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ALPs, the on-shell way

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Abstract: We study a set of conditions that must be imposed directly at the level of on-shell scattering amplitudes to obtain the coupling of axion-like particles (ALPs) to matter. We identify three conditions that allow to compute amplitudes that correspond to shift-symmetric Lagrangians, both at the level of dimension 5 operators and at the level of higher dimensional operators. In particular, one of the conditions allows one to link the 3-point amplitude between one ALP and two massive fermions to an high energy amplitude invariant under the Standard Model symmetry group, and highlight the connection with the Adler's zero. Finally, we discuss a phenomenological application, showing that in the process $\ell^+\ell^- \to \phi h$ (with $\ell^\pm$ two charged leptons, ϕ the ALP and h the Higgs boson), as a result of the structure of the 3-point and 4-point amplitudes, dimension-7 operators can dominate over the dimension-5 ones well before the energy reaches the cutoff of the theory.

Contents

1 Introduction

In recent years, the study of modern on-shell methods [1–5], together with their application to phenomenological issues, has been gaining much attention and giving fruitful results. Without a doubt, the most innovative feature of these methods consists of writing down scattering amplitudes by relying on nothing but the covariance of the S-matrix under little-group transformations of the Lorentz group [6], thus putting aside the need for fields and Lagrangians. The consequences and applications of the on-shell approach are far-reaching. On the phenomenological side, much progress has been made, for instance, in the computation of loop-integrals and anomalous dimensions [7–13], in the understanding of the Standard Model (SM) and of Effective Field Theories (EFTs) [14–32], in the study of the physics of higher-spin dark-matter [33] and also in the formulation of neutrino oscillations [34, 35]. One less pursued question is that of establishing a precise connection between the physical properties of infra-red (IR) on-shell amplitudes to the physics of the ultra-violet (UV) [14, 22, 28, 36–39]. In the Lagrangian approach, the different assumptions about the UV dynamics are translated in the IR to specific EFTs and power counting (*e.g.* SILH [40], HEFT [41]), giving us much more control over the properties of the low-energy amplitudes. The same exercise still needs to be carried out in a systematic way in the on-shell approach, in which the UV properties are reconstructed from the IR amplitudes.

Along these lines, an interesting problem that can be studied using on-shell methods is the one concerning the physics of axion-like particles (ALPs). From the usual quantum field theoretical perspective, it is well known that ALPs interactions must be invariant under a shift symmetry if the underlying global symmetry is exact. This has extensive physical consequences, among which the existence of *soft-theorems* and the appearance of the so-called *Adler's zero*, which state that amplitudes involving ALPs (or, equivalently, Nambu–Goldstone bosons) are either regular or vanish in the limit in which the ALP momentum becomes soft [42–45]. It is not straightforward, however, to invert the reasoning of the field theoretical approach and ask: *what are the physical properties that amplitudes should satisfy in order to recover shift-symmetry?* Given that on-shell methods allow us to write the amplitudes without assuming any Lagrangian or symmetry, they are the ideal framework to approach this question. Previous studies in the literature have partially tackled it, showing that amplitudes only involving exactly massless ALPs make manifest soft-theorems [46], as well as Adler's zero conditions [38, 39, 47–49]. In addition, it is possible to read off from IR properties of these amplitudes of ALPs the structure of the coset group associated to the spontaneous symmetry breaking in the UV [37].

In this paper, we continue to further explore this direction and investigate the coupling between ALPs and other Standard Model (SM) fields. We will first reproduce the well-known results of ALP 3-point couplings to massive fermions and vectors, which will then be uplifted to the study of 3- and 4-point amplitudes with SM particles in the unbroken electroweak phase. Understanding the properties of the simplest scattering amplitudes of ALPs and matter particles will allow us to generalize this procedure and construct higher-point functions in a systematic way.

The paper is organized as follows. In Section 2, we formulate our approach in terms of on-shell methods and apply it to generic 3-point amplitudes involving one ALP. In Section 3, we match the massive amplitudes in the IR to the massless ones in the UV, while specializing to the SM particle content, discussing also how to handle electroweak symmetry breaking effects. We also comment on the physical interpretation of the shift-symmetry breaking invariants introduced in Ref. [50]. In Section 4, we build higher-point contact amplitudes up to dimension 8 in the ALP scale, while some phenomenological applications are discussed in Section 5. In particular, we study the production of an ALP in association to a Higgs in a lepton collider, $\ell^-\ell^+ \to \phi h$, and show that higher-order contact operators can give the leading contribution to the cross-section at high-energies. Finally, we conclude in Section 6. We also add a number of appendices with more technical material: in Appendix A, we present our conventions for spinors; in Appendix B, we propose an alternative on-shell derivation of the connection between Yukawa couplings and fermions masses; finally, in Appendix C, we present detailed computations of the running of the 4-particle amplitude involving one ALP, a fermion-antifermion pair and one Higgs doublet.

2 ALP couplings to matter

2.1 General remarks

On-shell techniques have been previously used in the literature to study amplitudes involving massless scalar particles. More precisely, under the assumption that these amplitudes vanish as any of the momenta go soft, *i.e.* $p \to 0$, it is possible to derive a number of features of such scalars and in some cases even completely determined the underlying theory [14, 36–39, 46, 47]. However, these analyses are restricted to amplitudes with nothing but scalars, and therefore do not apply when they interact with other matter fields, for instance SM particles, making necessary the addition of extra assumptions. One of the goals of the present paper is then to extend this discussion and characterise the interactions of ALPs with other particles from an on-shell perspective.

Our starting point are amplitudes involving ALPs, hereafter denoted by ϕ, in the limit in which the ALP momentum $p_\phi \to 0$. We will thus be focusing on

$$\lim_{p_\phi \to 0} \mathcal{A}[\phi,\ \mathcal{O}]\,, \tag{2.1}$$

where $\mathcal{A}$ denotes the amplitude and $\mathcal{O}$ is a set of other arbitrary particles. In this section, we will restrict ourselves to 3-point amplitudes, $\mathcal{A}_3$, with one (massless) ALP and two other massive particles, which we will in turn take to have spin 0, 1/2 and 1. In on-shell language, 3-point amplitudes are special objects, since they are completely fixed by little-group covariance and have constant coefficients. Moreover, massless 3-point amplitudes can only make sense when considering complex momenta, since the three-body kinematics forces the amplitude to vanish for real momenta [2, 5]. 3-point amplitudes can also be used as building blocks to form the so-called "constructible" higher-point amplitudes [26, 51, 52]. Contact interactions, that cannot be constructed in this way, will be discussed in Section 4. For the 3-point amplitudes, we cannot apply the reasoning behind the limit (2.1) directly, because this would leave us with a non-physical 2-point amplitude. We must then consider them as part of a generic $(n+1)$-point amplitude $\mathcal{A}_{n+1}[\phi,\ \mathcal{O}]$, where now $\mathcal{O}$ is a set of at least $n \geq 3$ other particles. We will now focus on the coupling between one ALP and two massive particles of the same species (we will relax this condition in Section 2.5). The amplitudes we are interested in are written as

$$\mathcal{A}_3\Big[\phi\, \mathcal{P}_1^{I_1,\cdots,I_{2s}} \bar{\mathcal{P}}_2^{J_1,\ldots,J_{2s}}\Big]\,, \tag{2.2}$$

where $\mathcal{P}$ is a particle of spin s, $\bar{\mathcal{P}}$ its antiparticle, $\{I_1,\cdots I_{2s}\}$, $\{J_1,\cdots,J_{2s}\}$ are symmetrized massive little-group indices and the subscripts denote the labels of the momenta. For more details on the notation and conventions, we refer the reader to Appendix A. The soft-limit gives us

$$\lim_{p_\phi \to 0} \mathcal{A}_{n+1}\Big[\phi \cdots \mathcal{P}_{i,p}^{I_1,\cdots I_{2s}} \cdots\Big] = \\ \lim_{p_\phi \to 0} \sum_{i=1}^{n} \mathcal{A}_n\Big[\cdots \mathcal{P}_{i,p+p_\phi}^{K_1,\cdots K_{2s}} \cdots\Big] \frac{\epsilon_{K_1 J_1} \cdots \epsilon_{K_{2s} J_{2s}}}{(p+p_\phi)^2 - m_{\mathcal{P}_i}^2} \mathcal{A}_3\Big[\phi \mathcal{P}_{i,p}^{I_1,\cdots,I_{2s}} \bar{\mathcal{P}}_{i,-p-p_\phi}^{J_1,\ldots,J_{2s}}\Big]\,, \tag{2.3}$$

where we sum over all particles $\mathcal{P}$,[1] which are labeled by i in $\mathcal{A}_{n+1}$ and $\mathcal{A}_n$, and ϵ_{IJ} is the Levi–Civita tensor that takes into account the sum over spin configurations of the propagating particle (see Fig. 1). We can understand Eq. (2.3) as follows. Since the ALP couples to two particles of the same species, in the $p_\phi \to 0$ limit the momentum

[1]For simplicity, we consider only external particles $\mathcal{P}$, but analogous equations hold for external antiparticles $\bar{\mathcal{P}}$.

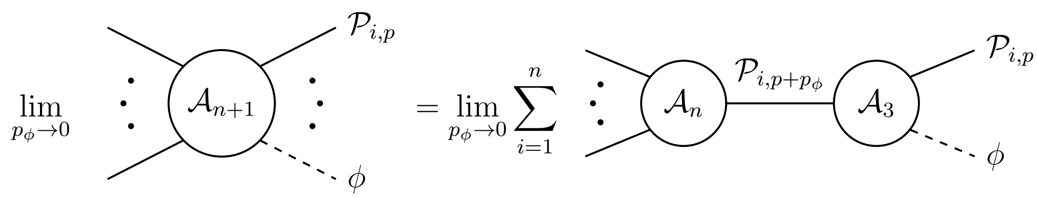

Figure 1. Diagramatic representation of Eq. (2.3). The index i labels the particle species.

of the particle in the propagator is very close to $p^2 = m^2_{\mathcal{P}_i}$, *i.e.* the particle is very close to its mass shell. According to *polology* [6], the total amplitude will then factorize into the product of the two sub-amplitudes multiplied by the intermediate propagator and hence we obtain Eq. (2.3). We can rewrite it more compactly as

$$\lim_{p_\phi\to 0}\mathcal{A}_{n+1}\Big[\phi\cdots\mathcal{P}^{I_1,\cdots I_{2s}}_{i,p}\cdots\Big] = \lim_{p_\phi\to 0}\sum_{i=1}^{n}\mathcal{A}_n\Big[\cdots\mathcal{P}^{K_1,\cdots K_{2s}}_{i,p}\cdots\Big]\times(\mathcal{S}_{\mathcal{P}_i})^{I_1,\cdots,I_{2s}}_{K_1,\cdots K_{2s}}, \tag{2.4}$$

with the soft factor given by

$$(\mathcal{S}_{\mathcal{P}_i})^{I_1,\cdots,I_{2s}}_{K_1,\cdots K_{2s}} = \frac{\epsilon_{K_1J_1}\cdots\epsilon_{K_{2s}J_{2s}}}{2p\cdot p_\phi}\mathcal{A}_3\Big[\phi\mathcal{P}^{I_1,\cdots,I_{2s}}_{i,p}\bar{\mathcal{P}}^{J_1,\ldots,J_{2s}}_{i,-p-p_\phi}\Big]\,. \tag{2.5}$$

In the last step we have simplified the propagator using the fact that ϕ is massless, which gives $(p+p_\phi)^2 - m^2_{\mathcal{P}_i} = 2p\cdot p_\phi$. We observe that, under our hypothesis, the amplitude $\mathcal{A}_{n+1}$ has the same kinematical configuration of $\mathcal{A}_n$ without the ALP, *i.e.* the ingoing particle $\mathcal{P}_i$ is on-shell in both cases. Having said this, we impose the following condition:

Soft factorization condition

When, in the limit $p_\phi\to 0$, the amplitude $\mathcal{A}_{n+1}[\phi,\ \mathcal{O}]$ factorizes as

$$\mathcal{A}_{n+1}[\phi,\ \mathcal{O}] \xrightarrow{p_\phi\to 0} \mathcal{A}_n[\mathcal{O}]\times\mathcal{S}, \tag{2.6}$$

then we demand that no poles appear in $\mathcal{S}$ in ***all*** *phase space for real momenta. This condition applies to 3-point amplitudes $\mathcal{A}_3$ in which the ALP interacts with the same species of particles, such that $\mathcal{A}_{n+1}$ has the same kinematical structure as $\mathcal{A}_n$ without the ALP.*

The motivation for the condition in Eq. (2.6) comes from the fact that we want the regularity of the soft-limit to be a general property of the amplitude, and not simply a characteristic of a particular point in the phase-space. Stated in another way: if the factorization above holds, *i.e.* if the set of particles $\mathcal{O}$ is the same in $\mathcal{A}_{n+1}$ and $\mathcal{A}_n$, the condition of regularity will be valid for any kinematical configuration of $\mathcal{O}$. As a consequence, the constraint we impose on $\mathcal{S}_\mathcal{P}$ will be independent of particular choices of momenta. Note that the soft factorization condition is valid after dropping subleading terms in p_ϕ coming from $\mathcal{A}_n$, that can be safely neglected as $\mathcal{A}_n$ is an arbitrary amplitude and so will not affect any of the results. Due to the singularities of the propagator in $\mathcal{S}_\mathcal{P}$, the soft factorization condition is expected to give non-trivial requirements on $\mathcal{A}_3$.[2]

To make progress we need to specify the particle content of the amplitude $\mathcal{A}_3$. We will now analyze in turn the cases in which $\mathcal{P}$ has spin 0, 1/2 and 1.

