B].$$

In the singular case $g = 1$ all λ-differentials have to be constructed from $\lambda = 0$. Here it is convenient to use Weierstrass σ-functions, like in [3]. For the case of the twice-punctured (super-)torus this was pursued and further elaborated in [13], where the central extension of the corresponding Krichever-Novikov algebra is also displayed.

The case $g = 0$ will be considered in detail in section IV.
If the set of punctures $\Gamma = \{P_1,...P_N\}$ admits only divisors which are λ-generic, and if those divisors, whose degrees and pole orders meet the conditions of the Corollary, are λ-generic for every enumeration of the punctures, then it is possible to write down the Krichever-Novikov basis $B_\lambda(Y)$ of holomorphic λ-differentials on Y which are meromorphic on X. This was done in [6,7].
In the notation introduced in [7] the result is:

A) $|(2\lambda-1)(g-1)| > g$:
$$B_\lambda(Y) = \{\omega_2(\beta);\ \beta \in \mathbb{Z}\} \cup \Big\{ \bigcup_{j=3}^{N} \{\omega_j(\beta);\ \beta < 0\}\Big\}$$

B) $\lambda = 0$:
$$B_0(Y) = \{f_2(\beta);\ \beta < -g \text{ or } \beta > 0\} \cup \Big\{ \bigcup_{j=3}^{N} \{f_j(\beta);\ \beta < -g\}\Big\} \cup \Big\{ \bigcup_{j=2}^{N} \{h_j(\beta);\ -g \le \beta < 0\}\Big\} \cup \{1\}$$

C) $\lambda = 1$:
$$B_1(Y) = \{\mu_2(\beta);\ \beta < -1 \text{ or } \beta \ge g\} \cup \Big\{ \bigcup_{j=3}^{N} \{\mu_j(\beta);\ \beta < -1\}\Big\} \cup \{\mu_0(\beta);\ 0 \le \beta < g\} \cup \Big\{ \bigcup_{j=2}^{N} \{\nu_j\}\Big\}$$

A simple proof of the completeness of these sets exploits Table 2 [7], and works in the following fashion: The linear independence is easily proved from the mismatch of pole orders. Furthermore, $\Omega^\lambda_{-E}(X) \subset \Omega^\lambda_{-D}(X)$ for $E > D$, and as any meromorphic differential on X which is holomorphic on Y is contained in $\Omega^\lambda_{-D}(X)$ with D chosen small enough, it is sufficient to observe that the $(m+1)$-dimensional space $\Omega^\lambda_{-D}(X)$ with
$$D = [2\lambda(g-1)-g-m-\sum_{j=2}^{N}\beta_j]P_1 + \sum_{j=2}^{N}\beta_j P_j$$
and
A) $|(2\lambda-1)(g-1)| > g$: $\beta_j < 0$, $m+\sum_{j=2}^{N}\beta_j \ge 0$

B) $\lambda = 0$: $\beta_j < -g$, $m+\sum_{j=2}^{N}\beta_j \ge 0$

C) $\lambda = 1$: $\beta_j < -1$, $m+\sum_{j=2}^{N}\beta_j \ge g$

is spanned by exactly $m+1$ differentials in $B_\lambda(Y)$.

III. Internal Time and Global Laurent Expansions on Punctured Riemann Surfaces

Punctured Riemann surfaces carry a natural notion of internal time [3,4,6,7].
To introduce this, it is convenient to switch from
$\{\mu_0(\beta) = \mu[g-1-\beta,\beta,0,...0],\ 0 \le \beta < g\}$
to the usual basis of holomorphic differentials defined by the homology basis chosen:
$$\mu_0(\beta) = \sum_{j=1}^{g} \omega_j \oint_{a_j}\mu_0(\beta)\ , \qquad \oint_{a_j}\omega_k = \delta^j{}_k\ , \qquad \oint_{b_j}\omega_k = \Omega^j{}_k$$
Furthermore, assume all abelian differentials $\nu_2 = \mu[-1,-1,g,0,...0]$ and $\nu_j = \mu[-1,g,0,...0,-1,0,...0]$ normalized to have residue 1 in P_1.

