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Jet Cross Sections in e^+e^- Annihilation

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Abstract: Using an elaborate partial fractioning procedure of all 4-parton matrix elements we calculate 3-jet cross sections in e^+e^- annihilation with mass-cut jet resolution. All subleading contributions from nonsingular terms are included. We give thrust distributions for $O(\alpha_s^2)$ 3- and 4-jets. We study integrated cross sections as a function of the cutoff and check the reconstruction of $O(\alpha_s^2)$ σ_{tot} using recently calculated 2-jet cross sections with the same partial fractioning of 4-parton terms.

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1. Introduction

It is well known now that the study of jet production in high energy e^+e^- annihilation is one of the best ways to test the validity of perturbative QCD /1/. Therefore in the past much effort has been put into the calculation of higher order QCD corrections to the e^+e^- annihilation total cross section /2/ and to various differential cross sections /3 - 13/. At the level of perturbation theory up to $O(\alpha_s^2)$, where $\alpha_s = g^2/4\pi$ and g is the quark-gluon coupling constant, e^+e^- annihilate into 2-, 3- and 4-parton final states ($q\bar{q}$, $q\bar{q}g$, $q\bar{q}gg$ and $q\bar{q}q\bar{q}$). Individually, the loop-corrected 2-parton and 3-parton diagrams (see fig. 1, 2) are infrared and collinear divergent. These divergencies are supposed to cancel if, for example in the case of the 2-jet cross section, the 3-parton (fig. 2) and 4-parton contributions (fig.3) integrated over the 2-jet region with one or two of the emitted gluons (or quarks) being soft and/or collinear are added to the loop contributions. Similarly, in the case of the various 3-jet differential cross sections the divergencies cancel if the $q\bar{q}g$ loop terms are taken together with the 4-parton terms where two unresolved partons (qg or $q\bar{q}$) are integrated over to produce one jet. This procedure is analogous to the well known Bloch-Nordsieck cancellation of infrared divergencies in QED and yields finite resolution parameter dependent jet cross sections. As resolution criteria for two or three partons,depending which cross section is being calculated, two methods have been applied in the past. First there exists the Sterman-Weinberg definition /14/ where two partons are considered irresolvable if either parton has energy less than $\varepsilon\sqrt{q^2}/2$ (q^2 being the total c.m. energy) or the angle between the two partons is less than δ . The second procedure for defining irresolvable partons is based on an invariant mass constraint. Here two (or three) partons are said to be unresolved if their invariant mass squared $(p_i+p_j)^2$ (or $(p_i+p_j+p_k)^2$)

is less than yq^2. Originally the 2-jet cross section was calculated up to $O(\alpha_s)$ by Sterman and Weinberg with the (ε,δ) cuts /14/. The equivalent calculation with the mass cut y is found in /15/. It has been extended by us up to $O(\alpha_s^2)$ in a recent paper /16/. $O(\alpha_s^2)$ differential 3-jet cross sections with (ε,δ)-resolution have been calculated in /4/ and with invariant mass resolution in /5/. These calculations were based on the most singular terms of the 4-parton production cross section which are responsible for the infrared and mass-singular divergencies appearing as negative powers of $2\varepsilon = 4-n$ in dimensional regularization with dimension n after integration over the unresolved 2-parton configurations. In this work it was assumed that the non-singular pieces give small contributions proportional to ε, δ or y respectively which could be neglected. We can expect this for very small values of the resolution parameters (ε,δ) or y. Therefore in some phenomenological analyses of e^+e^- annihilation data the resolution parameter was chosen very small. For example, in several analyses based on the work of Sjöstrand /17/, who incorporated the formulae of /5/ into the string fragmentation model of the Lund group the mass cut parameter was taken to be y = 0.015. But it was never checked whether this value of y is small enough to make the subleading terms negligible in the total sum of $O(\alpha_s^2)$ 3-parton and 4-parton terms.

A different and more reasonable route was taken by the TASSO Collaboration /18/ at PETRA using the 3-jet formulae with (ε,δ)-resolution /4/. They calculated $d\sigma_{3\text{-jet}}$ with very small (ε,δ)-values ($\varepsilon \simeq 0.01$) and added the 4-parton configurations between these small (ε,δ) constraints and the larger, considered more physical, (ε,δ) values which separated the real 3- and 4-jet events. When in these added 4-parton configurations two partons fell into the resolution criteria with the larger (ε,δ) they were classified as 3-jet events. It is the purpose

of this work to go beyond the singular approximation and to include also all subleading four-parton terms in the calculation of $d\sigma_{3\text{-jet}}$ [1]. In particular this is important when one wants to calculate the 3-jet cross section with all 3-jet variables integrated over. We have done the calculations analytically as far as possibe. We shall employ the invariant mass resolution criterion with parameter y. The calculation of the two-jet cross section $\sigma_{2\text{-jet}}(y)$ has been completed by us just recently /16/. This enables us to obtain the sum $\sigma_{2\text{-jet}}(y) + \sigma_{3\text{-jet}}(y) + \sigma_{4\text{-jet}}(y)$, which should be independent of y and yield the well known $O(\alpha_s^2)$ correction to σ_{tot} /2/. A first step for calculating the non-singular terms in $d\sigma_{3\text{-jet}}$ has been undertaken recently by Gottschlak and Shatz /20/. But this work is incomplete since not all of the finite terms have been considered. The organisation of the paper is as follows. In section 2 we present the result for $\sigma_{2\text{-jet}}(y)$ up to $O(\alpha_s^2)$ taken from our other work /16/. There we used the same complete 4-parton matrix elements as employed later to obtain $\sigma_{3\text{-jet}}(y)$. Furthermore we observed carefully that the 4-parton phase space was divided in 2-, 3-, 4-jet regions in such a way that these three regions had no overlaps and that their sum covered the whole four-parton phase space. The details of the calculation of $\sigma_{3\text{-jet}}$ and $\sigma_{4\text{-jet}}$ are given in section 3. First we calculate the sum of $\sigma_{3\text{-jet}}(y) + \sigma_{4\text{-jet}}(y)$ with some additional contributions to $\sigma_{2\text{-jet}}(y)$ included which are not taken into account in section 2. This sum is studied as a function of y. Then it is split into the real 2-jet, 3-jet and 4-jet pieces.

Knowing the separation into these three jet categories we calculate differential thrust distributions for 3- and 4-jets separately. In all calculations we divide the various terms into contributions to the three colour factors proportional to C_F^2, C_FN_c and C_FT_R in order to see their magnitude in the 3- and 4-jet cross sections as a function of y. This allows us to see the behaviour of these cross sections for a pure abelian theory (no $C_F N_c$ term). The final check concerning σ_{tot} is presented in section 4. A summary and some final remarks are contained in section 5. The rather lengthy formulae for the calculation of the non-singular terms are delegated to two appendices.

2. Two-Jet Cross Section

The calculation of the higher order 2-jet cross section with invariant mass cut is described in details in /16/. So we shall write down only the final formulae which will be needed for the final summation of all jet cross sections in our last section.

Before doing this we shall sketch the calculation of the lower order ($O(\alpha_s)$) 2-jet cross section /15/, which also explains the steps to be taken in the higher order calculation. In lowest order of α_s we have two types of diagrams. First we have the virtual correction to the 2-parton term which contributes to the 2-jet cross section and is shown in fig. 4. It produces an infrared divergent contribution proportional to α_s. This has to be combined with the bremsstrahlung contribution coming from the last two diagrams in fig. 4. Averaged over angles with respect to the beam direction this cross section has the familiar form:

$$\frac{d^2\sigma}{dy_{13}\,dy_{23}} = \frac{\alpha_s(\mu^2)}{2\pi} C_F \frac{\sigma^{(2)}}{\Gamma(1-\varepsilon)} \left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon} (y_{12}\, y_{13}\, y_{23})^{-\varepsilon}\, T(y_{13}, y_{23}) \tag{2.1}$$

where
$$T(y_{13}, y_{23}) = (1-\varepsilon)\left(\frac{y_{13}}{y_{23}} + \frac{y_{23}}{y_{13}}\right) + \frac{2y_{12}}{y_{13}\,y_{23}} - 2\varepsilon \tag{2.2}$$

Here $y_{ij} = s_{ij}/q^2 = 2p_i p_j/q^2$ and p_1, p_2 and p_3 are the momenta of quark, antiquark and gluon respectively. $y_{13} + y_{23} + y_{12} = 1$ so that T depends only on the two normalized invariants y_{13} and y_{23}. $2\varepsilon = 4-n$ with n the arbitrary dimension to regularize infrared singularities. $\sigma^{(2)} = \sigma_0 (4\pi\mu^2/q^2)^{\varepsilon} \frac{\Gamma(2-\varepsilon)}{\Gamma(2-2\varepsilon)}$ is the lowest order cross section in n dimensions with $\sigma_0 = \frac{4\pi\alpha^2}{3q^2} N_C \sum_f Q_f^2$. $C_F = 4/3$ and μ is the up to now arbitrary mass parameter to make g dimensionless also in n dimensions. The phase space for y_{13} and y_{23} is shown in fig. 5. T diverges for y_{13} and/or $y_{23} \to 0$. These singular regions of phase space up to some boundary parametrized by y have to be integrated over and added to the (α_s) virtual 2-parton term. The sum yields the total 2-jet cross section. It includes those terms of the 3-parton final state where two of the three partons are unresolved, which is given by the region

$$0 \leq y_{12},\, y_{13},\, y_{23} \leq y \tag{2.3}$$

shown in fig. 5. The region $0 \leq y_{12} \leq y$ is non-singular and contributes only terms O(y). Of course, y should be chosen such that $y \ll 1$ otherwise the 2-jet cross section exhausts almost all $O(\alpha_s)$ contributions. But in principle y is arbitrary in the interval $0 < y \leq \frac{1}{2}$. The integration over the y-strips yields infrared and collinear singular terms, evaluated for example in n-dimensional regularization, which cancel the identical singular terms from the virtual diagram in fig. 4. The result for the sum of the two terms is the 2-jet cross section

$$\sigma_{2\text{-jet}}(y) = \sigma_0 \Big\{ 1 + \frac{\alpha_s(\mu^2)}{2\pi} C_F$$

$$\Big[-2\ln^2 y - 3\ln y - 1 + 2\zeta_2 + 4y\ln y - y^2 \ln y$$

$$- 4 L_2(y) + 3\ln(1-y) - 4y\ln(1-y) + 5y$$

$$+ y^2 \ln(1-y) + \Delta(y) \Big] \Big\} \tag{2.4}$$

with

$$\Delta(y) = 4\ln y \ln(1-y) - 2\ln^2(1-y) + 2y\ln y$$

$$+ 3\ln(1-2y) - 3\ln(1-y) - 6y\ln(1-2y)$$

$$+ 4y\ln(1-y) - y^2\ln(1-y) + y^2 \ln y + y$$

$$+ \frac{9}{2} y^2 + 4 L_2(y) - 4 L_2\left(\frac{y}{1-y}\right) \tag{2.5}$$

Eq. (2.3) is exact and includes also all terms which vanish for $y \to 0$. The contribution $\Delta(y)$ comes from the region $y_{12} \leq y$. $\zeta_2 = \pi^2/6$. If we integrate (2.1) over the triangle y_{12}, y_{13}, $y_{23} \geq y$ in fig. 5 we obtain the integrated 3-jet cross section which is

$$\sigma_{3\text{-jet}}(y) = \sigma_{tot} - \sigma_{2\text{-jet}}(y) \tag{2.6}$$

where

$$\sigma_{tot} = \sigma_0 \left(1 + \frac{3C_F}{2} \frac{\alpha_s(\mu^2)}{2\pi} \right) \tag{2.7}$$

is the total e^+e^- annihilation cross section up to $O(\alpha_s)$. Of course, σ_{tot} must be independent of y. Knowing (2.7) it is, of course, much simpler to do the integration over the outer triangle and to calculate $\sigma_{3\text{-jet}}(y)$ from which $\sigma_{2\text{-jet}}(y)$ follows via (2.6).

From (2.4) we can calculate the magnitude of the subleading terms proportional to $O(y)$ which either come from the region $y_{12} \leq y$ or as correction terms in the singular region y_{13}, $y_{23} \leq y$. Numerical values of these corrections are given in /16/.

For the calculation of $O(\alpha_s^2)$ corrections to (2.4), which is described in detail in /16/, we use the partial fractioned 4-parton matrix elements to be introduced in the next section. Their general structure is as follows

$$M = C_F \left[\frac{R}{y_{13}} + (1-2) + (3-4) + (1-2,\, 3-4) \right]$$

$$+ N_C \left[\frac{S}{y_{34}} + \frac{T}{y_{13}} + (1-2) + (3-4) + (1-2,\, 3-4) \right] \tag{2.8}$$

$$+ T_R \frac{U}{y_{34}}$$

where R, S, T and U have the property that they remain finite for any one y_{ij} going to zero. (1-2) ects. denotes the interschange of indices in the variables y_{ij} occuring in (2.8) explicitly and as variables of R, S, T and U. The matrix element (2.8) must be integrated over regions of phase space to be specified later in order to extract the infrared/collinear singularities. For the first terms R/y_{13} it is most convenient to choose the 1-3-system, where $\vec{p}_1 + \vec{p}_3 = 0$ (for a full definition of variables in this system see /3/ and appendix B). In this system the 4-particle phase space /3/ is (see (A.4)):

$$dPS^{(4)} = \left(\frac{4\pi}{q^2}\right)^{3\varepsilon} \frac{q^4}{\Gamma(2-2\varepsilon)\,\Gamma(1-\varepsilon)\,N_S\;64\,(2\pi)^5}$$

$$(y_{123}\,y_{134} - y_{13})^{-\varepsilon}\, y_{24}^{-\varepsilon}\, y_{13}^{-\varepsilon}\,\theta(y_{13})\,\theta(y_{24})\,\theta(y_{123}y_{134}-y_{13})$$

$$(v(1-v))^{-\varepsilon}\;\sin^{-2\varepsilon}\theta'/N_{\theta'}\;\; d\theta'\, dv$$

$$dy_{123}\, dy_{134}\, dy_{13} \tag{2.9}$$

Here N_S is a statistical factor, $N_S = 2$ for $e^+e^- \rightarrow q\bar{q}gg$ and $N_S = 4$ for $e^+e^- \rightarrow q\bar{q}q\bar{q}$. $y_{ijk} = y_{ij} + y_{jk} + y_{ik}$. θ' is the azimuthal angle and θ the polar angle of $\vec{p}_1$ with $\vec{p}_2$ along the z-axis. $v = \frac{1}{2}(1-\cos\theta)$, $0 \leq \theta' \leq \pi$ and $0 \leq \theta \leq \pi$. Except for θ' and θ the phase space is described by invariants y_{123}, y_{134} and y_{13}. For fixed y_{134} the region allowed by (2.9) is shown in fig. 6. The region $y_{134} < y$ is certainly a 2-jet region with particle 2 being one jet and particles 1, 3 and 4 being the other jet. The notation of momenta is $q(p_1)$, $q(p_2)$, $g(p_3)$ and $g(p_4)$ in $e^+e^- \rightarrow q\bar{q}gg$. Thus 3 and 4 jets are always in the region $y_{134} > y$. If we look at the phase space in fig. 6 with $y_{134} > y$, we notice the division in three regions. Region II is y_{13}, $y_{24} \leq y$. This is also 2-jet. Then III: $y_{13} \leq y$, $y_{24} \geq y$ contains the 3-jet region and IV: $y_{13} \geq y$ contains the 4-jet region. In addition III has some 2-jet areas and IV has 2-jet and 3-jet regions. $y_{123} \lesseqgtr y$ is irrelevant because of lack of poles in y_{123}. Here we are interested only in region II, which we call <u>the</u> 2-jet region for $y_{134} > y$. III and IV will be considered in the next section. The integration of the terms R/y_{13} and T/y_{13} over $0 \leq v \leq 1$ and $0 \leq \theta' \leq \pi$

and over the regions $y_{134} \leq y$ and over II in $y_{134} > y$ constitutes the contribution to $\sigma_{2\text{-jet}}(y)$ considered in this section. The calculation is done only up to terms of order y. If we had wished to overcome this restriction we must extend the partial fractioning further including the variables y_{134} and y_{24}. With the decomposition (2.8) we cannot avoid to obtain terms proportional to y/ε which in total must cancel. But this has not been checked.

For the y_{34} pole term we adopt a different division of the 4-particle phase space. Here we go into a 3-4 system (see appendix B), where $\vec{p}_3+\vec{p}_4 = 0$. The angles θ and θ' are defined analogously to the 1-3 system. The other variables are y_{134}, y_{234} and y_{34}. The region $y_{134} \leq y$ is again a real 2-jet region. For $y_{134} \neq 0$ the phase space in y_{34} and y_{234} looks similar to that in fig. 6. It is plotted in fig. 7. Now the 2-jet region is II where $y_{234} \leq y$. The regions III and IV contain again the 3- and 4-jet regions and will be dealt with in the next section. V is included in $\sigma_{2\text{-jet}}$. It is clear that the permutation (1-2) etc. in (2.8) determine equivalent regions in the permutated coordinate systems. The integrations over these 2-jet regions, i.e. $y_{134} \leq y$ and II in fig. 6 for the y_{13}-pole terms and $y_{134} \leq y$ and II in fig. 7 for the y_{34}-pole terms are extremely lengthy and are discussed in /16/. We only quote the result for $\sigma_{2\text{-jet}}(y)$ which we have obtained after all infrared and collinear singularities had cancelled with the virtual corrections. We write the result as:

$$\sigma_{2\text{-jet}}(y) = \sigma_0 \Big[1 + \frac{\alpha_S(q^2)}{2\pi}\, C_F\, Z_1 + \left(\frac{\alpha_S(q^2)}{2\pi}\right)^2 C_F \left(C_F\, Z_C + N_C\, Z_N + T_R\, Z_T \right) \Big] \tag{2.10}$$

Z_1 is the $O(\alpha_s)$-term and is given in (2.4) and (2.5). The higher order terms are

$$Z_C = 2\ln^4 y + 6\ln^3 y + \left(\frac{13}{2} - 8\zeta_2\right)\ln^2 y - 2.094\ln y + 5.218 \tag{2.11}$$

$$Z_N = -\frac{1}{12}\ln^4 y + \frac{11}{3}\ln^3 y - \frac{169}{36}\ln^2 y - 10.40\ln y + 51.29 \tag{2.12}$$

$$Z_T = -\frac{4}{3}\ln^3 y + \frac{11}{9}\ln^2 y + 5\ln y + \frac{19}{9} - \frac{38}{9}\zeta_2 \tag{2.13}$$

For the T_R-term which has only a single pole partial fractioning is not needed. The 2-jet region is as in fig. 7.

In our last chapter we shall give the numerical values of these $O(\alpha_s^2)$ contributions to $\sigma_{2\text{-jet}}(y)$ and compare them with the sum of 3- and 4-jet cross section.

The remarkable result of (2.9) is that in (2.12) a term $\sim \ln^4 y$ appears which does not originate from the most singular terms. It comes from subleading terms according to our classification in the 3- and 4-jet calculation and is indeed important to reproduce σ_{tot} in the framework of the partial fractioning approach.

It is also unexpected from the point of view of the renormalization group /16/ and comes in only through the particular separation of y_{34}- and y_{13}-poles in equation (2.8). Remember that for y_{34}-poles the three jet variables are $y_{I\,III} = y_{134}$, $y_{II\,III} = y_{234}$, whereas for y_{13}-poles they are $y_{I\,III} = y_{134}$, $y_{II\,III} = y_{24}$. There is no simple interchange of partons connecting the two cases. For further discussion of the term $\sim \ln^4 y$ see /16/.

3. Three-Jet Cross Sections

In this section we consider the 3-jet production process

$$e^+e^- \to q(p_1) + \bar{q}(p_2) + g(p_3) \tag{3.1}$$

To $O(\alpha_s)$, q, $\bar{q}$ and g are elementary QCD quanta. At $O(\alpha_s^2)$ we have two contributions, which must be combined: the virtual corrections to (3.1) and the terms originating from the 4-parton matrix elements. The combination of both leads to the cut-dependent 3-jet cross sections. The kinematics of the 3-jet final state is described in terms of the invariant mass variables y_{12}, y_{13}, y_{23} introduced in section 2 which for convenience are denoted by $y_1 = y_{23}$, $y_2 = y_{13}$, $y_3 = y_{12}$.

The differential cross section for (3.1) is written as

$$\frac{1}{\sigma^{(2)}} \frac{d^2\sigma(3\text{-jet})}{dy_1\, dy_2} = \frac{\alpha_s}{2\pi} C_F \; T(y_1, y_2)$$

$$+ \left(\frac{\alpha_s}{2\pi}\right)^2 \Big\{ C_F^2 \, \sigma_C(y_1, y_2, y) + C_F \, f(y_1, y_2)$$

$$+ C_F N_C \, \sigma_N(y_1, y_2, y) + C_F T_R \, \sigma_T(y_1, y_2, y) \Big\} \tag{3.2}$$

$\sigma^{(2)}$ was defined in section 2. $T(y_1,y_2)$ is the $O(\alpha_s)$ 3-jet cross section

$$T(y_1, y_2) = \frac{y_1^2 + y_2^2 + 2y_3}{y_1 y_2} \tag{3.3}$$

y defines the cut value which is used to define the 3-jet cross section. It originates from defining the boundary over which the 4-parton matrix elements are integrated in order to incorporate the infrared-singular and mass-singular configurations. $f(y_1,y_2)$ stands for the infrared finite pieces of the virtual corrections as defined in our earlier papers /4, 5/. Its decomposition into terms with different colour factor can be obtained from /4 /. The $O(\alpha_s^2)$ contributions are decomposed into "singular" parts $\sigma^{(s)}$ and finite parts $\sigma^{(f)}$. In region III in fig. 6 and 7 the singular parts are sufficient to cancel the infrared and collinear singularities from the 4-parton and the $O(\alpha_s^2)$ corrections to the 3-parton cross section. This decomposition

$$\sigma_{C,N,T} = \sigma_{C,N,T}^{(s)} + \sigma_{C,N,T}^{(f)} \tag{3.4}$$

is not unique. Some of the finite pieces are included in $\sigma^{(s)}$. It is, however, important that the once chosen decomposition (3.4) originates from the same partial fractioning of the 4-parton cross section which is used also for calculating the 2-jet cross section. This calculation of $\sigma_{2\text{-jet}}$ is described in /16/. In the following we consider the contributions to the different colour factors C_F^2, $C_F N_c$ and $C_F T_R$. We choose $N_f = 1$, so that $C_F T_R = {}^2/3$. First we present results for the singular pieces defined below. We always consider first the integrated cross section in order to see how much they contribute to the total 3-jet cross section. Finite contributions to the 2-jet cross section originating from the same matrix elements are also given. Concerning differential distributions we present only the thrust-distribution $d\sigma/dT$ to facilitate comparison with our old approximate calculations /5/.

Colour Factor C_F^2

(a) Singular Contributions

The contributions of the virtual corrections will not be considered here. They are taken from our earlier work. In order to be able to integrate the 4-parton cross sections over degenerate configurations with arbitrary dimension n we must separate the most singular piece in the 4-parton matrix element. For the C_F^2-part this is given by (D.1) which must be integrated over ϑ and y_{13}. Before doing this we must identify the 3-jet variable. The choice of the 3-jet variables is not unique. In this sense the main aim of this paper is to reproduce the total cross section within one definite choice ((3.5)) of 3-jet variables. They may differ by terms which vanish for the singular configuration, in our case $y_{13} = 0$. We choose as 3-jet variables y_{134} and y_{24}. Then, according to 4-parton kinematics the third variable is

$$1 - y_{134} - y_{24} = y_{123} - y_{13} \tag{3.5}$$

This choice is more convenient than the choice y_{134} and y_{123}, where the third variable is $1 - y_{134} - y_{123} = y_{24} - y_{13}$. It has also the advantage, that part of the singular 2-jet region is defined in terms of y_{13} and y_{24}. The variables y_{134} and y_{24} are identified with y_2 and y_1 above when the cancellation of infrared and collinear singularities takes place. To facilitate this cancellation we replace $T(y_{134}, y_{13}+y_{24})$ in (D.1) by $T(y_{134}, y_{24})$ so that the same factor appears in the most singular 4-parton corrections as in the virtual corrections. This means that our most singular piece is given by:

$$T^{(s)}_{C_F^2} = C_F^2\, T(y_{134}, y_{24}) \frac{1}{y_{13}} \cdot \left(1 - v + \frac{2 y_{12}}{y_{13} + y_{23}} \right) \tag{3.6}$$

The difference between (3.6) and (D.1) is finite for $y_{13} = 0$ and will be taken into account when we calculate the subleading contributions. Now $T_{C_F^2}^{(s)}$ is integrated over v in the interval $0 \leq v \leq 1$ and over y_{13} in the 3-jet region III shown in fig. 6. This means that the upper limit of the y_{13}-integration varies depending whether the upper limit of the 4-parton phase space, i.e. $y_{123}y_{134}$, is larger or smaller than y. The limit $y_{13} \leq y_{123}y_{134}$ leads to the following limit in terms of the 3-jet variables y_{134} and y_{24}:

$$(y_{13})_{max} = \frac{y_{134}(1 - y_{134} - y_{24})}{1 - y_{134}} \tag{3.7}$$

We denote

$$c = \min\left(y,\ \frac{y_{134}(1 - y_{134} - y_{24})}{1 - y_{134}} \right) \tag{3.8}$$

and define

$$\Sigma^{(s)}_{C_F^2}(y_{134}, y_{24}) = T(y_{134}, y_{24}) \cdot \int_0^c dy_{13}\, y_{13}^{-1-\varepsilon} \int_0^1 dv\, (v(1-v))^{-\varepsilon} \cdot \left[-1 - v - \varepsilon(1-v) + \frac{2 y_{123}}{y_{13} + y_{23}} \right] \tag{3.9}$$

The variable $v = \frac{1}{2}(1 - \cos\theta)$ is defined in the 1-3 system. In this system $y_{23} = (y_{123} - y_{13})(1 - v) = (1 - y_{134} - y_{24})(1 - v)$. The integration is straightforward using formula (A.1) of /3/. The extra term proportional to ε in (3.9) comes from calculating the 4-parton cross section with arbitrary dimensions as collected in appendix E. Because of the partial fractioning (PF) approach all other terms proportional to ε in the matrix element can be neglected in the 3-jet problem. We obtain:

$$\begin{aligned} \Sigma^{(s)}_{C_F^2} = T(y_2, y_1) \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \Big\{ & \frac{1}{\varepsilon^2} - \frac{\ln y_3}{\varepsilon} + \frac{3}{2\varepsilon} \\ & - \ln^2 \frac{c}{y_3} + \frac{1}{2} \ln^2 y_3 - 2 \mathcal{L}_2\left(1 - \frac{c}{y_3}\right) - \frac{3}{2} \ln c + \frac{7}{2} \\ & - 2 \mathcal{L}_2\left(\frac{c}{y_3}\right) - 2 \mathcal{L}_2\left(-\frac{c}{y_3}\right) - 2 \ln\left(\frac{c}{y_3}\right) \ln\left(1 - \frac{c}{y_3}\right) \\ & + 2\left(1 + \frac{c}{y_3}\right) \ln\left(1 + \frac{c}{y_3}\right) \end{aligned} \tag{3.10}$$

In (3.10) we have put in the 3-jet variables $y_1 = y_{24}$, $y_2 = y_{134}$ and $y_3 = 1 - y_1 - y_2$. The expression with $y_2 = y_{134}$ and $y_3 = y_{123}$ as 3-jet variables is given in appendix F. (3.10) gives us the contribution of the y_{13}-pole term. To this we have to add the contributions corresponding to the interchanges $(1 - 2)(y_{23}$-pole), $(3 - 4)(y_{14}$-pole) and $(1 - 2,\ 3 - 4)(y_{24}$-pole). Except for the upper limit c in (3.10) all contributions are the same, since y_3 and T are symmetric in $1 - 2$. Summing up these 4 contributions and cancelling the singularities $^1/\varepsilon^2$ etc. against the virtual corrections /3, 4/ we obtain for the sum of the singular piece proportional to $\sum C_F^2$ (s) and the virtual corrections:

$$\sigma_C^{(S)}(y_1, y_2, y) = T(y_2, y_1)\Big\{ -\ln^2\frac{c_1}{y_3} - \ln^2\frac{c_2}{y_3} - \frac{3}{2}\ln c_1 - \frac{3}{2}\ln c_2 - 1 - 2\mathcal{L}_2\Big(-\frac{c_1}{y_3}\Big) - 2\mathcal{L}_2\Big(-\frac{c_2}{y_3}\Big) + 2\Big(1+\frac{c_1}{y_3}\Big)\ln\Big(1+\frac{c_1}{y_3}\Big) + 2\Big(1+\frac{c_2}{y_3}\Big)\ln\Big(1+\frac{c_2}{y_3}\Big)\Big\} \tag{3.11}$$

In (3.11) we have c_1 and c_2 coming from the upper limit of the y_{23} and y_{13} integration in the y_{23}-pole and the y_{13}-pole terms, resepctively, i.e.

$$c_i = \min\Big(y,\ \frac{y_i y_3}{1-y_i}\Big) \tag{3.12}$$

We emphasize that (3.11) is valid only for the choice $y_1 = y_{24}$, $y_2 = y_{134}$ as 3-jet variables. Had we chosen $y_2 = y_{134}$, $y_3 = y_{123}$ as variables then $\sigma_C^{(S)}$ looks differently. Of course, they agree in the limit $c_1 = c_2 \to 0$. The corresponding formula is given in appendix F.