2.2 Scalars

In the simple case in which $\mathcal{P} = S$, with S a spin-0 particle, there are no little group indices associated with S and $\mathcal{S}_S$ is simply given by

$$\mathcal{S}_S = \frac{1}{2p \cdot p_\phi} \mathcal{A}_3 \left[\phi S_p \bar{S}_{-p-p_\phi} \right] . \tag{2.7}$$

The 3-point amplitude among 3 scalars amounts just to a simple constant, therefore the only way to avoid $\mathcal{S}_S$ of diverging as $p_\phi \to 0$ is to set it to zero. This is nothing but a manifestation of the fact that in the usual quantum theoretical language the interactions of a ALP with two scalars, given by $(\partial_\mu \phi)(S^\dagger i \overleftrightarrow{\partial_\mu} S)$, gives a vanishing amplitude when all particles are taken on-shell.

2.3 Fermions

Moving to the case of fermions $\mathcal{P} = \psi$, we now have a non-trivial little-group scaling and the 3-point amplitude can be written as

$$\mathcal{A}_3 \left[\phi \psi_1^I \bar{\psi}_2^J \right] = g_L \langle \mathbf{12} \rangle + g_R \left[\mathbf{12} \right] , \tag{2.8}$$

with $g_{L,R}$ dimensionless constants that are related to the couplings to the left- and right-handed components of the fermions, respectively. In the previous amplitude we have used the *bold* notation introduced in Ref. [3], which amounts to simply bold the particle momenta instead of writing explicitly the little group indices. More specifically, in this

[2]We see that the factorization in Eq. (2.4) may in general affect the spin configuration of the remaining n-point amplitude when compared to the original $(n+1)$-point one. As we will see, however, the leading term in the soft expansion will always satisfy $(\mathcal{S}_\mathcal{P})^{I_1,\cdots,I_{2s}}_{K_1,\cdots K_{2s}} \propto \delta^{I_1}_{K_1} \cdots \delta^{I_{2s}}_{K_{2s}} + \text{symm}$, *i.e.* the spin configuration will remain the same, with a change in spin configuration arising only in the subleading terms. Here, "symm" indicates the symmetric combinations of little-group indices.

case the bolded spinor products are matrices in the $\{I, J\}$ space (see Appendix A). Also, we use the short-hand notation $|\boldsymbol{p_n}\rangle \equiv |\boldsymbol{n}\rangle$, and similar for other spinors, to label the momenta. Calling $p_1 = p$, $p_2 = -p - p_\phi$ and using Eqs. (A.13) and (A.16), the soft factor $\mathcal{S}_\psi$ becomes

$$(\mathcal{S}_\psi)^I_K = \frac{m_\psi}{2p \cdot p_\phi}(g_L + g_R)\delta^I_K + (\mathcal{S}_{p^0_\phi})^I_K, \tag{2.9}$$

where $\mathcal{S}_{p^0_\phi}$ denote terms that do not depend on p_ϕ and we will always assume masses to be real and positive. As we are going to see, these terms will play an important role in determining the form of g_L and g_R. To avoid the divergence of $\mathcal{S}_\psi$ in the soft limit we need $g_L = -g_R$,[3] which implies that the coupling of the ALP to the fermions must necessarily be axial. This condition is nevertheless not enough to guarantee the regularity for all p, as a divergence may still appear when the (real) 3-momenta $\vec{p}$ and $\vec{p}_\phi$ are collinear with $\vec{p} \cdot \vec{p}_\phi > 0$. In this case

$$\frac{1}{2p \cdot p_\phi} = \frac{1}{2|\vec{p}_\phi|\left(\sqrt{m^2_\psi + |\vec{p}\,|^2} - |\vec{p}\,|\right)} \xrightarrow{\frac{m_\psi}{|\vec{p}|} \ll 1} \frac{|\vec{p}\,|}{|\vec{p}_\phi| m^2_\psi}, \tag{2.10}$$

i.e. it diverges for very small masses. Note that the massless limit $m_\psi \to 0$ is precisely the limit that will allow us to match the massive amplitudes in the IR into amplitudes invariant under the SM symmetry group in the UV. To ensure that it is possible to perform this IR/UV matching, we must then require the soft factor $\mathcal{S}_\psi$ to be regular also in the collinear configuration. Since, with the condition $g_\psi = g_L = -g_R$, the first term in Eq. (2.9) vanishes, we turn to $\mathcal{S}_{p^0_\phi}$. In the collinear limit, this term is proportional to (see Eq. (A.18))

$$\lim_{\text{collinear}} (\mathcal{S}_{p^0_\phi})^I_K = \frac{g_\psi\, \epsilon_{JK}}{2m_\psi}\left(\delta^I_2 \delta^J_1 + \delta^J_2 \delta^I_1\right), \tag{2.11}$$

where the structure carrying the little group indices is diagonal traceless because of the contraction with the ϵ_{JK} tensor, $\epsilon_{JK}\left(\delta^I_2\delta^J_1 + \delta^J_2\delta^I_1\right) = (\sigma^3)^I_K$. The only way to guarantee a non-singular massless limit is to have the couplings to be proportional to the mass, since then Eq. (2.9) becomes independent of m_ψ in this limit. As a consequence, remembering that the couplings $g_{L,R}$ are dimensionless, we are forced to introduce a new scale f in order to correct their dimensionality. From the arguments above, we then find that $g_R = -g_L \propto m_\psi/f$.

We can obtain more information about the phase of the couplings by imposing CPT invariance and unitarity of the amplitude. The relation of the amplitude with their

[3] Since, under parity, angle and square brackets are exchanged, the condition $g_L = -g_R$ amounts to a parity-odd amplitude, *i.e.* to a pseudoscalar ALP.

CPT conjugate is given by Eq. (A.21), and implies in the present case that $g_L = g_R^*$. This, together with the previous conditions, leads to purely imaginary coefficients. In short, we conclude that the coefficients in Eq. (2.8) must be of the form

$$g_L = -g_R = C_\psi \frac{i m_\psi}{f}, \quad C_\psi \in \mathbb{R}. \tag{2.12}$$

This result agrees exactly with what we have from the usual quantum theoretical approach starting from an interaction given by $(C_\psi/2f)(\partial_\mu \phi)\bar{\psi}\gamma^5\gamma^\mu\psi$.

2.4 Vectors

We now move to the case $\mathcal{P} = V$, with V a spin 1 particle of mass m_V. Without any loss of generality we take $\bar{V} = V$. The 3-point amplitude reads

$$\mathcal{A}_3\Big[\phi V_1^{I_1,I_2} V_2^{J_1,J_2}\Big] = \frac{g_-}{f}\langle \mathbf{12}\rangle^2 + \frac{g_+}{f}[\mathbf{12}]^2 + \frac{g_0}{m_V}\langle \mathbf{12}\rangle [\mathbf{12}], \tag{2.13}$$

with $g_\pm, g_0$ dimensionless constants and f again a new scale needed to correct the dimension of the coefficients. We observe that, unlike $g_\pm$, the coupling g_0 is instead divided by a factor m_V^{-1} to ensure a well-defined high-energy limit [24, 28]. Similar to Eq. (2.8), we use the bold notation to leave the little-group indices implicit. It is worth stressing again that we need to symmetrize over the little-group indices, so for instance[4]

$$\langle \mathbf{12}\rangle^2 = \frac{1}{2}\left(\left\langle 1^{I_1} 2^{J_1}\right\rangle \left\langle 1^{I_2} 2^{J_2}\right\rangle + \left\langle 1^{I_2} 2^{J_1}\right\rangle \left\langle 1^{I_1} 2^{J_2}\right\rangle\right), \tag{2.14}$$

and analogous expressions for the other spinor structures. With this amplitude we compute the soft factor:

$$(\mathcal{S}_V)^{I_1,I_2}_{K_1,K_2} = \frac{m_V^2}{2p\cdot p_\phi}\left(\frac{g_- + g_+}{f} + \frac{g_0}{m_V}\right)\frac{1}{2}\left(\delta^{I_1}_{K_1}\delta^{I_2}_{K_2} + \delta^{I_2}_{K_1}\delta^{I_1}_{K_2}\right) + (\mathcal{S}_{p^0_\phi})^{I_1,I_2}_{K_1,K_2}. \tag{2.15}$$

As in the case of the fermions, regularity when $|\vec{p}_\phi| \to 0$ imposes that the term in brackets of the first term vanishes. Regularity in the collinear limit (when $m_V \to 0$), or more precisely, regularity of $\mathcal{S}_{p^0_\phi}$, requires $g_0 = 0$. Together with CPT and unitarity, the constraints on the couplings we obtain are

$$g_- = -g_+ = iC_V, \quad C_V \in \mathbb{R}. \tag{2.16}$$

Similarly to the case of fermions, the equation above implies that the couplings of the ALP to two vectors are purely imaginary and axial. It is possible to show that the amplitude with such couplings corresponds exactly to the operator $(C_V/2f)\phi V_{\mu\nu}\tilde{V}^{\mu\nu}$, where $\tilde{V}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}V^{\alpha\beta}$ (see Appendix A).

[4]Here we choose the normalization of 1/2 for all values of $I_{1,2}, J_{1,2}$ for simplicity, but other conventions can be useful in other contexts, for example in Refs. [18, 24].

2.5 Fermions - Many species case

So far we have analysed the couplings of the ALP assuming that only one species of particle $\mathcal{P}$ couples to ϕ. For the discussion in Section 3, it will also be necessary to consider the case in which the ALP couples to at least two non-degenerate fermion species. For this scenario, the first obstacle comes from the fact that the soft factorization condition (2.6) does not hold anymore, as the kinematics is changed due to the mass splitting. Stated in other terms: the external particles appearing in $\mathcal{A}_{n+1}$ and $\mathcal{A}_n$ are different and, as such, the two amplitudes cannot have the same kinematical configuration. To avoid this, we can work first in the high-energy limit, where all fermions are massless and Eq. (2.6) holds. On the one hand, at leading order in p_ϕ, the soft factor in Eq. (2.5) vanishes automatically as $\langle pp\rangle = [pp] = 0$ and regularity is trivially satisfied. On the other hand, the subleading terms that are constant in p_ϕ do not vanish and can be constrained in the collinear configuration. Considering that the angle between the 3-momenta is $\theta \ll 1$, we obtain for each helicity configuration (see Eq. (A.19))

$$\lim_{\theta \ll 1} \mathcal{S}_\psi \propto \frac{1}{\theta}. \tag{2.17}$$

Thus, using the regularity condition in the collinear limit, we arrive at the conclusion that the coefficients of the amplitude should vanish in the massless limit, *i.e.* they should be proportional to the mass of the fermions involved in the amplitude. The couplings, that are now matrices in the space of fermion species, can always be put into the form

$$g_{L,R} = \frac{i}{f}\left[M_\psi B_{L,R} - A_{L,R} M_\psi\right], \tag{2.18}$$

where M_ψ is the fermion mass matrix and $A_{L,R}$, $B_{L,R}$ are arbitrary matrices, that can depend on M_ψ as well. If there is any massless fermion in the spectrum, we must treat them separately as the form of the amplitude is different, meaning that we can always take M_ψ to be diagonal with positive entries. Since the parametrization above is redundant under the transformation $A_{L,R} \to A_{L,R} + M_\psi X$, $B_{L,R} \to B_{L,R} + X M_\psi$, with X some matrix, we can always choose $A_{L,R}$ to be hermitian by choosing $X = M_\psi^{-1} A_{L,R}^\dagger$. Imposing finally CPT and unitarity, which based on Eq. (A.21) amounts to $g_L = g_R^\dagger$, we obtain

$$g_{L,R} = \frac{i}{f}\left[M_\psi A_{R,L} - A_{L,R} M_\psi\right]. \tag{2.19}$$

This expression agrees with what one would expect from a shift-symmetric coupling of one ALP to fermions [50, 53–55], and precisely corresponds to the operator $(1/f)(\partial_\mu\phi)\bar{\psi}[C_V + C_A\gamma^5]\gamma^\mu\psi$, with $A_{L,R} = C_V \mp C_A$. In the 1-family case, it follows from Eq. (2.19) that $g_L = -g_R$, in agreement with Eq. (2.12), and hence the dependency on C_V drops out.

In principle we could also analyse the soft-limit of amplitudes with many species of scalars and vectors and try to obtain similar constraints as for fermions. However, we anticipate that, for the purpose of studying amplitudes with the SM particle content, it is not necessary to consider amplitudes with more than one type of scalar or vector.

We close this section by emphasising that the requirement of regularity of the soft factor $\mathcal{S}_{\mathcal{P}}$, combined with UV/IR compatibility of low- (massive) and high-energy (massless) amplitudes and CPT invariance plus unitarity, allowed us to fully determine the structure of the 3-point amplitudes. In addition, they all agree with the amplitudes one would obtain by imposing shift-symmetry at the level of Lagrangian.

3 ALP interactions with the SM particles

We turn now our attention to the following question: *how can we link the amplitudes derived so far with amplitudes invariant under the SM symmetry group* $\mathcal{G}_{SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$*?* Here in general, as in Section 2.5, particles of different species will couple to the ALP and we cannot simply apply the soft factorization condition of Eq. (2.6).

The case of the coupling between one ALP and to two Higgs doublets is identical to the one already discussed in Eq. (2.7), so we can skip directly to the case of the fermions.

3.1 Fermions and EWSB

Unlike what happens in Eq. (2.8), we cannot build an amplitude with one ALP and two fermions due to $\mathcal{G}_{\rm SM}$-invariance and fermion helicities. Nevertheless, it is possible to add a Higgs doublet and write

$$\mathcal{A}_4[\phi\psi^-_{L1}\bar{\psi}^-_{R2}\bar{H}] = \frac{\bar{C}_\psi}{f}\langle 12\rangle\,, \quad \mathcal{A}_4[\phi\bar{\psi}^+_{L1}\psi^+_{R2}H] = \frac{C_\psi}{f}[12]\,, \tag{3.1}$$

where $\psi_{L,R}$ denote the chiral fermions of the SM, while H is the Higgs doublet (in the case of couplings to up-quarks one should swap $H \leftrightarrow \bar{H}$). The $\pm$ superscripts show explicitly the helicities of the spin 1/2 particles. For simplicity, in the previous equation we have suppressed all indices and tensor structures related to $\mathcal{G}_{\rm SM}$. Also, it is important to notice that the dimensionless couplings C_ψ, $\bar{C}_\psi$ are matrices in fermion flavor space and the spinor structures $\langle 12\rangle$ and $[12]$ are flavor-independent since all fermions are massless. This also implies that we have the freedom to redefine C_ψ and $\bar{C}_\psi$, because in the massless limit they can be seen as tensors of the flavor group $U(3)_{\psi_L} \times U(3)_{\psi_R}$ [34]. At tree-level, CPT and unitarity enforces that $\bar{C}_\psi = C^\dagger_\psi$, that follows from Eq. (A.21).