Then every abelian differential of the kind
$$k = \sum_{j=2}^{N}\alpha^j\nu_j$$
defines an internal evolution parameter via
$$\tau(P,P_O) = \int_{P_O}^{P} d\tau$$
$$d\tau = \mathrm{Re}k - \mathrm{Re}\omega_i\cdot\oint_{a_i}\mathrm{Re}k - \mathrm{Im}\omega_i\cdot(\mathrm{Im}\Omega)^{-1\,i}{}_j\cdot[\oint_{b_j}\mathrm{Re}k - \mathrm{Re}\Omega^j{}_m\cdot\oint_{a_m}\mathrm{Re}k]$$
where summation from 1 to g is implied. If all the coefficients α^j and their sum are different from zero the time τ assumes the values $\pm\infty$ in the punctures. This notion of internal time implies, that there is another natural duality between λ-differentials and $(1-\lambda)$-differentials on Y, besides Serre duality [3,4,6]. If S_i denotes that set of indices for which
$\tau(P_j,P_O) = -\infty,\ j \in S_i$
and $C_\tau = \{P;\ \tau(P,P_O) = \tau\}$, then the dual pairing of a λ-differential ω and a $(1-\lambda)$-differential v is
$$\langle\omega,v\rangle = \frac{1}{2\pi i}\oint_{C_\tau}\omega\cdot v = \sum_{j\in S_i}\mathrm{Res}_{P_j}(\omega\cdot v)$$
To introduce global Laurent expansions on Y it is necessary to orthogonalize the bases of section II. This will be considered for $|(2\lambda-1)(g-1)| > g$:
For simplicity I assume all $\alpha^j > 0$ in the definition of k, Then the only initial point on X is P_1 and the scalar product reduces to the residue of the product in P_1. Like before, ω will be a λ-differential, while v is a $(1-\lambda)$-differential. Then, with a suitable normalization, $\omega_2(\beta)$ and $v_2(-1-\beta)$, $\beta \in \mathbb{Z}$, may serve as a starting point for the construction of orthonormal bases. This starting point is of course just the original Krichever-Novikov basis [3,4] of a twice-punctured surface, which is orthogonal from the very beginning. $B_\lambda(Y)$ can be orthonormalized via:
$$\Omega_2(\beta) = \omega_2(\beta)\ ,\ V_2(\beta) = v_2(-1-\beta)\big/\langle\omega_2(\beta),v_2(-1-\beta)\rangle$$
$$\Omega_j(\beta) = \omega_j(\beta) - \sum_{k=2}^{j-1}\sum_{\gamma}\langle\omega_j(\beta),V_k(\gamma)\rangle\ \Omega_k(\gamma)$$
$$V_j(\beta) = \Big(v_j(-1-\beta) - \sum_{k=2}^{j-1}\sum_{\gamma} V_k(\gamma)\ \langle\Omega_k(\gamma),v_j(-1-\beta)\rangle\Big):$$
$$:\Big(\langle\omega_j(\beta),v_j(-1-\beta)\rangle - \sum_{k=2}^{j-1}\sum_{\gamma}\langle\omega_j(\beta),V_k(\gamma)\rangle\langle\Omega_k(\gamma),v_j(-1-\beta)\rangle\Big)$$
Then the unique decomposition of any meromorphic λ-differential T on X which is holomorphic on Y reads
$$T = \sum_{j=2}^{N}\sum_{\gamma}\langle T,V_j(\gamma)\rangle\ \Omega_j(\gamma)$$

Because of its relevance for the analysis of initial value problems, equations like this are called equal time decomposition in physics.
As T is meromorphic, the sum over γ stops after finitely many terms. However, our experience with harmonic analysis suggests, that the above equation holds pointwise on Y, or in the mean, or in a distributional sense also for suitable classes of holomorphic λ-differentials on Y with essential singularities in some of the punctures.
If, in case $\lambda = 2$, T is interpreted as a classical energy momentum tensor, the modes

$$L_j(\gamma) = \langle T, V_j(\gamma) \rangle$$

reduce for $N = 2$ and $g = 0$ to the familiar Virasoro generators.