We emphasize that $\sigma_C^{(S)}(y_1,y_2,y_3)$ differs depending whether $(y_{13})_{max} \lessgtr y$ and/or $(y_{23})_{max} \lessgtr y$. The region $(y_{23})_{max} \leqslant y$ corresponds in the y_1-y_2-plane to

$$y_2 \geqslant (1-y_1)(y_1-y)/y_1 \tag{3.13}$$

and similarly for $(y_{13})_{max} \leqslant y$. As already mentioned these boundaries originate from the 4-parton kinematics. They disappear for $y \to 0$. Therefore they are part of the corrections O(y) and their effect had been neglected in our earlier work /5/.

To make further contact with our earlier calculations /5/ we note that (3.11) includes only the 3-jet region $0 \leqslant y_{13} \leqslant y$ and $0 \leqslant y_{23} \leqslant y_{123} - y_{13} = y_3$, i.e. the infrared singular region y_{13} and $y_{23} \to 0$ and the $p_1 \parallel p_3$ collinear region $y_{13} \to 0$. If we include also the $p_2 \parallel p_3$, i.e. $y_{23} \to 0$, collinear region we must integrate (3.12) over the region: $y \leqslant y_{13} \leqslant c_1$, $1 - {}^y/y_3 \leqslant v \leqslant 1$. This is a completely finite integral because of the partial fractioning and can be evaluated with $\varepsilon = 0$. Since it is a contribution coming from the singular term we denote the result by $\sigma_C^{(s_{23})}(y_1,y_2,y)$. It includes also the contributions originating from the interchanges $(1 - 2)$, $(3 - 4)$ and $(1 - 2,\ 3 - 4)$.

$$\sigma_C^{(S_{23})}(y_1, y_2, y) = 2\,T(y_2, y_1) \Big\{ \frac{c_1}{y_3}\ln\frac{c_1+y}{c_1} + \frac{y}{y_3}\ln(c_1+y) - \frac{2y}{y_3}\ln 2y + \frac{y}{y_3}\ln y + \frac{\zeta_2}{2} - \mathcal{L}_2\Big(\frac{y}{y+c_1}\Big) - \frac{1}{2}\ln^2\Big(\frac{c_1+y}{c_1}\Big) - \frac{y}{y_3}\ln\frac{c_1}{y} + \frac{y^2}{4y_3}\ln\frac{c_1}{y}\Big\} + \Big\{c_1 \to c_2\Big\} \tag{3.14}$$

We note that for $y \to 0$

$$\sigma_C^{(S_{23})}(y_1, y_2, y) = 2\,T(y_2, y_1)\,\zeta_2 \tag{3.15}$$

Adding this contribution to that in (3.11) recovers our old result for $y \to 0$ /5/. Actually in the framework of partial fractioning the contribution (3.14) is part of the subleading terms which will be calculated numerically. If (3.14) would be taken together with the singular contribution (3.11) we must subtract it again when we calculate all the finite terms in the region $y_{13} \geq y$. For the case that y_{134} and y_{123} are chosen as 3-jet variables the formula for $\sigma_C^{(s_{23})}(y_1,y_2,y)$ is different. It will be given in appendix F.

In the following we shall disregard (3.14) and include its contribution in the finite terms which yield $\sigma_C^{(f)}(y_1,y_2,y)$.

(b) Nonsingular Contributions

The finite terms consist of two parts: (i) The difference between the complete C_F^2-terms with the y_{13} pole minus the singular part $T^{(s)}_{C_F^2}$ in (3.6) in the strip $y_{13} \leq y$, y_{134}, $y_{24} \geq y$. (ii) All other contributions for $y_{13} \geq y$. σ_C^f is for 3-jet so in (ii) we must avoid the 4-jet region $y_{ij} \geq y$ for all i,j. Whereas the contribution (i) is a genuine 3-jet contribution except for small terms in the region $y_3 = 1-y_{134}-y_{24} \leq y$ the terms from (ii) include genuine 2-jet, 3-jet and 4-jet contributions. Before disentangling these terms we shall calculate first the total integrated cross sections in these regions without specifying whether they belong to 2-, 3- or 4-jet. This will be added to the integrated cross section coming from $\sigma_C^{(s)}$ and the 2-jet cross section calculated in section 2 in order to see that they add up to the expected C_F^2-contribution to σ_{tot}. Of course, we can expect this only for rather small y values since σ(2-jet) as calculated in section 2 is correct only up to terms O(y). However, we were able to disentagle all other terms to yield σ_{tot} quite accurately.

The finite terms in the strip $0 \leq y_{13} \leq y$ are computed with the help of (D.33) in appendix D, in which the integral of $(AB_{13} - AB^s_{13})/y_{13}$ integrated over

$0 \leq \theta' \leq \pi$ and $0 \leq v \leq 1$ has been given. To this we add (D.41) with T replaced by $T_1 = T(y_{134}, y_{13}+y_{24}) - T(y_{134}, y_{24})$ which accounts for the difference of the singular term AB^s_{13}/y_{13} and (3.6). The formulas in appendix D can also be used to obtain the total integrated cross section for $y_{13} \geq y$. Integrating (D.31) with $\zeta = 1$ in the region $y_{13} \geq y$, $y_{134} \geq y$ and $0 \leq y_{123} \leq 1$ with the 4-parton phase space boundaries in fig. 6 produces the contribution of the singular part. The integration of (D.33) with $\zeta = 1$ over the same region gives the contribution of the nonsingular (for $y_{13} \to 0$) part. In addition we have integrated $\sigma^s(y_1,y_2,y)$ over (a) the total three-jet phase space $y_1,y_2,y_3 \geq y$ and (b) over the region $y_1,y_2 \geq y$ which includes the 2-jet contribution $0 \leq y_3 \leq y$. We did these integrations for various cut values y ranging between 0.05 and 0.001. The results are shown in Table 1. There σ_3^s(a) stands for the singular contribution for $y_1,y_2,y_3 \geq y$, σ_3^s(b) stands for the integral over $y_1,y_2 \geq y$, σ_3^f stands for the finite terms in $y_{13} \leq y$, σ_4^s(σ_4^f) is the singular (nonsingular) part in the region $y_{13} \geq y$. In all these cross sections we included the factor C_F^2 from (3.2) and left out the factor $(\alpha_s/2\pi)^2$, i.e. these cross sections must be multiplied by $(\alpha_s/2\pi)^2$ in order to obtain their contribution for a definite α_s value. The results in Table 1 show the following. As to be expected σ_3^s is negative and decreases with decreasing cut value y. Also σ_3^f becomes negative for small y. σ_4^s and σ_4^f increase with decreasing y. But the sum of $\sigma_3^s(b) + \sigma_3^f + \sigma_4^s + \sigma_4^f$ is negative for $y < 0.04$. It changes sign between y = 0.04 and y = 0.02. Concerning σ_4 the most singular part σ_4^s is always of the same order of magnitude as the finite part σ_4^f, so that σ_4 cannot be approximated by σ_4^s. On the other hand σ_3^f becomes small compared to σ_3^s for small enough y's. We must emphasize that σ_3 and σ_4 are not genuine 3- and 4-jet cross sections. σ_3^s and σ_3^f contain small 2-jet pieces. For the singular part it is given by $\sigma_3^s(b) - \sigma_3^s(a)$ and originates from the region $y_3 \leq y$. As can be seen from Table 1 it is rather small. Similarly σ_3^f contains a small 2-jet contribution from $y_3 \leq y$.

$\sigma_4 = \sigma_4^s + \sigma_4^f$ contains the genuine 4-jet cross section, 3-jet contributions from the strips $y_{ij} \leqslant y$ other than $y_{13} \leqslant y$ and 2-jet contributions. These will be disentangled by a separate calculation for y values between 0.01 and 0.05. The sum shown in the last column of Table 1 must be compensated by the contribution coming from the $q\bar{q}q\bar{q}$ interference terms and the 2-jet cross section given in section 2.

The integrations necessary to obtain the results in Table 1 have been done by two independent integration routines in the case of σ_3^f, σ_4^s and σ_4^f, where still integrations over three variables had to be done. $\sigma_3^s(a)$ and $\sigma_3^s(b)$ are integrals over two variables only. They are obtained with high accuracy. σ_3^f, σ_4^s and σ_4^f are also very accurate for the larger y's but for cut values below y = 0.02 the error is presumably larger than the last significant figure given. To calculate σ_4^s and σ_4^f with an accuracy smaller than 10^{-4} for y = 0.001 and y = 0.002 an additional effort would be necessary.

For y = 0.05, 0.04, 0.02 and 0.01 we have calculated the thrust distribution in the 3-jet and 4-jet region. The 4-jet region is defined by all $y_{ij} \geqslant y$ and the thrust in this region is the four-parton thrust. The thrust distribution of the singular region $y_{13} \leqslant y$ follows from (3.2) and (3.11). It is negative and gives the integrated cross section $\sigma_3^s(a)$ in Table 1, if integrated in the region $^2/3 \leqslant T \leqslant 1-y$. The corresponding nonsingular contribution in $y_{13} \leqslant y$ is not very important compared to the singular contribution, in particular with decreasing y. Its integral is equal to σ_3^f minus some small 2-jet terms. The sum of $\sigma_4^s + \sigma_4^f$ consists of 4-jet, 3-jet and 2-jet contributions. The 3-jet region has the following 5 parts (always $y_{13} \geqslant y$, i.e. also $y_{134} \geqslant y$)

$$
\begin{array}{lll}
\text{(i)} & y_{24} \leqslant y \;; & y_{234},\ y_{124} - y_{24} \geqslant y \\
\text{(ii)} & y_{23} \leqslant y \;; & y_{24},\ y_{14} \geqslant y \\
\text{(iii)} & y_{14} \leqslant y \;; & y_{24},\ y_{23} \geqslant y \\
\text{(iv)} & y_{34} \leqslant y \;; & y_{24},\ y_{23},\ y_{14},\ y_{12} \geqslant y \\
\text{(v)} & y_{12} \leqslant y \;; & y_{24},\ y_{23},\ y_{14},\ y_{34} \geqslant y
\end{array}
\tag{3.16}
$$

All other regions in $y_{134} \geqslant y$ which are not contained in (i) to (v) belong either to the 4-jet region (all $y_{ij} \geqslant y$) or to 2-jet regions, as for example the regions y_{23}, $y_{14} \leqslant y$ or $y_{234} \leqslant y$ or $y_{124}-y_{24} \leqslant y$ etc. In the regions (i) to (v) we have chosen as 3-jet variables: (i) y_{13}, y_{234}; (ii) y_{14}, y_{234}; (iii) y_{23}, y_{134}; (iv) y_{12}, y_{134} and in (v) y_{34}, y_{124}. Then the thrust variable T is always

$$T = max\left(x_1,\ x_2,\ x_3\right) \tag{3.17}$$

where x_1,x_2 and $x_3 = 2-x_1-x_2$ are calculated from the 3-jet variables above, for example in (i) $x_1 = 1-y_{234}$, $x_2 = 1-y_{13}$ and similarly in the other regions.

The thrust-distributions of the nonsingular terms have been calculated by a Monte-Carlo integration routine. Therefore these results are less accurate than the numbers in Table 1. The sum of all these 3-jet distributions including the contribution from the singular term is shown in fig. 8 for y = 0.05 and 0.01. For y = 0.05 the distribution is positive except for $T \geqslant 0.9$, whereas for y = 0.01

the distribution is negative throughout. The corresponding 4-jet distributions in fig. 9 are such that for y = 0.05 the 4-jet distribution peaks around T = 0.8 and is of the same order of magnitude as the 3-jet distribution. For y = 0.01 the 4-jet distribution is a factor of 20 larger than for y = 0.05. It has its maximum near T = 0.95. In both cases, 3-jet and 4-jet, the thrust distribution is written as

$$\frac{1}{\sigma_0}\ \frac{d\sigma}{dT} = \left(\frac{\alpha_s}{2\pi}\right)^2 A_2(T) \tag{3.18}$$

and it is $A_2(T)$ which is plotted as a function of T in fig. 8 and fig. 9. The sum of $A_2(T)_{3\text{-jet}}$ and $A_2(T)_{4\text{-jet}}$ for y = 0.05 and = 0.01 is shown in fig. 10. It is positive except for T's near T = 1. This inclusive thrust distribution depends strongly on the cut value y. It increases with decreasing y in such a way that the maximum is shifted to larger T's if y is decreased. This cut dependence of the C_F^2 contribution, i.e. the higher order thrust distribution of an abelean theory with all $q\bar{q}q\bar{q}$ final states neglected, was found some time ago by Gottschalk /21/.

In this connection it might be of interest to known which of the 3-jet regions (i) to (v) makes the largest contribution to the nonsingular part of $A_2(T)_{3\text{-jet}}$. The integrated sum of these contributions for y = 0.05 is equal to σ = 98.3 to be compared to $\sigma_4^s + \sigma_4^f$ = 140.1 in Table 1. To this term (i) to (v) contribute respectively 52.0, 18.4, 16.1, 11.2 and 0.6. This means that the region (i): $y_{24} \ll y$ is the most important one.

Knowing the 3-jet part and the 4-jet part in $\sigma_4^s + \sigma_4^f$ allows to calculate the 2-jet contribution in $\sigma_4^s + \sigma_4^f$. In Table 2 we present the results for the separation of the "sum" of Table 1 into 2-jet(σ_2), 3-jet(σ_3) and 4-jet cross sections (σ_4). We see that for y = 0.05

the 3-jet cross section is small and there is an appreciable 2-jet contribution. With decreasing y the negative σ_3 decreases whereas σ_2 and σ_4 increase.

Colour Factor C_FN_c

(a) Singular Contributions

The C_FN_c-part is more complicated since it receives contributions from the y_{13}-pole and the y_{34}-pole terms. After partial fractioning the y_{13}-pole term comes from (C.14), i.e. the terms proportional to C_FN_c in B_{13}/y_{13} and from (C.20). The singular part from these two terms was called BC^s_{13}/y_{13} and is given in (D.44). To make further integrations easier BC^s_{13}/y_{13} is slightly modified and as singular contribution the following expression was used

$$T^{(s)}_{C_FN_c} = C_FN_c\ T(y_{134}, y_{24}) \left(\frac{y_{134}}{y_{13} + (y_{134} - y_{13})(1-v)} - \frac{y_{123}}{y_{13} + y_{23}}\right) \tag{3.19}$$

The difference between (3.19) and $BC^s_{13}/48y_{13}$ is finite for y_{13} = 0 and will be included in the nonsingular contributions. v is related to the θ variable in the 1-3-system as usual. We choose as 3-jet variables y_{134} and y_{24} as for the C_F^2-contribution. (3.19) is integrated over v in the interval $0 \leqslant v \leqslant 1$ and over y_{13} in the 3-jet region III in fig. 6. Then the upper limit of the y_{13} integration is (3.8) and the integral to be evaluated is

$$\Sigma^{(S_{13})}_{C_F N_C} = \int_0^c dy_{13}\, y_{13}^{-1-\varepsilon} \int_0^1 dv\,(v(1-v))^{-\varepsilon}\, T(y_{134}, y_{24})$$

$$\left[\frac{y_{134}}{y_{13} + (y_{134} - y_{13})(1-v)} - \frac{y_{123}}{y_{13} + (y_{123} - y_{13})(1-v)} \right] \tag{3.20}$$

with $y_{123} = 1-y_{134}-y_{24}+y_{13}$.

The singular term originating from the y_{34}-pole is written down in (D.83) and has the following form if integrated over y_{34} and v:

$$\Sigma^{(S_{34})}_{C_F N_C} = \int_0^c dy_{34}\, y_{34}^{-1-\varepsilon} \int_0^1 dv\,(v(1-v))^{-\varepsilon}\, T(y_{134}, y_{24})$$

$$\left[\frac{y_{134}}{y_{34} + (y_{134} - y_{34})(1-v)} - \frac{1}{2} \right] \tag{3.21}$$

In (3.21) v is related to the angle θ in the 3-4-system.

In addition we add the contribution of the class C_b which consists of the pure "QCD"-matrix elements given in (C.27). They are integrated over y_{34}, θ and θ'. If we add all these three singular terms (together with the nonsingular terms in Σ_5) and cancel the singular terms proportional to ε^{-2} and ε^{-1} with the corresponding terms in the well-known virtual corrections /3, 4/ we obtain for the total singular term of the N_c-part:

$$\sigma_N^{(s)}(y_1, y_2, y) = T(y_1, y_2, y)$$

$$\Big\{ \frac{1}{2}\ln^2\frac{c_1}{y_3} + \frac{1}{2}\ln^2\frac{c_2}{y_3} + \mathcal{L}_2\left(-\frac{c_1}{y_3}\right) + \mathcal{L}_2\left(-\frac{c_2}{y_3}\right)$$

$$- \left(1+\frac{c_1}{y_3}\right)\ln\left(1+\frac{c_1}{y_3}\right) - \left(1+\frac{c_2}{y_3}\right)\ln\left(1+\frac{c_2}{y_3}\right)$$

$$-\frac{1}{2}\ln^2\frac{c_1}{y_1} - \frac{1}{2}\ln^2\frac{c_2}{y_2} - \mathcal{L}_2\left(1-\frac{c_1}{y_1}\right)$$

$$-\mathcal{L}_2\left(1-\frac{c_2}{y_2}\right) + 4\zeta_2 - \frac{1}{2}\ln^2\frac{d}{y_1} - \frac{1}{2}\ln^2\frac{d}{y_2}$$

$$-\mathcal{L}_2\left(1-\frac{d}{y_1}\right) - \mathcal{L}_2\left(1-\frac{d}{y_2}\right) + \frac{67}{18} - \frac{11}{6}\ln d \Big\}$$

$$- \frac{10}{3y_1 y_2}\left\{ \frac{1}{2}d^2 + d\left(y_3 - \frac{(y_1-y_2)^2}{2y_1 y_2}\right)\right\} \tag{3.22}$$

where

$$d = \min(y,\ y_1 y_2) \tag{3.23}$$

is the upper limit originating from the y_{34} integration in $\Sigma^{(S_{34})}_{C_F N_C}$ and from the y_{34} integration in Σ_5. In the terms originating from $\Sigma^{(S_{13})}_{C_F N_C}$ we have chosen $y_1 = y_{24}$ and $y_2 = y_{134}$ as 3-jet variables as in the C_F^2-term. This explains the somewhat unsymmetrical appearance of $\mathcal{L}_2$-functions. In (3.23) we have included also the terms coming from the interchanges (1 — 2), (3 — 4) and (1 — 2, 3 — 4). Therefore (3.22) is symmetric concerning $y_1 \leftrightarrow y_2$.

Similar to the $C_F^{\,2}$ term we can add also other collinear regions. For example in the y_{13}-pole contribution we integrate over $y \leqslant y_{13} \leqslant c_1$ and $1-{}^{y}/y_3 \leqslant v \leqslant 1$. This is again a finite integral. We denote it by $\sigma_N^{(S_{23})}(y_1,y_2,y)$. The result of the integration is:

$$\sigma_N^{(S_{23})}(y_1,y_2,y) = T(y_1,y_2)$$

$$\Big\{ \frac{c_1}{y_1}\ln\frac{c_1+y}{c_1} + \frac{y}{y_1}\ln(c_1+y) - \frac{2y}{y_1}\ln 2y$$

$$+ \frac{y}{y_1}\ln y - \frac{y}{y_1}\ln\frac{c_1}{y} + \frac{1}{4}\frac{y^2}{y_1}\ln\frac{c_1}{y}$$

$$- \frac{c_1}{y_3}\ln\frac{c_1+y}{c_1} - \frac{y}{y_3}\ln(c_1+y) + \frac{2y}{y_3}\ln 2y$$

$$- \frac{y}{y_3}\ln y + \frac{y}{y_3}\ln\frac{c_1}{y} - \frac{1}{4}\frac{y^2}{y_3}\ln\frac{c_1}{y}\Big\} + \{1\leftrightarrow 2\} \tag{3.24}$$

In the singular term of the y_{34}-pole contribution we can integrate analytically the $p_3 \parallel p_1$, i.e. $y_{13}\to 0$, collinear region (and similar for $(1\leftrightarrow 2)$, $(3\leftrightarrow 4)$ and $(1\leftrightarrow 2,\ 3\leftrightarrow 4)$ exchange terms). In this case the result differs somewhat from (3.24) since for the y_{34}-pole the variables y_{134} and y_{234} were chosen as 3-jet variables. The result is denoted by $\sigma_N^{(S_{13})}(y_1,y_2,y)$ and is given by

$$\sigma_N^{(S_{13})}(y_1,y_2,y) = T(y_1,y_2)$$

$$\Big\{ \mathcal{L}_2\Big(\frac{y_1-d}{y_1+y}\Big) - \mathcal{L}_2\Big(\frac{y_1-y}{y_1+y}\Big)$$

$$+ \mathcal{L}_2\Big(1-\frac{y}{y_1}\Big) - \mathcal{L}_2\Big(1-\frac{d}{y_1}\Big)$$

$$+ \mathcal{L}_2\Big(-\frac{y}{d}\Big) + \frac{\zeta_2}{2} - \ln\Big(1+\frac{y}{y_1}\Big)\ln\Big(\frac{y_1-d}{y_1-y}\Big)$$

$$- \Big(\frac{y}{y_1} - \frac{1}{4}\frac{y^2}{y_1^2}\Big)\Big(\ln\Big(\frac{y_1-y}{y_1-d}\Big) + \ln\frac{d}{y}\Big)$$

$$+ \frac{1}{4}\,\frac{y^2(d-y)}{y_1(y_1-y)(y_1-d)}\Big\} + \{1\leftrightarrow 2\} \tag{3.25}$$

For $y\to 0$, we have

$$\sigma_N^{(S_{23})} + \sigma_N^{(S_{13})} = T(y_1,y_2)\,\zeta_2 \tag{3.26}$$

in agreement with our old result /5/ for the sum of $\sigma_N^{(S)} + \sigma_N^{(S_{23})} + \sigma_N^{(S_{13})}$.

As for the $C_F^{\,2}$-term it is more convenient to include these collinear regions in the finite parts which will be evaluated numerically. So we make no use of the formula (3.25) and (3.26). They have been included here only to point out their origin in our old approach.

(b) Nonsingular Contributions

The calculation of the nonsingular terms proceeds in an analogous fashion to that of the C_F^2-term. First we consider the terms coming from the y_{13}-pole. This is denoted $^{BC_{13}}/y_{13}$ in appendix D. After subtracting the singular part $^{BC^s_{13}}/y_{13}$ from it we integrate over the strip $y_{13} \leqslant y$; y_{134}, $y_{24} \geqslant y$; $0 \leqslant \vartheta \leqslant 1$ using the formulas in appendix D, i.e. (D.46) - (D.82). The result is shown in Table 3 as σ_3^f in the column with the heading "$^1/y_{13}$". This σ_3^f is positive. It increases as a function of y. Compared to the contribution of the singular part (3.22) which is $\sigma_3^s(b)$ in Table 3, the σ_3^f varies between 7% and 30% of $\sigma_3^s(b)$. It is less important at large y and more important for small y values. This was not expected. Second we integrate the total y_{13}-pole, i.e. the singular and the nonsingular part over the region $y_{13} \geqslant y$. This contains 2-, 3- and 4-jet terms. In Table 3 this is denoted σ_4 in the column with "$^1/y_{13}$". It is negative and decreases with decreasing y. For the larger y's the total 4-jet contribution to the N_c-part is rather small, so that most of σ_4 in this column is 3-jet. For example, for y = 0.05 the 3-jet contribution is $\sigma_3 = -26.1$ which cancels almost completely σ_3^f in the $^1/y_{13}$-term.

Similarly the y_{34}-pole contribution is divided in σ_3^f and σ_4. σ_3^f comes from the region $y_{34} \leqslant y$ and is obtained from integrating (D.99) over the region y_{134}, $y_{234} \geqslant y$. σ_4 is the sum of the singular part (D.83) and of the nonsingular part (D.99) over the region $y_{34} \geqslant y$. The results for these cross sections are in Table 3 in the column "$^1/y_{34}$". Both contributions are positive and increase with decreasing cut value y. For the pure "QCD"-terms the region $y_{34} \leqslant y$ is fully taken into account with $\sigma_N^{(s)}$ in (3.22). The contribution for the region $y_{34} \geqslant y$ is given as σ_4 in Table 3 in the column with the label "QCD". This σ_4 is negative and only a small fraction of the other terms. The real 4-jet cross section is

contained in $\sigma_4(^1/y_{13})$, $\sigma_4(^1/y_{34})$ and σ_4(QCD). To get an idea about of how much of the sum of all subleading terms in Table 3 plus $\sigma_3^s(b)$, which is in the column denoted by "sum" in Table 3, is not $\sigma_{4\text{-jet}}$, i.e. 3-jet and 2-jet cross section, we have calculated $\sigma_{4\text{-jet}}$ separately and show it in the last column of Table 3. We see that the N_c-part contributes dominantly to the 3-jet cross section and very little to the 4-jet cross section. Even for y = 0.001, $\sigma_{4\text{-jet}}$/sum is only of the order of 5%. We emphasize that $\sigma_{4\text{-jet}}$ is evaluated with MC-integration and therefore is less accurate than the other "cross sections" in Table 3. The σ_2 in Table 3 is the contribution of the singular part in the y_{34}-pole (including the pure "QCD" terms) integrated over the small triangle in fig. 7 which is not accounted for in any of the other "cross sections" in Table 3. It belongs to the 2-jet region. It is small compared to the other terms and converges to zero with decreasing y. By comparing $\sigma_3^{(s)}(b)$ with the numbers following from sum- $\sigma_{4\text{-jet}}$ we notice that the difference between these two numbers is only a fraction of $\sigma_3^{(s)}(a)$. This means that the N_c-part of the 3-jet cross section is only moderately influenced by the nonsingular terms in contrast to the situation observed for the C_F^2-part.

The calculation of the thrust distributions on the 3-jet and 4-jet region is analogous to that of the C_F^2-part described earlier. The only change is that the y_{13}-pole and the y_{34}-pole contributions have to be treated differently. For the y_{13}-pole term we adopt the procedure as for the C_F^2-part. The 3-jet variables are chosen as described above. Then the 3-jet thrust distribution for $y_{13} \geqslant y$ is calculated in the five strips as given above. For the y_{34}-pole terms including the pure "QCD"-terms the procedure is different. In the strip $y_{34} \leqslant y$ the 3-jet variables are y_{134}, y_{234} and $1-y_{134}-y_{234} = y_{12}-y_{34}$. The thrust T is calculated as usual from $T = \max(x_1,x_2,x_3)$ where $x_1 = 1-y_{234}$, $x_2 = 1-y_{134}$, $x_3 = 2-x_1-x_2$. The region $y_{34} \geqslant y$ is divided into the following five strips:

(i) $y_{12} \leq y \; ; \; y_{123}, \; y_{124}, \; y_{34} - y_{12} \geq y$

(ii) $y_{23} \leq y \; ; \; y_{123}, \; y_{234}, \; y_{14} - y_{23} \geq y$

(iii) $y_{14} \leq y \; ; \; y_{124}, \; y_{134}, \; y_{23} - y_{14} \geq y$ (3.27)

(iv) $y_{13} \leq y \; ; \; y_{123}, \; y_{134}, \; y_{24} - y_{13} \geq y$

(v) $y_{24} \leq y \; ; \; y_{124}, \; y_{234}, \; y_{13} - y_{24} \geq y$

In addition the 4-jet region with all $y_{ij} \geq y$ and the 2-jet regions y_{123}, y_{124}, $y_{34}-y_{12} \leq y$ etc. were excluded. The thrust T in these five regions was computed from the 3-jet variables, for example in (i), from y_{123} and y_{124} and so on. The sum of all these 3-jet thrust distributions including the contribution from the singular term (3.22) is plotted for y = 0.05 and = 0.01 in fig. 11. The thrust distribution for y = 0.01 is smaller than that for y = 0.05. This means $A_2(T)$ decreases with decreasing y as to be expected. The corresponding 4-jet distributions with the 4-parton thrust are shown in fig. 12. For y = 0.05 $A_2(T)_{4\text{-jet}}$ is negligible compared to $A_2(T)_{3\text{-jet}}$, whereas for y = 0.01 they are of the same order of magnitude. If we compute the sum of $A_2(T)_{3\text{-jet}}$ and $A_2(T)_{4\text{-jet}}$, i.e. the inclusive distributions, we see that it decreases slightly with decreasing y except for $T \geq 0.925$. This is in contrast to the $C_F{}^2$-part, where the inclusive sum increases with decreasing y (see fig. 19).

In Table 4 we show the result for the separation of the sum of Table 3 into real 2-jet(σ_2), 3-jet(σ_3) and 4-jet cross sections (σ_4). We see that the dominant term is always σ_3. σ_4, as remarked earlier, is really small compared to σ_3.

The Colour Factor $C_F T_R$.

The contributions proportional to $C_F T_R$ originate from the diagonal terms of the $q\bar{q}q\bar{q}$ final state. These contribtuions have only single pole terms. Their structure is similar to that of the pure "QCD" terms in the $C_F N_c$-term. The finite pieces in the region $y_{12} \leq y$ are combined with the singular term. They are easily calculated from (C.34) by doing the ϑ and θ' integration. Together with the singular term they yield

$$\sigma_T(y_1, y_2, y) = T(y_2, y_1)\left\{\frac{2}{3}\ln d - \frac{10}{9}\right\} + \frac{4}{3\, y_1 y_2}\left\{\frac{1}{2} d^2 + d\left(y_3 - \frac{(y_1 - y_2)^2}{2\, y_1 y_2}\right)\right\} \qquad (3.28)$$

where

$$d = \min(y, y_1 y_2) \qquad (3.29)$$

is the upper limit of the y_{12} integration.