The fact that the amplitude in Eq. (3.1) is now a 4-point amplitude brings two changes to the analysis. First, the coefficients C_ψ can now depend on the kinematics through kinematical invariants, which we will always assume to be regular, *i.e.* given by a power expansion. Second, the reasoning that led us to the soft factorization condition (2.6) fails, because taking $p_\phi \to 0$ does not guarantee that the particle exchanged in the propagator goes on-shell and that the amplitude factorizes. The key to understand the soft limit for these amplitudes lies in the Brout–Englert–Higgs mechanism. In general, to connect the massless (high-energy) amplitudes with the massive (low-energy) amplitudes without the Higgs, one uses

$$\lim_{p_H \to 0} \mathcal{A}[H, \cdots] = \lim_{\text{high-energy}} \frac{1}{v} \mathcal{A}[\cdots] \,, \tag{3.2}$$

where v is the scale at which the Higgs becomes non-dynamical ("frozen" in the language of Ref. [22]), which is introduced by dimensional analysis. The limit $p_H \to 0$ is to be understood also as the limit in which the Higgs becomes non-dynamical, which amounts to removing it from the amplitude, and the right hand side is to be taken as the high-energy limit of the corresponding massive amplitude. What is stated in Eq. (3.2) is nothing but the UV/IR compatibility for amplitudes involving the Higgs.

In short, we learn from the discussion above that the soft factorization condition (2.6) is not directly applicable to amplitudes involving Higgs. Therefore, we are led to impose an extra condition for such amplitudes:

3-point Higgs obstruction

If a low-energy 3-point amplitude $\mathcal{A}_3$ involving one ALP cannot be associated to a non-vanishing gauge invariant 3-point amplitude in the UV, but only to a 4-point one with an extra Higgs, then we impose the following condition:

$$\lim_{p_\phi, p_H \to 0} \mathcal{A}_4[\phi\, H \cdots] = \lim_{p_\phi \to 0} \lim_{\text{high-energy}} \frac{1}{v} \mathcal{A}_3[\phi \cdots]. \tag{3.3}$$

The prescription above guarantees that the result we obtained previously in Eq. (2.19) holds, as the double soft-limit $p_\phi, p_H \to 0$ assures that the factorization in Eq. (2.3) takes place. Therefore, from Eq. (2.19), after applying Eq. (3.2) to (3.1), we obtain

$$C_\psi \left(p_H = 0\right) = i \left(Y_\psi \tilde{A}_R - \tilde{A}_L Y_\psi \right) , \tag{3.4}$$

where Y_ψ the Yukawa couplings, $\tilde{A}_{L,R} = \lim_{v \to 0} U_{L,R} A_{L,R} U^\dagger_{L,R}$ (with $A_{L,R}$ the original couplings appearing in Eq. (2.19)), and $U_{L,R} \in U(3)_{\psi_{L,R}}$ are flavor transformations.

The equation above needs clarification. The first point regards the appearance of flavor transformations. As mentioned previously, when the fermions are massless we are free to perform flavor transformations, while this freedom is lost when they become massive. Hence, when performing the UV/IR matching of both regimes they will in general agree up to a flavor transformation [34]. Secondly, the Yukawa matrices can be expressed here as $vY_\psi = U_L M_\psi U_R^\dagger$, where the relation to the mass from an on-shell perspective was shown in Refs. [22, 56] and we provide an alternative derivation in Appendix B. Lastly, from the two previous points we can derive the expressions for $\tilde{A}_{L,R}$, where the high-energy limit was changed to the equivalent limit $v \to 0$. At zero Higgs momentum, that is, for constant coupling, Eq. (3.4) is identical to what one would expect from ALPs coupled to fermions in a shift-symmetric manner [50, 53–55].

Before moving on to other amplitudes, it is worth to explore in more detail some features of the amplitude $\mathcal{A}_3[\phi\psi\bar{\psi}]$ and its relation to the 3-point Higgs obstruction. At face value, the high-energy limit of this amplitude with couplings given in Eq. (2.19) leads to a vanishing result, since $C_\psi \sim M_\psi$. This naive result is inconsistent with $\mathcal{G}_{\rm SM}$-invariance and, for this reason, we had to promote the UV amplitude to a 4-point amplitude with an additional Higgs. One could, however, imagine an alternative route. Using the Weyl equations in Eq. (A.11), it is possible to trade the mass factor for a momentum insertion in the spinor structures, *e.g.* $M_\psi \langle \mathbf{12} \rangle \sim \langle \mathbf{1} | p_\phi | \mathbf{2}]$ for the case of angle brackets, with an analogous identity holding for square brackets. This form highlights that the amplitude does not vanish in the high-energy limit and should be directly matched into $\mathcal{G}_{\rm SM}$-invariant 3-point amplitudes $\mathcal{A}_3[\phi\psi_{L,R}\bar{\psi}_{L,R}]$. But these amplitudes nonetheless vanish identically due to the 3-particle kinematics, leaving us with the 4-point amplitude already discussed as the unique route to match the massive amplitude in the UV.

Another interesting feature of the coefficient in Eq. (3.4) is its connection with the invariants that parameterise the breaking of shift-symmetry defined in Ref. [50]. In this reference, a total of 3 and 10 invariants were constructed in the lepton and quark sector, respectively, using the mathematical properties of $C_\psi(p_H = 0)$. From our perspective, the only way to break the coupling structure of Eq. (3.4) is to violate the scaling with the Yukawa matrices (which correspond to a breaking of the dependence on the particle mass in Eq. (2.18)), as the phase of the amplitude and the hermiticity of $A_{L,R}$ are fixed by CPT and unitarity. We thus conclude that the maximum number of independent parameters that can break the shift-symmetry (in our language, that gives a singular massless/high energy limit) amounts to the number of independent parameters in the Yukawa matrices, *i.e.* the number of parameters left after all possible flavor transformations are applied. For leptons, taking massless neutrinos, this implies 3 independent parameters; for quarks (with two Yukawa matrices that are correlated in the UV because we have only one left handed doublet), the number increases to 10 (6 masses plus the 4 parameters of the Cabibbo–Kobayashi–Maskawa matrix[5]). Hence, in the general case in which the Yukawas have the maximal amount of parameters, this counting exactly corresponds to the one found in Ref. [50]. Since, however, the invariants are constructed based on the exact form of the flavor group, we cannot conclude that by reducing the physical parameters of the Yukawas will necessarily reduce the amount of non-vanishing invariants. In general, their number will only decrease once the flavor group is enlarged.

3.2 ϕVV amplitudes

For couplings to spin 1 particles we do not face such conceptual difficulties, as invariance under $\mathcal{G}_{\rm SM}$ allows us to build the 3-point amplitudes at high-energies:

$$\begin{aligned} \mathcal{A}_3\left[\phi B_1^- B_2^-\right] &= \frac{g_B^-}{f}\langle 12\rangle^2\,,\; \mathcal{A}_3\left[\phi B_1^+ B_2^+\right] = \frac{g_B^+}{f}[12]^2\,, \\ \mathcal{A}_3\left[\phi W_1^- W_2^-\right] &= \frac{g_W^-}{f}\langle 12\rangle^2\,,\; \mathcal{A}_3\left[\phi W_1^+ W_2^+\right] = \frac{g_W^+}{f}[12]^2\,, \end{aligned} \tag{3.5}$$

ignoring again $\mathcal{G}_{\rm SM}$ indices. At low-energies we apply the soft-limit to 3-point amplitudes with two charged $W^\pm$'s and two Z's, for which the soft factorization condition (2.6) holds. In this case the result given in Eq. (2.16) follows. For amplitudes with two photons or one photon and one Z the same reasoning does not work, because the photon remains massless at low-energies and the amplitude with one photon and one Z does not respect Eq. (2.6). Nevertheless, since γ, Z are connected to the massless electroweak bosons W, B through spontaneous symmetry breaking [56, 57], using $\mathcal{G}_{\rm SM}$-invariance we can relate the couplings obtained for the charged W's and Z's to the ones of Eq. (3.5). We thus conclude that $g^\pm_{B,W}$ also satisfy Eq. (2.16), that is $g^\mp_{B,W} = -g^\pm_{B,W} = iC_{B,W}$, with $C_{B,W} \in \mathbb{R}$.

Gluons, on the other hand, cannot be related to other gauge bosons via symmetry breaking, nor they are massive at low energy. At all energies at which perturbation theory is valid, the coupling between the ALP and gluons will be

$$\mathcal{A}_3\left[\phi G_1^- G_2^-\right] = \frac{g_G^-}{f}\langle 12\rangle^2\,,\; \mathcal{A}_3\left[\phi G_1^+ G_2^+\right] = \frac{g_G^+}{f}[12]^2\,, \tag{3.6}$$

as usual omiting $\mathcal{G}_{\rm SM}$ structures. The way we found to extract some information out of these amplitudes is to go to the 1-loop level. More precisely, we compute the anomalous

[5]The emergence of the Cabibbo–Kobayashi–Maskawa and the Pontecorvo–Maki–Nakagawa–Sakata matrices from the on-shell perspective was studied in Ref. [34].

dimension of the couplings C_ψ in Eq. (3.1) induced by Eq. (3.6). Then, the specific structure (3.4) will give some constraints on $g_G^\pm$. We carry out this computation in Appendix C and show that we can obtain the same results for gluons, namely $g_G^- = -g_G^+ = iC_G$, with C_G real. It is worth stressing that the reasoning above makes the amplitudes in Eq. (3.6) qualitatively different from the others, as we were only able to arrive at the constraints above by using other amplitudes.

4 Constructing higher-point functions

So far we dealt with 3-point functions, *i.e.* amplitudes with couplings proportional to $1/f$ that, at leading order, can be matched onto dimension $d = 5$ operators. We now proceed to extend our discussion to amplitudes of higher-order and build explicitly the amplitude basis up to $1/f^4$. In the literature, only $d = 6$ (for instance in Refs. [58, 59]) and a couple of $d = 7$ operators [58, 60–62] were considered and studied at the phenomenological level. Our present work provides the complete amplitudes, and a corresponding operator basis, up to $d = 8$ consistent with the ALP properties.

The higher-point amplitudes we are interested in are contact ones, *i.e.* amplitudes that are regular in the kinematical invariants. These necessarily involve 4 or more particles and, as a consequence, the factorization in Eqs. (2.3) and (2.6) does not hold anymore. However, since they are not 3-point amplitudes, we can study their soft limit directly, without the need of embedding them in an auxiliary amplitude. Given that they are contact amplitudes, by definition they are regular as $p_\phi \to 0$, so in order to make progress we need an additional physical constraint on them. We impose:

Soft contact condition

If $\mathcal{A}_n[\cdots]$ is a contact amplitude with $n \geq 4$ involving at least one ALP, then

$$\lim_{p_\phi \to 0} \mathcal{A}_n[\cdots] = 0, \tag{4.1}$$

for each ALP ϕ present in the amplitude.

If we look back at the results from Section 2, the only reason that the soft-limit did not give a zero was because of the p_ϕ-independent terms, that only appeared because of the singular propagator in Eq. (2.5). Since, now, we do not have such singular terms as we are looking directly at the contact amplitudes, the natural extension of the previous requirement is to have the amplitudes to vanish in the soft-limit. The

Dimension 6

Particle content	$\mathcal{A} \times f^2$	$\mathcal{O} \times f^2$
$\phi^2 H \bar{H}$	$p_{\phi_1} \cdot p_{\phi_2}$	$(\partial_\mu \phi)(\partial^\mu \phi)|H|^2$

Table 1. Amplitudes suppressed by $1/f^2$ and corresponding $d = 6$ operators. All the amplitudes are stripped of overall coefficients.

only exceptions we found to the condition above are the amplitudes (3.1), that have a constant coefficient in Eq. (3.4). However, as discussed in Section 3, these amplitudes must be treated according to the 3-point Higgs obstruction (3.3) and thus must not necessarily respect the soft contact condition (4.1) at order $\mathcal{O}(1/f)$.

We stress that the regularity conditions of Eqs. (2.6) and (4.1) are, in most cases, equivalent to the Adler's zero condition [43]. In Quantum Field Theoretical language the Adler's zero condition states that, in the soft limit $p_\phi \to 0$, the soft factor is regular when ϕ is emitted from some external leg and vanishes when ϕ is emitted from the interior of the diagram. The only exceptions are, as explained above, those amplitudes involving the Higgs doublet that match into 3-point amplitudes at low energy.

We will now systematically build amplitudes involving ALPs and SM particles in the unbroken electroweak phase, imposing Eq. (4.1). In practice, these amplitudes must scale with some positive power of the ALP momentum. This can either appear in the coefficients through Lorentz invariant combinations such as $p \cdot p_\phi$, or in the spinor structures themselves, for instance as $\langle p|p_\phi|p']$, with p, p' the momenta of other particles in the amplitude. To have any chance to contribute to amplitudes suppressed at most by f^4, the spinor structures appearing must have dimension ≤ 4. Additionally, we also have to impose Bose symmetry when there is more than one identical particle. For simplicity, we assume only one species of ALP ϕ.