IV. The Virasoro Algebra for N > 2.

It is well known, that the Virasoro algebra without central extension is the algebra $Vec(S^1)$ of vector fields on the circle. From the work of Krichever and Novikov we learn, however, that it appears in string theory as the Lie algebra of the basis of meromorphic vector fields on S^2 which are holomorphic on the twice punctured sphere, and how it can be generalized to twice-punctured Riemann surfaces of arbitrary genus. The resulting Lie algebra of the basis of meromorphic vector fields holomorphic outside the two punctures was then analyzed qualitatively by calculations near the punctures [3]. This procedure was applied to the case of more punctures in [6]. Finally, of course, the structure coefficients should be calculated explicitly from the formulas of section II.
For technical reasons, I will concentrate here on the case of the N-punctured sphere [6,7,14]. This is interesting in its own, not only as a sphere approximation to the global formalism described here, but also in view of the local formalism [1,2], which comes into play by considering a single patch on Y which contains all the punctures.
Because it is not more complicated, let us consider from the beginning the action on the module of λ-differentials:

$$\mathcal{L}_v \omega = \lambda \cdot \omega \cdot \partial_z v^z + v^z \partial_z \omega \ .$$

Here and in the following equations, v will be a vector.
I will use charts U_j around the punctures, such that $z_j(P_j) = 0$, $z_1(P_j) = c_j$ for $j \geq 2$, and in the overlap regions $z_1 = c_j/(1+c_j z_j)$. The unique λ-differentials of section II are in these coordinates:

$$\omega[-2\lambda - \sum_{j=2}^{N} \beta_j, \beta_2, \ldots \beta_N] = \prod_{j=2}^{N} \left(\frac{1}{z_1} - \frac{1}{c_j} \right)^{\beta_j} \left(\frac{dz_1}{z_1^2} \right)^{\lambda}$$

and the action of the basis of meromorphic vector fields on the sphere, which are holomorphic outside the N punctures, on the corresponding basis of λ-differentials is explicitly

$$\mathcal{L}_{v_j(\beta)} \omega_j(\gamma) = -(\gamma + \lambda\beta) \cdot \omega_j(\beta + \gamma - 1)$$

$j \neq k$, $\beta < 0$, $\gamma < 0$:

$$\mathcal{L}_{v_j(\beta)} \omega_k(\gamma) = - \sum_{n=\beta-1}^{-1} [n+1+(\lambda-1)\beta] \binom{\gamma}{n-\beta+1} \left(\frac{c_k - c_j}{c_j \cdot c_k} \right)^{\beta+\gamma-n-1} \cdot \omega_j(n) - \sum_{n=\gamma-1}^{-1} [\lambda(n+1-\gamma)+\gamma] \binom{\beta}{n-\gamma+1} \left(\frac{c_j - c_k}{c_j \cdot c_k} \right)^{\beta+\gamma-n-1} \cdot \omega_k(n)$$

$k \neq 2$, $\beta \geq 0$:

$$\mathcal{L}_{v_2(\beta)} \omega_k(\gamma) = - \sum_{n=\gamma-1}^{\beta+\gamma-1} [\lambda(n+1-\gamma)+\gamma] \binom{\beta}{n-\gamma+1} \left(\frac{c_2 - c_k}{c_2 \cdot c_k} \right)^{\beta+\gamma-n-1} \cdot \omega_k(n)$$

$j \neq 2$, $\gamma \geq 0$:

$$\mathcal{L}_{v_j(\beta)} \omega_2(\gamma) = - \sum_{n=\beta-1}^{\beta+\gamma-1} [n+1+(\lambda-1)\beta] \binom{\gamma}{n-\beta+1} \left(\frac{c_2 - c_j}{c_2 \cdot c_j} \right)^{\beta+\gamma-n-1} \cdot \omega_j(n)$$

If $\beta + \gamma > 0$ part of the last equations must be decomposed further according to $j > 2$, $n \geq 0$:

$$\omega_j(n) = \sum_{m=0}^{n} \binom{n}{m} \left(\frac{c_j - c_2}{c_j \cdot c_2} \right)^{n-m} \cdot \omega_2(m)$$

For a digression on central extensions, see [14].
A closer look at other representations of this algebra and its role in conformal field theory on punctured surfaces remains for future work.

A better handling of punctured Riemann surfaces offers three paths to follow: The conceptually most clear, but maybe also most intricate, path might start off from the "explicit" formulas of section II, considering from the beginning global Riemann surfaces of arbitrary genus.
Another approach would employ the local formalisms already mentioned and, presumably, lead to an intense investigation of the Virasoro algebras on N-punctured spheres.
In a similar manner, holomorphic λ-differentials on N-punctured spheres as dynamical degrees of freedom, and as correlation functions, might be discussed in the framework of a "sewing approach" to higher genus string vertices.
Of course, much of the necessary technology has already been developed in a considerable common effort of many physicists and mathematicians, and much remains to be done.

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