The contribution (3.28) has been integrated over the region $y_1, y_2 \geq y$. The result, $\sigma_3{}^s(b)$, is collected in Table 5 for y = 0.05 to y = 0.001. This contribution is negative and decreases with decreasing y. We emphasize that the results in Table 5 are just for one flavour (N_f = 1). If the region $y_3 \leq y$ is excluded in the integration we obtain $\sigma_3{}^s(a)$ in Table 5. It differs only slightly from $\sigma_3{}^s(b)$, the difference is O(y). The contribution of (C.34) in the region $y_{12} \geq y$ is denoted by σ_4. It consists of contributions to the 2-, 3- and 4-jet cross section. It is positive and small compared to $\sigma_3{}^s(b)$. By comparing σ_4 with $\sigma_{4\text{-jet}}$ in Table 5 we see that the relative contribution of 2- plus 3-jet terms in σ_4 diminishes with decreasing y. The term σ_2 in Table 5 is a small term coming from the "singular term" (3.28) in the region V of fig. 7. It is O(y) and not significant. In Table 6 we have disentangled σ_4' into its 2-, 3- and 4-jet contribution (denoted by σ_2, σ_3, σ_4, where σ_3 includes also $\sigma_3^{s}(a)$). We see that for $y \simeq 0.05$ the 4-jet cross section is small compared to the 3-jet cross section. The thrust distributions for 3-jets and 4-jets are plotted in fig. 13 and 14, respectively, for $y = 0.05$ and 0.01. We see that the $C_F T_R$-term makes always a negative contribution to the $O(\alpha_s^2)$ corrections to $(d\sigma/dT)_{3\text{-jet}}$. Of course, $(d\sigma/dT)_{4\text{-jet}}$ must be positive.

The $q\bar{q}q\bar{q}$-Interference Contribution.

The $q\bar{q}q\bar{q}$-interference contribution is proportioanl to $C_F(C_F-N_c/2)$ and therefore ought to be included in σ_C and σ_N of (3.2). We consider it separately here, since no separation in singular and nonsingular pieces is necessary. The formulas for this interference term were given in appendix C. From these formulas we have calculated the contribution to the 3- and 4-jet cross section. The 3-jet cross section consists of several pieces. First the contribution $y_{34} \leqslant y$, y_{134}, $y_{234} \geqslant y$. This is called σ_3 and is collected in Table 7 for several y's. The region $y_{34} \geqslant y$, which again has 2-, 3- and 4-jet contributions, is denoted σ_4', also given in Table 7. σ_3 and σ_4' are of the same order of magnitude. These cross sections are multiplied with $C_F(C_F-N_c/2)$. If one wants to know their contribution to the (C_F^2) or the $(C_F N_c)$-term one must multiply with - 8 or 9, respectively. Then we see that they add only insignificant terms to σ_C and σ_N in particular for small y's. In Table 7 we give also in the last column the real 4-jet cross section $\sigma_{4\text{-jet}}$. For the larger y's the larger part of σ_4' must be $\sigma_{2\text{-jet}} + \sigma_{3\text{-jet}}$. The division of σ_3 and σ_4' in Table 7 into physical 2-jet-(σ_2), 3-jet-(σ_3) and 4-jet-(σ_4) cross sections is in Table 8. Of some importance is only σ_3', which is negative and increases in absolute value with decreasing y. σ_4, which is very small for $y = 0.05$, also increases in absolute value with decreasing y, as

one would expect. We emphasize, that all these cross sections are completely finite. In the $q\bar{q}q\bar{q}$-interference term only the region y_{134}, $y_{234} \leqslant y$ is singular. This region is included in the 2-jet cross section /16/ which is reported in section 2. The corresponding thrust distributions are in fig. 15 and 16 for $y = 0.05$ and 0.01. Both are negative for all T. The sum of 3- and 4-jet distribution varies only very little if we change y between 0.05 and 0.01.

In fig. 17 and 18 the sum of all contributions (C_F^2, $C_F N_c$, $C_F T_R$ with $N_f = 5$ and $q\bar{q}q\bar{q}$-interference term) to the 3-jet and 4-jet thrust distribution has been plotted for $y = 0.05$ and $y = 0.01$. One observes the expected pattern, that $d\sigma/dT$ becomes negative for $y = 0.01$ for the larger T-values, whereas for $y = 0.05$ it stays positive for all T's. If we sum up $\frac{1}{\sigma_0} d\sigma/dT$ for 3- and 4-jets we see that the resulting inclusive thrust distribution varies with y. Near $T = 0.8$ the two distributions are almost identical whereas for $T < 0.75$ and $T > 0.85$ the distribution for $y = 0.01$ lies above the distribution for $y = 0.05$ [F3]. This y-dependence of the inclusive distribution was also observed in our earlier work based on the singular terms for $(d\sigma/dT)_{3\text{-jet}}$ /5/. There, however, the y-dependence was much stronger. This was due to the fact that the nonsingular terms were not included in the 3-jet distributions. This leads to an underestimate of $(d\sigma/dT)_{3\text{-jet}}$. For example, taking $y = 0.04$ and $\alpha_s = 0.16$ the 3-jet distribution calculated from the singular terms is between 10% to 20% smaller, depending on the T value considered, than the 3-jet distribution with all subleading terms taken into account [F4]. It is clear that this leads to an equivalent adjustment of the α_s values if experimental thrust distributions are fitted to the complete theory.

With the now known decomposition of $\sigma_{3\text{-jet}}$ and $\sigma_{4\text{-jet}}$ into the three colour factors one can study whether the experimental data distinguish between an abelean or a nonabelean quark-gluon theory. We shall not consider this in detail and make only some qualitative remarks. It follows from Tables 2, 4 and 6 that the $O(\alpha_s^2)$ part of $\sigma_{3\text{-jet}}$ comes essentially from the $C_F N_c$ contribution. This means that the nonabelean nature of QCD (or the 3-gluon coupling) produces the rather large and positive higher-order corrections in $\sigma_{3\text{-jet}}$. In an abelean theory, where the $C_F N_c$-part does not exist, the higher-order correction is also large, how large depends how one compensates for the three colours by enlarging the number of flavours (for example $T_R = 6N_f$, see /15/), but it is now negative. On the other hand, the 4-jet cross section receives its main contribution from the C_F^2-part. Therefore, $\sigma_{4\text{-jet}}$ does not change very much if we evaluate it for an abelean theory with roughly the same coupling α_s as in QCD. Thus, considering all three jet cross sections, $\sigma_{2\text{-jet}}$, $\sigma_{3\text{-jet}}$ and $\sigma_{4\text{-jet}}$ it appears possible to single out nonabelean QCD as the correct theory.

So far we studied only the case where the 3-jet variables for compensating the infrared singularities between virtual and real contributions were chosen as $y_{I\ III} = y_{134}$ and $y_{II\ III} = y_{24}$ so that $y_{I\ II} = y_{123} - y_{13}$. This choice is in no way unique. In order to see how the integrated 3-jet cross sections and the differential distributions depend on the selection of the 3-jet variables we studied another possibility. For this we have chosen $y_{I\ III} = y_{134}$, $y_{I\ II} = y_{123}$ so that $y_{II\ III} = y_{24} - y_{13}$. This way we influence the separation of the 3-and 2-jet region. Instead of $y_{13} \leqslant y$, $y_{24} \leqslant y$ the 2-jet region (in case of the $^1/y_{13}$-pole) is now given by $y_{13} \leqslant y$, $y_{24} - y_{13} \leqslant y$. From this we expect a larger portion given to 2-jets as compared to the other choice with y_{134}, y_{24} as 3-jet variables. The result of the calculation with 3-jet variables y_{134} and y_{123} is exhibited in Table 12 for the C_F^2-part and in Table 13 for the $C_F N_c$-part. Compared to the results in Table 2 and 4 we see that σ_3 is now appreciably smaller for the C_F^2-part and somewhat smaller in the case of the $C_F N_c$-part. Of course, the 2-jet contributions

are correspondingly increased. That the $C_F N_c$-part is influenced less is explained by the fact that in this case the main contribution comes from the $^1/y_{34}$-pole terms where the 3-jet variables were unchanged (y_{134}, y_{234}). Second the $^1/y_{13}$-pole in the $C_F N_c$-part contains several contributions which appear with opposite sign and compensate each other. The corresponding thrust distributions are shown in fig. 19 (C_F^2-part) and fig. 20 (sum of C_F^2-, $C_F N_c$-, $C_F T_R$- and $q\bar{q}q\bar{q}$-interference part). The $C_F N_c$-part is not shown separately since it is not changed significantly as compared to fig. 11. The main change appears in the C_F^2-part as is seen if we compare the curves in fig. 8 and fig. 19. The $O(\alpha_s^2)$ 3-jet distribution $A_2(T)$ in fig. 19 is smaller than in fig. 8 in particular for $y = 0.05$. This is also seen in the $A_2(T)$ where all four terms, C_F^2, $C_F N_c$ etc. are combined. Here we must compare fig. 20 with fig. 17. It is clear that the change of $A_2(T)$ for 3-jets influences also the inclusive distribution $A_2(T)_{3+4\text{-jet}}$ which is obtained by adding to $A_2(T)_{4\text{-jet}}$ in fig. 18 the $A_2(T)_{3\text{-jet}}$ in fig. 20. With the new variables y_{134}, y_{123} the curves for $y = 0.05$ and $y = 0.01$ are split much more than it was the case for the variables y_{134} and y_{24}.

From the numbers in Table 6, 8, 12 and 13 we can obtain σ_3 for the second choice of variables. This is $\sigma_3 = 238.79$ for $N_f = 5$ as compared to $\sigma_3 = 327.85$ with the first choice. This leads to a 3-jet multiplicity with $O(\alpha_s)$-term included of $m_3 = 29.5\%$ (variables y_{134}, y_{123}) as compared to $m_3 = 32.7\%$ (variables y_{134}, y_{24}) if $\alpha_s = 0.12$, $y = 0.05$ and $N_f = 5$.

So we must conclude that the cross sections for a fixed number of jets depend on the choice of variables used in the process of cancelling infrared and collinear singularities. This is unavoidable. A priori it is not clear which choice is the right one. One might think that one can avoid this indeterminacy by going to particular small cut values for y. There the difference diminishes as for example can be seen by comparing the y = 0.01 curves if fig. 17 and fig. 20. In order to obtain 3- and 2-jet cross sections for realistic, i.e. larger y values, one must combine the 4 partons again to 3 jets. This recombination depends again on the algorithm used for recombination /20/. This recombination dependence has been studied in /18, 19, 25/. In particular in /25/ it was found that $O(\alpha_s^2)$ corrections to 3-jet cross sections depend on the recombination scheme. This dependence on the recombination procedure is equivalent to the dependence on the choice of 3-jet variables found above in our approach.

4. Check with σ_{tot}.

The total inclusive e^+e^- annihilation cross section σ_{tot} up to $O(\alpha_s^2)$ has been calculated from the imaginary part of the vacuum polarization by several groups already some time ago. In the $\overline{MS}$-renormalization scheme it is /2, 15/

$$\sigma_{tot} = \sigma_0 \left\{ 1 + \frac{3}{2} C_F \frac{\alpha_s(q^2)}{2\pi} + \left(\frac{\alpha_s(q^2)}{2\pi}\right)^2 \left[-\frac{3}{8} C_F^2 + \left(\frac{123}{8} - 11\zeta_3\right) C_F N_c + \left(4\zeta_3 - \frac{11}{2}\right) C_F T_R \right] \right\} \tag{4.1}$$

From (4.1) we see that in the $O(\alpha_s^2)$ the C_F^2 and the $C_F T_R$-terms have coefficients smaller than 1, whereas the coefficient of the $C_F N_c$-term is roughly 2. We cannot

expect to reproduce these small coefficients of $(\alpha_s/2\pi)^2$ by summing $\sigma_{2\text{-jet}}(y) + \sigma_{3\text{-jet}}(y) + \sigma_{4\text{-jet}}(y)$ for very small y since our results for these cross sections have still errors. First, $\sigma_{2\text{-jet}}(y)$ from section 2 is only correct up to O(y) terms. Second our $\sigma_{3\text{-jet}}(y) + \sigma_{4\text{-jet}}(y)$ have some numerical errors, in particular for very small y's, where the comparison is most meaningful.

In Table 9 and 10 we show the comparison for the C_F^2- and the $C_F N_c$-term respectively. The values for σ_2 follow from (2.10) with (2.11) and (2.12), respectively. $\sigma_3 + \sigma_4$ is the sum of Table 1 and 3. It consists of all 3- and 4-jet contributions and those 2-jet terms not considered in section 2. As already mentioned, we carefully observed that no double-counting occured. The only problem is that in σ_2 terms of O(y) had been neglected. The sum obtained in Table 9 decreases with decreasing y. For very small y it is supposed to converge to $-\frac{3}{8}C_F^2 = -\frac{2}{3}$ (see (4.1)). Our results in Table 9 at very small y are consistent with this. The behaviour of the remaining difference could be due to a term $\sim y \ln^2 y$. According to (4.1) the sum in Table 10 is supposed to converge to $(\frac{123}{8} - 11\zeta_3) C_F N_c = 8.61$. Our results are consistent with this although even at y = 0.001 our value is still much larger than this number. But we have to keep in mind that our results for $\sigma_3 + \sigma_4$ have numerical errors of the order of 0.5%. In addition a general feature of N_c-type contributions is that its O(y) corrections are large (cf. /16/). The behaviour of the final sum in Table 10 could be due to a term $\sim y \ln^3 y$. The results in Table 11 represent the test of the $C_F T_R$-term. σ_2 is obtained from (2.10) and (2.13) and $\sigma_3 + \sigma_4$ from Table 5. The sum should converge to $(4\zeta_3 - \frac{11}{2}) C_F T_R = -0.46$ which apparently is the case. The remaining difference is a term consistent with a behaviour $\sim y \ln^2 y$.

We consider the results presented in Table 9, 10, 11 as sufficient proof that our $O(\alpha_s^2)$ integrated cross sections for 2-, 3- and 4-jet production are consistent with σ_{tot} in (4.1). The terms not accounted for grow with increasing y, since in σ_2 terms of O(y) had been neglected. At y = 0.01 they are of the order of 5%. For y-cuts above 0.01 these terms are larger, in particular for the C_F^2-term. Here cancellation of many different terms take place which causes the change of sign in " σ_2" and in " $\sigma_3 + \sigma_4$" in the y-range between 0.04 and 0.02.

5. Summary and Final Remarks.

In this work we have described in detail the calculation of the O(α_s^2) contribution to the 3-jet cross section with an invariant mass squared resolution y. By using an elaborate partial fractioning procedure we were able to calcuate all subleading terms including those which vanish if the resolution parameter y goes to zero. All these corrections are nonnegligible and must be taken into account if $\sigma_{3-jet}(y)$ is computed for physical relevant y's. We have compared our results for the sum of $\sigma_{3-jet}(y)$ and $\sigma_{4-jet}(y)$ and results for $\sigma_{2-jet}(y)$ obtained in some earlier work /16/ with the O(α_s^2) contribution to σ_{tot} /2/. Within the limitations of our numerical accuracy and neglecting order y terms we find that all cross sections are consistent.

One may ask whether such complicated computations as done here are really necessary in order to obtain a fairly accurate result for the O(α_s^2) corrections to the 3-jet cross section. We have seen that for our procedure first the partial fractioning of all 4-parton matrix elements is needed and second a careful identification of all 3-jet regions in terms of the chosen 3-jet variables must be looked for. Compared to this the procedure by the TASSO-Collaboration /18/ is much simpler and accurate enough.

We have found that the O(α_s^2) results depend on the choice of 3-jet variables which are used to perform the cancellation of infrared and collinear singularities between higher order virutal and real corrections. This influences the separation of the 4-parton contributions into 2- and 3-jet cross sections and therefore changes also differential 3-jet cross sections. This jet-variable dependence is equivalent to the recombination dependence found in /18, 19, 25/.

Appendix A: Phase Space Formulae

The normalization of the hadronic tensor $H_{\mu\nu}$ for $e^+e^- \to$ 4-partons is taken from /21/. Then the differential cross section is

$$d\sigma = \frac{e^4}{2q^6 N_S} \prod_{i=1}^{4} \frac{d^4p_i}{(2\pi)^3}\, \delta_+(p_i^2)(2\pi)^4 \delta^{(4)}(q - \sum_{i=1}^{4} p_i)$$
$$\cdot\left(-\frac{q^2}{3} g^{\mu\nu}\right) H_{\mu\nu} \qquad \text{(A.1)}$$

The hadron tensor contains summation over the final spin, colour and flavour states including the appropriate quark charge factors Q_k^2. N_S is a statistical factor due to the identity of final-state particles. We have integrated over the orientation of the parton production plane with respect to the $e^\pm$ beam direction.

According to /21/ the tensor $H_{\mu\nu}$ is written

$$H_{\mu\nu} = (4\pi\alpha_s)^2 \sum_{k=1}^{N_f} Q_k^2 \sum_{m \geq n-1}^{8} A(m,n)_{\mu\nu} \qquad \text{(A.2)}$$

The 4-particle phase space (in n dimensions)

$$PS^{(4)} = \prod_{i=1}^{4} \int \frac{d^np_i}{(2\pi)^{n-1}}\, \delta_+(p_i^2)(2\pi)^n \delta^{(n)}(q - \sum_{i=1}^{4} p_i) \qquad \text{(A.3)}$$

has in the 1-3-system (see appendix B) the following form ($4-n = 2\varepsilon$):

$$PS^{(4)} = \frac{q^4}{64(2\pi)^5}\left(\frac{4\pi}{q^2}\right)^{3\varepsilon} \frac{1}{\Gamma(2-2\varepsilon)\,\Gamma(1-\varepsilon)}$$
$$\int dy_{123}\, dy_{134}\, dy_{13} \left(y_{123}y_{134} - y_{13}\right)^{-\varepsilon} y_{24}^{-\varepsilon} y_{13}^{-\varepsilon}$$
$$\cdot\,\Theta(y_{13})\,\Theta(y_{24})\,\Theta(y_{123}y_{134} - y_{13})$$
$$\cdot \int_0^1 dv\, (v(1-v))^{-\varepsilon} \int_0^{\pi} d\theta' \frac{1}{N_{\theta'}} \sin^{-2\varepsilon}\theta' \qquad \text{(A.4)}$$

where $y_{24} = 1-y_{123}-y_{134}+y_{13}$ and $N_{\theta'} = \pi\, 2^{2\varepsilon} \dfrac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)}$

In this system the differential 4-parton cross section is:

$$d\sigma = \sigma_0 \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\left(\frac{4\pi}{q^2}\right)^{\varepsilon} \frac{1}{24}$$
$$\cdot\left(\frac{4\pi}{q^2}\right)^{2\varepsilon} \frac{1}{\Gamma^2(1-\varepsilon)\, N_S}\left(\frac{\alpha_s}{2\pi}\right)^2 dy_{123}\, dy_{134}\, dy_{13}$$
$$\cdot\, y_{13}^{-\varepsilon}\, y_{24}^{-\varepsilon}\left(y_{123}y_{134} - y_{13}\right)^{-\varepsilon}\Theta(y_{13})\,\Theta(y_{24})\,\Theta(y_{123}y_{134} - y_{13})$$
$$dv\,(v(1-v))^{-\varepsilon}\, d\theta' \frac{1}{N_{\theta'}} \sin^{-2\varepsilon}\theta' \left(-q^2 g^{\mu\nu}\right)$$
$$\cdot \sum_{m \geq n-1}^{8} A(m,n)_{\mu\nu} \qquad \text{(A.5)}$$

Appendix B: 1-3-System

The 1-3-system is defined by $\vec{p}_1+\vec{p}_3 = 0$ and $p_2 \parallel e_z$. Besides the three invariants y_{13}, y_{123} and y_{134} two angles θ and θ' are needed for the 4 momenta $p_1, \ldots, p_4$. They are:

$$
\begin{aligned}
p_1 &= \frac{\sqrt{y_{13}}}{2}\sqrt{q^2}\left(1, \ldots\ldots, \sin\theta\cos\theta', \cos\theta\right) \\
p_2 &= \frac{y_{123}-y_{13}}{2\sqrt{y_{13}}}\sqrt{q^2}\left(1, \ldots\ldots, 0, 1\right) \\
p_3 &= \frac{\sqrt{y_{13}}}{2}\sqrt{q^2}\left(1, \ldots\ldots, -\sin\theta\cos\theta', -\cos\theta\right) \\
p_4 &= \frac{y_{134}-y_{13}}{2\sqrt{y_{13}}}\sqrt{q^2}\left(1, \ldots\ldots, \sin\beta, \cos\beta\right)
\end{aligned}
\qquad (B.1)
$$

where

$$1 - \cos\beta = \frac{2\, y_{13}\, y_{24}}{(y_{123}-y_{13})(y_{134}-y_{13})} \qquad (B.2)$$

We introduce $v = \frac{1}{2}(1-\cos\theta)$.

In terms of these variables the remaining invariants have the following form

$$
\begin{aligned}
y_{12} &= (y_{123}-y_{13})\, v \\
y_{23} &= (y_{123}-y_{13})(1-v) \\
y_{14} &= (y_{134}-y_{13})\left[v(1-\gamma) + (1-v)\gamma - 2\cos\theta'\sqrt{v(1-v)\gamma(1-\gamma)}\right] \\
y_{34} &= (y_{134}-y_{13})\left[(1-v)(1-\gamma) + v\gamma + 2\cos\theta'\sqrt{v(1-v)\gamma(1-\gamma)}\right]
\end{aligned}
\qquad (B.3)
$$

where

$$\gamma = \frac{y_{13}\, y_{24}}{(y_{123}-y_{13})(y_{134}-y_{13})} \qquad (B.4)$$

and $y_{24} = 1-y_{123}-y_{134}+y_{13}$.

Appendix C: Four-Parton Matrix Elements

In this appendix we describe the decomposition of the various four-parton matrix elements into partial fractions. These matrix elements have been calculated in /22/ and listed in the appendix A of /21/ for $e^+e^- \rightarrow q\bar{q}gg$ and in the appendix of /23/ for the channel $e^+e^- \rightarrow q\bar{q}q\bar{q}$. We follow the nomenclature of /22/. The contributions to the four-parton cross sections are divided in four classes: (A) Uncrossed QED-like graphs proportional to C_F^2, (B) Crossed QED-like graphs proportional to $C_F(C_F-N_c/2)$, (C) QCD-like graphs proportional to C_FN_c and (D) four-quark production graphs proportional to C_FT_R and $C_F(C_F-N_c/2)$ (class E), respectively. The constants C_F^2, $C_F(C_F-N_c/2)$, C_FN_c and C_FT_R are the colour factors associated with the graphs ($C_F = 4/3$, $N_c = 3$ and $T_R = N_f/2$ for N_f flavours). Interference terms to $e^+e^- \rightarrow q_\alpha \bar{q}_\alpha q_\beta \bar{q}_\beta$ for identical flacours $\alpha = \beta$ contribute to terms proportional to $C_F(C_F-N_c/2)$.

The contributions of the various graphs were denoted $A(m,n)_{\mu\nu}$ in /21/ where m,n denote the ordering in fig. 1 of /21/. So m,n = 1, 2, 3 are the QED-like graphs which contribute to group (A) etc. We contract these matrix elements with $(-g_{\mu\nu}q^2)$ which is needed for the angular integrated cross section. We define

$$\mathcal{H}(m,n) = -q^2 g^{\mu\nu} A(m,n)_{\mu\nu} \qquad (C.1)$$

The needed H(m,n) expressed in terms of the invariants y_{ik} are listed below.

A. Colour Factor C_F^2.

$$\mathcal{H}(1,1) = 48\, C_F^2 \frac{y_{34}}{y_{13}\, y_{24}} \qquad (C.2)$$

$$\mathcal{H}(2,1) = 48\, C_F^2 \frac{2\, y_{23}(1-y_{13})}{y_{13}\, y_{24}\, y_{134}} \qquad (C.3)$$

$$\mathcal{H}(2,2) = 48\, C_F^2 \frac{y_{24}\, y_{34} + y_{12}\, y_{13} - e_1}{y_{13}\, y_{134}^2} \qquad (C.4)$$

$$\begin{aligned}\mathcal{H}(3,2) = 48\, C_F^2 \frac{1}{y_{134}\, y_{234}} \Big\{ & \frac{2y_{12}}{y_{13}\, y_{24}} + 2y_{12} \\ & - \frac{2y_{12}}{y_{13}}(1+y_{134}) - \frac{2y_{12}}{y_{24}}(1+y_{234}) \\ & + \frac{(1-y_{12})\, e_1}{y_{13}\, y_{24}} - \frac{e_1}{y_{13}} - \frac{e_1}{y_{24}} \Big\}\end{aligned} \qquad (C.5)$$

where

$$e_1 = y_{14}\, y_{23} - y_{12}\, y_{34} - y_{13}\, y_{24} \qquad (C.6)$$

e_1 vanishes whenever any of y_{12}, y_{13}, y_{24} or y_{34} vanishes. We give here the 4 dimensional expressions. The O(ε)-contributions are needed only for the pole parts as given in appendix E. H(1,1), H(2,1), H(2,2) and H(3,2) have poles for $y_{13} = 0$ and $y_{24} = 0$. Another matrix element with poles at $y_{13} = 0$ and $y_{24} = 0$ is H(3,1) = $= \{H(2,1)\}_{p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4}$. The matrix elements with poles at $y_{14} = 0$ and $y_{23} = 0$ are generated term by term by interchanging momenta. So for $p_1 \leftrightarrow p_2$ we have H(2,1), H(2,2) $\rightarrow$ H(6,4), H(6,6) and $p_3 \leftrightarrow p_4$ generates the matrix elements H(1,1), H(2,1), H(2,2), H(3,2) $\rightarrow$ H(4,4), H(5,4), H(5,5), H(6,5). The simultaneous interchange of p_1 with p_2 and p_3 with p_4 generates the remaining matrix elements of class A: H(2,1), H(2,2) $\rightarrow$ H(3,1), H(3,3).

The sum of the matrix elements of class A is now written as a sum of pole terms in y_{13}, y_{23}, y_{14} and y_{24}, i.e.

$$\sum_{m,n} \left\{ H(m,n) \right\}_{\text{class } A} = \frac{A_{13}}{y_{13}} + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\ 3 \leftrightarrow 4) \tag{C.7}$$

The residue functions are not unique. They differ depending how non-pole terms are distributed between the various A_{ij}. Our result after partial fractioning of the terms proportional to $^1/y_{13}y_{24}$ is:

$$\begin{aligned} A_{13} = 48 C_F^2 \Big\{ & \frac{y_{34}}{y_{13}+y_{24}} + \frac{2 y_{23}(1-y_{13})}{(y_{13}+y_{24})\, y_{134}} \\ & + \frac{y_{24} y_{34} + y_{12} y_{13} - e_1}{y_{134}^2} + \frac{2 y_{14}(1-y_{24})}{(y_{13}+y_{24})\, y_{234}} \\ & + \frac{2 y_{12}}{(y_{13}+y_{24})\, y_{134}\, y_{234}} + \frac{y_{12} y_{13}}{y_{134}\, y_{234}} - \frac{2 y_{12}(1+y_{134})}{y_{134}\, y_{234}} \\ & + \frac{(1-y_{12})\, e_1}{(y_{13}+y_{24})\, y_{134}\, y_{234}} - \frac{e_1}{y_{134}\, y_{234}} \Big\} \end{aligned} \tag{C.8}$$

B. Colour Factor $C_F(C_F - \frac{N_c}{2})$.

For the contributions of class B only the matrix elements $H(4,1)$, $H(4,2)$, $H(5,2)$ and $H(6,2)$ are needed. All others of this class are obtained by interchanging momenta. They are:

$$H(4,1) = 48 C_F \left(C_F - \frac{N_c}{2}\right) \frac{2 y_{12}\, y_{123}\, y_{124}}{y_{13}\, y_{14}\, y_{23}\, y_{24}} \tag{C.9}$$

$$\begin{aligned} H(4,2) = 48 C_F \left(C_F - \frac{N_c}{2}\right) \frac{1}{y_{134}} \Big\{ & -\frac{e_2}{y_{13}\, y_{23}} - \frac{y_{24}\, e_2}{y_{13}\, y_{14}\, y_{23}} \\ & - \frac{2 y_{12}}{y_{14}} + \frac{2 y_{24}\, y_{12}}{y_{13}\, y_{23}} + \frac{2 y_{24}}{y_{13}} + \frac{2 y_{24}}{y_{23}} - \frac{y_{34}\, y_{12}}{y_{13}\, y_{14}} + \frac{y_{23}}{y_{13}} - \frac{y_{24}}{y_{14}} \Big\} \end{aligned} \tag{C.10}$$

$$H(5,2) = 96 C_F \left(C_F - \frac{N_c}{2}\right) \frac{1}{y_{134}} \left\{ \frac{y_{34}\, y_{12}}{y_{13}\, y_{14}} - \frac{y_{24}+y_{23}}{y_{13}} - \frac{y_{24}+y_{13}}{y_{14}} \right\} \tag{C.11}$$

$$H(6,2) = 96 C_F \left(C_F - \frac{N_c}{2}\right) \frac{y_{12}\, y_{123}\, (1+y_{34})}{y_{13}\, y_{23}\, y_{134}\, y_{234}} \tag{C.12}$$

where $e_2 = y_{14}y_{23} - y_{12}y_{34} + y_{14}y_{24}$

e_2 vanishes for any of y_{13}, y_{14}, y_{23} or $y_{24} \to 0$. Therefore the terms proportional to e_2 have no poles in these variables. All other matrix elements in the class B are again obtained by interchanging momenta. So interchanging p_1 and p_2 we generate $H(6,1)$ from $H(4,2)$. The interchange $p_3 \leftrightarrow p_4$ generates $H(4,2)$, $H(6,2) \rightarrow H(5,1)$, $H(5,3)$ and the simultaneous interchange of p_1 and p_2 and p_3 with p_4 generates the remaining matrix elements $H(4,2)$, $H(5,2) \rightarrow H(4,3)$, $H(6,3)$. The pole terms in y_{13} appear only in $H(4,1)$, $H(4,2)$, $H(5,1)$, $H(5,2)$, $H(6,1)$ and $H(6,2)$. As above the sum of the matrix elements of class B is written in terms of pole terms in y_{13}, y_{23}, y_{14} and y_{24}:

$$\sum_{m,n} \left\{ \mathcal{H}(m,n) \right\}_{\text{class } B} = \frac{B_{13}}{y_{13}} + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\ 3 \leftrightarrow 4) \tag{C.13}$$

B_{13} has the following form:

$$\begin{aligned} B_{13} = 48\, C_F \left(C_F - \frac{N_c}{2} \right) \Bigg\{ & \frac{2\, y_{12}\, y_{123}\, y_{124}}{(y_{13}+y_{24})(y_{13}+y_{23})(y_{13}+y_{24})} \left(1 + \frac{2 y_{13}}{y_{14}+y_{24}} \right) \\ & - \frac{2(y_{12}+y_{23})}{y_{134}} + \frac{2\, y_{14}}{y_{234}} + \frac{2 y_{12}\, y_{24}}{(y_{13}+y_{23})\, y_{134}} + \frac{2\, y_{12}\, y_{14}}{(y_{13}+y_{23})\, y_{234}} \\ & - \frac{e_2}{y_{23}\, y_{134}} - \frac{y_{23}\, e_2}{y_{14}\, y_{24}\, y_{134}} + \frac{2 y_{12}\, y_{123}\, (1+y_{34})}{y_{13}\, (y_{13}+y_{23})\, y_{134}\, y_{234}} \Bigg\} \end{aligned} \tag{C.14}$$

C. Colour Factor $C_F N_c$.

The contributions of class C consist of two parts. First we have the interference term between the diagrams 1-6 with the triple-gluon diagrams 7 and 8, which we call the QED-QCD interference terms and second there are the pure QCD-like diagramm 7 and 8 interfering with each other which produce H(7,7), H(8,7) and H(8,8). We consider now the first class (C_a).