Let us illustrate the method with an example, $\phi^2 V_1 V_2$, for which we have four helicity configurations for the vectors: $(-,-)$, $(+,+)$, $(+,-)$ and $(-,+)$. For the $(-,-)$ configuration, little-group covariance and dimensional analysis allow us to write only the spinor structure $\langle 12\rangle^2$. This can satisfy Eq. (4.1) only if multiplied by invariants proportional to $p_{\phi_{1,2}}$, appropriately symmetrized to satisfy Bose symmetry. The only possibility at the mass dimension of interest is $p_{\phi_1} \cdot p_{\phi_2}$ and the amplitude is

$$\mathcal{A}_4 \left[\phi\phi V_1^- V_2^-\right] \propto \frac{p_{\phi_1} \cdot p_{\phi_2}}{f^4} \langle 12\rangle^2 . \tag{4.2}$$

Note that we could have built different kinematical invariants with more powers of momenta, but they would induce an amplitude of higher order in f. An identical reasoning

Dimension 7

Particle content	$\mathcal{A}\times f^3$	$\mathcal{O}\times f^3$
$\phi\psi_1\bar{\psi}_2 H$	$(p_\phi\cdot p_1)\langle 12\rangle$	$(\partial_\mu\phi)\bar{\psi}H(D^\mu\psi)$
	$(p_\phi\cdot p_2)\langle 12\rangle$	$(\partial_\mu\phi)(D^\mu\bar{\psi})H\psi$
$\phi\psi_1\bar{\psi}_2 V_3$	$\langle 13\rangle\langle 3\|p_\phi\|2]\,,\langle 23\rangle\langle 3\|p_\phi\|1]$	$(\partial^\mu\phi)\bar{\psi}\gamma^\nu\psi V_{\mu\nu}$ $(\partial^\mu\phi)\bar{\psi}\gamma^\nu\psi\tilde{V}_{\mu\nu}$ $(\partial^\mu\phi)\bar{\psi}\gamma^\nu T^A\psi V^A_{\mu\nu}$ $(\partial^\mu\phi)\bar{\psi}\gamma^\nu T^A\psi\tilde{V}^A_{\mu\nu}$
$\phi H_1\bar{H}_2 V_3$	$\langle 3\|\,p_\phi(p_1-p_2)\,\|3\rangle$	$(\partial_\mu\phi)(H^\dagger i\overleftrightarrow{D}_\nu H)V^{\mu\nu}$ $(\partial_\mu\phi)(H^\dagger i\overleftrightarrow{D}_\nu H)\tilde{V}^{\mu\nu}$ $(\partial_\mu\phi)(H^\dagger i\overleftrightarrow{D}_\nu^A H)V^{A,\mu\nu}$ $(\partial_\mu\phi)(H^\dagger i\overleftrightarrow{D}_\nu^A H)\tilde{V}^{A,\mu\nu}$
$\phi H\bar{H}\psi_1\bar{\psi}_2$	$\langle 1\|p_\phi\|2]$	$(\partial_\mu\phi)(\bar{\psi}\gamma^\mu\psi)\|H\|^2$ $(\partial_\mu\phi)(\bar{\psi}\gamma^\mu T^A\psi)(H^\dagger T^A H)$
$\phi H_1\bar{H}_2 H_3\bar{H}_4$	$p_\phi\cdot(p_1-p_2)$ + symm.	$(\partial^\mu\phi)(H^\dagger i\overleftrightarrow{D}_\mu H)\|H\|^2$

Table 2. Amplitudes suppressed by $1/f^3$ and corresponding $d=7$ operatores. The symbol ψ denotes SM fermions, while $V=B$ and $V^A=W^A$, G^A denote the abelian and non-abelian SM gauge bosons, respectively. Each operator must be invariant under the SM gauge group and this restricts the type of fields that can appear. The symbol 'symm' indicates that it is necessary to symmetrize the momenta of identical particles according to Bose symmetry. Additional spinor structures can be obtained from the ones shown by swapping angle and square brackets. The operator $\overleftrightarrow{D}_\mu^A$ is defined as $\overleftrightarrow{D}_\mu^A = T^A\overrightarrow{D}_\mu - \overleftarrow{D}_\mu T^A$ and the dual field strength as $\tilde{V}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}V^{\alpha\beta}$.

applies to the $(+,+)$ helicity configuration, for which the amplitude has exactly the same form, provided we exchange the angle brackets with square brackets. We thus obtain a second independent spinor structure. The $(-,+)$ configuration requires, instead, a momentum insertion of the ALP in between the brackets, as we need to connect an

Dimension 8

Particle content	$\mathcal{A} \times f^4$	$\mathcal{O} \times f^4$
ϕ^4	$(p_{\phi_1} \cdot p_{\phi_2})(p_{\phi_3} \cdot p_{\phi_4})$ + symm.	$(\partial_\mu \phi\, \partial^\mu \phi)^2$
$\phi^2 H_1 \bar{H}_2$	$(p_{\phi_1} \cdot p_{\phi_2})^2$	$(\partial_\mu \partial_\nu \phi\, \partial^\mu \partial^\nu \phi)\|H\|^2$
	$(p_{\phi_1} \cdot p_1)\,(p_{\phi_2} \cdot p_2)$ + symm.	$(\partial^\mu \phi\, \partial^\nu \phi)(D_\mu H^\dagger D_\nu H)$
$\phi^2 V_1 V_2$	$(p_{\phi_1} \cdot p_{\phi_2})\,\langle 12 \rangle^2$	$(\partial_\alpha \phi\, \partial^\alpha \phi) V_{\mu\nu} V^{\mu\nu}$ $(\partial_\alpha \phi\, \partial^\alpha \phi) V_{\mu\nu} \tilde{V}^{\mu\nu}$ $(\partial_\alpha \phi\, \partial^\alpha \phi) V^A_{\mu\nu} V^{A,\mu\nu}$ $(\partial_\alpha \phi\, \partial^\alpha \phi) V^A_{\mu\nu} \tilde{V}^{A,\mu\nu}$
	$\langle 1 \| p_{\phi_1} \| 2]\,\langle 1 \| p_{\phi_2} \| 2]$	$(\partial^\mu \phi\, \partial_\nu \phi) V_{\mu\alpha} V^{\alpha\nu}$ $(\partial^\mu \phi\, \partial_\nu \phi) V^A_{\mu\alpha} V^{A,\alpha\nu}$
$\phi^2 \psi_1 \bar{\psi}_2$	$(p_{\phi_1} \cdot p_1)\,\langle 1 \| p_{\phi_2} \| 2]$ + symm.	$(\partial_\mu \phi\, \partial_\nu \phi)(\bar{\psi} \gamma^\mu D^\nu \psi)$
$\phi^2 \psi_1 \bar{\psi}_2 H$	$(p_{\phi_1} \cdot p_{\phi_2})\,\langle 12 \rangle$	$(\partial_\mu \phi\, \partial^\mu \phi) \bar{\psi} H \psi$
$\phi^2 H^4$	$p_{\phi_1} \cdot p_{\phi_2}$	$(\partial_\mu \phi\, \partial^\mu \phi)\|H\|^4$

Table 3. Same as Tables 1 and 2 for amplitudes of order $1/f^4$ and the correspondent $d = 8$ operators.

angle with a square spinor. The only symmetric combination in $p_{\phi_{1,2}}$ is then

$$\mathcal{A}_4 \left[\phi\phi V_1^- V_2^+\right] \propto \frac{1}{f^4} \langle 1|p_{\phi_1}|2]\,\langle 1|p_{\phi_2}|2]\,. \tag{4.3}$$

The $(+,-)$ configuration can be obtained from the equation above just swapping angle for squared brackets. Unlike the previous case, this does not produce an independent spinor structure since, due to Bose symmetry, once we exchange angle and square brackets, re-label the momenta as $1 \leftrightarrow 2$ and use the identity $[p|q|k\rangle = \langle k|q|p]$, we obtain exactly the same spinor combination we began with. As a consequence, for each vector V we can reconstruct only 3 independent operators:

$$\mathscr{L}_{d=8} \supset \frac{C}{f^4}(\partial_\alpha \phi)(\partial^\alpha \phi) V_{\mu\nu} V^{\mu\nu} + \frac{C'}{f^4}(\partial_\alpha \phi)(\partial^\alpha \phi) V_{\mu\nu} \tilde{V}^{\mu\nu} + \frac{C''}{f^4}(\partial^\mu \phi)(\partial_\nu \phi) V_{\mu\alpha} V^{\alpha\nu}, \tag{4.4}$$

with C, C', C'' dimensionless coefficients. Note that in the list above we do not have the operator $(\partial^\mu\phi)(\partial_\nu\phi)V_{\mu\alpha}\tilde{V}^{\alpha\nu}$. Although not trivial, it can be shown that this operator is redundant [63]. From our on-shell construction, however, the non-redundant operators were singled out automatically.[6]

We present the list of higher-point amplitudes suppressed by f^{-2}, f^{-3} and f^{-4} and the correspondent dimension 6, 7 and 8 operators in Tables 1, 2 and 3, respectively. For each amplitude we only show independent spinor structures, with the exception of those that can be obtained exchanging angle with square brackets which are left implicit. We also do not write explicitly overall coefficients that may depend on SM gauge group structures like group generators[7]. These coefficients will depend on the particles appearing in the amplitude and their quantum numbers. We show explicit examples in Appendix A for the case of 3-point amplitudes with SM particles. For the higher dimensional operators, we only show the general particle content, with ψ representing SM fermions, $V = B$ the abelian SM gauge boson and $V^A = W^A$ or G^A the SM non-abelian gauge bosons. Each operator should be invariant under the SM gauge group and, as usual, this imposes restrictions on the particles that can appear in each operators. We observe that some spinor structure involving spin 1 particles may involve both abelian or non-abelian gauge bosons. The difference in this case lies in the overall coefficient that we do not write explicitly, but the amplitude corresponds to different operators depending on the nature of the spin 1 particle involved. To avoid confusion, we make explicit the dependence on the gauge bosons when we write the operators corresponding to the amplitude under consideration. For instance, the $\langle 1|p_{\phi_1}|2]\langle 1|p_{\phi_2}|2]$ amplitude may correspond to the $d = 8$ operators $(\partial^\mu\phi\partial_\nu\phi)V_{\mu\alpha}V^{\nu\alpha}$ or $(\partial^\mu\phi\partial_\nu\phi)V^A_{\mu\alpha}V^{A,\nu\alpha}$, depending on the vector considered.

The construction of amplitudes suppressed by higher powers of f follows in similar fashion, with their number growing exponentially. Among all of them, one interesting class of amplitudes is $\mathcal{A}\left[\phi VVH^n\bar{H}^n\right]$, with $n \geq 1$, that start only at dimension 9. In the non-dynamical limit of the Higgs, these amplitudes are the only ones that can contribute to the 3-point couplings of the ALPs to gauge bosons, thus they can give us an insight of how the low-energy amplitudes are affected by the higher-dimensional ones. According to the soft contact condition (4.1), we must have an insertion of the ALP momentum and either a gauge boson and/or a Higgs momentum. Structures with the latter, however, vanish in the non-dynamical limit (3.2), so only insertions of

[6] On-shell techniques have been used to systematically construct higher-dimensional operators in general effective theories [27, 64].

[7] In Eqs. (4.2) and (4.3), for example, if V is an abelian boson the proportionality factor is just a constant, while if it is non-abelian then $\mathcal{A}_4[\phi\phi V_1^A V_2^B] \propto \delta^{AB}$, where δ^{AB} takes into account the $\mathcal{G}_{\rm SM}$ tensor structure.

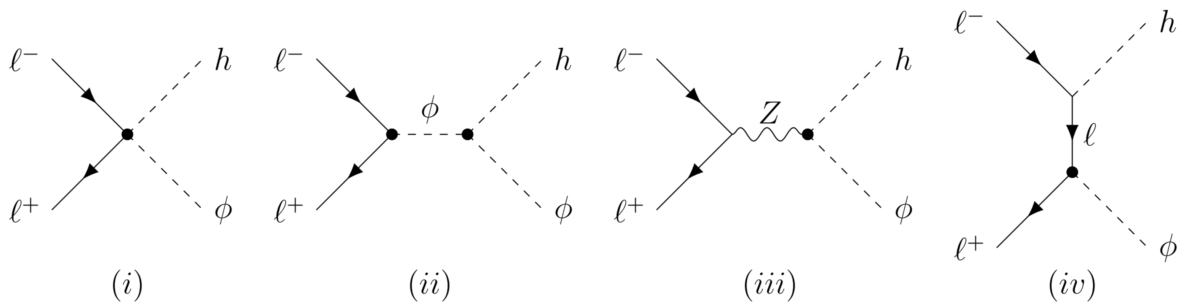

Figure 2. Diagrams contributing to $\ell^-\ell^+ \to \phi h$ generated by the operators described in the text. Black circles denote the insertion of operators containing the ALP. For the last diagram, also the crossed contribution in which the h and ϕ legs are exchanged should be considered.

gauge boson momenta are relevant. We can show that at low energies, after $p_{H,\bar{H}} \to 0$, all possible amplitudes reduce to the same spinor structure as in Eqs. (3.5) and (3.6) multiplied by powers of p_ϕ^2. If the ALP is exactly massless, it means that the low-energy couplings are not corrected by higher-dimensional operators, while if we allow for a small mass $p_\phi^2 = m_\phi^2$, it implies that all corrections are suppressed by at least the ALP mass squared. This conclusion agrees with what was previously argued in Refs. [65–67] and follows straightforwardly from our formalism.

We conclude stressing that the coefficients that multiply the amplitudes in Tables 1, 2 and 3 are not further constrained by Eq. (4.1), in sharp contrast to the amplitudes suppressed by f^{-1} discussed in Section 3. At $d = 8$ we do not consider amplitudes that violate baryon or lepton number.

5 A phenomenological application : $\ell^-\ell^+ \to \phi h$

The phenomenological impact of higher-dimensional ALP operators have been previously considered in Refs. [58, 60–62], where the effects of the d=6 operator $(\partial_\mu\phi)(\partial^\mu\phi)|H|^2$ and of the d=7 operator $(\partial^\mu\phi)(H^\dagger i\overleftrightarrow{D}_\mu H)|H|^2$ were studied in the context of collider physics. In this section we will turn our attention to the d=7 operators $(\partial_\mu\phi)(D^\mu\bar{\psi})H\psi$, $(\partial_\mu\phi)\bar{\psi}H(D^\mu\psi)$ and explore their phenomenology at lepton colliders. More precisely, we will study the impact of these operators in the process $\ell^-\ell^+ \to \phi h$, with $\ell = e, \mu$ and h the physical Higgs, that can in principle be tested at future lepton colliders.

The full set of operators that we will consider is

$$\mathscr{L}_{\rm int} = \frac{C_{\phi^2H^2}}{2f^2}(\partial_\mu\phi)(\partial^\mu\phi)|H|^2 + \frac{C_{\phi H^4}}{f^3}(\partial^\mu\phi)(H^\dagger i\overleftrightarrow{D}_\mu H)|H|^2 + \\ + \sum_{\ell=e,\mu} i\frac{C_{\phi\ell^2H}}{f}y_\ell\phi\bar{L}_\ell H\ell_R + \frac{C^{(1)}_{\phi\ell^2HD^2}}{f^3}(\partial_\mu\phi)(D^\mu\bar{L}_\ell)H\ell_R + \frac{C^{(2)}_{\phi\ell^2HD^2}}{f^3}(\partial_\mu\phi)\bar{L}_\ell H(D^\mu\ell_R) + h.c., \tag{5.1}$$

where $L_{e,\mu}$ are the left-handed doublets of the first and second families, while e_R and μ_R are the corresponding right-handed fields. We denote by $C_{\phi^2H^2}, C_{\phi H^4}, C_{\phi\ell^2H}, C^{(1,2)}_{\phi\ell^2HD^2}$ the Wilson coefficients, noticing that $C_{\phi\ell^2H}$ is real and we have already factorized the Yukawa y_ℓ explicitly[8]. We do not consider lepton flavor violating couplings and will always assume that the $d = 5$ coupling has the structure given by Eq. (3.4). Also, we only consider ALP effective interactions and do not include any SMEFT operators.

After electroweak symmetry breaking, the operators in Eq. (5.1) will generate the diagrams contributing to $\ell^-\ell^+ \to \phi h$ shown in Fig. 2. More in detail, diagrams (i) and (iv) receive contributions from both $C_{\phi\ell^2H}$ and $C^{(1,2)}_{\phi\ell^2HD^2}$, diagram (ii) gets contributions from $C_{\phi\ell^2H}$, $C^{(1,2)}_{\phi\ell^2HD^2}$ and $C_{\phi^2H^2}$ while diagram (iii) is only generated by $C_{\phi H^4}$. To compute the corresponding cross-section, we generate and manipulate the total amplitude using `FeynRules` [68], `FeynArts` [69] and `FeynCalc` [70, 71]. In the limit of very high-energies, $\sqrt{s} \gg v$, with $\sqrt{s}$ the center-of-mass energy and v the Higgs vacuum expectation value, the differential cross-section in the center-of-mass frame simplifies to the following expression:

$$\frac{{\rm d}\sigma^{\rm tot}(\ell^-\ell^+ \to \phi h)}{{\rm d}\cos\theta} \simeq \frac{1}{512\pi f^6}\Big[(u^2+t^2)\left(|C^{(1)}_{\phi\ell^2HD^2}|^2 + |C^{(2)}_{\phi\ell^2HD^2}|^2\right) + \\ + 4tu{\rm Re}\left(C^{(1)}_{\phi\ell^2HD^2}C^{(2)*}_{\phi\ell^2HD^2}\right)\Big], \tag{5.2}$$

where $t = (p_{\ell^-} - p_h)^2$ and $u = (p_{\ell^-} - p_\phi)^2$. Inspection of Eq. (5.2) shows that, in the high energy limit, the only relevant contributions are those coming from $C^{(1,2)}_{\phi\ell^2HD^2}$ and, more specifically, from diagram (i), due to the fact that only this diagram give a contribution that grows quadratically with the center-of-mass energy. The conclusion is that, even though d=7 operators are suppressed by more powers of f, they can still give the dominant contribution to observables for sufficiently high energies. This strongly relies on the fact that the coefficient of the $d = 5$ operator is Yukawa suppressed, while

[8]Having in mind the structure of Eq. (3.4), our parametrization amounts to suppose that $Y_\ell\tilde{A}_R - \tilde{A}_L Y_\ell$ equals the Yukawa coupling times an order one factor that we denote by $C_{\phi\ell^2H}$, as expected by power counting.

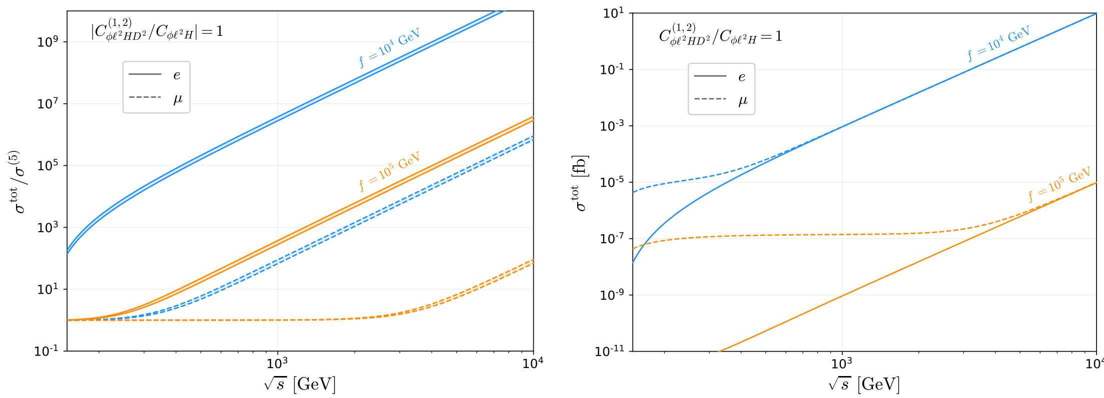

Figure 3. *Left*: Ratio between the cross-sections with and without dimension 7 operators as a function of the center of mass energy. We take the modulus of the ratio of $d = 7$ and $d = 5$ operators equal to 1. The bands are obtained by varying the sign between $C^{(1,2)}_{\phi\ell^2HD^2}$. *Right*: Total cross-section with the inclusion of $d = 7$ operators as a function of the center of mass energy, taking $C^{(1,2)}_{\phi\ell^2HD^2} = C_{\phi\ell^2H} = 1$. All cross-sections are computed taking $m_\phi = 0$. For both panels the solid curves denote $\ell = e$ and dashed $\ell = \mu$, while blue (orange) means $f = 10^4$ (10^5) GeV.

an analog suppression does not exist for the $d = 7$ operators. We also observe that Eq. (5.2) is still the leading contribution even if effects from $d = 8, 9$ operators are included. This is because, to give a cross section at the same order in $1/f$, they would need to interfere with $d = 6, 5$ operators, respectively, and would thus be suppressed by powers of masses and v.

To quantify the effects of the $d = 7$ operators and compare them to those of the $d = 5$ operator, we show in Fig. 3 the complete cross-section (in which particle masses and the Higgs vacuum expectation value are properly taken into account) as a function of $\sqrt{s}$, with $\sigma^{\rm tot,(5)}$ denoting, respectively, the total cross-section and the cross-section computed with only $d = 5$ operators. Since $C^{(1,2)}_{\phi\ell^2HD^2}$ give the dominant contribution at high-energies, we set $C_{\phi^2H^2} = C_{\phi H^4} = 0$. Furthermore, we take $C_{\phi\ell^2H} = |C^{(1,2)}_{\phi\ell^2HD^2}| = 1$, while we allow for a different signs between $C^{(1)}_{\phi\ell^2HD^2}$ and $C^{(2)}_{\phi\ell^2HD^2}$. On the left panel, we plot the ratio $\sigma^{\rm tot}/\sigma^{(5)}$ for $\ell = e, \mu$ and $f = 10^4$ GeV (blue lines) and $f = 10^5$ GeV (orange lines). Continuous lines refer to $e^+e^- \to \phi h$ while dashed lines refer to $\mu^+\mu^- \to \phi h$. The bands are obtained varying the sign between $C^{(1)}_{\phi\ell^2HD^2}$ and $C^{(2)}_{\phi\ell^2HD^2}$. As we can see, at high energies $\sigma^{\rm tot}$ can be larger than $\sigma^{(5)}$ by many orders of magnitude. The effect is larger for electrons, since $\sigma^{(5)}$ is suppressed by smaller Yukawa couplings.

On the right panel we instead show how $\sigma^{\rm tot}$ grows with energy, with the same conventions used in the left panel. Inspecting the two muon cross-sections shown, we clearly see the transition from the low energy regime (dominated by the $d = 5$ operator and essentially independent on energy [9]) and the high energy energy regime, in which the $d = 7$ operator dominates and the cross-section scales as s^2 as predicted by Eq. (5.2). It is interesting to notice that for $f = 10^4$ GeV the dimension 7 effects can push the cross-section to reasonable values ($\sigma^{\rm tot} \sim 1$ fb), while it would remain out of reach considering only the $d = 5$ operator ($\sigma^{(5)} \lesssim 10^{-5}$ fb).

From the EFT perspective, it is to be expected that higher-dimensional operators dominate the cross-section when the energy is close to the cut-off f. What makes the present case more special and interesting is that the effects of the $d = 7$ operators are relevant much before $\sqrt{s} \sim f$. This is a direct consequence of the fact that the soft factorization condition (2.6) required the coefficient of the $d = 5$ amplitude to be proportional to the Yukawa, while the soft contact condition (4.1) did not impose such selection rule on the $d = 7$ operators. We see from Fig. 3 that, at the phenomenological level, this distinction between $d = 5$ and $d = 7$ operators might be extremely relevant, since, for instance for electrons, the contribution from $d = 5$ is always subdominant.

6 Conclusions

Given a scattering amplitude involving ALPs, what is the set of physical properties that it must satisfy to recover shift-symmetry? Of course, starting from a Lagrangian and requiring it to be shift-symmetry invariant is not a complete answer to this question, as we are not making statements about the amplitudes themselves. Taking the on-shell approach, that bypasses fields and Lagrangians, makes the question even less obvious, but provides a natural framework for working it out. In this paper, using on-shell methods, we have identified three conditions that allow us to construct amplitudes with the desired shift-symmetric properties. All three conditions rely on the properties of the amplitude in the limit of soft ALP momentum.

The first condition, that we call *soft factorization condition*, Eq. (2.6), enables to reproduce the correct amplitudes when the ALP is interacting with two massive particles, *i.e.*, in the case of amplitudes suppressed by f^{-1}, with f the scale associated with the ALP (Section 2). In addition to constructing these amplitudes, we also discuss at length how they can be connected with amplitudes that involve SM particles and are invariant under the SM symmetry group at high energies. While the case of the coupling of an ALP to Higgs doublets or gauge bosons is straightforward, the case of fermions is much more subtle, due to an interesting obstruction that emerges because

[9]When $\sqrt{s}$ is comparable to the Higgs mass, the cross section is not flat with respect to $\sqrt{s}$ because of the threshold.

of a combination of the kinematic of 3-point amplitudes and invariance under the SM symmetry group. In Section 3 we identify a second condition, the *3-point Higgs obstruction*, that deals with this latter amplitude by relying on electroweak symmetry breaking and on the fact that, in the limit in which the Higgs momentum becomes soft, the 4-point amplitude which is invariant under the SM symmetry becomes a 3-point amplitude with massive fermions. The consequence is that the scattering amplitude ends up being proportional to the Yukawa coupling of the fermions involved.

The third condition, that we call *soft contact condition*, Eq. (4.1), allows to determine the correct amplitudes when the ALP is interacting with more than two other particles. Since these higher-point amplitudes correspond to effective operators suppressed by higher powers of f, we determine in Section 4, solely using scattering amplitudes, the complete list of operators up to $d = 8$. We also study a phenomenological application in which these higher dimensional operators can dominate over the $d = 5$ ones: the process $\ell^+\ell^- \to \phi h$ at high energy lepton colliders (Section 5). We find that, due to the Yukawa suppression of the $d = 5$ operator, the $d = 7$ ones can dominate already at energies $\sqrt{s} \ll f$, where they are typically expected to give only subdominant contributions with respect to the $d = 5$ operators.

We observe that the *soft factorization condition* and the *soft contact condition* are equivalent to requiring the amplitude to manifest the Adler's zero in the limit of soft ALP limit. Nevertheless, this procedure is not completely universal, since they cannot be applied to 4-point amplitudes involving one Higgs doublet. As we have shown, in this case we must resort to the *3-point Higgs obstruction* condition of Eq. (3.3), which can be seen as kind of a generalization of the Adler's zero to this case.

Our work can be extended in several directions. First of all, having derived the shift-symmetric ALP amplitudes, it follows immediately that all additional terms that do not satisfy our conditions must break the shift symmetry. An interesting point would be to investigate how some sort of power counting could be applied to such coefficients without resorting to Lagrangians. A second aspect that can be explored would be the generalization to amplitudes involving several different ALPs, *i.e.* to the case in which we have spontaneous symmetry breaking of a non-abelian group. Possible applications would be to the study of amplitudes involving light mesons and matter particles (like nucleons or vector mesons) in Chiral Perturbation Theory, or the coupling between a Composite Higgs and SM fermions and gauge bosons. Moreover, since on-shell methods can be used to derive the renormalization group equations of effective operators [10–13], amplitudes with more than one ALP can be used to study the effects of light ALPs in the running of Wilson coefficients of operators of the Standard Model Effective Field Theory. Finally, the on-shell scattering amplitudes formalism extended using the techniques of Ref. [72] can, in principle, be used to compute the interactions between ALPs and photons when magnetic monolopoles are present [73, 74]. We leave the study of these aspects to future work.