(C_a) QED-QCD Interference.

We need to write down only the matrix element H(7,1), H(7,2) and H(8,2). They are:

$$\begin{aligned} \mathcal{H}(7,1) = 48\, C_F N_c \Bigg\{ & \frac{y_{12}\,(y_{124}+y_{234})}{2\, y_{13}\, y_{24}\, y_{34}} + \frac{y_{12}\,(y_{124}+y_{234})}{2\, y_{34}\, y_{24}\, y_{134}} \\ & + \frac{e_1}{4\, y_{34}\, y_{134}} \left[\frac{2\, y_{23}}{y_{13}\, y_{24}} + \frac{3}{y_{13}} + \frac{2}{y_{24}} \right] + \frac{y_{23}\, y_{234}}{y_{24}\, y_{34}\, y_{134}} \\ & + \frac{1}{2\, y_{34}\, y_{134}} \left(1 - 2\, y_{14} - 2\, y_{34} - 4\, y_{12} \right) \end{aligned} \tag{C.15}$$

$$\begin{aligned} \mathcal{H}(7,2) = 48\, C_F N_c \Bigg\{ & \frac{2\, y_{14}\, y_{24} + e_1}{2\, y_{13}\, y_{34}\, y_{134}} + \frac{y_{13}}{y_{34}\, y_{134}^2} \\ & + \frac{4\, y_{24} + y_{134} - 2 y_{13} - 1}{2\, y_{34}\, y_{134}} \Bigg\} \end{aligned} \tag{C.16}$$

$$\begin{aligned} \mathcal{H}(8,2) = 48\, C_F N_c \Bigg\{ & \frac{y_{12}\,(y_{123} + y_{134} + 2\, y_{234})}{2\, y_{13}\, y_{34}\, y_{234}} \\ & + \frac{e_1\,(3\, y_{234} + 2\, y_{14} - 2\, y_{24})}{4\, y_{13}\, y_{34}\, y_{134}\, y_{234}} - \frac{y_{12}\, y_{123}}{y_{13}\, y_{134}\, y_{234}} \\ & - \frac{y_{12}}{2\, y_{34}\, y_{134}\, y_{234}} \left(2 + y_{13} + y_{14} + y_{23} - y_{24} \right) \Bigg\} \end{aligned} \tag{C.17}$$

All other matrix elements are obtained from H(7,1), H(7,2) and H(8,2) by interchanging momenta. It is

$$\mathcal{H}(8,4),\ \mathcal{H}(8,6),\ \mathcal{H}(7,6) = \left\{ \mathcal{H}(7,1),\ \mathcal{H}(7,2),\ \mathcal{H}(8,2) \right\}_{p_1 \leftrightarrow p_2}$$

$$\mathcal{H}(7,4),\ \mathcal{H}(7,5),\ \mathcal{H}(8,5) = \left\{ \mathcal{H}(7,1),\ \mathcal{H}(7,2),\ \mathcal{H}(8,2) \right\}_{p_3 \leftrightarrow p_4}$$

$$\mathcal{H}(8,1),\ \mathcal{H}(8,3),\ \mathcal{H}(7,3) = \left\{ \mathcal{H}(7,1),\ \mathcal{H}(7,2),\ \mathcal{H}(8,2) \right\}_{p_1 \leftrightarrow p_2,\ p_3 \leftrightarrow p_4} \tag{C.18}$$

All these matrix elements have poles in y_{34} and in y_{13}, y_{23}, y_{14} and y_{24} depending which group we consider. For example, poles in y_{13} appear only in H(7,1), H(7,2), H(8,2) and H(8,1). By partial fractioning of the terms with simultaneous poles in y_{13} and y_{34} we separate the poles in y_{13} and y_{34}. Then we obtain for the sum of all the matrix elements of class C_a:

$$\sum_{m,n} \left\{ \mathcal{H}(m,n) \right\}_{\text{class } C_a} = \left\{ \frac{C_{13}}{y_{13}} + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\ 3 \leftrightarrow 4) \right\} + \frac{C_{34}}{y_{34}} \tag{C.19}$$

C_{13} is obtained from H(7,1), H(7,2), H(8,1) and H(8,2). It has the following form:

$$\begin{aligned} C_{13} = 48\, C_F N_C \Bigg\{ & \frac{y_{12}}{(y_{13}+y_{24})(y_{13}+y_{34})} + \frac{y_{12}\, y_{13}}{(y_{13}+y_{24})(y_{13}+y_{34})(y_{24}+y_{34})} \\ & + \frac{y_{12}(y_{123}+y_{134})}{(y_{13}+y_{34})\, y_{234}} + \frac{y_{14}\, y_{134}}{(y_{13}+y_{34})\, y_{234}} + \frac{y_{14}\, y_{24}}{(y_{13}+y_{34})\, y_{134}} \\ & + \frac{y_{12}}{y_{13}+y_{34}} - \frac{y_{12}\, y_{123}}{y_{134}\, y_{234}} + \frac{2 e_1}{y_{34}\, y_{134}} \\ & + \frac{y_{23}\, e_1}{2 y_{34}\, y_{24}\, y_{134}} \Bigg\} \end{aligned} \tag{C.20}$$

We remark that the definition of C_{13} and C_{34} is certainly not unique since non-pole parts can either be written in C_{13} or C_{34} or distributed in both. C_{34} consists of four parts which follow from each other by interchanging momenta $1 \leftrightarrow 2$ etc. Therefore it is sufficient to write explicitly only one term.

$$\begin{aligned} C_{34} = 48\, C_F N_C \Bigg\{ & \frac{y_{12}}{2(y_{13}+y_{34})(y_{24}+y_{34})} - \frac{y_{12}}{2(y_{13}+y_{34})\, y_{234}} \\ & - \frac{y_{12}}{2 y_{134}(y_{24}+y_{54})} + \frac{y_{12}\, y_{34}}{2 y_{134}\, y_{234}} + \frac{2 y_{34}}{y_{234}(y_{13}+y_{34})} \\ & - \frac{y_{34}}{2 y_{134}^2} - \frac{2 y_{34}}{y_{134}} - \frac{y_{34}}{y_{134}\, y_{234}} \end{aligned}$$

$$+\left[\frac{1}{y_{134}\,y_{234}}-\frac{2}{y_{134}}-\frac{1}{y_{234}}\right]\left[\frac{e_1+y_{12}\,y_{34}}{y_{13}+y_{34}}\right.$$

$$\left.-\,y_{34}-\frac{y_{24}\,y_{34}}{y_{13}+y_{34}}\right]-\frac{y_{24}\,y_{34}}{y_{234}(y_{13}+y_{34})}-\frac{y_{34}^2}{y_{234}(y_{13}+y_{34})}$$

$$+\frac{e_1}{2\,y_{13}\,y_{234}}+\frac{(y_{14}-y_{24})\,e_1}{2\,y_{13}\,y_{134}\,y_{234}}$$

$$+\,T(y_{134},y_{234})\left[\frac{y_{134}}{y_{13}+y_{34}}-\frac{1}{2}\right]\Bigg\}+(1\leftrightarrow 2)+(3\leftrightarrow 4)+(1\leftrightarrow 2,\,3\leftrightarrow 4) \qquad \text{(C.21)}$$

We have written C_{34} already in the form that the most singular term had been separated.

C_b: Pure QCD Diagrams

The contributions of class C_b consists of the product of the diagrams n = 7 and 8 and represent the contributions of the three-gluon coupling. They depend in the following way on the invariants y_{ij}

$$H(7,7)=-6\,C_F N_C\,\frac{1}{y_{34}^2\,y_{134}^2}\Big\{16\,y_{24}\,y_{34}\,y_{14}+8\,y_{24}\,y_{34}\,y_{13}$$

$$+8\,y_{24}\,y_{14}\,y_{13}-8\,y_{24}\,y_{13}^2+8\,y_{34}\,y_{23}\,y_{14}+16\,y_{23}\,y_{34}\,y_{13}$$

$$-8\,y_{23}\,y_{14}^2+8\,y_{23}\,y_{14}\,y_{13}-3\,y_{34}^2\,y_{12}+18\,y_{34}\,y_{14}\,y_{12} \qquad \text{(C.22)}$$

$$+18\,y_{34}\,y_{12}\,y_{13}+5\,y_{14}^2\,y_{12}+26\,y_{14}\,y_{12}\,y_{13}+5\,y_{12}\,y_{13}^2\Big\}$$

$$H(8,7)=-12\,C_F N_C\,\frac{1}{y_{34}^2\,y_{134}\,y_{234}}\Big\{-4\,y_{24}^2\,y_{12}$$

$$+4\,y_{24}\,y_{23}\,y_{14}+4\,y_{24}\,y_{23}\,y_{13}+7\,y_{24}\,y_{34}\,y_{12}$$

$$-5\,y_{24}\,y_{14}\,y_{12}+4\,y_{24}\,y_{14}\,y_{13}-13\,y_{24}\,y_{12}\,y_{13}$$

$$-4\,y_{24}\,y_{13}^2-4\,y_{23}^2\,y_{14}+7\,y_{23}\,y_{34}\,y_{12}$$

$$-4\,y_{23}\,y_{14}^2-13\,y_{23}\,y_{14}\,y_{12}+4\,y_{23}\,y_{14}\,y_{13}$$

$$-5\,y_{23}\,y_{12}\,y_{13}+19\,y_{34}^2\,y_{12}+7\,y_{34}\,y_{14}\,y_{12}$$

$$+16\,y_{34}\,y_{12}^2+7\,y_{34}\,y_{12}\,y_{13}\Big\} \qquad \text{(C.23)}$$

$$H(8,8)=\{H(7,7)\}_{1\leftrightarrow 2} \qquad \text{(C.24)}$$

The sum of these three terms has only a single pole in y_{34}. Therefore partial fractioning is not necessary. To see the single pole structure we introduce

$$e_3 = y_{13}\, y_{24} - y_{14}\, y_{23} \tag{C.25}$$

The sum Σ_5 = H(7,7) + H(8,7) + H(8,8) is given by the following expression:

$$\begin{aligned}
\Sigma_5 = {} & 96\, C_F N_C \frac{1}{y_{34}^2\, y_{134}^2\, y_{234}^2} \Big\{ 3\, e_3^2\, y_{34} \\
& + e_3^2 \left(y_{12} + y_{13} + y_{14} + y_{23} + y_{24} \right) - e_3\, y_{34}^3 \\
& - e_3\, y_{34}^2 \left(4 y_{23} + 4 y_{14} + 7 y_{12} \right) \\
& - e_3\, y_{34} \big(2 y_{24} y_{23} + y_{24} y_{12} + 3 y_{23}^2 + 5 y_{23} y_{12} + 3 y_{14}^2 \\
& \quad + 5 y_{14} y_{12} + 2 y_{14} y_{13} + 2 y_{12}^2 + y_{12} y_{13} \big) - 2 y_{34}^2 y_{12} \\
& - 2 y_{34}^3 \big(y_{24} y_{14} + 2 y_{24} y_{12} + y_{23} y_{14} + 2 y_{23} y_{12} \\
& \quad + y_{23} y_{13} + 2 y_{14} y_{12} + y_{12}^2 + 2 y_{12} y_{13} \big)
\end{aligned}$$

$$\begin{aligned}
& - y_{34}^2 \big(2 y_{24}^2 y_{14} + y_{24}^2 y_{12} + 4 y_{24} y_{23} y_{14} \\
& \quad + 3 y_{24} y_{23} y_{12} + 2 y_{24} y_{14}^2 + 8 y_{24} y_{14} y_{12} \\
& \quad + 2 y_{24} y_{12}^2 + 4 y_{23}^2 y_{14} + y_{23}^2 y_{12} + 2 y_{23}^2 y_{13} \\
& \quad + 4 y_{23} y_{14}^2 + 14 y_{23} y_{14} y_{12} + 4 y_{23} y_{14} y_{13} + 2 y_{23} y_{12}^2 \\
& \quad + 8 y_{23} y_{12} y_{13} + 2 y_{23} y_{13}^2 + y_{14}^2 y_{12} + 2 y_{14} y_{12}^2 \\
& \quad + 3 y_{14} y_{12} y_{13} + 2 y_{12}^2 y_{13} + y_{12} y_{13}^2 \big) \\
& - y_{34} \big(y_{24}^3 y_{14} + 3 y_{24}^2 y_{23} y_{14} + 2 y_{24}^2 y_{14} y_{12} \\
& \quad + 4 y_{24} y_{23}^2 y_{14} + 6 y_{24} y_{23} y_{14} y_{12} + y_{24} y_{14}^3 \\
& \quad + 2 y_{24} y_{14}^2 y_{12} + 2 y_{24} y_{14} y_{12}^2 + 3 y_{23}^3 y_{14} \\
& \quad + y_{23}^3 y_{13} + 6 y_{23}^2 y_{14} y_{12} + 2 y_{23}^2 y_{12} y_{13} + 3 y_{23} y_{14}^3 \\
& \quad + 6 y_{23} y_{14}^2 y_{12} + 4 y_{23} y_{14}^2 y_{13} + 4 y_{23} y_{14} y_{12}^2 \\
& \quad + 6 y_{23} y_{14} y_{12} y_{13} + 3 y_{23} y_{14} y_{13}^2 + 2 y_{23} y_{12}^2 y_{13} \\
& \quad + 2 y_{23} y_{12} y_{13}^2 + y_{23} y_{13}^3 \big) \Big\}
\end{aligned} \tag{C.26}$$

To evaluate Σ_5 further we go into the 3-4 system, the angle variables are $v = \frac{1}{2}(1+\cos\theta)$ and θ'. In this system Σ_5 has the following simple form

$$\Sigma_5 = 48 C_F N_C \Big\{ \frac{1}{y_{34}} T(y_{134}, y_{234}) \big(2v(1-v) - 2 \big)$$

$$+ \frac{2}{y_{134} y_{234}} (1 - v + v^2) \left[\frac{y_{134}}{y_{234}} + \frac{y_{234}}{y_{134}} - 2 - 2 y_{12} \right]$$

$$+ \frac{4v(1-v)}{y_{134} y_{234}} \left(2\cos^2\theta' - 1 \right)$$

$$+ \frac{2 y_{12} (y_{234} + y_{12} - y_{34})^2}{y_{134}^2 (y_{234} - y_{34})^2} \left[(1-2v)^2 - 4v(1-v)\cos^2\theta' \right]$$

$$- \frac{2C}{y_{134}^2 y_{234} (y_{234} - y_{34})^2 y_{34}} \Big[2 y_{34} (1 - y_{134})^2$$

$$+ (y_{234} - y_{34}) \left[y_{12} - y_{34} + y_{34} y_{134} + (1 - y_{134})^2 \right] \Big] \Big\} \tag{C.27}$$

with

$$C = 2(1-2v)\sqrt{v(1-v)} \sqrt{y_{34} y_{12} (y_{134} y_{234} - y_{34})} \cos\theta' \tag{C.28}$$

The first term in (C.27) is the most singular term with just one pole in y_{34}. The last three terms vanish after integration over θ' and v. The second term is a contribution to the finite terms. This can be combined with the most singular term and its contribution has been accounted for already in $\sigma_N^{(s)}$ as given in (3.22) in section 3.

D. Colour Factor $C_F T_R$.

The contributions with colour factor $C_F T_R$ come from the final state $q\bar{q}q\bar{q}$. These are the diagrams fig. 1b of /21/. They lead to 36 matrix elements H(m,n). Only 12 of the these contribute to the $C_F T_R$-term. In the following we write explicitly the sum of H(1,1), H(2,1) and H(2,2) which have a pole in y_{12}. The other three groups: m,n = 33, 43, 44; 55, 65, 66; 77, 87, 88 follow by interchanging momenta. We denote the total contributions to the $T_R C_F$-term as class D. Then we have

$$\sum_{m,n} \{ H(m,n) \}_{\text{class D}} = S_1 + S_2 + S_3 + S_4 + (1 \leftrightarrow 3) + (2 \leftrightarrow 4) + (1 \leftrightarrow 3, 2 \leftrightarrow 4) \tag{C.29}$$

where

$$S_1 = -96 C_F T_R \frac{1}{y_{12}^2 y_{123}^2 y_{124}^2} (y_{13} y_{24} - y_{23} y_{14})^2 (1 - y_{12}) \tag{C.30}$$

$$S_2 = -48 C_F T_R \frac{1}{y_{12} y_{123}^2 y_{124}^2} \Big\{ 6 (y_{13} y_{24} - y_{23} y_{14})^2$$

$$- (y_{13} y_{24} - y_{23} y_{14}) \left(y_{23}^2 - y_{13}^2 + y_{14}^2 - y_{24}^2 + 2 y_{34} (y_{14} - y_{13}) \right.$$

$$\left. + 2 y_{34} (y_{23} - y_{24}) \right) \Big\} \tag{C.31}$$

$$S_3 = 48 C_F T_R \frac{1}{y_{12}} \Big\{ \left(y_{24} y_{23} + y_{13} y_{14} - y_{34} y_{12} \right) \cdot \left(\frac{1}{y_{123}^2} + \frac{1}{y_{124}^2} \right) + \frac{2 y_{34} (1 + y_{12})}{y_{123} \, y_{124}} \Big\} \qquad (C.32)$$

$$S_4 = 96 \, C_F T_R \frac{1}{y_{123}^2 \; y_{124}^2} \Big\{ \left(y_{13} y_{24} - y_{23} y_{14} \right) \cdot \left(y_{14} - y_{13} + y_{23} - y_{24} \right) + y_{34} \left(y_{13} - y_{14} \right) \left(y_{23} - y_{24} \right) \Big\} \qquad (C.33)$$

Since $y_{13} y_{24} - y_{23} y_{14} \sim y_{12}$ after integration over angles only S_1 and S_3 contribute to the most singular terms. We evaluate these terms in the 1-2 system. The angles in this system are $v = \frac{1}{2}(1-\cos\theta)$ and θ'. We obtain :

$$\begin{aligned} H(1,1) + H(2,1) + H(2,2) = \; & 48 \, C_F T_R \frac{1}{y_{12}} \Big\{ \\ & \left(v^2 + (1-v)^2 \right) T \left(y_{123}, y_{124} \right) \\ & + \left(v^2 + (1-v)^2 \right) \frac{y_{12}}{y_{123}^2 \, y_{124}^2} \left[2 y_{34} y_{123} y_{124} - \left(y_{123} - y_{124} \right)^2 \right] \\ & + 4 v (1-v) \left(1 - \cos^2\theta' \right) \frac{y_{34}}{y_{123} \, y_{124}} \\ & + \left(4 v (1-v) \cos^2\theta' - (1-2v)^2 \right) \frac{2 y_{12} \, y_{34} \left(1 - y_{124} \right)^2}{y_{124}^2 \, \left(y_{123} - y_{12} \right)^2} \\ & + \frac{2B}{y_{124}^2 \left(y_{123} - y_{12} \right)^2} \left[\left(1 - y_{124}^2 \right) + y_{34} - \frac{y_{12} \, y_{34} \, y_{124}}{y_{123}} \right] \end{aligned} \qquad (C.34)$$

with

$$B = 2 (1-2v) \sqrt{v(1-v)} \sqrt{y_{12} y_{34} \left(y_{123} y_{124} - y_{12} \right)} \; \cos\theta' \qquad (C.35)$$

Similar to the contributions of class C_b the $C_F T_R$-term has at most a single pole. The first term in (C.34) is the most singular term. The last three terms vanish after integration over θ' and v. The second term contributes to the finite terms which had been neglected in the past. This term can be combined again with the most singular term and was taken into account already in σ_T (3.28) of section 3.

E. The $q\bar{q}q\bar{q}$-Interference Term.

The non-vanishing $q\bar{q}q\bar{q}$ interference terms come in 4 groups which are related by interchange of momenta. As seen in fig. 1b of /21/ the amplitudes $H(m,n)$ with m,n = 3,1; 3,2; 4,1; 4,2; 5,1; 5,2; 6,1; 6,2 have a single pole in y_{12} which cancels in appropriate sums whereas the other two groups $H(m,n)$ with m,n = 7,3; 7,4; 7,5; 7,6; 8,3; 8,4; 8,5; 8,6 have a pole in y_{34}. We pick the sum of $H(7,3)$, $H(7,4)$, $H(8,3)$ and $H(8,4)$ which have poles in y_{34}, y_{14}, y_{134}, y_{234} and y_{124}. In the sum of these amplitudes only the poles in y_{134} and y_{234} remain. Therefore for y_{134} and $y_{234} \geq y$ we can calculate the 3- and 4-jet cross sections by integrating over all y_{34} without having to subtract any singular terms. Only in the 2-jet region y_{134} or $y_{234} \leq y$ we encounter singular terms. The $q\bar{q}q\bar{q}$ interference terms are proportional to $C_F(C_F - N_c/2)$. The particular $H(7,3)$, $H(7,4)$ etc. have the following form:

$$H(7,3) = 96 \, C_F \left(C_F - \frac{N_c}{2} \right) \frac{y_{13}}{y_{14} \, y_{34} \, y_{134}^2} \cdot \Big\{ y_{12} \left(y_{14} + y_{34} \right) + y_{23} \, y_{34} + y_{14} y_{23} - y_{13} \, y_{24} \Big\} \qquad (C.36)$$

$$\mathcal{H}(7,4) = 48\, C_F \left(C_F - \frac{N_c}{2}\right) \frac{1}{y_{14}\, y_{34}\, y_{124}\, y_{134}}$$

$$\cdot \Big\{ y_{12}\, y_{13}\, y_{24} + y_{12}\, y_{14}\, y_{23} + y_{13}\, y_{14}\, y_{24} - y_{13}\, y_{23}\, y_{24}$$

$$+ y_{14}^2\, y_{23} + y_{14}\, y_{23}^2 - y_{12}\, y_{34}^2 - y_{12}^2\, y_{34}$$

$$- y_{12}\, y_{14}\, y_{34} - y_{12}\, y_{23}\, y_{34} + y_{13}\, y_{24}\, y_{34} + y_{14}\, y_{23}\, y_{34} \Big\} \tag{C.37}$$

$$\mathcal{H}(8,3) = \Big\{ \mathcal{H}(7,4) \Big\}_{(1 \leftrightarrow 3)} \tag{C.38}$$

$$\mathcal{H}(8,4) = -96\, C_F \left(C_F - \frac{N_c}{2}\right) \frac{(y_{12} + y_{14})(y_{23} + y_{34})}{y_{14}\, y_{34}\, y_{124}\, y_{234}} \tag{C.39}$$

Appendix D. Finite Terms for Three Jets

As was explained in section 3, it is sufficient to integrate the term A_{13}/y_{13} in the decomposition (6.7) over the 3-jet region, where the "3-jet-region" is as defined in section 3. One has a 3-jet-region, where $y_{13} \leqslant y$ (region III in fig. 6) and 3-jet regions where $y_{13} \geqslant y$ is guaranteed. The latter contributions are calculated numerically as explained in section 3.

In the region $y_{13} \leqslant y$ one has singular contributions $\frac{A_{13}(y_{13}=0)}{y_{13}}$ and finite ones $(A_{13}-A_{13}(y_{13} = 0))/y_{13}$. It is the purpose of this appendix to calculate these finite contributions. Actually we include some finite terms already in $A_{13}(y_{13} = 0)$, i.e. we do not take the plaine residue. This is necessary in order to regulate integrations over other variables than y_{13}, for instance the integration over y_{23} (or ϑ). So the term which is subtracted from A_{13} is actually the most singular piece with some y_{13} dependence included which is defined below. Of course, there is a certain arbitrariness in defining this "most singular" piece. We choose it in such a way that the expressions have no singularity outside the region $y_{13} \leqslant y$ and thus has a meaning also for the 2- and 4-jet case. This is in contrast to the work of Gottschalk and Shatz /20/.

We start with the terms of class A and B, which are proportional to C_F^2. The sum of these terms with the pole y_{13}^{-1} is denoted by AB_{13}. The singular term of this sum is AB_{13}^S which is defined as

$$AB_{13}^S = 48\, C_F^2\, T(y_{134}, y_{13} + y_{24}) \left[1 - \vartheta + \frac{2 y_{12}}{y_{13} + y_{23}} \right] \tag{D.1}$$

with $\vartheta = y_{12}/(y_{12}+y_{23})$ and T is the Born term matrix element (2.2). If (D.1) is subtracted from the full matrix element AB_{13} the difference is finite for

$y_{13} \to 0$. So it can be integrated over the strip $0 \leqslant y_{13} \leqslant y$ without introducing any regulator. In (D.1) the pole terms $(y_{13}+y_{23})^{-1}$ and $(y_{13}+y_{24})^{-1}$ originate from the partial fractioning in (C.7) and (C.13). Therefore $AB_{13}-AB^{S}_{13}$ can be integrated over the full range of y_{23} (or ϑ) up to the kinematical limit $y_{23} = 0$ (or $\vartheta = 1$). This was the purpose of the expansion in partial fractions and we must make sure that also $AB_{13}-AB^{S}_{13}$ has this property. This is the reason for choosing the singular term in the form of (D.1). We notice that the variables in $T(y_{134}, y_{13}+y_{24})$ are such that we do not have the factorizing form of a genuine 3-jet matrix elements times the Altarelli-Parisi factor. For this we need $T(y_{134}, y_{24})$ or $T(y_{134}, y_{24}-y_{13})$ as factor as was explained in section 3 depending which 3-jet variables were chosen for calculating $\sigma_C^{(s)}$ and $\sigma_N^{(s)}$. Since the difference is only an additional finite term we shall compute these terms at the end of this appendix.

The easiest method for calculating the finte terms in the strip $0 \leqslant y_{13} \leqslant y$ is to evaluate $AB_{13}-AB^{S}_{13}$ by Monte-Carlo integration. This should not produce any problem since the difference is finite for $y_{13} \to 0$ and has no pole for $y_{23} = 0$ as long as we take y_{134} and $y_{24} > 0$.

In this section we develop another method which allows us to calculate these finite terms by numerical integration. We integrate over θ' and ϑ (the two angles in the 1-3 system) analytically. The remaining integration over y_{13} can be done with numerical routines. The result is then a 3-jet contribution which depends on y_{134} and y_{24} or y_{134} and y_{123} depending whether we choose y_{134} and y_{24} or y_{134} and y_{123} as 3-jet variables.

Before we present the result of this integration which is straightforward but leads to a very lengthy expression we define some integrals which give us the result of the θ' and ϑ integration for several pole terms and constants. We define $\int \equiv \frac{1}{\pi} \int_0^{\zeta} d\vartheta \int_0^{\pi} d\theta'$ where $\vartheta = \frac{1}{2}(1-\cos\theta)$ and θ and θ' are the angles in the 1-3 system. We have defined the integration region of ϑ as $0 \leq \vartheta \leq \zeta$. So we can use the results also for other integrals in which ϑ is not integrated up to $\vartheta = 1$ (or $y_{23} = 0$). The invariants y_{12}, y_{23}, y_{14} and y_{34} are given in terms of the 1-3 variables y_{123}, y_{134}, ϑ and θ' in appendix B.