Note added: While completing this paper, Refs. [75, 76] appeared, in which the list of $d = 8$ operators with ALPs is presented. Our list agrees with the one shown in these papers.

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A Conventions

A.1 Spinor variables

In this appendix, we set the notation for the spinor variables and summarize the identities we use. The starting point is the covariance of the S-matrix under little-group, that allows us to write scattering amplitudes as a sum of all possible kinematical structures that carry the correct little-group transformation, each multiplied by a corresponding coupling. The building blocks for these kinematical structures are 2-component spinors. For massless momenta we define the spinor variables as

$$|p\rangle \equiv |p\rangle_\alpha\,, \quad |p] \equiv |p]^{\dot\alpha}\,, \quad \langle p| \equiv \langle p|^\alpha\,, \quad [p| \equiv [p|_{\dot\alpha}\,, \tag{A.1}$$

where $\alpha, \dot\alpha$ are $SL(2, C)$ indices for left- and right-handed spinors, respectively. The spinors $|p\rangle\,, \langle p|$ are referred as *angle* spinors, while $|p]\,, [p|$ as *square* spinors. They transform under the $U(1)$ little-group with opposite phases:

$$|p\rangle \to e^{-i\phi} |p\rangle \,, \quad |p] \to e^{i\phi} |p] \,, \quad \phi \in \mathbb{R}. \tag{A.2}$$

All indices can be raised and lowered with the Levi–Civita tensor that is defined by $\epsilon^{12} = -\epsilon_{12} = 1$, for instance $|p\rangle_\alpha = \epsilon_{\alpha\beta} \langle p|^\beta$. The spinors above satisfy

$$p_\mu (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv p^{\dot{\alpha}\alpha} \equiv |p]^{\dot{\alpha}} \langle p|^\alpha \,, \quad p_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} \equiv p_{\alpha\dot{\alpha}} \equiv |p\rangle_\alpha [p|_{\dot{\alpha}} \,, \tag{A.3}$$

$$p\,|p\rangle = p\,|p] = \langle p|\, p = [p|\, p = 0. \tag{A.4}$$

Equation (A.3) is simply the defining equation of the spinors, where $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$, while in Eq. (A.4) we have the Weyl equations. For a massless 4-momentum $(p^\mu) = p(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, with $p \equiv |\vec{p}|$, one possible realization for the spinors in Eq. (A.3) is given by

$$\begin{aligned} (\langle p|^\alpha) &= -\sqrt{2p} \begin{pmatrix} c \\ s^* \end{pmatrix}, \quad (|p]^{\dot{\alpha}}) = -\sqrt{2p} \begin{pmatrix} c \\ s \end{pmatrix}, \\ (|p\rangle_\alpha) &= \sqrt{2p} \begin{pmatrix} s^* \\ -c \end{pmatrix}, \quad ([p|_{\dot{\alpha}}) = \sqrt{2p} \begin{pmatrix} s \\ -c \end{pmatrix}, \end{aligned} \tag{A.5}$$

where $c \equiv \cos\frac{\theta}{2}$, $s \equiv e^{i\phi} \sin\frac{\theta}{2}$. We can also define the usual anti-symmetric Lorentz invariant products of spinors:

$$\langle pq\rangle \equiv \langle p|^\alpha |q\rangle_\alpha \,, \quad [pq] \equiv [p|_{\dot{\alpha}} |q]^{\dot{\alpha}} \,, \tag{A.6}$$

with the contraction of up- and down-indices performed with the Levi–Civita tensor. These products can be computed explicitly using Eq. (A.5).

For massive momenta we adopt the notation of Ref. [3] and define bold angle and square spinors as

$$|\boldsymbol{p}\rangle \equiv \left|p^I\right\rangle_\alpha, \quad |\boldsymbol{p}] \equiv \left|p^I\right]^{\dot{\alpha}}, \quad \langle\boldsymbol{p}| \equiv \left\langle p^I\right|^\alpha, \quad [\boldsymbol{p}| \equiv \left[p^I\right|_{\dot{\alpha}}, \tag{A.7}$$

where now $I = 1, 2$ is the $SU(2)$ index of the little-group, that are also raised and lowered through the Levi–Civita tensor. Contrary to the massless case, they transform in the same way under little-group as

$$\left|p^I\right\rangle \to W^I{}_J \left|p^J\right\rangle, \quad \left|p^I\right] \to W^I{}_J \left|p^J\right], \quad W \in SU(2), \tag{A.8}$$

and analogous to $\langle\boldsymbol{p}|$ and $[\boldsymbol{p}|$. The massive spinors satisfy similar relations as Eqs. (A.3) and (A.4), but with the inclusion of the $SU(2)$ little-group indices:

$$p^{\dot{\alpha}\alpha} \equiv \epsilon_{IJ} \left|p^I\right]^{\dot{\alpha}} \left\langle p^J\right|^\alpha, \quad p_{\alpha\dot{\alpha}} \equiv -\epsilon_{IJ} \left|p^I\right\rangle_\alpha \left[p^J\right|_{\dot{\alpha}}. \tag{A.9}$$

With the same parametrization as in Eq. (A.5), we can represent the massive spinors in terms of the components of the 4-momentum as

$$
\begin{aligned}
(\langle p^I|^\alpha) &= -\left(\begin{array}{c|c} c\sqrt{E+p} & -s\sqrt{E-p} \\ s^*\sqrt{E+p} & c\sqrt{E-p} \end{array}\right), \quad (|p^I]^{\dot\alpha}) = -\left(\begin{array}{c|c} s^*\sqrt{E-p} & c\sqrt{E+p} \\ -c\sqrt{E-p} & s\sqrt{E+p} \end{array}\right), \\
(|p^I\rangle_\alpha) &= +\left(\begin{array}{c|c} s^*\sqrt{E+p} & c\sqrt{E-p} \\ -c\sqrt{E+p} & s\sqrt{E-p} \end{array}\right), \quad ([p^I|_{\dot\alpha}) = +\left(\begin{array}{c|c} -c\sqrt{E-p} & s\sqrt{E+p} \\ -s^*\sqrt{E-p} & -c\sqrt{E+p} \end{array}\right),
\end{aligned}
\tag{A.10}
$$

where the first (second) column refers to $I = 1$ (2), while the rows are for different $SL(2,C)$ indices. Also, $E = \sqrt{m^2+p^2}$ is the energy and, as before, $p \equiv |\vec{p}|$. In writing Eq. (A.10), we have assumed that momenta, and in particular the mass m, are real. For the expressions with complex momenta, we refer the reader to Ref. [3] and references therein. From the expressions above it is trivial to check that

$$
\begin{aligned}
p_{\alpha\dot\alpha}\,|p^I]^{\dot\alpha} &= -m\,|p^I\rangle_\alpha\,, \quad p^{\dot\alpha\alpha}\,|p^I\rangle_\alpha = -m\,|p^I]^\alpha\,, \\
\langle p^I|^\alpha\,p_{\alpha\dot\alpha} &= m\,[p^I|_{\dot\alpha}\,, \quad [p^I|_{\dot\alpha}\,p^{\dot\alpha\alpha} = m\,\langle p^I|^\alpha\,,
\end{aligned}
\tag{A.11}
$$

which are the massive version of the Weyl equations in (A.4). We notice that in the massive Weyl equations (A.11) the two types of spinors are related by the mass m. The anti-symmetric spinor product is defined in the same way as before

$$
\langle \boldsymbol{pq}\rangle \equiv \langle p^I q^J\rangle = \langle p^I|^\alpha\,|q^J\rangle_\alpha\,, \quad [\boldsymbol{pq}] \equiv [p^I q^J] = [p^I|_{\dot\alpha}\,|q^J]^{\dot\alpha}\,. \tag{A.12}
$$

From the Weyl equations we can also derive

$$
\langle p^I p^J\rangle = -m\epsilon^{IJ}, \quad [p^I p^J] = m\epsilon^{IJ}. \tag{A.13}
$$

We notice that spinor variables associated to massless particles can be defined as the high-energy limit of the massive spinors. As a practical rule of thumb, this amounts to simply *un-bold* the massive spinors

$$
|\boldsymbol{p}\rangle \to |p\rangle\,, \quad |\boldsymbol{p}] \to |p]\,. \tag{A.14}
$$

The *un-bolding* in Eq. (A.14) states that only one linear combination of the $SU(2)$ little-group indices is selected in the massless limit, while the orthogonal combination is always proportional to the mass. This can be seen more explicitly from the explicitly expressions (A.5) and (A.10), where

$$
\begin{aligned}
(\langle p^I|^\alpha) &\to (\langle p^1|^\alpha) = (\langle p|^\alpha), \quad (|p^I]^{\dot\alpha}) \to (|p^2]^{\dot\alpha}) = (|p]^{\dot\alpha}), \\
(|p^I\rangle_\alpha) &\to (|p^1\rangle_\alpha) = (|p\rangle_\alpha), \quad ([p^I|_{\dot\alpha}) \to ([p^2|_{\dot\alpha}) = ([p|_{\dot\alpha}),
\end{aligned}
\tag{A.15}
$$

that makes manifest that the high-energy limit selects $I = 1$ for angle and $I = 2$ for square spinors. It is important to notice that the limit above is for up-index little-group indices.

It is also useful to consider spinors with negative momenta, for which the analytic continuation compatible with Eq. (A.11) reads

$$|-\boldsymbol{p}\rangle = |\boldsymbol{p}\rangle\,, \quad |-\boldsymbol{p}] = -\,|\boldsymbol{p}]\,. \tag{A.16}$$

Other useful identities are

$$\begin{gathered}\langle p|q|k] = [k|q|p\rangle\,,\\ \langle p|\sigma^\mu|q]\,\langle k|\sigma_\mu|l] = -2\,\langle pk\rangle\,[ql]\end{gathered} \tag{A.17}$$

where $\langle p|q|k] \equiv \langle p|^\alpha\, q_{\alpha\dot\alpha}\, |k]^{\dot\alpha}$ and similar for $[k|q|p\rangle$. The same relations also hold for massive spinors.

Finally, to compute the soft factors in Section 2, more precisely the p_ϕ-independet piece in Eq. (2.9), we can use the explicit representations for the spinors in Eqs.(A.5) and (A.10). Starting with a massive momentum p and a massless one p_ϕ, we have:

$$\begin{aligned}\left\langle p^I\;(p+p_\phi)^J\right\rangle &= \begin{pmatrix} 0 & -m \\ \sqrt{(E-p)(E+p+2p_\phi)} & 0\end{pmatrix}^{IJ}\,,\\ \left[p^I\;(p+p_\phi)^J\right] &= \begin{pmatrix} 0 & \sqrt{(E-p)(E+p+2p_\phi)} \\ -m & 0\end{pmatrix}^{IJ}\,,\end{aligned} \tag{A.18}$$

where we are assuming that the angle between the 3-momenta $\vec{p}$ and $\vec{p}_\phi$ is zero. We can see that in the limit $p_\phi \to 0$ we recover Eq. (A.12). If instead we had started with both massless p and p_ϕ, such that the angle between the 3-momenta is $\theta \ll 1$, we would have obtained

$$\langle p\; p+p_\phi\rangle = -\,[p\; p+p_\phi] = -p_\phi\sqrt{\frac{p}{p+p_\phi}}\,\theta + \mathcal{O}\left(\theta^2\right)\,, \tag{A.19}$$

which are relevant quantities for the computation of Eq. (2.17).

A.2 Construction of amplitudes

To write down amplitudes, we use the fact that the S-matrix is covariant under little-group transformations. This implies that we can express the amplitudes as a sum of all possible Lorentz-invariant combinations of spinor variables that have the correct little-group transformation.

For massless particles the little-group transformation is given by Eq. (A.2) and, according to the covariance of the S-matrix, the amplitude must transform as $\mathcal{A} \to$

$e^{-2ih}\mathcal{A}$, with h the helicity of the corresponding particle. Specializing to the case of 3-point amplitudes, little-group covariant gives us three constraints, while the amplitude can be built of 6 spinor products, namely $\langle 12\rangle, \langle 23\rangle, \langle 13\rangle$ and the same with square brackets. We use here the short-hand notation $|p_n\rangle \equiv |n\rangle$ to label the momenta in the spinors. Due to 3-particle kinematics, however, all these products vanish if the momenta are real. Relaxing this hypothesis, we find that we can either have the angle contractions or the square ones to be non-vanishing, thus leaving us with only three independent Lorentz-invariant products. Up to coefficients, this allows us to completely fix the amplitude [1, 2, 5]:

$$\mathcal{A}_3 \propto \begin{cases} \langle 12\rangle^{h_3-h_1-h_2} \langle 23\rangle^{h_1-h_2-h_3} \langle 13\rangle^{h_2-h_1-h_3}, & \sum_i h_i \leq 0 \\ [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [13]^{h_1+h_3-h_2}, & \sum_i h_i \geq 0 \end{cases}, \tag{A.20}$$

where $h_{1,2,3}$ are the helicities of the particles. For higher-point functions, we cannot completely fix the amplitude, since we have more spinor contractions than little-group transformation rules.

When considering massive particles the discussion is more involved, because both angle and square spinors transform in the same way under little-group. Nevertheless, one can simply write the amplitude as a sum of all independent spinor structures, each with a different coefficient (see Ref. [32] for a classification). For a massive particle of spin s, the transformation of the amplitude is given by the completely symmetric $2s$ tensor representation, which is equivalent to the usual representation in terms of the total spin and its projection [3]. Hence, each term in the amplitude must contain exactly $2s$ spinors, angle and/or square, of this given particle.