In $AB_{13}-AB^{S}_{13}$ we encounter the following integrals:

$$\mathcal{D}_1(\zeta) = \int \frac{1}{(y_{13}+y_{23})(y_{13}+y_{14})}$$

$$= \frac{1}{y_{123}-y_{13}} \; \mathcal{J}\left(a_1, b_1, c_1, \frac{y_{123}}{y_{123}-y_{13}}, \zeta\right) \tag{D.2}$$

where

$$\mathcal{J}(a,b,c,\delta,\zeta) = \frac{1}{\sqrt{a\delta^2+2b\delta+c}} \left\{ \ln \frac{\delta}{\delta-\zeta} \right.$$

$$\left. + \ln \frac{(a\delta+b)\zeta + b\zeta + c + \sqrt{a\zeta^2+2b\zeta+c}\,\sqrt{a\delta^2+2b\delta+c}}{b\delta + c + \sqrt{c}\,\sqrt{a\delta^2+2b\delta+c}} \right. \tag{D.3}$$

and

$$a_1 = (y_{134} - y_{13})^2$$

$$b_1 = \tfrac{1}{2}(\alpha_1 + \beta_1)^2 - \tfrac{1}{2}(a_1 + \alpha_1^2)$$

$$c_1 = \alpha_1^2$$

$$\alpha_1 = (y_{134} - y_{13})\,\gamma + y_{13}$$

$$\beta_1 = (y_{134} - y_{13})(1 - 2\gamma) \;; \quad \gamma = \frac{y_{13}\, y_{24}}{(y_{134} - y_{13})(y_{123} - y_{13})} \tag{D.4}$$

$$D_2(\zeta) = \int \frac{1}{y_{13} + y_{14}} = \mathcal{J}_1(a_1, b_1, c_1, \zeta) \tag{D.5a}$$

with

$$\mathcal{J}_1(a, b, c, \zeta) = \frac{1}{\sqrt{a}} \ln \frac{a\zeta + b + \sqrt{a}\,\sqrt{a\zeta^2 + 2b\zeta + c}}{b + \sqrt{a}\,\sqrt{c}} \tag{D.5b}$$

$$D_3(\zeta) = \int \frac{v}{y_{13} + y_{14}} = \mathcal{J}_1(a_1, b_1, c_1, \zeta) - \mathcal{J}_2(a_1, b_1, c_1, \zeta) \tag{D.6}$$

$$D_4(\zeta) = \int \frac{v^2}{y_{13} + y_{14}} = \mathcal{J}_1(a_1, b_1, c_1, \zeta) - 2\,\mathcal{J}_2(a_1, b_1, c_1, \zeta) + \mathcal{J}_4(a_1, b_1, c_1, \zeta) \tag{D.7}$$

where

$$\mathcal{J}_2(a, b, c, \zeta) = \frac{a + b}{a}\,\mathcal{J}_1(a, b, c, \zeta) - \frac{1}{a}\left(\sqrt{a\zeta^2 + 2b\zeta + c} - \sqrt{c}\right)$$

$$\mathcal{J}_4(a, b, c, \zeta) = \frac{\zeta}{2a}\sqrt{a\zeta^2 + 2b\zeta + c} + \left(2 + \frac{3b}{a}\right)\mathcal{J}_2(a, b, c, \zeta) - \left(1 + \frac{3b}{2a} + \frac{c}{2a}\right)\mathcal{J}_1(a, b, c, \zeta) \tag{D.8}$$

Some very simple integrals are:

$$D_5(\zeta) = \int \frac{1}{y_{13} + y_{23}} = -\frac{1}{y_{123} - y_{13}} \ln\left(1 - \frac{y_{123} - y_{13}}{y_{123}}\,\zeta\right) \tag{D.9}$$

$$D_6(\zeta) = \int y_{34} = (y_{134} - y_{13})\left(\zeta - \tfrac{1}{2}\zeta^2 - \gamma\zeta(1 - \zeta)\right) \tag{D.10}$$

$$D_7(\zeta) = \int \frac{y_{34}}{y_{13} + y_{23}} = (y_{134} - y_{13})\left[-\frac{(2\gamma - 1)\zeta}{y_{123} - y_{13}} + \frac{(1 - 2\gamma)\,y_{123} - (1 - \gamma)(y_{123} - y_{13})}{(y_{123} - y_{13})^2} \ln\left(1 - \frac{y_{123} - y_{13}}{y_{123}}\,\zeta\right)\right] \tag{D.11}$$

$$D_8(\zeta) = \int 1 = \zeta \tag{D.12}$$

$$D_9(\zeta) = \int v = \tfrac{1}{2}\zeta^2 \tag{D.13}$$

$$D_{e_1}(\zeta) = \int \frac{e_1}{y_{13}} = -y_{24}\,\zeta^2 \tag{D.14}$$

$$D_{11}(\zeta) = \int \frac{y_{12}}{y_{13}+y_{23}} = y_{123}\, D_5(\zeta) - D_8(\zeta) \tag{D.15}$$

$$D_{e_2}(\zeta) = \int \frac{e_2}{y_{13}\, y_{23}} = \frac{2\, y_{24}\, \zeta}{y_{123} - y_{13}} \tag{D.16}$$

$$\begin{aligned} D^{w}_{e_2}(\zeta) &= \int \frac{y_{23}\, e_2}{y_{14}\, y_{24}\, y_{13}} \\ &= \frac{(y_{123}-y_{13})^2}{y_{13}\, y_{24}} \Big\{ \zeta - \tfrac{1}{2}\zeta^2 + (\gamma-1)\left(\gamma - \sqrt{\zeta^2 - 2\gamma\zeta + \gamma^2}\right) \\ &\quad + \left(2 - \tfrac{3}{2}\gamma\right)\gamma - \left(2 - \tfrac{3}{2}\gamma - \tfrac{1}{2}\zeta\right)\sqrt{\zeta^2 - 2\gamma\zeta + \gamma^2} \Big\} \end{aligned} \tag{D.17}$$

$$D_{14}(\zeta) = \int \frac{y_{14}}{y_{13}+y_{23}} = (y_{134}-y_{13})\, D_5(\zeta) - D_7(\zeta) \tag{D.18}$$

$$D_{15}(\zeta) = \int y_{14} = (y_{134}-y_{13})\, D_8(\zeta) - D_6(\zeta) \tag{D.19}$$

$$D_{26}(\zeta) = \int \frac{1}{y_{14}+y_{24}} = J_1(a_2, b_2, c_2, \zeta) \tag{D.20}$$

$$D^{v}_{26}(\zeta) = \int \frac{v}{y_{14}+y_{24}} = J_1(a_2, b_2, c_2, \zeta) - J_2(a_2, b_2, c_2, \zeta) \tag{D.21}$$

with

$$\begin{aligned} a_2 &= (y_{134} - y_{13})^2 \\ b_2 &= \tfrac{1}{2}(\alpha_2+\beta_2)^2 - \tfrac{1}{2}(a_2+\alpha_2^2) \\ c_2 &= \alpha_2^2 \\ \alpha_2 &= y_{24} + (y_{134}-y_{13})\gamma \\ \beta_2 &= (y_{134}-y_{13})(1-2\gamma) \end{aligned} \tag{D.22}$$

$$\begin{aligned} D_{28}(\zeta) &= \int \frac{1}{(y_{13}+y_{23})(y_{14}+y_{24})} \\ &= \frac{1}{y_{123}-y_{13}}\, J\left(a_2, b_2, c_2, \frac{y_{123}}{y_{123}-y_{13}}, \zeta\right) \end{aligned} \tag{D.23}$$

$$B_1(\zeta) = \int \frac{1}{y_{234}} = J_1(a_3, b_3, c_3, \zeta) \tag{D.24}$$

$$B_2(\zeta) = \int \frac{1-v}{y_{234}} = J_2(a_3, b_3, c_3, \zeta) \tag{D.25}$$

$$B_4(\zeta) = \int \frac{(1-v)^2}{y_{234}} = J_4(a_3, b_3, c_3, \zeta) \tag{D.26}$$

with

$$\begin{aligned} a_3 &= (1 - y_{24} + y_{13})^2 - 4\, y_{13} \\ b_3 &= \tfrac{1}{2}(\alpha_3+\beta_3)^2 - \tfrac{1}{2}(a_3+\alpha_3^2) \\ c_3 &= \alpha_3^2 \quad , \quad \alpha_3 = 1 - y_{13} - (y_{134}-y_{13})\gamma \\ \beta_3 &= y_{13} - 1 + \frac{y_{24}(y_{123}+y_{13})}{y_{123} - y_{13}} \end{aligned} \tag{D.27}$$

$$E_{e_1}(\xi) = \int \frac{e_1}{y_{234}} = -(y_{123} - y_{13})\, D_8(\xi)$$

$$+ y_{24}(y_{123} - 2y_{13})\, B_1(\xi) + (y_{123} - y_{13})(y_{134} + y_{123} - 2y_{13})\, B_2(\xi)$$

(D.28)

$$E^{v}_{e_1}(\xi) = \int \frac{v e_1}{y_{234}} = -(y_{123} - y_{13})\, D_9(\xi)$$

$$+ y_{24}(y_{123} - 2y_{13})\big(B_1(\xi) - B_2(\xi)\big)$$

$$+ (y_{123} - y_{13})(y_{134} + y_{123} - 2y_{13})\big(B_2(\xi) - B_4(\xi)\big)$$

(D.29)

$$F(\xi) = \int \frac{1}{(y_{13} + y_{23})\, y_{234}}$$

$$= \frac{1}{y_{123} - y_{13}}\; J\left(a_3, b_3, c_3, \frac{y_{123}}{y_{123} - y_{13}}, \xi\right)$$

(D.30)

$$\mathcal{E}(\xi) = \int \frac{y_{14}\, y_{23} - y_{12}\, y_{34}}{(y_{13} + y_{23})\, y_{234}}$$

$$= (y_{134} - y_{13})\, B_1(\xi) - \Big[(y_{134} - 2y_{13} + y_{24})\, y_{13} + y_{123}(y_{13} - y_{24})\Big] F(\xi)$$

$$+ (y_{123} - y_{13})\big(B_1(\xi) - D_5(\xi)\big)$$

(D.31)

$$E(\xi) = \int \frac{y_{14}\, y_{23} - y_{12}\, y_{34}}{y_{234}}$$

$$= (y_{123} - y_{13})\Big[(y_{134} - y_{13})\, B_2(\xi) - D_8(\xi)$$

$$+ (y_{123} - y_{13})\, B_2(\xi) + y_{24}\, B_1(\xi)\Big]$$

(D.32)

It is obvious that the results for the double pole terms are more complicated than for the single pole terms. We also notice that the poles in y_{234} occur only in the finite term. The singular contribution AB^S_{13} does not have such poles. They cancel in the sum of A_{13} and B_{13} for the terms proportional to C_F^2.

The result of the integral over θ' and v is written as a sum of seven terms:

$$\frac{1}{\pi}\int_0^{\pi} d\theta' \int_0^{\xi} dv \left(\frac{AB_{13}}{y_{13}} - \frac{AB^S_{13}}{y_{13}}\right) = K_{11} + K_{21} + K_{22} + K_{32}$$

$$+ K_{41} + K_{42} + K_{61} \qquad \text{(D.33)}$$

The notation with indices i,j in K_{ij} is related to the original decomposition of the C_F^2 matrix elements H(i,j). But because of the partial fractioning and the subtraction of the singular term they cannot be directly connected. The result is obtained from A_{13} and B_{13} if the singular term AB^S_{13} has been subtracted. We get

$$
\begin{aligned}
K_{41} = 48 C_F^2 \Big\{ & \frac{4}{y_{134}(y_{13}+y_{24})} \Big(y_{123} D_5(\zeta) - D_8(\zeta) \Big) \\
& + \Big(\frac{4}{y_{134}} - \frac{4 y_{123}}{y_{134}(y_{13}+y_{24})} \Big) D_8(\zeta) \\
& + \frac{2 y_{123}}{y_{13}(y_{13}+y_{24})} \Big\{ y_{123} \big(y_{123} - y_{13} + y_{24} \big) D_1(\zeta) - y_{123} \big(D_2(\zeta) - D_5(\zeta) \big) \\
& - (y_{123} - y_{13}) D_3(\zeta) - D_8(\zeta) - (y_{24} - y_{13}) D_2(\zeta) \\
& - \frac{y_{123}}{y_{134}} \Big[(y_{123} + y_{24}) D_5(\zeta) + D_{14}(\zeta) - D_8(\zeta) \Big] + \frac{y_{123} - y_{13}}{y_{134}} D_9(\zeta) \\
& + \frac{1}{y_{134}} D_{15}(\zeta) + \frac{y_{24}}{y_{134}} D_8(\zeta) - \frac{y_{123} - y_{13}}{y_{123}} \Big((y_{123} - y_{13}) D_4(\zeta) + D_9(\zeta) \\
& + (y_{24} - y_{13}) D_3(\zeta) \Big) \Big\} + \frac{2 y_{123}}{y_{13} y_{134} (y_{24} + y_{13})} \Big[- y_{123} \big(D_8(\zeta) + D_7(\zeta) \big) \\
& + (y_{123} - y_{13}) \big(D_8(\zeta) - D_9(\zeta) \big) + y_{13} D_8(\zeta) + D_6(\zeta) \Big] \\
& + \frac{2}{y_{13}(y_{13}+y_{24})} \Big[(y_{123} - y_{13})^2 D_4(\zeta) + (y_{123} - y_{13}) \big(D_9(\zeta) + (y_{24} - y_{13}) D_3(\zeta) \big) \Big] \\
& - \frac{2 y_{123} (y_{24} - y_{13})}{y_{134} \, y_{13} (y_{24} + y_{13})} D_8(\zeta) + \frac{4 y_{123}}{y_{24} + y_{13}} \Big[\frac{y_{123}^2}{y_{24} - y_{13}} \big(D_1(\zeta) - D_{28}(\zeta) \big) \\
& - \frac{y_{123}}{y_{24} - y_{13}} \big(D_2(\zeta) - D_{20}(\zeta) \big) - \frac{y_{123} - y_{13}}{y_{24} - y_{13}} \big(D_3(\zeta) - D_{26}^{v}(\zeta) \big) \\
& + y_{123} D_1(\zeta) - D_2(\zeta) \Big] \Big\}
\end{aligned}
\qquad \text{(D.34)}
$$

It is easy to find out that K_{41} has no poles in y_{13} and $y_{24}-y_{13}$ if the results for the integrals $D_i(\zeta)$ etc. are substituted.

$$
\begin{aligned}
K_{11} = 48 C_F^2 \frac{1}{y_{13}(y_{13}+y_{24})} \Big[& - \gamma \, y_{13} \big(D_8(\zeta) - 2 D_9(\zeta) \big) \\
& - y_{13} \Big((1-\gamma) D_8(\zeta) + (2\gamma - 1) D_9(\zeta) \Big) \Big]
\end{aligned}
\qquad \text{(D.35)}
$$

$$
K_{21} = 48 C_F^2 \frac{1 + y_{13} - y_{123}}{y_{134}(y_{13}+y_{24})} \, 2 \big(D_8(\zeta) - D_9(\zeta) \big) \qquad \text{(D.36)}
$$

$$
\begin{aligned}
K_{22} = 48 C_F^2 \Big\{ & \frac{1}{y_{13} y_{134}^2} \Big[- y_{24} y_{134} \gamma \big(D_8(\zeta) - 2 D_9(\zeta) \big) \\
& - y_{24} y_{13} \Big((1-\gamma) D_8(\zeta) + (2\gamma - 1) D_9(\zeta) \Big) - y_{13} y_{134} \big(D_8(\zeta) - D_9(\zeta) \big) \\
& + \frac{y_{123} - y_{13}}{y_{134}^2} D_9(\zeta) - \frac{1}{y_{134}^2} D_{e_1}(\zeta) \Big\}
\end{aligned}
\qquad \text{(D.37)}
$$

$$
\begin{aligned}
K_{32} = 48 C_F^2 \frac{1}{y_{134}} \Big\{ & (y_{123} - y_{13}) \big(B_1(\zeta) - B_2(\zeta) \big) \\
& + \Big(\frac{1}{y_{13}(y_{13}+y_{24})} - \frac{1}{y_{13}} \Big) E_{e_1}(\zeta) - \frac{y_{123} - y_{13}}{y_{13}(y_{13}+y_{24})} E_{e_1}^{v}(\zeta) \Big\}
\end{aligned}
\qquad \text{(D.38)}
$$

$$\mathcal{K}_{42} = -48 C_F^2 \frac{2}{y_{134}} \left\{ \mathcal{D}_8(\zeta) + \mathcal{D}_{11}(\zeta) + \mathcal{D}_{e_2}(\zeta) + \mathcal{D}_{e_2}^{w}(\zeta) \right\}$$

(D.39)

$$\mathcal{K}_{61} = 48 C_F^2 \left\{ \frac{2}{y_{13}+y_{24}} \left[\left(y_{134} - 2y_{13} + y_{24}\right) F(\zeta) - \mathcal{D}_5(\zeta) + \mathcal{B}_1(\zeta) \right] \right.$$

$$+ \frac{2\,\mathcal{C}(\zeta)}{y_{13}\left(y_{13}+y_{24}\right)} + \frac{2}{y_{13}+y_{24}} \left[\left(y_{134} - y_{13} + y_{24}\right) \mathcal{B}_1(\zeta) - \mathcal{D}_8(\zeta) \right.$$

$$\left. + \left(y_{123} - y_{13}\right) \mathcal{B}_2(\zeta) \right] + \frac{4\left(1+y_{123}\right)}{y_{134}\left(y_{13}+y_{24}\right)} \left(y_{123}\, \mathcal{D}_5(\zeta) - \mathcal{D}_8(\zeta) \right)$$

$$+ 2\, \frac{2y_{123}\left(1+y_{13}-y_{24}\right) - y_{134}\left(y_{24}-y_{13}\right)}{y_{134}\left(y_{13}+y_{24}\right)} \left(y_{123}\, F(\zeta) - \mathcal{B}_1(\zeta) \right)$$

$$- \frac{2y_{123}\left(y_{123}-y_{13}\right)}{y_{134}\left(y_{13}+y_{24}\right)} \left(\mathcal{B}_1(\zeta) - \mathcal{B}_2(\zeta) \right) + \frac{2\left(y_{123}+2y_{134}\right)}{y_{134}\left(y_{13}+y_{24}\right)} \Big[$$

$$y_{123} \left(\left(y_{134} + y_{24} - 2y_{13} \right) F(\zeta) - \mathcal{D}_5(\zeta) + \mathcal{B}_1(\zeta) \right)$$

$$\left. - \left(\left(y_{134} + y_{24} - y_{13} \right) \mathcal{B}_1(\zeta) - \mathcal{D}_8(\zeta) + \left(y_{123} - y_{13} \right) \mathcal{B}_2(\zeta) \right) \right]$$

$$+ \frac{2y_{123}}{y_{13}\, y_{134}\left(y_{13}+y_{24}\right)} \left(y_{123}\, \mathcal{C}(\zeta) - E(\zeta) \right)$$

$$\left. - \frac{4\left(y_{123}-y_{13}\right)\left(1+y_{134}\right)}{y_{134}\left(y_{13}+y_{24}\right)} \left(\mathcal{B}_1(\zeta) - \mathcal{B}_2(\zeta) \right) \right\}$$

(D.40)

It is clear that some of the terms could be combined in particular for $\zeta = 1$, where some of the integrals over θ' and v are very simple. The result (C.33) can also be used to calculate to integrate $AB_{13}-AB_{13}^{S}$ over regions $y_{13} \geqslant y$ which is in the 4-jet region. For this we need also the integral over the singular part. This is:

$$\frac{1}{\pi} \int_0^{\pi} d\theta' \int_0^{\zeta} dv\, \frac{AB_{13}^{S}}{y_{13}} = 48 C_F^2\, T\left(y_{134}, y_{24}+y_{13}\right)$$

$$\frac{1}{y_{13}} \left\{ \mathcal{D}_8(\zeta) - \mathcal{D}_9(\zeta) + 2\left(y_{123}\, \mathcal{D}_5(\zeta) - \mathcal{D}_8(\zeta) \right) \right\}$$

(D.41)

For the finite 3-jet cross section in the strip $0 \leqslant y_{13} \leqslant y$ we also need the difference of (D.41) as compared to $T(y_{134}, y_{24})$ (with y_{134}, y_{24} as 3-jet variables) or to $T(y_{134}, y_{24}-y_{13})$ (with y_{134}, y_{123} as 3-jet variables). In the first case we substitute $\Delta T_1 = T(y_{134}, y_{13}+y_{24}) - T(y_{134}, y_{24})$ instead of T in (D.41). In the second case we substitute $\Delta T_2 = T(y_{134}, y_{13}+y_{24}) - T(y_{134}, y_{24}-y_{13})$ for T in (D.41). These two functions are:

$$\Delta T_1 = \frac{y_{13}}{y_{134}} - \frac{y_{13}\, y_{134}}{\left(y_{24}+y_{13}\right) y_{24}} - \frac{2y_{13}\left(1-y_{134}\right)}{y_{134}\left(y_{24}+y_{13}\right) y_{24}}$$

(D.42)

$$\Delta T_2 = \frac{2y_{13}}{y_{134}} - \frac{2y_{13}\, y_{134}}{\left(y_{24}+y_{13}\right)\left(y_{24}-y_{13}\right)} - \frac{4y_{13}\left(1-y_{134}\right)}{y_{134}\left(y_{24}+y_{13}\right)\left(y_{24}-y_{13}\right)}$$

(D.43)

The computation of the finite contributions originating from the terms proportional to $C_F N_c$ in B_{13} and from C_{13} is even lengthier. The most singular term in these contributions has the following form

$$BC_{13}^{S} = 48\, C_F N_c \; T(y_{134},\, y_{24}+y_{13}) \left\{ \frac{y_{14}}{y_{13}+y_{34}} - \frac{y_{12}}{y_{13}+y_{23}} \right\} \tag{D.44}$$

In the nonsingular region $y_{13} > 0$ this can easily be integrated over θ' and v :

$$\frac{1}{\pi}\int_0^{\pi} d\theta' \int_0^{5} dv \; BC_{13}^{S} = -48\, C_F N_c \; T(y_{134},\, y_{13}+y_{24}) \left\{ y_{123}\, D_5(5) - y_{134}\, D_{39}(5) \right\} \tag{D.45}$$

The integral D_{39} will be given later, D_5 was given in (D.9). We notice that the most singular term BC_{13}^{S} has no singularities for $y_{234} \to 0$, i.e. they must cancel, when the most singular parts of B_{13} and C_{13} are added. But such terms appear in the "finite" contributions. This is written in the following form:

$$\frac{BC_{13} - BC_{13}^{S}}{y_{13}} = F_{in} + \Delta_3 + \hat{K}_{41} + \hat{K}_{42} \tag{D.46}$$

$\hat{K}_{41}$ and $\hat{K}_{42}$ have been calculated already in connection with the terms of class A and B proportional to C_F^2 and were written down in (D.34) and (D.40). $\hat{K}_{41}$ and $\hat{K}_{42}$ differ from K_{41} and K_{42} by the factor $(-N_c/2C_F)$, i.e.

$$\hat{K}_{41} = -\frac{N_c}{2C_F}\, K_{41} \tag{D.47}$$

$$\hat{K}_{42} = -\frac{N_c}{2C_F}\, K_{42} \tag{D.48}$$

Δ_3 contains all the finite terms originating from C_{13}:

$$\begin{aligned}\Delta_3 = 48\, C_F N_c \Bigg\{ & \frac{v}{(y_{13}+y_{34})(y_{13}+y_{24})} \\ & + \frac{y_{12}}{(y_{13}+y_{24})(y_{24}-y_{13})} \left(\frac{1}{y_{13}+y_{34}} - \frac{1}{y_{24}+y_{34}} \right) \\ & + \frac{v(2y_{123}+1-3y_{13}-y_{24})}{(y_{13}+y_{34})\, y_{234}} + \frac{y_{134}(y_{14}-y_{134} v)}{y_{13}(y_{13}+y_{34})\, y_{234}} \\ & + \frac{y_{24}(y_{14}-v\, y_{134})}{y_{13}(y_{13}+y_{34})\, y_{134}} + \frac{v(4y_{13}-3y_{123})}{y_{134}\, y_{234}} \Bigg\}\end{aligned} \tag{D.49}$$

The separate terms in (D.49) are written in such a way that the integration over θ' and v is easily performed. The result is:

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi} d\theta' \int_0^{5} dv\, \Delta_3 = 48\, C_F N_c \Bigg\{ & \frac{1}{y_{13}+y_{24}}\, D_{40}(5) \\ & + \frac{y_{123}-y_{13}}{y_{24}^2 - y_{13}^2} \left(D_{40}(5) - D_{45}(5) \right) \\ & + \left(2y_{123} + 1 - 3y_{13} - y_{24} \right) D_{41}(5)\end{aligned}$$

$$+ \frac{y_{134}}{y_{13}} \left[y_{134} \left(D_{42}(\xi) - D_{41}(\xi) \right) - B_1(\xi) \right]$$

$$+ \frac{y_{24}}{y_{13}\, y_{134}} \left[y_{134} \left(D_{39}(\xi) - D_{40}(\xi) \right) - D_8(\xi) \right]$$

$$+ \frac{4 y_{13} - 3 y_{123}}{y_{134}} \left(B_1(\xi) - B_2(\xi) \right)$$

$$- \frac{4 y_{24}}{y_{134} (y_{134} - y_{13})} D_8(\xi) - \frac{y_{123} - y_{13}}{(y_{134} - y_{13})\, y_{134}} \Big(D_9(\xi)(2\gamma - 1)$$

$$+ D_8(\xi)(1-\gamma) \Big) + \frac{(y_{123} - y_{13})^2 \gamma (1-\gamma)}{y_{13}\, y_{24}\, y_{134}} D_9(\xi) \Bigg\} \tag{D.50}$$

Of course, there are no poles for $y_{24} = \pm y_{13}$, since the terms multiplying the poles have zeros.

The contribution F_{in}, which is

$$F_{in} = 48 C_F N_c \left\{ \Delta_3^8 - \frac{y_{14}}{y_{13} (y_{13} + y_{34})} T(y_{134}, y_{24} + y_{13}) \right.$$

$$\left. + \frac{y_{12}}{y_{13} (y_{13} + y_{23})} \left(\frac{y_{123} - 2 y_{13}}{y_{134} (y_{13} + y_{24})} + \frac{y_{134}}{y_{13} + y_{24}} \right) \right\} + \hat{K}_{61} + \hat{K}_{62} \tag{D.51}$$

In (D.51) the terms $\hat{K}_{61}$ and $\hat{K}_{62}$ come from B_{13} and were orignally in H(6,1) and H(6,2) in (C.13). The third term is the corresponding singular terms of $\hat{K}_{61}$ and $\hat{K}_{62}$, Δ_3^8 contains the singular terms in $^{C_{13}}/y_{13}$. The second term in (D.51) is the singular term which is contained in BC^S_{13}/y_{13}. Therefore F_{in} has no pole in y_{13}. The computation of F_{in} needs many steps. We write it as a sum of two terms

$$F_{in} = F_c + F_f \tag{D.52}$$

They are integrated over θ' and ϑ. The result is

$$\frac{1}{\pi} \int_0^{\pi} d\theta' \int_0^{\xi} d\vartheta\, F_c = -48 C_F N_c \left\{ \left(-2 y_c + y_c y_{134} + 2 y_c y_b + y_c^2 \right. \right.$$

$$\left. + \frac{y_c^2}{y_{134}} + \frac{y_c^3}{y_{134}} - 2 y_{134}^2 \right) D_a^1(\xi)$$

$$+ \left(y_c y_{134} + \frac{y_c^2}{y_{134}} + \frac{y_c^3}{y_{134}} \right) D_b^1(\xi)$$

$$+ \left(y_c - y_c y_b - \frac{y_c^3}{y_{134}} + y_{134}^2 \right) D_a^2(\xi) - \frac{y_c^3}{y_{134}} D_b^2(\xi)$$

$$- y_c^2 D_a^3(\xi) + \frac{2 y_c^2}{y_{134} (y_{123} - y_{13})} \left[y_{123} F(\xi) - B_1(\xi) \right]$$

$$- \frac{y_c^2}{y_{134}} \left[\frac{y_{123}^2}{(y_{123} - y_{13})^2} F(\xi) - \frac{y_{123}}{(y_{123} - y_{13})^2} B_1(\xi) - \frac{1}{y_{123} - y_{13}} \left(B_1(\xi) - B_2(\xi) \right) \right]$$

$$+ \left(-\frac{y_c y_{134}}{y_b} - \frac{y_c y_b}{y_{134}} + \frac{4 y_c}{y_b} - \frac{2 y_c^2}{y_{134} y_b} + \frac{2 y_{134}^2}{y_b} + 2 y_b \right) D_{46}(\xi)$$

$$+\left(-2y_c+\frac{y_c y_b}{y_{134}}-\frac{4y_c}{y_b}+\frac{y_c^2}{y_{134}y_b}-\frac{y_{134}^2}{y_b}-3y_b\right)$$

$$\cdot\frac{1}{y_{123}-y_{13}}\left(y_{123}D_{46}(\xi)-D_{39}(\xi)\right)+\left(\frac{y_c y_b}{y_{134}}+\frac{y_c^2}{y_{134}y_b}\right)D_{47}^b(\xi)$$

$$+\left(y_c+\frac{y_c}{y_b}+y_b\right)\left[\frac{y_{123}^2}{(y_{123}-y_{13})^2}D_{46}(\xi)-\frac{y_{123}}{(y_{123}-y_{13})^2}D_{39}(\xi)\right.$$

$$\left.-\frac{1}{y_{123}-y_{13}}D_{40}(\xi)\right]+\left(-\frac{y_c y_{134}}{y_b}-\frac{y_c y_b}{y_{134}}-\frac{2y_c^2}{y_{134}y_b}\right)D_{46}^b(\xi)$$

$$+\left(-\frac{4y_c}{y_{134}y_b}-\frac{2y_{134}}{y_b}-\frac{2y_b}{y_{134}}\right)D_5(\xi)$$

$$+\left(\frac{2y_c}{y_{134}y_b}+\frac{y_{134}}{y_b}+\frac{y_b}{y_{134}}\right)\frac{1}{y_{123}-y_{13}}$$

$$\left.\cdot\left(y_{123}D_5(\xi)-D_8(\xi)\right)\right\}$$

(D.53)

where we introduced

$$y_b = y_{13}+y_{24} \tag{D.54}$$

$$y_c = y_{123}-2y_{13} = 1-y_{134}-y_{123}-y_{13} \tag{D.55}$$

$$F_f = -48\,C_F N_c\left\{\frac{(3y_{123}-4y_{13})}{y_{134}(y_{123}-y_{13})}\left(y_{123}F(\xi)-B_1(\xi)\right)\right.$$

$$+\frac{y_{123}^2}{y_{13}\,y_{134}(y_{123}-y_{13})}\Big[y_{123}\Big(D_5(\xi)-B_1(\xi)$$

$$+(y_{13}-y_{24})F(\xi)-\frac{y_{134}}{y_{123}-y_{13}}B_1(\xi)+\frac{y_{13}y_{134}}{y_{123}-y_{13}}F(\xi)\Big)$$

$$-\left(D_8(\xi)-(y_{123}-y_{13})B_2(\xi)-y_{24}B_1(\xi)-y_{134}B_2(\xi)\right)\Big]$$

$$-\frac{y_{123}}{y_{134}(y_{123}-y_{13})}\Big[y_{123}\left(D_5(\xi)-B_1(\xi)-(y_{24}-y_{13})F(\xi)\right)$$

$$-\left(D_8(\xi)-(y_{123}-y_{13})B_2(\xi)-y_{24}B_1(\xi)\right)\Big]$$

$$+\,4\left[\frac{y_{123}}{y_{123}-y_{13}}\left(B_1(\zeta)-y_{13}\,F(\zeta)\right)-B_2(\zeta)\right]$$

$$+\,\frac{y_{123}}{y_{13}}\Big[\left(y_{134}-2y_{13}+y_{24}\right)F(\zeta)-D_5(\zeta)+B_1(\zeta)$$

$$-\,\frac{y_{134}}{y_{123}-y_{13}}\left(y_{123}\,F(\zeta)-B_1(\zeta)\right)\Big]+\frac{2y_{134}}{y_{123}-y_{13}}\left(y_{123}F(\zeta)-B_1(\zeta)\right)$$

$$+\left[\frac{y_{123}-2y_{13}}{y_{34}\,(y_{13}+y_{24})(y_{123}-y_{13})}+\frac{y_{134}}{(y_{13}+y_{24})(y_{123}-y_{13})}\right]$$

$$\left(D_8(\zeta)-y_{123}\,D_5(\zeta)\right) \qquad \text{(D.56)}$$

In these formula, namely (D.50), (D.53) and (D.56) we encounter some new integrals, which are listed below.