A.3 CPT invariance and unitarity

To extract information on the phases of the amplitudes, we need to relate them to their complex conjugate, which can be done using CPT invariance and unitarity. In terms of the transfer matrix T, the amplitude for a state $\mathcal{O}$ is written as $\mathcal{A}[\mathcal{O}] = \langle 0|T\mathcal{O}\rangle$, where we take all particles to be incoming and 0 is the vacuum. Using CPT invariance of the S-matrix, one can show that $\langle 0|T\mathcal{O}\rangle = \langle \mathcal{O}_\Theta|T0\rangle$, with Θ representing the action of CPT in the multi-particle states, that amounts to reversing the spin (*i.e.* changing up and down little-group indices in the massive case, or flipping the helicity for massless particles) and swapping particles with anti-particles [6]. In addition, when applying Θ we must also reverse the ordering of the particles in the amplitude, that can lead to extra minus signs in the case of fermions. Then, unitarity of the S-matrix imposes that $T \simeq T^\dagger$ up to corrections of order $T^\dagger T$. We therefore obtain at leading order

$$\mathcal{A}[\mathcal{O}] \simeq \mathcal{A}[\mathcal{O}_\Theta]^*. \tag{A.21}$$

The complex conjugation of spinor structures can be found to yield

$$\left\langle p^I q^J \right\rangle^* = -\left[p_I q_J\right], \tag{A.22}$$

that holds when considering real momenta with positive energy [78, 79].

A.4 Feynman rules with spinor variables

It is also instructive to show how some Feynman rules for effective operators are translated in terms of spinor variables. To do so, we first need to express the external wave-functions in terms of spinors. For spin 1/2 and spin 1 particles we have, respectively,

$$\begin{aligned} v^I(p) &= \begin{pmatrix} \left|p^I\right\rangle \\ \left|p^I\right] \end{pmatrix}, \quad \bar{v}_I(p) = \left(-\left\langle p_I\right|, \ \left[p_I\right|\right), \\ u^I(p) &= \begin{pmatrix} \left|p^I\right\rangle \\ -\left|p^I\right] \end{pmatrix}, \quad \bar{u}_I(p) = \left(\left\langle p_I\right|, \ \left[p_I\right|\right), \end{aligned} \tag{A.23}$$

$$\epsilon_\mu(p) = \frac{\langle \boldsymbol{p}|\sigma_\mu|\boldsymbol{p}]}{\sqrt{2}m_V}, \quad \text{or} \quad \epsilon(p)_{\alpha\dot{\alpha}} = \sqrt{2}\frac{|\boldsymbol{p}\rangle\,[\boldsymbol{p}|}{m_V}, \tag{A.24}$$

where $u, v, \bar{u}, \bar{v}$ are the usual solutions to the Dirac equation, and ϵ_μ is the massive polarization vector, with m_V the mass of the spin 1 particle. Considering a Yukawa interaction of the form

$$\mathscr{L}_{\text{Yukawa}} = -\phi\left(g_L \psi_R^\dagger \psi_L + g_R \psi_L^\dagger \psi_R\right), \tag{A.25}$$

the corresponding on-shell amplitude with all-in convention is

$$\mathcal{A}_3\left[\phi\psi_1^I\bar{\psi}_2^J\right] = -\bar{u}^J(-p_2)\begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix} u^I(p_1) = g_L\left\langle 1^I 2^J\right\rangle + g_R\left[1^I 2^J\right]. \tag{A.26}$$

For vectors the relevant interactions are

$$\mathscr{L}_V = \frac{m_V c_0}{2}\phi V_\mu V^\mu + g\phi V_{\mu\nu}V^{\mu\nu} + \tilde{g}\phi V_{\mu\nu}\tilde{V}^{\mu\nu}. \tag{A.27}$$

The corresponding amplitude is

$$\begin{aligned} \mathcal{A}_3\left[\phi V_1^{I_1,I_2} V_2^{J_1,J_2}\right] &= \left(2\frac{m_V c_0}{2}\eta_{\mu\nu} + 4g[p_1^\nu p_2^\mu - (p_1\cdot p_2)\eta_{\mu\nu}] + 4\tilde{g}\epsilon_{\mu\nu\alpha\beta}p_1^\alpha p_2^\beta\right)\epsilon^\mu(p_1)\epsilon^\nu(p_2) \\ &= 2(i\tilde{g} - g)\langle \mathbf{12}\rangle^2 - 2(i\tilde{g} + g)[\mathbf{12}]^2 - \frac{c_0}{m_V}\langle \mathbf{12}\rangle[\mathbf{12}], \end{aligned} \tag{A.28}$$

where we note explicitly the $1/m_V$ scaling of the last term. To manipulate the Levi–Civita we have used

$$\epsilon_{\mu\nu\alpha\beta} = \frac{1}{4i}\Big(\text{Tr}[\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\alpha\sigma_\beta] - \text{Tr}[\sigma_\mu\bar{\sigma}_\nu\sigma_\alpha\bar{\sigma}_\beta]\Big). \tag{A.29}$$

A.5 SM amplitudes

For completeness, we present here as well the SM 3-point amplitudes we use to derive our results in Section 3:

$$
\begin{aligned}
\mathcal{A}_3\left[Q_1^{ibn}\bar{u}^a_{m2}H^j\right] &= (Y_u^\dagger)^{ab}\delta^n_m\epsilon^{ij}\langle 12\rangle\,, & \mathcal{A}_3\left[\bar{Q}^b_{in1}u_2^{am}\bar{H}_j\right] &= (Y_u)^{ba}\delta^m_n\epsilon_{ij}[12]\,,\\
\mathcal{A}_3\left[Q_1^{ibn}\bar{d}^a_{m2}\bar{H}_j\right] &= (Y_d^\dagger)^{ab}\delta^n_m\delta^i_j\langle 12\rangle\,, & \mathcal{A}_3\left[\bar{Q}^b_{in1}d_2^{am}H^j\right] &= (Y_d)^{ba}\delta^m_n\delta^j_i[12]\,,\\
\mathcal{A}_3\left[Q_1^{ibn}\bar{Q}^a_{jm2}G_3^{A+}\right] &= -\sqrt{2}g_sT^A_{mn}\delta^i_j\delta^b_a\frac{[23]^2}{[12]}, & \mathcal{A}_3\left[Q_1^{ibn}\bar{Q}^a_{jm2}G_3^{A-}\right] &= \sqrt{2}g_sT^A_{mn}\delta^i_j\delta^b_a\frac{\langle 13\rangle^2}{\langle 12\rangle},\\
\mathcal{A}_3\left[u_1^{bn}\bar{u}^a_{m2}G_3^{A+}\right] &= \sqrt{2}g_sT^A_{mn}\delta^b_a\frac{[13]^2}{[12]}, & \mathcal{A}_3\left[u_1^{bn}\bar{u}^a_{m2}G_3^{A-}\right] &= -\sqrt{2}g_sT^A_{mn}\delta^b_a\frac{\langle 23\rangle^2}{\langle 12\rangle},\\
\mathcal{A}_3\left[H_1^i\bar{H}_{j2}W_3^{A-}\right] &= -\sqrt{2}gT^A_{ji}\frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}, & \mathcal{A}_3\left[H_1^i\bar{H}_{j2}W_3^{A+}\right] &= -\sqrt{2}gT^A_{ji}\frac{[13][23]}{[12]},\\
\mathcal{A}_3\left[H_1^i\bar{H}_{j2}B_3^{-}\right] &= -\sqrt{2}g'y_H\delta^i_j\frac{\langle 13\rangle\langle 23\rangle}{\langle 12\rangle}, & \mathcal{A}_3\left[H_1^i\bar{H}_{j2}B_3^{+}\right] &= -\sqrt{2}g'y_H\delta^i_j\frac{[13][23]}{[12]},
\end{aligned}
\tag{A.30}
$$

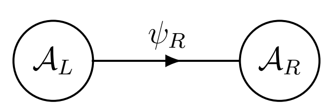
where the expressions are analogous for other fermions and vector bosons, g', g, g_s are the gauge couplings and y_H the Higgs hypercharge. In the equations above a, b denote flavor indices, i, j indices from the fundamental of $SU(2)_L$ and m, n for the fundamental of $SU(3)_c$. We use T^A for the generators of both non-abelian groups. Here, the conventions are chosen such to correspond to covariant derivatives defined as $D_\mu = \partial_\mu - igV_\mu$, and the Yukawas as $\mathscr{L}_{\text{Yukawa}} = -Y_\psi\bar{\psi}_L H\psi_R + h.c.$.

B Yukawa - Mass connection

In this appendix we present an alternative way to understand the relation between fermion masses in the broken electroweak phase and the Yukawa matrices in the unbroken phase. This relation was previously studied in Ref. [22] (and more recently in Ref. [56]) by analysing how the "freezing" of external Higgs particles can make massless spinors to become massive. Here instead, we show explicitly that we can understand the Higgsing as a modification to the dispersion relation of the fermions, which is very similar to the analysis performed in Ref. [34] for neutrino propagation in matter.

Consider a generic amplitude $\mathcal{A}$ that has a factorization channel in ψ_R for $p^2 \to 0$, with ψ_R being one of the right-handed fermions of the SM. More precisely,

$$\lim_{p^2\to 0}\mathcal{A} = \lim_{p^2\to 0}\mathcal{A}^a_L\frac{\delta^{ab}}{p^2}\mathcal{A}^b_R =$$



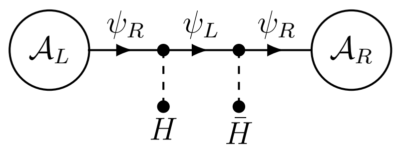
, (B.1)

where $\mathcal{A}^a_{L,R}$ are the sub-amplitudes, a, b are flavor indices and we leave implicit spinorial indices and the Levi–Civita tensor. Consider now the same amplitude with the addition of an Higgs - anti-Higgs pair in the $p_H, p_{\bar{H}} \to 0$ limit, in such a way that the Higgs become non-dynamical. Given that the Higgs interacts with the fermions through Yukawa interactions,

$$\mathcal{A}\left[\psi^b_{L,1}\bar{\psi}^a_{R,2}\bar{H}\right] = (Y^\dagger_\psi)^{ab}\langle 12\rangle\,, \quad \mathcal{A}\left[\bar{\psi}^a_{L,1}\psi^b_{R,2}H\right] = (Y_\psi)^{ab}[12]\,, \tag{B.2}$$

where $H \leftrightarrow \bar{H}$ in the case of u-quarks and ignoring SM group indices, we can make recursive use of *polology* to show that the amplitude will feature extra factorization channels as $p^2, p_H, p_{\bar{H}} \to 0$. Schematically[10],

$$\lim_{p_H, p_{\bar{H}}\to 0}\lim_{p^2\to 0}\mathcal{A}\left[H, \bar{H}\right] = \tag{B.3}$$



.

Note that we need one Higgs and one anti-Higgs, because otherwise we would not obtain a factorization to the same amplitudes $\mathcal{A}^a_{L,R}$. Evaluating the amplitude above we obtain

$$\begin{aligned}&\lim_{p_H, p_{\bar{H}}\to 0}\lim_{p^2\to 0} v^2\mathcal{A}\left[H, \bar{H}\right]\\ &= \lim_{p_H, p_{\bar{H}}\to 0}\lim_{p^2\to 0}\mathcal{A}^a_L\frac{1}{p^2}\frac{v(Y_\psi)^{ca}\left[(-p-p_H)p\right]}{(p+p_H)^2}\frac{v(Y^\dagger_\psi)^{bc}\langle(p+p_H)(-p-p_H-p_{\bar{H}})\rangle}{(p+p_H+p_{\bar{H}})^2}\mathcal{A}^b_R,\end{aligned} \tag{B.4}$$

where we insert one factor of v for each Higgs becoming non-dynamical. The numerator of the expression above can be rewritten as

$$\left[(-p-p_H)p\right]\langle(p+p_H)(-p-p_H-p_{\bar{H}})\rangle \simeq \left[pp_H\right]\langle p_H p\rangle = (p+p_H)^2, \tag{B.5}$$

where we have neglected higher-order terms in Higgs momenta. With the simplification above the limit becomes

$$\lim_{p_H, p_{\bar{H}}\to 0}\lim_{p^2\to 0} v^2\mathcal{A}\left[H, \bar{H}\right] = \lim_{p^2\to 0}\mathcal{A}^b_R\frac{1}{p^2}\frac{v^2(Y^\dagger_\psi Y_\psi)^{ba}}{p^2}\mathcal{A}^a_L. \tag{B.6}$$

It is clear that Eq. (B.6) contributes to the original amplitude without the Higgs - anti-Higgs pair, as the latter are removed from the external states. Hence, after we take all

[10] Here we ignore the contributions from the Higgs being emitted by the other external legs, that contribute to them becoming massive and therefore are not relevant to our computations.

Higgs non-dynamical we arrive at

$$\lim_{p_H,p_{\bar{H}}\to 0}\lim_{p^2\to 0}\left(\mathcal{A}+v^2\mathcal{A}\left[H,\bar{H}\right]+\cdots\right)=\lim_{p^2\to 0}\mathcal{A}_R^b\frac{1}{p^2}\left[\delta^{ba}+\frac{v^2(Y_\psi^\dagger Y_\psi)^{ba}}{p^2}+\cdots\right]\mathcal{A}_L^a, \tag{B.7}$$

where the dots denote similar amplitudes computed with more Higgs insertions, which are given by analogous expressions. It is possible to resum all the contributions:

$$\frac{1}{p^2}\left[\delta^{ba}+\frac{v^2(Y_\psi^\dagger Y_\psi)^{ba}}{p^2}+\cdots\right]=\left[\frac{1}{p^2-v^2Y_\psi^\dagger Y_\psi}\right]^{ba}, \tag{B.8}$$

hence it is clear that the Higgs "background" modifies the dispersion relations of the fermions by inducing an effective mass. Note that the propagator above does not respect locality manifestly, due to the matrix structure of the Yukawas. Nevertheless, it is always possible to find basis in which the propagator becomes diagonal and we therefore restore manifest locality. The necessity to perform such rotation is nothing but a realization of the mismatch between the massless and massive flavor basis. The mass matrix is given by

$$M_\psi^2=v^2U_R^\dagger Y_\psi^\dagger Y_\psi U_R=v^2\left(Y_\psi^\dagger Y_\psi\right)_{\text{diagonal}}, \tag{B.9}$$

where $U_R\in U(3)_{\psi_R}$, the right-handed flavor group of the massless phase of the theory. To be able to write directly the mass matrix in terms of the Yukawa, we can repeat the same steps above with a factorization channel on a left-handed fermion and obtain similarly $M_\psi^2=U_L^\dagger Y_\psi Y_\psi^\dagger U_L$, where now $U_L\in U(3)_{\psi_L}$. In order for both expressions to match, the mass must be given by

$$M_\psi=vU_L^\dagger Y_\psi U_R. \tag{B.10}$$

It is interesting to notice that for the neutrino sector, in the absence of a Yukawa interaction involving right-handed neutrinos, we can still generate masses via the Weinberg amplitude

$$\mathcal{A}_4\left[H^2L^2\right]=c_W\langle 12\rangle, \tag{B.11}$$

with L the lepton doublet. Repeating the reasoning above for this interaction, we obtain

$$M_\nu^2=v^4(c_W^\dagger c_W)_{\text{diagonal}}, \tag{B.12}$$

and we can thus see explicitly the different scaling with the scale v when compard to Eq. (B.10).