$$D_{39}(\zeta) = \int\frac{1}{y_{13}+y_{34}} = \mathcal{I}_1(a_4, b_4, c_4, \zeta) \qquad \text{(D.57)}$$

with

$$a_4 = (y_{134}-y_{13})^2(2\gamma-1)^2+4\,(y_{134}-y_{13})^2\gamma(1-\gamma)$$

$$b_4 = \tfrac{1}{2}\,(\alpha_4+\beta_4)^2-\tfrac{1}{2}\,(a_4+\alpha_4^2)$$

$$c_4 = \alpha_4^2$$

$$\alpha_4 = y_{13}+(y_{134}-y_{13})(1-\gamma)$$

$$\beta_4 = (y_{134}-y_{13})\,(2\gamma-1) \qquad \text{(D.58)}$$

$$D_{89}(\zeta) = \int\frac{1}{y_{13}+(y_{134}-y_{13})(1-v)}$$

$$= -\frac{1}{y_{134}-y_{13}}\ln\left(1-\zeta\,\frac{y_{134}-y_{13}}{y_{134}}\right) \qquad \text{(D.59)}$$

$$D_{40}(\zeta) = \int\frac{v}{y_{13}+y_{34}} = \mathcal{I}_1(a_4, b_4, c_4, \zeta)$$

$$-\,\mathcal{I}_2\,(a_4, b_4, c_4, \zeta) \qquad \text{(D.60)}$$

$$D_{57}(\zeta) = \int\frac{1}{y_{24}+y_{34}} = \mathcal{I}_1(a_5, b_5, c_5, \zeta) \qquad \text{(D.61)}$$

with

$$a_5 = (y_{134}-y_{13})^2$$

$$b_5 = \tfrac{1}{2}\,(\alpha_5+\beta_5)^2-\tfrac{1}{2}\,(a_5+\alpha_5^2)$$

$$c_5 = \alpha_5^2$$

$$\alpha_5 = y_{24}+(y_{134}-y_{13})(1-\gamma)$$

$$\beta_5 = (y_{134}-y_{13})\,(2\gamma-1) \qquad \text{(D.62)}$$

$$D_{45}(\zeta) = \int\frac{v}{y_{24}+y_{34}} = \mathcal{I}_1(a_5, b_5, c_5, \zeta)$$

$$-\,\mathcal{I}_2\,(a_5, b_5, c_5, \zeta) \qquad \text{(D.63)}$$

$$D_{42}(\xi) = \int \frac{1}{(y_{13}+y_{34})\,y_{234}}$$

$$= -\frac{1}{y_{123}-y_{13}}\Big\{ \mathcal{J}(a_3, b_3, c_3, \delta_4, \xi) - \mathcal{J}(a_4, b_4, c_4, \delta_4, \xi)\Big\} \qquad \text{(D.64)}$$

with

$$\delta_4 = 1 + \frac{y_{24}-y_{13}}{y_{123}-y_{13}} \qquad \text{(D.65)}$$

$$D_{41}(\xi) = \int \frac{v}{(y_{13}+y_{34})\,y_{234}} = -\frac{1}{y_{123}-y_{13}}\Big\{ \delta_4 \big[\mathcal{J}(a_3,b_3,c_3,\delta_4,\xi) - \mathcal{J}(a_4,b_4,c_4,\delta_4,\xi)\big] - (B_1(\xi) - D_{39}(\xi))\Big\} \qquad \text{(D.66)}$$

$$D_{49}(\xi) = \int \frac{v^2}{(y_{13}+y_{34})\,y_{234}}$$

$$= -\frac{1}{y_{123}-y_{13}}\Big\{ \delta_4^2\big(\mathcal{J}(a_3,b_3,c_3,\delta_4,\xi) - \mathcal{J}(a_4,b_4,c_4,\delta_4,\xi)\big) - \delta_4\big(B_1(\xi) - D_{39}(\xi)\big) - \big(B_1(\xi) - B_2(\xi) - D_{40}(\xi)\big)\Big\} \qquad \text{(D.67)}$$

$$D_{46}(\xi) = \int \frac{1}{(y_{13}+y_{34})(y_{13}+y_{23})}$$

$$= \frac{1}{y_{123}-y_{13}}\, \mathcal{J}(a_4, b_4, c_4, \delta, \xi) \qquad \text{(D.68)}$$

$$D_{46}^{b}(\xi) = \int \frac{y_{34} - y_{134}(1-v)}{y_{13}\,(y_{13}+y_{34})(y_{13}+y_{23})}$$

$$= \frac{1}{y_{13}}\Big\{ D_5(\xi) - y_{13}\, D_{46}(\xi) + \frac{y_{13}\,y_{134}}{y_{123}-y_{13}}\, D_{46}(\xi) - \frac{y_{134}}{y_{123}-y_{13}}\, D_{39}(\xi) \qquad \text{(D.69)}$$

$$D_{47}^{b}(\xi) = \int \frac{(y_{34} - y_{134}(1-v))\,v}{y_{13}\,(y_{13}+y_{34})(y_{13}+y_{23})}$$

$$= \frac{y_{123}}{y_{123}-y_{13}}\, D_{46}^{b}(\xi) - \frac{1}{(y_{123}-y_{13})\,y_{13}}\Big\{ D_8(\xi) - y_{13}\, D_{39}(\xi) - y_{134}\, D_{39}(\xi) + y_{134}\, D_{40}(\xi)\Big\} \qquad \text{(D.70)}$$

We define

$$D_a^{n}(\xi) = \int \frac{v^n}{(y_{13}+y_{34})(y_{13}+y_{23})\,y_{234}} \qquad \text{(D.71)}$$

which we need for n = 0, 1, 2, 3. These integrals are:

$$D_a^0(\xi) = \frac{1}{2y_{13}-y_{24}}\left(F(\xi) - D_{46}(\xi) + D_{42}(\xi)\right) \qquad (D.72)$$

$$D_a^1(\xi) = \frac{1}{y_{123}-y_{13}}\left(y_{123}\, D_a^0(\xi) - D_{42}(\xi)\right) \qquad (D.73)$$

$$D_a^2(\xi) = \frac{1}{y_{123}-y_{13}}\left(y_{123}\, D_a^1(\xi) - D_{41}(\xi)\right) \qquad (D.74)$$

$$D_a^3(\xi) = \frac{1}{y_{123}-y_{13}}\left(y_{123}\, D_a^2(\xi) - D_{49}(\xi)\right) \qquad (D.75)$$

Similarly we define

$$D_b^n(\xi) = \int \frac{v^n\left(y_{34} - y_{134}(1-v)\right)}{y_{13}\,(y_{13}+y_{34})(y_{13}+y_{23})\,y_{234}} \qquad (D.76)$$

which we need for n = 1.2, They are given by:

$$D_b^1(\xi) = \frac{1}{y_{13}}\left\{\frac{1}{y_{123}-y_{13}}\left(y_{123}\,F(\xi) - B_1(\xi)\right) - (y_{13}+y_{134})\,D_a^1(\xi) + y_{134}\,D_a^2(\xi)\right\} \qquad (D.77)$$

$$D_b^2(\xi) = \frac{1}{y_{13}}\left\{\frac{y_{123}^2}{(y_{123}-y_{13})^2}\,F(\xi) - \frac{y_{123}}{(y_{123}-y_{13})^2}\,B_1(\xi) \right. \qquad (D.78)$$

$$\left. - \frac{1}{y_{123}-y_{13}}\left(B_1(\xi) - B_2(\xi)\right) - (y_{13}+y_{134})\,D_a^2(\xi) + y_{134}\,D_a^3(\xi)\right\}$$

The singular term BC_{13}^S/y_{13} as given in (D.44) is not identical with the singular part used in section 3. The difference is obtained by writing:

$$\frac{BC_{13}^S}{y_{13}} = 48 C_F N_C \frac{1}{y_{13}}\, T(y_{134}, y_{13}+y_{24})\left\{\frac{y_{134}}{y_{13}+y_{34}} - \frac{y_{123}}{y_{13}+y_{23}}\right\}$$

$$= 48 C_F N_C \frac{1}{y_{13}}\left\{T(y_{134}, y_{24}) + y_{13}\,\Delta T_1\right\}$$

$$\cdot\left\{\frac{y_{134}}{y_{13}+(y_{134}-y_{13})(1-v)} + X_1 - \frac{y_{123}}{y_{13}+y_{23}}\right\} \qquad (D.79)$$

where

$$X_1 = \frac{y_{134}}{y_{13}+y_{34}} - \frac{y_{134}}{y_{13}+(y_{134}-y_{13})(1-v)} \qquad (D.80)$$

The singular term in section 3 is

$$\frac{\widehat{BC}_{13}^S}{y_{13}} = 48\, C_F N_C\, T(y_{134}, y_{24})\frac{1}{y_{13}}\left\{\frac{y_{134}}{y_{13}+(y_{134}-y_{13})(1-v)} - \frac{y_{123}}{y_{13}+y_{23}}\right\} \qquad (D.81)$$

The difference compared to (D.79) and integrated over θ' and v is an additional finite term:

$$\frac{1}{\pi}\int_0^{\pi} d\theta' \int_0^{\zeta} d\zeta \; \frac{BC_{13}^S - \hat{BC}_{13}^S}{y_{13}}$$

$$= 48\, C_F N_C \Big\{ -\Delta T_1 \,(y_{123}\, D_5(\zeta) - D_{39}(\zeta)$$

$$+ \frac{y_{134}}{y_{13}}\, T(y_{134}, y_{24}) \,\big(D_{39}(\zeta) - D_{89}(\zeta)\big)\Big\} \qquad \text{(D.82)}$$

This completes the calculation of the finite terms originating from B_{13} and C_{13}.

Now we come to the y_{34}-pole term in the class C_a. The integration is done in the 3-4 system. θ (or $v = \frac{1}{2}(1+\cos\theta)$) and θ' are the angles in this system which are integrated out. The "most singular" term which must be subtracted originates from C_{34} in (C.21). It has the following form

$$C_{34}^S = 48\, C_F N_C \left\{ \frac{y_{134}}{y_{13}+y_{34}} - \frac{1}{2} \right\} T(y_{134}, y_{234})$$

$$+ (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\, 3 \leftrightarrow 4) \qquad \text{(D.83)}$$

The difference $\Delta_4 = C_{34} - C_{34}^S$ can immediately be read off from (C.21). Δ_4 contains all the contributions induced by permutations of momenta. Now Δ_4 is integrated over θ' and v for $0 \le v \le \zeta$. Some of the integrals which we encounter are easily obtained from integrals calculated in the 1-3-system. As above we denote by $\int \equiv \frac{1}{\pi}\int_0^{\pi} d\theta' \int_0^{\zeta} dv$, then

$$D_{2a}(\zeta) = \int v^2 = \frac{1}{3}\zeta^3 \qquad \text{(D.84)}$$

The integral

$$P_{13}(\zeta) = \int \frac{1}{y_{13}+y_{34}}$$

can be obtained from the integral (D.5a) which is $D_2(\zeta)$ calculated in the 1-3-system. With the interchange $1 \leftrightarrow 4$ and then $1 \leftrightarrow 3$ the integrand of P_{13} has the form of (D.5a). Therefore P_{13} is equal to $D_2(\zeta)$ with $1 \leftrightarrow 4$ and $3 \leftrightarrow 4$ interchanged in the labels of the invariants present in $D_2(\zeta)$. To come back to the 3-4 system we need to interchange $3 \leftrightarrow 4$ which is $v \leftrightarrow (1-v)$ and $\cos\theta' \leftrightarrow -\cos\theta'$ Therefore

$$P_{13}(\zeta) = \Big[D_2(1) - D_2(1-\zeta) \Big]_{1\leftrightarrow 4} \qquad \text{(D.85)}$$

In a similar way we have obtained the following integrals which are needed for the integration of Δ_4.

$$P_{14}(\zeta) = \int \frac{1}{y_{14}+y_{34}} = \big[D_2(\zeta) \big]_{1\leftrightarrow 4} \qquad \text{(D.86)}$$

$$P_{23}(\zeta) = \int \frac{1}{y_{23}+y_{34}} = \big[D_5(\zeta) \big]_{1\leftrightarrow 4} \qquad \text{(D.87)}$$

$$P_{24}(\zeta) = \int \frac{1}{y_{24}+y_{34}} = \big[D_5(1) - D_5(1-\zeta) \big]_{1\leftrightarrow 4} \qquad \text{(D.88)}$$

$$Q_{13}(\xi) = Q_{24}(\xi) = \int \frac{1}{(y_{13}+y_{34})(y_{24}+y_{34})} = \left[D_1(1) - D_1(1-\xi)\right]_{1\leftrightarrow 4} \tag{D.89}$$

$$Q_{14}(\xi) = Q_{23}(\xi) = \int \frac{1}{(y_{14}+y_{34})(y_{23}+y_{34})} = \left[D_1(\xi)\right]_{1\leftrightarrow 4} \tag{D.90}$$

$$\begin{aligned} \mathcal{H}_{13}(\xi) &= \int \frac{y_{24}}{y_{13}+y_{34}} \\ &= \left[(y_{123}-y_{13}) \left(D_2(1) - D_2(1-\xi) \right) - D_3(1) + D_3(1-\xi) \right]_{1\leftrightarrow 4} \\ &= (y_{234}-y_{34}) \left[\mathcal{I}_2(a_1,b_1,c_1,1) - \mathcal{I}_2(a_1,b_1,c_1,1-\xi) \right]_{1\leftrightarrow 4} \end{aligned} \tag{D.91}$$

$$\mathcal{H}_{14}(\xi) = \int \frac{y_{23}}{y_{14}+y_{34}} = (y_{234}-y_{34}) \left[\mathcal{I}_2(a_1,b_1,c_1,\xi) \right]_{1\leftrightarrow 4} \tag{D.92}$$

$$\mathcal{H}_{23}(\xi) = \int \frac{y_{14}}{y_{23}+y_{34}} = \left[D_{14}(\xi) \right]_{1\leftrightarrow 4} \tag{D.93}$$

$$\mathcal{H}_{24}(\xi) = \int \frac{y_{13}}{y_{24}+y_{34}} = \left[D_{14}(1) - D_{14}(1-\xi) \right]_{1\leftrightarrow 4} \tag{D.94}$$

$$\begin{aligned} \mathcal{F}_{13}(\xi) &= \int \frac{y_{14}\,y_{23} - y_{13}\,y_{24}}{y_{34}(y_{13}+y_{34})} \\ &= \frac{y_{134}}{y_{34}} (y_{234}-y_{34}) \left[D_3(1) - D_3(1-\xi) \right]_{1\leftrightarrow 4} \\ &\quad - \frac{y_{234}-y_{34}}{y_{34}} D_8(\xi) + \mathcal{H}_{13}(\xi) \end{aligned} \tag{D.95}$$

$$\begin{aligned} \mathcal{F}_{14}(\xi) &= \int \frac{y_{13}\,y_{24} - y_{14}\,y_{23}}{y_{34}(y_{14}+y_{34})} \\ &= \frac{y_{134}}{y_{34}} (y_{234}-y_{34}) \left[D_3(\xi) \right]_{1\leftrightarrow 4} \\ &\quad - \frac{y_{234}-y_{34}}{y_{34}} D_8(\xi) + \mathcal{H}_{14}(\xi) \end{aligned} \tag{D.96}$$

$$\begin{aligned} \mathcal{F}_{23}(\xi) &= \int \frac{y_{24}\,y_{13} - y_{23}\,y_{14}}{y_{34}(y_{23}+y_{34})} \\ &= \frac{y_{234}}{y_{34}} \left[D_7(\xi) \right]_{1\leftrightarrow 4} - \frac{y_{134}-y_{34}}{y_{34}} D_8(\xi) + \mathcal{H}_{23}(\xi) \end{aligned} \tag{D.97}$$

$$\begin{aligned} \mathcal{F}_{24}(\xi) &= \int \frac{y_{23}\,y_{14} - y_{24}\,y_{13}}{y_{34}(y_{24}+y_{34})} \\ &= \frac{y_{234}}{y_{34}} \left[D_7(1) - D_7(1-\xi) \right]_{1\leftrightarrow 4} - \frac{y_{134}-y_{13}}{y_{34}} D_8(\xi) + \mathcal{H}_{24}(\xi) \end{aligned} \tag{D.98}$$

Having these integrals to our disposal we can write down the integral of Δ_4

$$\frac{1}{\pi}\int_0^\pi d\theta' \int_0^\xi dv\, \Delta_4 = 48\, C_F N_C \Big\{ \frac{y_{12}}{2}\Big(Q_{13}(\xi) - \frac{P_{13}(\xi)}{y_{234}} - \frac{P_{24}(\xi)}{y_{134}}\Big) + \frac{y_{12}\,y_{34}}{2\, y_{134}\,y_{234}} D_8(\xi)$$

$$+ \frac{2y_{34}}{y_{234}} P_{13}(\xi) - \frac{y_{34}}{2y_{134}^2} D_8(\xi) - \frac{2y_{34}}{y_{134}} D_8(\xi) - \frac{y_{34}}{y_{134}\,y_{234}} D_8(\xi)$$

$$+ \Big[\frac{1}{y_{134}\,y_{234}} - \frac{2}{y_{134}} - \frac{1}{y_{234}}\Big] y_{34}\Big(F_{13}(\xi) - D_8(\xi) - H_{13}(\xi)\Big)$$

$$- \frac{y_{34}}{y_{234}} H_{13}(\xi) - \frac{y_{34}^2}{y_{234}} P_{13}(\xi) - \frac{y_{12}\,y_{34}}{y_{234}(y_{134}-y_{34})} D_8(\xi)$$

$$- \frac{y_{12}\,y_{34}}{y_{134}\,y_{234}} D_9(\xi) + \frac{y_{12}\,y_{34}\,(y_{234}-y_{34})}{y_{134}\,y_{234}\,(y_{134}-y_{34})}$$

$$\cdot\Big(D_8(\xi)\,\hat{\gamma} + D_9(\xi)\,(1-2\hat{\gamma})\Big)$$

$$+ \frac{\hat{\gamma}(1-\hat{\gamma})\,(y_{234}-y_{34})^2}{y_{134}\,y_{234}} D_9(\xi)\Big\}$$

$$+ (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2,\ 3 \leftrightarrow 4) \qquad \text{(D.99)}$$

$\hat{\gamma}$ is now the equivalent expression to γ in the 3-4 system, namely

$$\hat{\gamma} = \frac{y_{12}\,y_{34}}{(y_{134}-y_{34})(y_{234}-y_{34})}$$

In (D.99) the interchange $1 \leftrightarrow 2$, $3 \leftrightarrow 4$ means that for example $Q_{13}(\xi)$ is replaced by $Q_{24}(\xi)$ and so on.

The most singular C^S_{34} can be integrated over v and θ' in the region $y_{34} > 0$. The result is

$$\frac{1}{\pi}\int_0^\pi d\theta' \int_0^\xi dv\, \frac{C^S_{34}}{y_{34}} = 48\, C_F N_C\ T(y_{134}, y_{234})$$

$$\cdot \frac{1}{y_{34}} \Big\{ y_{134}\, P_{13}(\xi) + y_{234}\, P_{23}(\xi) + y_{134}\, P_{14}(\xi) + y_{234}\, P_{24}(\xi) - 2\, D_8(\xi)\Big\} \qquad \text{(D.100)}$$

This formula is useful for integrating the finite contributions in the region $y_{34} \geq y$.

We now come to the $q\bar{q}q\bar{q}$ interference contribution. We define

$$H_I = H(7,3) + H(7,4) + H(8,3) + H(8,4)$$

and integrate H_I over v and θ' in the 3-4 system. The result depends on y_{34}, y_{134} and y_{234}. We write the result in such form that it can be applied to the case that v is integrated over the whole range $0 \leq v \leq 1$ or only over part of the v range $\xi \leq v \leq 1$. We obtain:

$$H_I = 48\, C_F \left(C_F - \frac{N_c}{2}\right) \Big\{ A_0 \Big[- \frac{4 y_{12}}{y_{34}\, y_{234}} - \frac{y_{12}}{y_{34}\, y_{134}}$$

$$+ \frac{2 y_{12}}{y_{134}\, y_{234}} - \frac{4 y_{12}^2}{y_{134}^2} + \frac{4 y_{34}}{y_{134}^2} - \frac{2}{y_{34}} - \frac{4 y_{234}}{y_{134}^2} - \frac{2}{y_{134}} \Big]$$

$$+ \frac{A_1}{y_{134a}} \Big[- \frac{2 y_{12}\, y_{34}}{y_{134}^2} - \frac{2 y_{12}}{y_{34}} + \frac{4 y_{12}}{y_{134}} - \frac{2 y_{12}^2}{y_{34}\, y_{234}}$$

$$+ \frac{y_{12}^2}{y_{134}\, y_{234}} - \frac{2 y_{34}}{y_{134}^2} - \frac{4 y_{34}}{y_{134}} + \frac{2 y_{34}^2}{y_{134}^2}$$

$$- \frac{y_{234}}{y_{34}} + \frac{3 y_{234}}{y_{134}} + 2 \Big]$$

$$+ A_3 \Big[- \frac{2 y_{12}}{y_{34}\, y_{134}} + \frac{2 y_{12}}{y_{134}^2} - \frac{y_{34}}{y_{134}\, y_{234}} - \frac{2 y_{34}}{y_{134}^2} - \frac{3}{y_{34}}$$

$$- \frac{2 y_{234}}{y_{34}\, y_{134}} - \frac{2 y_{134}}{y_{34}\, y_{234}} + \frac{2 y_{234}}{y_{134}^2} + \frac{3}{y_{234}} + \frac{5}{y_{134}} \Big]$$

$$+ A_4 \Big[\frac{2 y_{12}}{y_{34}\, y_{134}} - \frac{2 y_{12}}{y_{134}\, y_{234}} + \frac{2 y_{34}}{y_{134}\, y_{234}} + \frac{3}{y_{34}}$$

$$+ \frac{3 y_{234}}{y_{34}\, y_{134}} - \frac{3}{y_{234}} - \frac{5}{y_{134}} \Big] + A_{14} \frac{y_{12}}{y_{134a}} \Big[\frac{1}{y_{34}} - \frac{1}{y_{234}} \Big]$$

$$+ A_{15} \Big[- \frac{y_{34}}{y_{134}\, y_{234}} - \frac{y_{234}}{y_{34}\, y_{134a}} + \frac{y_{234}}{y_{134}\, y_{134a}} + \frac{2}{y_{134}} \Big]$$

$$+ B_1 \Big[-2 + \frac{4 y_{12}}{y_{34}} + \frac{2 y_{12}\, y_{234}}{y_{34}\, y_{134}} - \frac{2 y_{12}}{y_{134}} + \frac{6 y_{12}^2}{y_{34}\, y_{234}}$$

$$+ \frac{2 y_{12}^2}{y_{34}\, y_{134}} + \frac{2 y_{34}}{y_{134}} + \frac{y_{234}}{y_{34}} + \frac{y_{234}^2}{y_{34}\, y_{134}} - \frac{2 y_{234}}{y_{134}} \Big]$$

$$+ B_2 \Big[-2 + \frac{4 y_{12}}{y_{34}} + \frac{6 y_{12}\, y_{134}}{y_{34}\, y_{234}} - \frac{6 y_{12}}{y_{234}} - \frac{4 y_{12}}{y_{134}}$$

$$+ \frac{2 y_{234}}{y_{34}} + \frac{2 y_{134}}{y_{34}} - \frac{2 y_{234}}{y_{134}} \Big]$$

$$+ \frac{B_3}{y_{134a}} \Big[\frac{y_{12}\, y_{234}}{y_{34}} - \frac{2 y_{12}\, y_{234}}{y_{134}} - 2 y_{12} + \frac{2 y_{12}\, y_{34}}{y_{134}}$$

$$+ \frac{2 y_{12}^2}{y_{34}} - \frac{2 y_{12}^2}{y_{134}} + \frac{2 y_{12}^3}{y_{34}\, y_{134}} \Big] + B_4 \Big[-4 + \frac{2 y_{34}}{y_{234}}$$

$$+ \frac{2 y_{34}}{y_{134}} + \frac{2 y_{134}^2}{y_{34}\, y_{234}} + \frac{2 y_{134}}{y_{34}} - \frac{4 y_{134}}{y_{234}} \Big] \Big\} \qquad \text{(D.101)}$$

Here

$$y_{12} = 1 - y_{134} - y_{234} + y_{34}$$
$$y_{134a} = y_{134} - y_{34} \qquad \text{(D.102)}$$

and for in the integration over the total v region : $0 \leqslant v \leqslant 1$ the coefficients A_o, A_1, etc. are:

$$A_0 = 1 \quad , \quad A_1 = 0 \, , \quad A_3 = A_4 = \frac{1}{2}$$

$$A_{14} = 1 - 2\gamma \quad , \quad A_{15} = \frac{1}{2} + \gamma - 3\gamma^2$$

$$B_1 = \frac{1}{\sqrt{a}} \ln \frac{2\alpha + \beta + \sqrt{a}}{2\alpha + \beta - \sqrt{a}}$$

$$B_2 = \frac{a+b}{a} B_1 - \frac{\beta}{a}$$

$$B_3 = \frac{1}{\alpha+\beta} \ln \frac{2(\alpha+\beta)^2}{\alpha^2 + \alpha(\alpha+\beta) + b}$$

$$B_4 = \frac{\alpha+\beta}{2a} + \left(2 + \frac{3b}{2a}\right) B_2 - \left(1 + \frac{3b}{2a} + \frac{c}{2a}\right) B_1 \tag{D.103}$$

where

$$a = (y_{134} + y_{234})^2 - 4 y_{34}$$

$$b = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(a + \alpha^2)$$

$$\alpha = 1 - y_{34} - y_{234a}\gamma$$

$$\alpha + \beta = \frac{y_{12}\, y_{134}}{y_{134a}}$$

$$\gamma = \frac{y_{12}\, y_{34}}{y_{134a}\, y_{234a}} \quad , \quad y_{234a} = y_{234} - y_{34} \tag{D.104}$$

For the other case that we integrate over the interval $v \in [\xi, 1]$, the coefficient functions are:

$$A_0 = 1 - \xi \quad , \quad A_1 = \ln(1-\xi) \quad , \quad A_3 = \frac{1}{2}(1-\xi)^2$$

$$A_4 = (1-2\gamma)\, \frac{1}{2}(1-\xi)^2 + \gamma(1-\xi)$$

$$A_{14} = (1-2\gamma)(1-\xi) + \gamma \ln(1-\xi)$$

$$A_{15} = \frac{1}{2}(1-\xi)^2(1-2\gamma)^2 + 2\gamma(1-\gamma)(1-\xi) + \gamma^2 \ln(1-\xi) + 2\left(1-\xi - \frac{1}{2}(1-\xi)^2\right)\gamma(1-\gamma)$$

$$B_1 = \frac{1}{\sqrt{a}} \ln \frac{a + b + \sqrt{a}\sqrt{a+2b+c}}{a\xi + b + \sqrt{a}\sqrt{a\xi^2 + 2b\xi + c}}$$

$$B_2 = \frac{a+b}{a} B_1 - \frac{1}{a}\left[\sqrt{a+2b+c} - \sqrt{a\xi^2 + 2b\xi + c}\right]$$

$$B_3 = \frac{1}{\sqrt{a+2b+c}} \left[\ln \frac{2(a+2b+c)}{a\xi + b(1+\xi) + c + \sqrt{a+2b+c}\sqrt{a\xi^2+2b\xi+c}} + \ln(1-\xi) \right] \tag{D.105}$$

$$B_4 = \frac{1}{2a}\sqrt{a+2b+c} - \frac{\xi}{2a}\sqrt{a\xi^2 + 2b\xi + c} + \left(2 + \frac{3b}{2a}\right) B_2 - \left(1 + \frac{3b}{2a} + \frac{c}{2a}\right) B_1$$

Appendix F: Cross Section with 3-jet Variables y_{134} and y_{123}.

In this appendix we collect the formulae which are needed for the case that instead of $y_1 = y_{24}$, $y_2 = y_{134}$, $y_3 = y_{123}-y_{13}$ the 3-jet variables are chosen as $y_1 = y_{24}-y_{13}$, $y_2 = y_{134}$ and $y_3 = y_{123}$. Then instead of (3.10) we obtain

$$\Sigma^{(s)}_{C_F^2} = T(y_2, y_1)\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left\{\frac{1}{\varepsilon^2}\right.$$

$$-\frac{\ln y_3}{\varepsilon} + \frac{3}{2\varepsilon} - \ln^2\frac{c}{y_3} + \frac{1}{2}\ln^2 y_3$$

$$\left. -\frac{3}{2}\ln c + \frac{7}{2} - 2\,\mathcal{L}_2\left(1-\frac{c}{y_3}\right)\right\} \qquad \text{(F.1)}$$

In the limit $c \to 0$ it agrees with (3.10) as it should. The expression $\sigma_c(y_1, y_2, y)$ for the collinear region $p_2 \parallel p_3$ is also different. Instead of (3.14) we have:

$$\sigma_c^{(S_{23})}(y_1, y_2, y) = 2\,T(y_2, y_1)$$

$$\left\{ \mathcal{L}_2\left(\frac{y_3-c_1}{y_3+y}\right) - \mathcal{L}_2\left(\frac{y_3-y}{y_3+y}\right)\right.$$

$$+\mathcal{L}_2\left(1-\frac{y}{y_3}\right) - \mathcal{L}_2\left(1-\frac{c_1}{y_3}\right)$$

$$+\mathcal{L}_2\left(-\frac{y}{c_1}\right) + \frac{S_2}{2}$$

$$-\ln\left(1+\frac{y}{y_3}\right)\ln\frac{y_3-c_1}{y_3-y}$$

$$-\left(\frac{y}{y_3} - \frac{1}{4}\frac{y^2}{y_3^2}\right)\left(\ln\frac{y_3-y}{y_3-c_1} + \ln\frac{c_1}{y}\right)$$

$$\left.+\frac{1}{4}\,\frac{y^2(c_1-y)}{y_3(y_3-y)(y_3-c_1)}\right\} + \left\{c_1 \to c_2\right\} \qquad \text{(F.2)}$$

This agrees with (3.25) if appropriate changes of variables are made.

Appendix G: n-Dimensional Four Parton Matrix Elements.

In this appendix we give the n-dimensional extensions of equations (C.2-5), (C.9-12), (C.15-17), (C.22-24), (C.30-33) and (C.36-39). For the calculations presented in this paper they are not needed in full. We include them for completeness, because they have never appeared in the literature. Writing $H^{(n)}(i,j) = H(i,j) + \Delta H(i,j)$ we find

$$\Delta H(1,1) = (-2\varepsilon + \varepsilon^2)\, H(1,1) \tag{G.1}$$

$$\Delta H(2,1) = -\varepsilon\, H(2,1) + 48 C_F^2 \frac{2\varepsilon(1-\varepsilon)}{y_{13}\, y_{134}} \left(\frac{e_1}{y_{24}} - 2 y_{34} \right) \tag{G.2}$$

$$\Delta H(2,2) = (-2\varepsilon + \varepsilon^2)\, H(2,2) \tag{G.3}$$

$$\Delta H(3,2) = 48 C_F^2 \frac{\varepsilon}{y_{13}\, y_{24}\, y_{134}\, y_{234}}$$

$$\Big\{ 2 y_{24} y_{14} (y_{12} + y_{34} + y_{14}) + 2 y_{13} y_{23} (y_{12} + y_{34} + y_{23})$$

$$+ e_2 (1 + y_{12} + y_{23} + y_{14}) + e_3 (-y_{24} - y_{13})$$

$$+ \varepsilon \left[e_1 (1 + y_{23} + y_{14}) + e_2 (y_{12} + y_{34}) \right]$$

$$+ 2\varepsilon^2 \left[e_1 (y_{23} + y_{14}) + e_2 (y_{12} + y_{34}) \right] \Big\} \tag{G.4}$$

where now in contrast to (C.25) e_3 is defined as

$$e_3 = y_{24} y_{13} - y_{23} y_{14} - y_{12} y_{34} \tag{G.5}$$

$$\Delta H(4,1) = 48 C_F \left(C_F - \frac{N_c}{2} \right) \frac{1}{y_{13}\, y_{14}\, y_{23}\, y_{24}}$$

$$\Big\{ \varepsilon \left[2 e_2 (y_{12} + y_{34}) - 2 y_{12} y_{24} y_{14} - 2 y_{12} y_{23} y_{13} \right.$$

$$\left. + e_1 (y_{23} + y_{14}) + e_3 (y_{13} + y_{14}) \right]$$

$$- \varepsilon^2 e_2 (y_{34} - 2 y_{12}) - 2\varepsilon^3 e_2 (y_{34} - y_{12}) \Big\} \tag{G.6}$$

$$\Delta \mathcal{H}(4,2) = 48\, C_F \left(C_F - \frac{N_c}{2}\right) \frac{\varepsilon}{y_{13}\, y_{14}\, y_{23}\, y_{134}}$$

$$\Big\{ e_1 y_{14} + e_2 (2 y_{13} + y_{34}) + e_3 (y_{24} - 2 y_{13})$$

$$- 2 y_{12} y_{14} y_{24} + 2 y_{13} y_{23} y_{34} + 2 y_{13}^2 y_{23}$$

$$- \varepsilon (1 + 2\varepsilon) (e_3 y_{13} + e_2 y_{34}) \Big\} \qquad \text{(G.7)}$$

$$\Delta \mathcal{H}(5,2) = 96\, C_F \left(C_F - \frac{N_c}{2}\right) \frac{\varepsilon}{y_{13}\, y_{14}\, y_{134}^2}$$

$$\Big\{ y_{12} y_{13} y_{14} + y_{14} y_{24} y_{34} + y_{24} y_{14}^2 + y_{13} y_{23} y_{34}$$

$$+ y_{23} y_{13}^2 + \frac{e_2}{2} (y_{34} + 3 y_{13} + 3 y_{14})$$

$$- e_3 y_{13} - e_1 y_{14}$$

$$+ \varepsilon \Big[\frac{e_2}{2} y_{34} + \frac{e_3}{2} y_{13} + \frac{e_1}{2} y_{14} - y_{12} y_{13} y_{14} \Big] \Big\} \qquad \text{(G.8)}$$

$$\Delta \mathcal{H}(6,2) = 96\, C_F \left(C_F - \frac{N_c}{2}\right) \frac{\varepsilon}{y_{13}\, y_{23}\, y_{134}\, y_{234}}$$

$$\Big\{ - y_{12} (y_{14} y_{24} + 2 y_{13} y_{23}) + e_2 (y_{34} - y_{12})$$

$$+ \frac{e_3}{2} y_{24} + \frac{e_1}{2} y_{14}$$

$$+ \varepsilon \Big[- y_{12} y_{13} y_{23} - e_2 \Big(y_{12} + \frac{y_{34}}{2} \Big) - \frac{e_1}{2} y_{23} - \frac{e_3}{2} y_{13} \Big]$$

$$- \varepsilon^2 \Big[e_2 (y_{12} + y_{34}) + e_1 y_{23} + e_3 y_{13} \Big] \Big\} \qquad \text{(G.9)}$$

$$\Delta \mathcal{H}(7,1) = 48\, C_F N_c \frac{\varepsilon}{y_{13}\, y_{24}\, y_{34}\, y_{134}}$$

$$\Big\{ - y_{14} y_{24} y_{34} - y_{24} y_{14}^2 - y_{13} y_{23}^2 - y_{13} y_{23} y_{34}$$

$$- e_2 \Big(\frac{y_{24}}{4} + \frac{y_{34}}{2} + \frac{y_{13}}{4} + \frac{3}{4} y_{23} + \frac{3}{4} y_{14} \Big)$$

$$+ e_1 \Big(y_{14} + \frac{y_{13}}{4} + \frac{y_{23}}{4} + \frac{y_{34}}{4} \Big)$$

$$+ \varepsilon (1 + \varepsilon) (e_1 y_{14} + e_2 y_{34}) \Big\} \qquad \text{(G.10)}$$

$$\Delta \mathcal{H}(7,2) = 48\, C_F N_C \frac{\varepsilon}{y_{13}\, y_{34}\, y_{134}^2}$$

$$\Big\{ \tfrac{1}{2}\, y_{12}\, y_{13}\, y_{14} - \tfrac{1}{2}\, y_{12}\, y_{13}^2 - y_{14}\, y_{24}\, y_{34} - y_{24}\, y_{14}^2$$

$$+ \tfrac{1}{2}\, y_{13}\, y_{23}\, y_{34} - \tfrac{1}{2}\, y_{23}\, y_{13}^2 - e_2 \left(\tfrac{3}{4}\, y_{34} + y_{13} + \tfrac{5}{4}\, y_{14} \right)$$

$$+ e_1 \left(\frac{y_{34}}{4} + \frac{y_{13}}{2} + \frac{3}{4}\, y_{14} \right) - \frac{e_3}{2}\, y_{13} \Big\} \tag{G.11}$$

$$\Delta \mathcal{H}(8,2) = 48\, C_F N_C \frac{\varepsilon}{y_{13}\, y_{34}\, y_{134}\, y_{234}}$$

$$\Big\{ \tfrac{1}{2}\, y_{12}\, y_{13}\, y_{23} - y_{14}\, y_{24}\, y_{34} - y_{24}\, y_{14}^2 + \tfrac{1}{2}\, y_{23}\, y_{34}\, y_{13}$$

$$- \tfrac{1}{2}\, y_{23}\, y_{13}^2 - e_2 \left(\frac{y_{12}}{4} + \tfrac{3}{4}\, y_{34} + \tfrac{5}{4}\, y_{14} + y_{13} \right)$$

$$+ \frac{e_1}{4} \left(y_{12} + y_{24} + y_{13} - y_{23} + y_{34} \right)$$

$$+ \frac{e_3}{2}\, y_{13} \Big\} \tag{G.12}$$

$$\Delta \mathcal{H}(7,7) = -6\, C_F N_C \frac{\varepsilon}{y_{34}^2\, y_{134}^2} \Big\{ 12\, y_{12}\, y_{34}^2 - 8\, y_{12}\, y_{13}\, y_{14}$$

$$+ 4\, y_{12}\, y_{13}^2 + 4\, y_{12}\, y_{14}^2 - 8\, y_{13}\, y_{24}\, y_{34} - 16\, y_{14}\, y_{24}\, y_{34}$$

$$- 8\, y_{13}\, y_{14}\, y_{24} + 8\, y_{24}\, y_{13}^2 - 16\, y_{13}\, y_{23}\, y_{34} - 8\, y_{14}\, y_{23}\, y_{34}$$

$$- 8\, y_{13}\, y_{14}\, y_{23} + 8\, y_{23}\, y_{14}^2 \Big\} \tag{G.13}$$

$$\mathcal{H}(8,7) = -6\, C_F N_C \frac{\varepsilon}{y_{34}^2\, y_{134}\, y_{234}} \Big\{ 8\, y_{12}\, y_{13}\, y_{24}$$

$$- 8\, y_{12}\, y_{14}\, y_{24} - 8\, y_{12}\, y_{13}\, y_{23} + 8\, y_{12}\, y_{14}\, y_{23}$$

$$+ 24\, y_{12}\, y_{34}^2 - 8\, y_{13}\, y_{23}\, y_{24} - 8\, y_{14}\, y_{23}\, y_{24} - 16\, y_{13}\, y_{24}\, y_{34}$$

$$- 32\, y_{14}\, y_{24}\, y_{34} - 8\, y_{13}\, y_{14}\, y_{24} + 8\, y_{24}\, y_{13}^2 + 8\, y_{24}^2\, y_{13}$$

$$- 32\, y_{13}\, y_{23}\, y_{34} - 16\, y_{14}\, y_{23}\, y_{34} - 8\, y_{13}\, y_{14}\, y_{23}$$

$$+ 8\, y_{23}\, y_{14}^2 + 8\, y_{14}\, y_{23}^2 \Big\} \tag{G.14}$$

The n-dimensional corrections to the T_R-term are

$$\Delta(S_1+S_2+S_3+S_4) = 48\, C_F T_R\, \varepsilon$$

$$\left\{ \frac{e_2 - y_{14}y_{13} - y_{23}y_{24}}{y_{12}\, y_{124}^2} - \frac{e_2 + (y_{13}y_{14} + y_{23}y_{24})}{y_{12}\, y_{123}^2} \right.$$

$$\left. + 2\, \frac{\varepsilon^2(1+\varepsilon)\, e_3 - y_{13}y_{14} - y_{23}y_{24}}{y_{12}\, y_{123}\, y_{124}} \right\}$$

(G.15)

The n-dimensional corrections to the $q\bar{q}q\bar{q}$ interference terms are

$$\Delta H(7,3) = 96\, C_F\left(C_F - \frac{N_C}{2}\right) \frac{\varepsilon}{y_{14}\, y_{34}\, y_{134}^2}$$

$$\left\{ - y_{12}y_{13}y_{14} - y_{14}y_{24}y_{34}(1-e) - y_{13}y_{23}y_{34} \right.$$

$$- (1-\varepsilon)\frac{e_2}{2} y_{34} + (1+\varepsilon)\frac{e_3}{2} y_{13}$$

$$\left. + (1-\varepsilon)\frac{e_1}{2} y_{14} \right\}$$

(G.16)

$$\Delta H(7,4) = 96\, C_F\left(C_F - \frac{N_C}{2}\right) \frac{\varepsilon}{y_{14}\, y_{34}\, y_{124}\, y_{134}}$$

$$\left\{ - y_{12}y_{13}y_{14} + \frac{e_3}{2}\left(y_{13} + y_{24} + y_{14}\right) - \frac{e_1}{2} y_{23} + \right.$$

$$+ \varepsilon\left[y_{14}y_{24}y_{34} - \frac{e_2}{2} y_{34} - \frac{e_3}{2} y_{13} - \frac{e_1}{2} y_{14} \right]$$

$$\left. - \varepsilon^2\left(e_2 y_{34} + e_3 y_{13}\right) \right\}$$

(G.17)

$$\Delta H(8,4) = 96\, C_F\left(C_F - \frac{N_C}{2}\right) \frac{\varepsilon}{y_{14}\, y_{34}\, y_{124}\, y_{234}}$$

$$\left\{ y_{14}y_{24}y_{34}(1+\varepsilon) + \frac{e_2}{2}\left(y_{12} + y_{34}\right) - \frac{e_1}{2}\left(y_{14} + y_{23}\right) \right.$$

$$+ e_3 y_{13} + \varepsilon\left[\frac{e_1}{2} y_{14} - \frac{e_2}{2} y_{34} - e_3\left(\frac{y_{13}}{2} + y_{24}\right)\right]$$

$$\left. + \varepsilon^2\left[e_1 y_{14} - e_3\left(y_{13} + y_{24}\right) - e_2\, y_{34}\right] \right\}$$

(G.18)

Foot Notes:

(1) These subleading terms are incorporated by numerical integration in the work of /3, 7 - 10, 19/. Except in /19/ no full separation into 2-, 3- and 4-jet cross sections is attempted.

(2) This limits all attempts to give O(y) corrections to our old calculations /5/ based on the most singular terms.

(3) Our distribution for y = 0.05 is somewhat smaller than the $O(\alpha_s^2)$ thrust distribution in /7, 9, 24/ but approaches them if y is decreased.

(4) In ref. /5/ some subleading terms proportional to ylny which evolved from the singular terms were included. They are incomplete and have the effect to diminish the $O(\alpha_s^2)$ corrections further as compared to the pure singular approximation.

Figure Captions:

Fig. 1: Two-loop diagram with $q\bar{q}$ in the final state.

Fig. 2: One-loop diagrams with $q\bar{q}g$ in the final state.

Fig. 3: Diagrams with four partons, $q\bar{q}gg$ and $q\bar{q}q\bar{q}$, in the final state.

Fig. 4: Diagrams for $q\bar{q}$ and $q\bar{q}g$ production in order α_s.

Fig. 5: $q\bar{q}g$ phase space in terms of y_{13} and y_{23} with 2-jet (shaded) and 3-jet region.

Fig. 6: Four parton phase space in terms of y_{134}, y_{123} and y_{13}. Regions II, III and IV yield cross sections σ_2, σ_3, σ_4 (see text).

Fig. 7: Four parton phase space in terms of y_{134}, y_{234} and y_{34}. Regions II and V yield σ_2, III and IV yield σ_3 and σ_4, respectively.

Fig. 8: C_F^2-part of $O(\alpha_s^2)$ 3-jet distribution $A_2(T)$ as a function of T for y = 0.05 and y = 0.01.

Fig. 9: C_F^2-part of 4-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01.

Fig. 10: C_F^2-part of sum of 3- and 4-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01.

Fig. 11: C_FN_c-part of $O(\alpha_s^2)$ 3-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01.

Fig. 12: C_FN_c-part of 4-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01.

Dif. 13: C_FT_R-part of $O(\alpha_s^2)$ 3-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01 (N_f = 1).

Fig. 14: C_FT_R-part of 4-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01 (N_f = 1).

Fig. 15: 3-jet distribution $A_2(T)$ of $q\bar{q}q\bar{q}$-interference part for y = 0.05 and y = 0.01.

Fig. 16: 4-jet distribution $A_2(T)$ of $q\bar{q}q\bar{q}$-interference part for y = 0.05 and y = 0.01.

Fig. 17: Total $O(\alpha_s^2)$ 3-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01 with $N_f = 5$.

Fig. 18: Total 4-jet distribution $A_2(T)$ for y = 0.05 and y = 0.01 with $N_f = 5$.

Fig. 19: Same as fig. 8 with 3-jet variables $y_{I\ III} = y_{134}$ and $y_{I\ II} = y_{123}$.

Fig. 20: Same as fig. 17 with 3-jet variables $y_{I\ III} = y_{134}$ and $y_{I\ II} = y_{123}$.

Table Captions:

Table 1: Integrated cross sections as a function of y for C_F^2-part.

Table 2: Separation of cross sections in table 1 into physical 2-jet(σ_2), 3-jet(σ_3) and 4-jet(σ_4) cross sections.

Table 3: Integrated cross sections as a function of y for $C_F N_c$-part.

Table 4: Separation of cross sections in table 3 into 2-jet(σ_2), 3-jet(σ_3) and 4-jet(σ_4) cross sections.

Table 5: Integrated cross sections as a function of y for $C_F T_R$-part ($N_f = 1$).

Table 6: Separation of cross sections in table 5 into 2-jet(σ_2), 3-jet(σ_3) and 4-jet(σ_4) cross sections.

Table 7: Integrated cross sections as a function of y for $q\bar{q}q\bar{q}$-interference part.

Table 8: Separation of cross sections in table 7 into 2-jet(σ_2), 3-jet(σ_3) and 4-jet(σ_4) cross sections.

Table 9: Summation of all $O(\alpha_s^2)$ 2-jet, 3-jet and 4-jet pieces for C_F^2-part and y values between 0.05 and 0.001.

Table 10: Summation of all $O(\alpha_s^2)$ 2-jet, 3-jet and 4-jet pieces for $C_F N_c$-part and y values between 0.05 and 0.001.

Table 11: Summation of all $O(\alpha_s^2)$ 2-jet, 3-jet and 4-jet pieces for $C_F T_R$-part ($N_f = 1$) and y values between 0.05 and 0.001.

Table 12: Separation of cross sections in table 1 into physical 2-jet(σ_2) and 3-jet(σ_3) cross sections with $y_{I\ III} = y_{134}$ and $y_{I\ II} = y_{123}$ as 3-jet variables.

Table 13: Separation of cross sections in table 3 into 2-jet(σ_2) and 3-jet(σ_3) cross sections with $y_{I\ III} = y_{134}$ and $y_{I\ II} = y_{123}$ as 3-jet variables.

Table 1

y	$\sigma_3^s(a)$	$\sigma_3^s(b)$	σ_3^f	σ_4^s	σ_4^f	sum	$\sigma_{4\text{-jet}}$
0.05	- 104.96	- 103.10	12.43	58.24	81.77	49.34	11.21
0.04	- 168.46	- 167.08	12.42	86.83	105.79	37.96	21.61
0.02	- 535.43	- 535.20	5.95	250.31	206.40	- 72.54	98.52
0.01	- 1287.39	- 1287.78	- 12.49	588.31	347.76	- 364.20	283.70
0.005	- 2623.54	- 2624.16	- 44.44	1200.49	530.07	- 938.04	655.60
0.002	- 5697.91	- 5698.65	- 107.69	2635.18	834.65	- 2336.51	1560.00
0.001	- 9379.13	- 9379.80	- 171.31	4378.13	1107.30	- 4065.68	2737.50

Table 2

y	σ_2	σ_3	σ_4
0.05	33.92	4.21	11.21
0.04	39.92	- 23.07	21.61
0.02	50.04	- 221.10	98.52
0.01	71.96	- 719.85	283.70

Table 3

			$^1/y_{13}$		$^1/y_{34}$		QCD			
y	$\sigma_3^s(a)$	$\sigma_3^s(b)$	σ_3^f	σ_4	σ_3^f	σ_4	σ_4	σ_2	sum	$\sigma_{4\text{-jet}}$
0.05	369.22	367.46	24.22	- 38.59	47.04	23.40	- 5.58	4.33	422.28	1.48
0.04	463.02	460.43	39.07	- 57.32	65.91	33.80	- 7.69	4.06	538.26	2.69
0.02	833.09	829.09	112.32	- 159.82	165.90	91.27	- 17.93	3.10	1023.93	11.78
0.01	1335.25	1331.07	235.98	- 362.28	338.35	208.00	- 35.47	2.21	1717.86	34.22
0.005	1986.53	1982.84	421.82	- 714.19	628.88	418.86	- 62.61	1.49	2677.09	81.16
0.002	3102.33	3099.96	780.60	- 1514.66	1225.06	917.73	- 117.14	0.83	4392.38	202.42
0.001	4154.72	4153.13	1161.99	- 2465.87	1885.48	1531.01	- 175.44	0.52	6090.82	373.74

Table 4

y	σ_2	σ_3	σ_4
0.05	- 6.51	427.32	1.48
0.04	- 10.28	545.85	2.69
0.02	- 11.97	1024.12	11.78
0.01	- 24.10	1704.71	34.22

Table 5

y	$\sigma_3{}^s(a)$	$\sigma_3{}^s(b)$	σ_4	σ_2	sum	σ_{4-jet}
0.05	- 21.131	- 21.598	0.7446	0.07155	- 20.782	0.1766
0.04	- 27.316	- 27.744	1.0254	0.06428	- 26.654	0.3226
0.02	- 53.384	- 53.696	2.3902	0.04344	- 51.262	1.2864
0.01	- 91.412	- 91.624	4.7296	0.02758	- 86.867	3.2212
0.005	- 143.440	- 143.580	8.3484	0.01678	- 135.215	6.6401
0.002	- 237.180	- 237.256	15.6184	0.00833	- 221.629	13.360
0.001	- 329.612	- 329.656	23.3924	0.00477	- 306.259	21.256

Table 6

y	σ_2	σ_3	σ_4
0.05	- 0.3086	- 20.656	0.1766
0.04	- 0.2806	- 26.696	0.3226
0.02	- 0.2368	- 52.312	1.2864
0.01	- 0.1438	- 89.944	3.2212

Table 7

y	σ_3	σ_4	σ_{4-jet}
0.05	- 0.3993	- 0.1236	- 0.01199
0.04	- 0.4373	- 0.1568	- 0.02275
0.02	- 0.5356	- 0.2843	- 0.08516
0.01	- 0.6024	- 0.4423	- 0.2018
0.005	- 0.6500	- 0.6226	- 0.3496
0.002	- 0.6825	- 0.3835	- 0.6164
0.001	- 0.6989	- 1.0916	- 0.8000

Table 8

y	σ_2	σ_3	σ_4
0.05	- 0.1099	- 0.4010	- 0.01199
0.04	- 0.1032	- 0.4682	- 0.02275
0.02	- 0.07314	- 0.6616	- 0.08516
0.01	- 0.04760	- 0.7971	- 0.2018

Table 9

y	σ_2	$\sigma_3+\sigma_4$	$I(q\bar{q}q\bar{q})$	sum
0.05	- 86.22	49.34	4.18	- 32.70
0.04	- 75.45	37.96	4.75	- 32.74
0.02	36.80	- 72.54	6.56	- 29.18
0.01	332.73	- 364.20	8.37	- 23.10
0.005	912.09	- 938.04	10.18	- 15.77
0.002	2318.40	- 2336.51	12.53	- 5.58
0.001	4049.78	- 4065.68	14.32	- 1.58

Table 10

y	σ_2	$\sigma_3+\sigma_4$	$I(q\bar{q}q\bar{q})$	sum
0.05	- 259.88	422.28	- 4.71	157.69
0.04	- 380.40	538.26	- 5.35	152.51
0.02	- 875.60	1023.93	- 7.38	148.33
0.01	- 1583.80	1717.86	- 9.42	124.64
0.005	- 2545.64	2677.09	- 11.45	120.00
0.002	- 4278.8	4392.38	- 14.09	99.49
0.001	- 5996.8	6090.82	- 16.11	77.91

Table 11

y	σ_2	$\sigma_3+\sigma_4$	sum
0.05	18.00	- 20.78	- 2.78
0.04	24.14	- 26.65	- 2.51
0.02	49.42	- 51.26	- 1.84
0.01	85.52	- 86.87	- 1.35
0.005	134.20	- 135.22	- 1.02
0.002	220.88	- 221.63	- 0.75
0.001	305.63	- 306.26	- 0.63

Table 12

y	σ_2	σ_3
0.05	115.01	- 76.88
0.04	138.49	- 122.14
0.02	220.42	- 391.48
0.01	315.36	- 963.26

Table 13

y	σ_2	σ_3
0.05	1.46	419.35
0.04	- 1.23	536.80
0.02	- 6.65	1018.80
0.01	- 11.45	1692.06

References:

/1/ For reviews see: G. Wolf, Proceedings of the XIV Int. Symposium on Multiparticle Dynamics, Lake Tahoe, 1983;
Sau Lan Wu, Phys. Rep. 107 (1984) 59;
B. Adeva et al., Phys. Rep. 109 (1984) 131;
L. Criegee and G. Knies, Phys. Rep. 83 (1982) 152;
G. Kramer, Springer Tracts in Modern Physics, Vol. 102.
Berlin, Heidelberg, New York: Springer 1984

/2/ M. Dine and J. Sapirstein, Phys. Rev. Lett. 43 (1979) 668;
W. Celmaster and R.J. Gonsalves, Phys. Rev. Lett. 44 (1980) 560, Phys. Rev. D21 (1980) 3112;
K.G. Chetyrkin, A.L. Kataev and F.V. Tachov, Phys. Lett. 85B (1979) 277

/3/ R.K. Ellis, D.A. Ross and A.E. Terrano, Phys. Rev. Lett. 45 (1980) 1126, Nucl. Phys. B178 (1981) 421

/4/ K. Fabricius, I. Schmitt, G. Schierholz and G. Kramer, Phys. Lett. 97B (1980) 431, Z. Phys. C11 (1982) 315

/5/ F. Gutbrod, G. Kramer and G. Schierholz, Z. Phys. C21 (1984) 235

/6/ B. Lampe and G. Kramer, Commun. Math. Phys. 97 (1985) 257

/7/ Z. Kunszt, Phys. Lett. 99B (1981) 429, Phys. Lett. 107B (1981) 123

/8/ J.A.M. Vermaseren, K.-J.M. Gaemers and S.J. Oldham, Nucl. Phys. B187 (1981) 301

/9/ A. Ali, Phys. Lett. 110B (1982) 67

/10/ A. Ali and F. Barreiro, Phys. Lett. 118B (1982) 155, Nucl. Phys. B236 (1984) 269

/11/ S.D. Ellis, D.G. Richards and W.J. Stirling, Phys. Lett. 136B (1984) 99

/12/ D.G. Richards, W.J. Stirling and S.D. Ellis, Phys. Lett. 119B (1982) 193

/13/ H.N. Schneider, G. Kramer and G. Schierholz, Z. Phys. C22 (1984) 201

/14/ G. Sterman and S. Weinberg, Phys. Rev. Lett. 39 (1977) 1436

/15/ G. Kramer, ref. /1/

/16/ G. Kramer and B. Lampe, DESY-report 86-103

/17/ T. Sjöstrand, Comp. Phys. Comm. 28 (1983) 299, Z. Phys. C26 (1984) 93

/18/ M. Althoff et al., Z. Phys. C26 (1984) 157

/19/ R.Y. Zhu, Thesis, Massachusetts Institute of Technology, 1983

/20/ Th.D. Gottschalk, M.P. Shatz, California Institute of Technology report CALT-68-1172 (1984)

/21/ T. Gottschalk, Phys. Lett. 109B (1982) 331

/22/ A. Ali, J.G. Körner, Z. Kunszt, J. Willrodt, G. Kramer, G. Schierholz, E. Pietarinen, Nucl. Phys. B167 (1980) 454

/23/ J.G. Körner, G. Schierholz, J. Willrodt, Nucl. Phys. B185 (1981) 365

/24/ R.K. Ellis, D.A. Ross, Phys. Lett. 106B (1981) 88

/25/ G. Rudolph, Habilitationsschrift, Innsbruck, 1986 (unpublished)