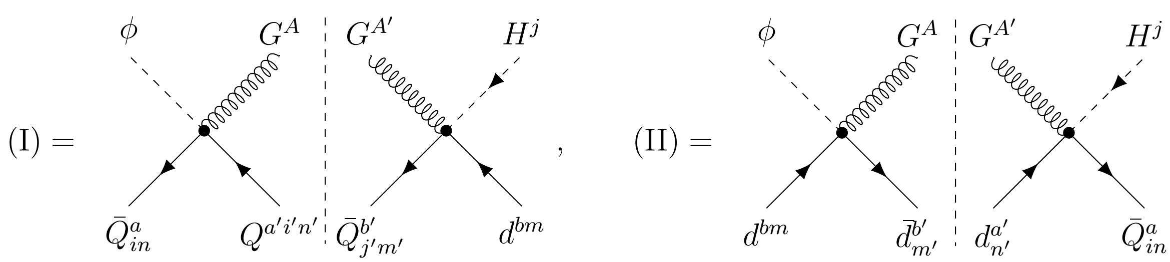
C One-loop running of $\mathcal{A}[\phi\psi\bar{\psi}H]$

In this appendix we describe in more detail the computation of the beta function used in Section 3 using the methods of generalized unitarity [5, 7–12, 80–88]. For 1-loop beta functions, the master formula reads [10, 84]

$$\left(\gamma_{\mathcal{O}_i} - \gamma_{\rm IR}^{(\mathcal{O}_i)}\right) \mathcal{A}\left[\mathcal{O}_i\right] = \frac{i}{8\pi^4} \sum_a \int \mathrm{d}\Pi[\ell_1, \ell_2] \oint_{z=\infty} \frac{\mathrm{d}z}{z} \mathcal{A}_L^{(a)} \left[\hat{\ell}_1, \hat{\ell}_2\right] \mathcal{A}_R^{(a)} \left[-\hat{\ell}_1, -\hat{\ell}_2\right], \tag{C.1}$$

where $\gamma_{\mathcal{O}_i}$ is the anomalous dimension of the amplitude $\mathcal{A}\left[\mathcal{O}_i\right]$. The formula above is obtained by selecting bubble diagrams from the Passarino–Veltman decomposition [89], which is the only topology that can contain UV divergences. This is achieved by first performing all 2-cuts (labeled by a), that result in the product of two tree amplitudes $\mathcal{A}_{L,R}^{(a)}$ integrated over the 2-body phase-space $\mathrm{d}\Pi[\ell_1, \ell_2]$ of the two cutted momenta $\ell_{1,2}$. Doing so does not select only bubble, since 2-cuts get contributions from triangles and boxes as well. To remove them, we shift $\ell_{1,2}$ to the complex plane using a BCFW-like shift [51, 52], $\hat{\ell}_{1,2} = \ell_{1,2} \pm z\ell_{2,1}$ and integrate over $\mathrm{d}z/z$ around $z = 0$. Then, we deform the contour around the origin as a sum over contours around poles and at $z = \infty$. The poles can only come from triangles and boxes, as they have un-cutted propagators. So dropping them and keeping only the residue at infinity guarantees that we are selecting only the bubbles. The numerical factor $i/8\pi^4$ arises from collecting the divergent piece of the bubble $(-1/8\pi^2)$, normalizing the phase-space integral $(2/\pi)$ and from Cauchy's theorem $(1/2\pi i)$. Also, $\gamma_{\rm IR}^{(\mathcal{O}_i)}$ denotes the IR contribution to the anomalous dimension that must be subtracted, and that depends only on the external particles of $\mathcal{A}\left[\mathcal{O}_i\right]$.

We are interested in computing the contribution of the amplitude $\mathcal{A}_3\left[\phi GG\right]$ to the running of $\mathcal{A}_4\left[\phi\psi\bar{\psi}H\right]$. For concreteness, let us choose $\mathcal{A}_4\left[\phi\bar{Q}dH\right]$ and $\mathcal{A}_4\left[\phi Q\bar{d}\bar{H}\right]$. For the first amplitude we have only two possible 2-cuts:



(C.2)

where we ignore other cuts that do not involve $\mathcal{A}_3\left[\phi GG\right]$. In addition, A, A' and n, n', m, m' are indices of the adjoint and fundamental of $SU(3)_c$, respectively, a, a', b, b' of flavor and i, i', j, j' of the fundamental of $SU(2)_L$. To compute the 2-cuts above, we need the corresponding 4-point amplitudes. They can be computed using Eqs. (3.6) and (A.30):

$$\begin{aligned}
\mathcal{A}_4\left[\phi Q_1^{a'i'n'}\bar{Q}_{in2}^{a}G_3^{A+}\right] &= -\frac{\sqrt{2}g_s g_G^+}{f}T^A_{nn'}\delta_i^{i'}\delta^{aa'}\frac{[23]^2}{[12]},\\
\mathcal{A}_4\left[\phi d_1^{a'n'}\bar{d}_{i2}^{a}G_3^{A+}\right] &= \frac{\sqrt{2}g_s g_G^+}{f}T^A_{nn'}\delta^{aa'}\frac{[13]^2}{[12]},\\
\mathcal{A}_4\left[\phi Q_1^{a'i'n'}\bar{Q}_{in2}^{a}G_3^{A-}\right] &= \frac{\sqrt{2}g_s g_G^-}{f}T^A_{nn'}\delta_i^{i'}\delta^{aa'}\frac{\langle 13\rangle^2}{\langle 12\rangle},\\
\mathcal{A}_4\left[\phi d_1^{a'n'}\bar{d}_{n2}^{a}G_3^{A-}\right] &= -\frac{\sqrt{2}g_s g_G^-}{f}T^A_{nn'}\delta^{aa'}\frac{\langle 23\rangle^2}{\langle 12\rangle},\\
\mathcal{A}_4\left[\bar{Q}_{in1}^{a}d_2^{a'n'}G_3^{A-}H^{i'}\right] &= -\sqrt{2}g_s T^A_{nn'}(Y_d)^{aa'}\delta_i^{i'}\frac{[12]^2}{[23]\,[13]},\\
\mathcal{A}_4\left[Q_1^{ain}\bar{d}_{n'2}^{a'}G_3^{A+}\bar{H}_{i'}\right] &= -\sqrt{2}g_s T^A_{nn'}(Y_d^\dagger)^{a'a}\delta_{i'}^{i}\frac{\langle 12\rangle^2}{\langle 23\rangle\,\langle 13\rangle},
\end{aligned} \tag{C.3}$$

where T^A denotes the generators of $SU(3)_c$ and $g_G^\pm$ the ALP-gluon couplings. The product of the two amplitudes in the left 2-cut in Eq. (C.2) reads

$$\begin{aligned}
\text{(I)} &= (-1)\mathcal{A}_4\left[\phi Q_{\ell_1}^{a'i'n'}\bar{Q}_{in1}^{a}G_{\ell_2}^{A+}\right]\mathcal{A}_4\left[\bar{Q}^{b'}_{j'm'(-\ell_1)}d_2^{bm}G^{A'-}_{(-\ell_2)}H_4^j\right]\delta^{AA'}\delta^{b'a'}\delta_{i'}^{j'}\delta_{n'}^{m'}\\
&= -\frac{2\mathcal{C}_A(\mathbf{3})g_s^2 g_G^+}{f}(Y_d)^{ab}\delta_i^j\delta_n^m\frac{[1\ell_2]^2\,[\ell_1 2]^2}{[1\ell_1]\,[\ell_1\ell_2]\,[2\ell_2]},
\end{aligned} \tag{C.4}$$

with $\mathcal{C}_A(\mathbf{3})$ the Casimir of the adjoint of $SU(3)_c$. The extra minus sign above takes into account fermion ordering, *i.e.* to arrange the amplitudes as we have defined in Eqs. (A.30) and (C.3) we need to anti-commute some fermions, leading to an extra minus. Note that only one helicity configuration of the gluons contribute, as the SM amplitude $\mathcal{A}_4\left[\bar{Q}dGH\right]$ is non-zero for only one choice of gluon helicity. Then, we shift $\ell_{1,2}$ to the complex plane as

$$|\ell_1\rangle \to |\ell_1\rangle + z\,|\ell_2\rangle\,,\quad |\ell_2] \to |\ell_2] - z\,|\ell_1]\,, \tag{C.5}$$

while $|\ell_1]$ and $|\ell_2\rangle$ remain unchanged. After performing the shift above and selecting the residue at $z=\infty$ we obtain

$$\text{(I)} \to -2\pi i\frac{2\mathcal{C}_A(\mathbf{3})g_s^2 g_G^+}{f}(Y_d)^{ab}\delta_i^j\delta_n^m\frac{2\,[1\ell_2]\,[2\ell_1]-[1\ell_1]\,[2\ell_2]}{[\ell_1\ell_2]}. \tag{C.6}$$

Now comes the phase-space integration. To this end, we write $\ell_{1,2}$ as linear combinations of external momenta [11, 22, 80, 82],

$$\begin{pmatrix}|\ell_1]\\|\ell_2]\end{pmatrix} = \begin{pmatrix}\cos\theta & -e^{i\phi}\sin\theta\\ e^{-i\phi}\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}|4]\\|2]\end{pmatrix}, \tag{C.7}$$

where ϕ is the azimutal and θ the half-polar angles. The angle spinors are related in similar way, but with an extra complex conjugation. Inserting Eq. (C.7) in (C.6) and integrating over phase-space

$$\int \mathrm{d}\Pi[\ell_1, \ell_2]\ (\mathrm{I}) = -2\pi i \frac{2\mathcal{C}_A(\mathbf{3}) g_s^2 g_G^+}{f} (Y_d)^{ab} \delta_i^j \delta_n^m \left(-\frac{3\pi}{4} [12] \right). \tag{C.8}$$

The 2-cut on the right in Eq. (C.2) is computed analogously. The product of the amplitudes is

$$\begin{aligned}(\mathrm{II}) &= \mathcal{A}_4 \left[\phi d_2^{bm} \bar{d}^{b'}_{m'\ell_1} G^{A+}_{\ell_2} \right] \mathcal{A}_4 \left[\bar{Q}^a_{in1} d^{a'}_{n'(-\ell_1)} G^{A'-}_{(-\ell_2)} H^j_4 \right] \delta^{AA'} \delta^{b'a'} \delta^{n'}_{m'} \\ &= \frac{2\mathcal{C}_A(\mathbf{3}) g_s^2 g_G^+}{f} (Y_d)^{ab} \delta_i^j \delta_n^m \frac{[2\ell_2]^2 [\ell_1 1]^2}{[2\ell_1][\ell_1 \ell_2][1\ell_2]},\end{aligned} \tag{C.9}$$

where now we do not have any extra minus from fermion ordering. Note that the spinor structure from expression above is identical to the one in Eq. (C.4) by changing $1 \leftrightarrow 2$. Thus,

$$\int \mathrm{d}\Pi[\ell_1, \ell_2]\ (\mathrm{II}) = 2\pi i \frac{2\mathcal{C}_A(\mathbf{3}) g_s^2 g_G^+}{f} (Y_d)^{ab} \delta_i^j \delta_n^m \left(\frac{3\pi}{4} [12] \right). \tag{C.10}$$

The result of both 2-cuts must be then plugged in Eq. (C.1) and compared to $\mathcal{A}_4 \left[\phi \bar{Q} d H \right]$. It is important to notice that the 2-cuts produced the same kinematical structure as the original amplitude. Besides, there is no contribution from IR divergences in this case [25, 26]. We arrive at

$$\frac{\mathrm{d}C_d}{\mathrm{d}\log\mu} = -\frac{g_s^2 g_G^+}{\pi^2} Y_d, \quad \frac{\mathrm{d}\bar{C}_d}{\mathrm{d}\log\mu} = -\frac{g_s^2 g_G^-}{\pi^2} Y_d^\dagger, \tag{C.11}$$

where we included the result also for the conjugate amplitude, that follows in a very similar way. The results above, properly translated to the usual language by Eq. (A.28), agree with previous computations in the literature [53–55]. At leading order, the couplings $g_G^\pm$ are related by complex conjugation according to Eq. (A.21), which implies that

$$\frac{\mathrm{d}\bar{C}_d}{\mathrm{d}\log\mu} \simeq \frac{\mathrm{d}C_d^\dagger}{\mathrm{d}\log\mu}, \tag{C.12}$$

up to two-loop effects. As a consequence, we can use the results of Section 3 and, to be consistent with the 1-flavor limit, must satisfy $g_G^- = -g_G^+$. Therefore,

$$g_G^- = -g_G^+ = iC_G, \quad C_G \in \mathbb{R}, \tag{C.13}$$

that corresponds solely to the $\phi G\tilde{G}$ coupling, as we wanted to demonstrate.

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