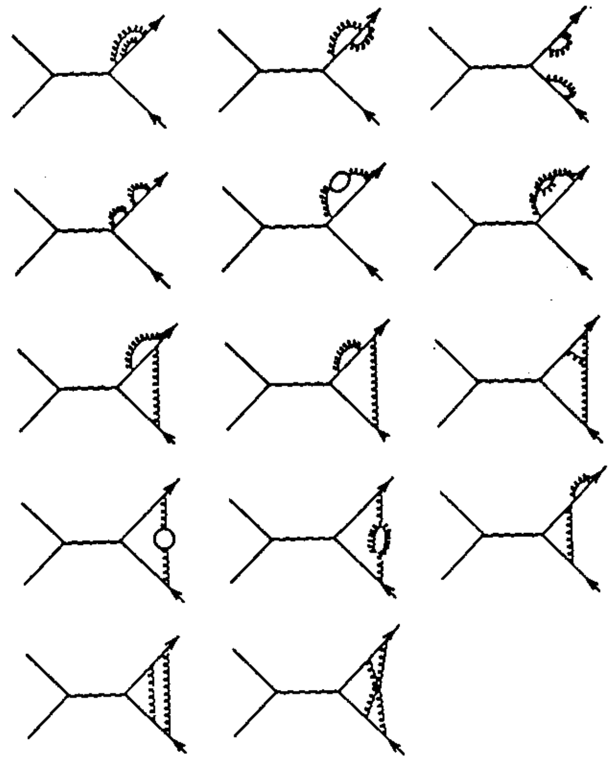


Fig. 1

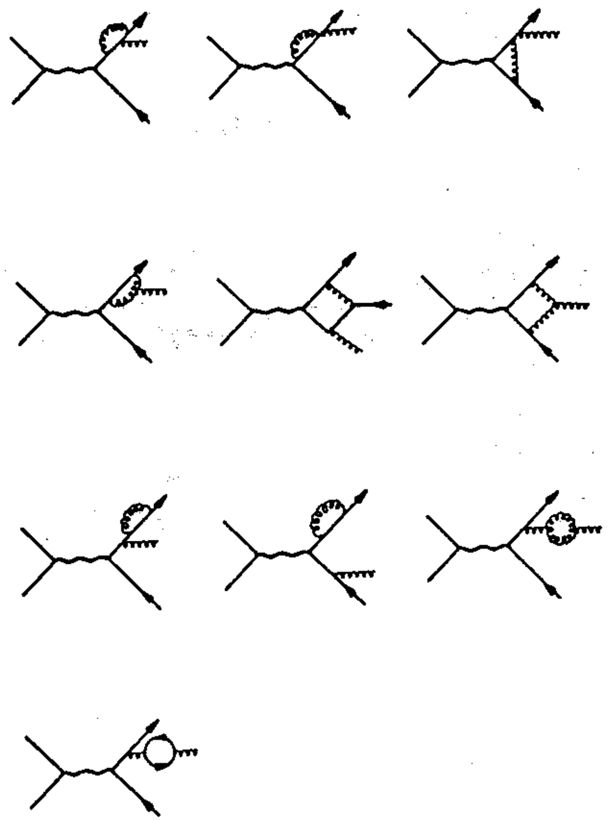


Fig. 2

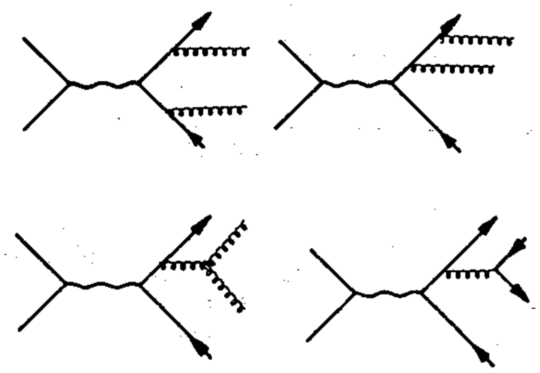


Fig. 3

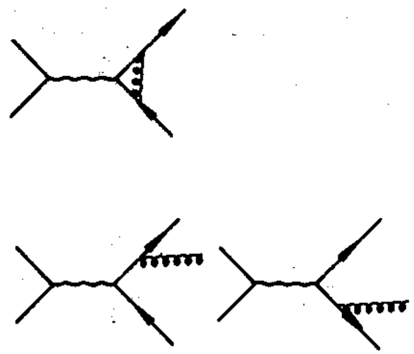


Fig. 4

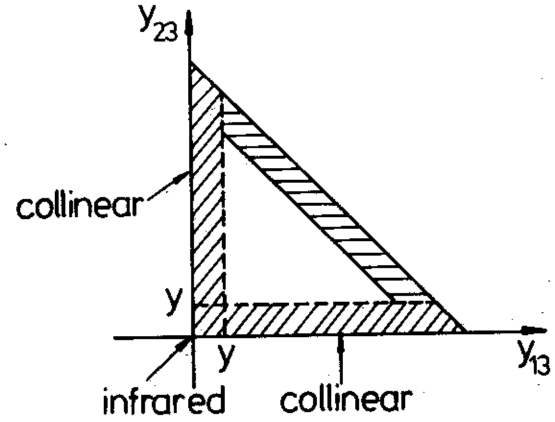


Fig. 5

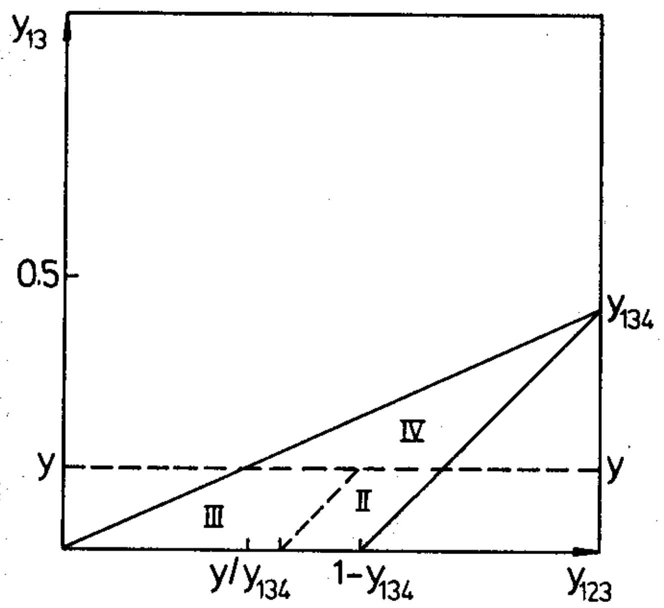
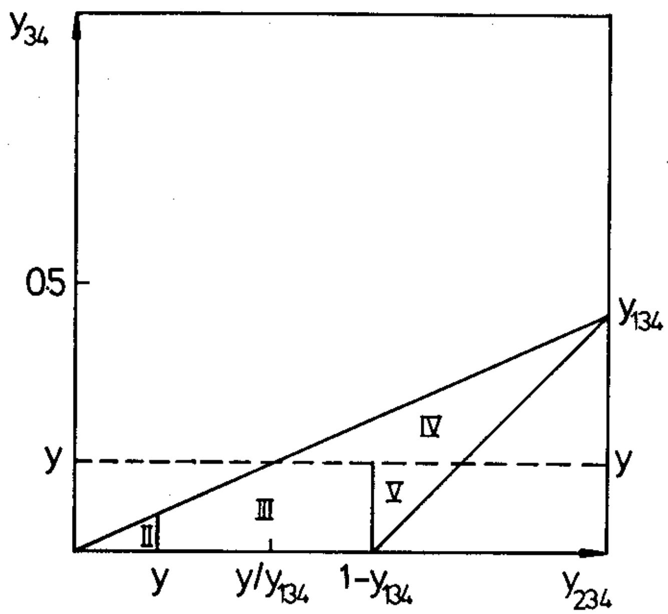

y_{13}
0.5
y_{134}
y
y
IV
II
III
y/y_{134}
$1-y_{134}$
y_{123}

Fig. 6


y_{34}
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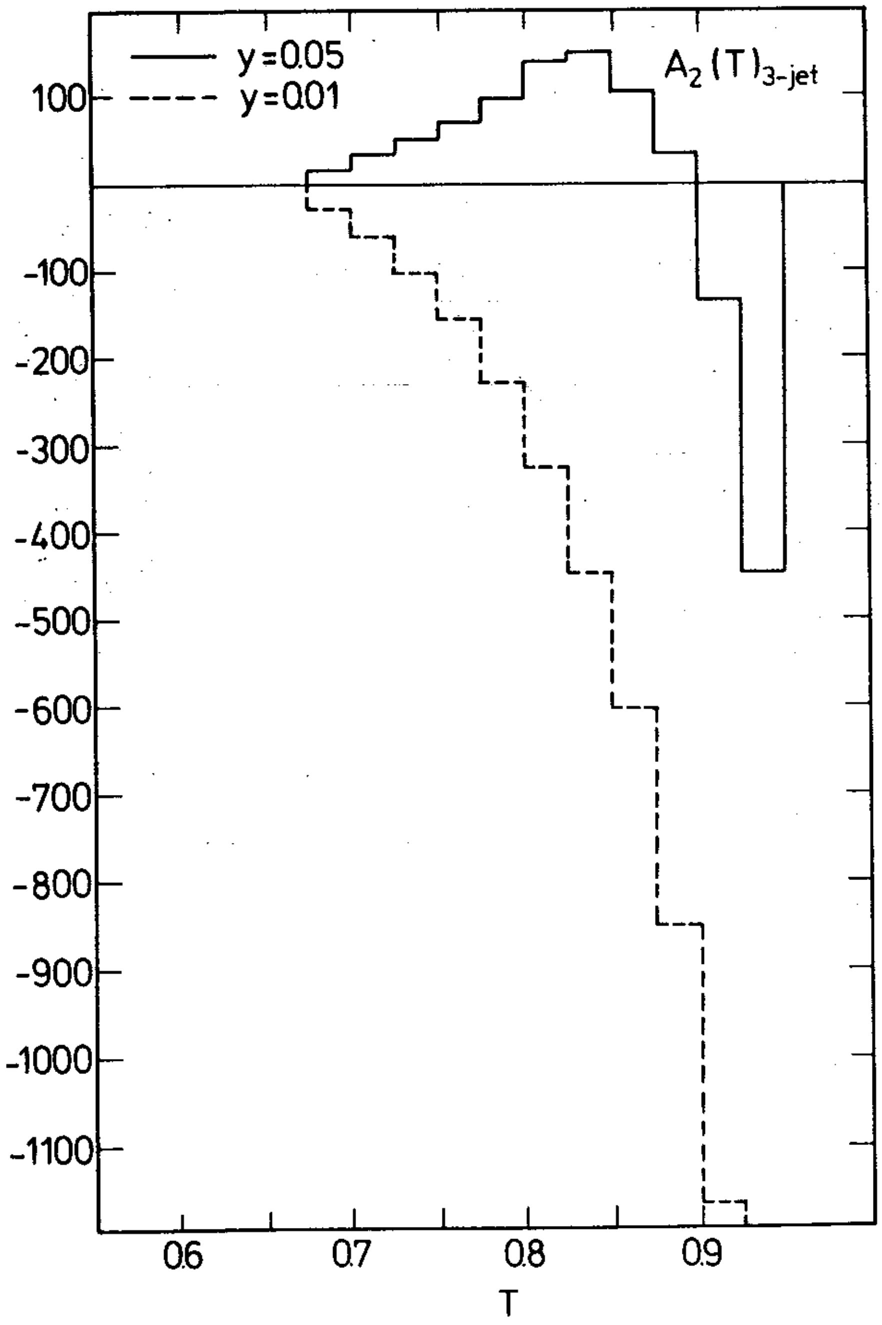
y_{134}
y
y
IV
V
III
II
y
y/y_{134}
$1-y_{134}$
y_{234}

Fig. 7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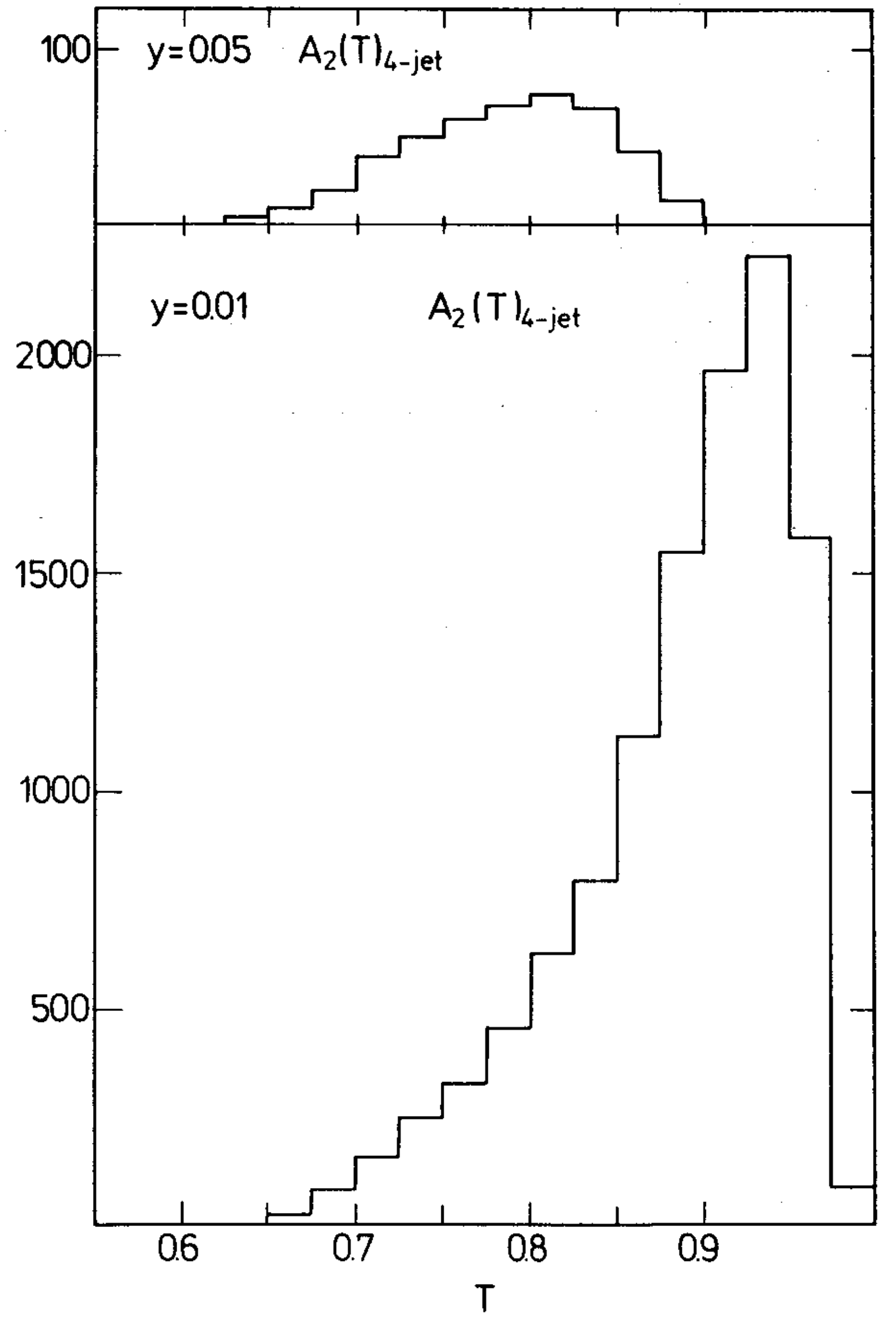
y=0.01
$A_2(T)_{3\text{-jet}}$
100
-100
-200
-300
-400
-500
-600
-700
-800
-900
-1000
-1100
0.6
0.7
0.8
0.9
T

Fig. 8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$A_2(T)_{4\text{-jet}}$
100
y=0.01
$A_2(T)_{4\text{-jet}}$
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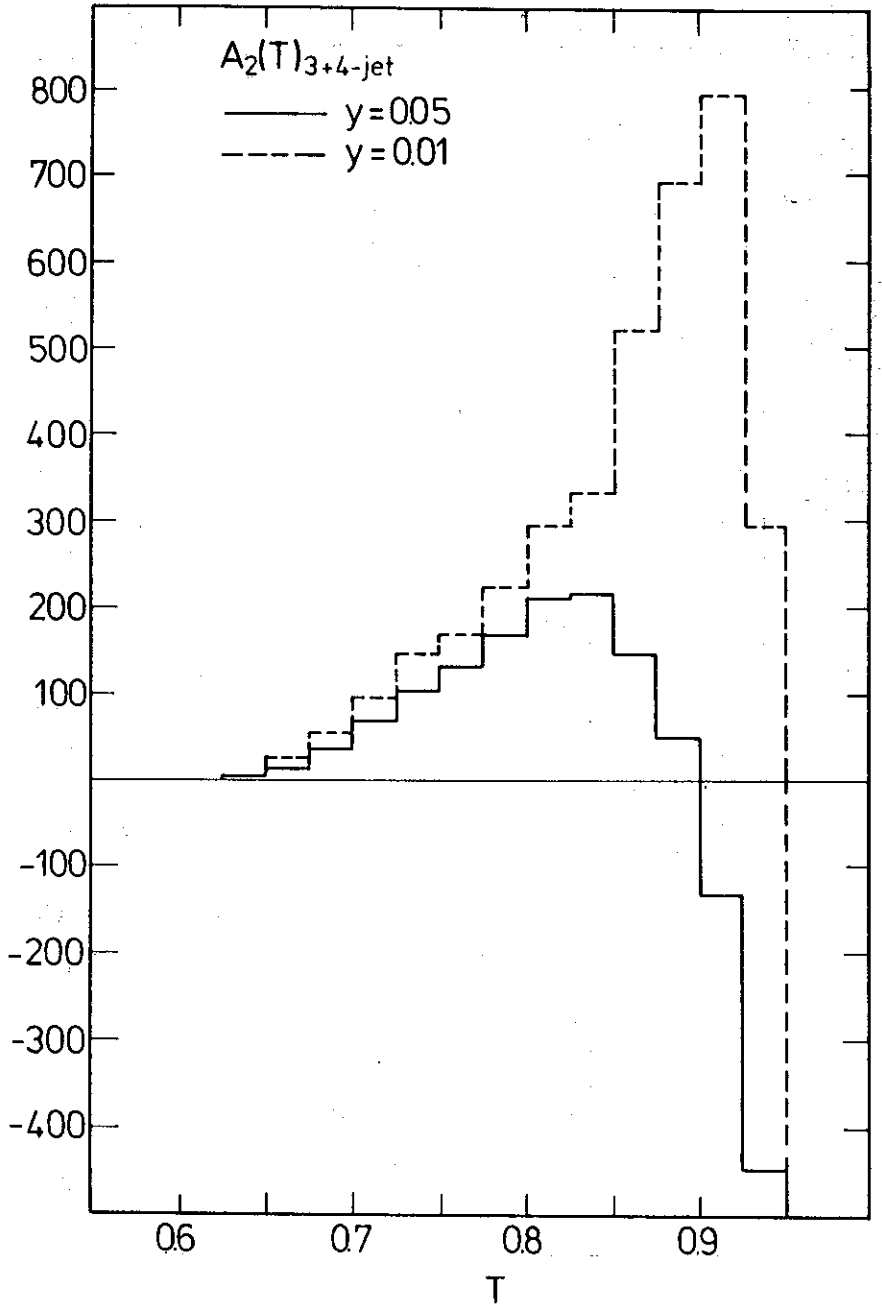
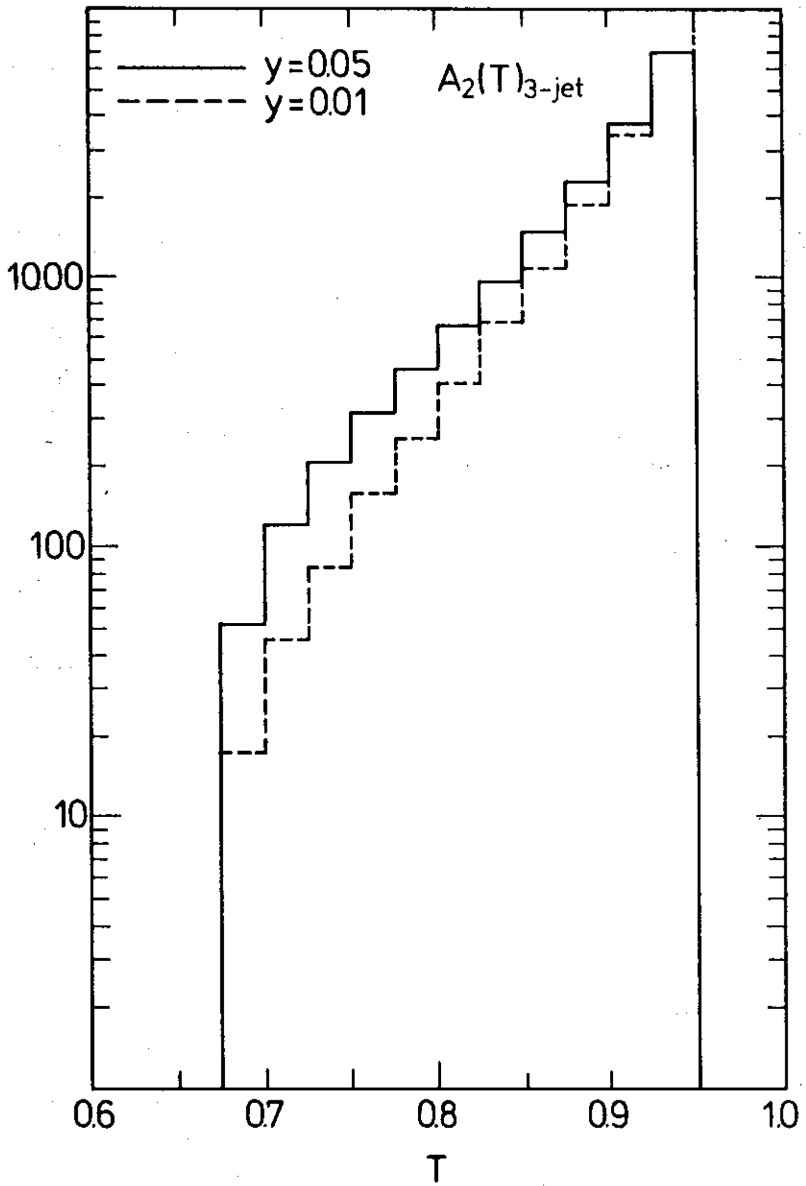
1500
1000
500
0.6
0.7
0.8
0.9
T

Fig. 9



$A_2(T)_{3+4\text{-jet}}$
y = 0.05
y = 0.01
800
700
600
500
400
300
200
100
-100
-200
-300
-400
0.6
0.7
0.8
0.9
T

Fig. 10



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y = 0.01
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100
10
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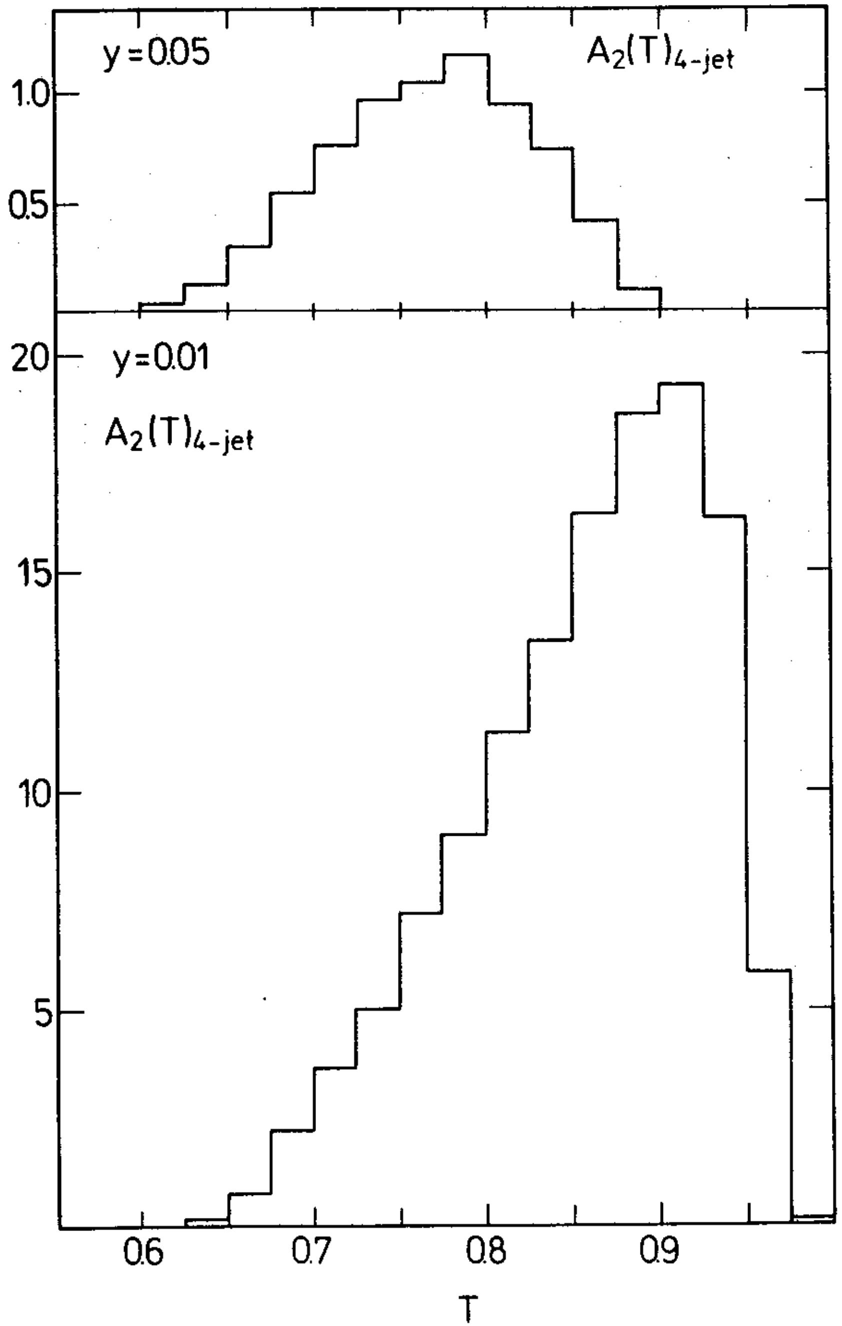
0.7
0.8
0.9
1.0
T

Fig. 11



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0.5
20
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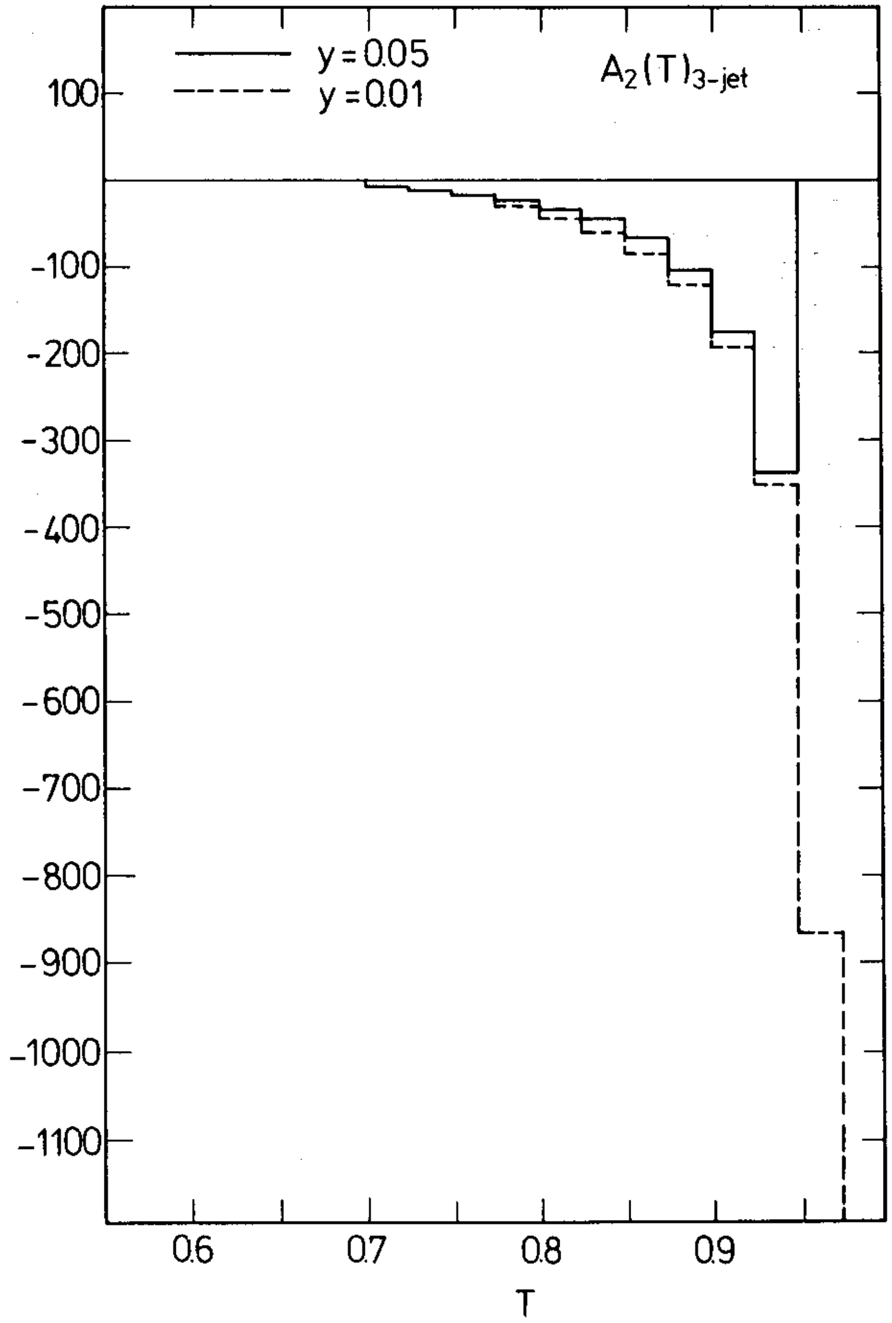
A2(T)4-jet
15
10
5
0.6
0.7
0.8
0.9
T

Fig. 12



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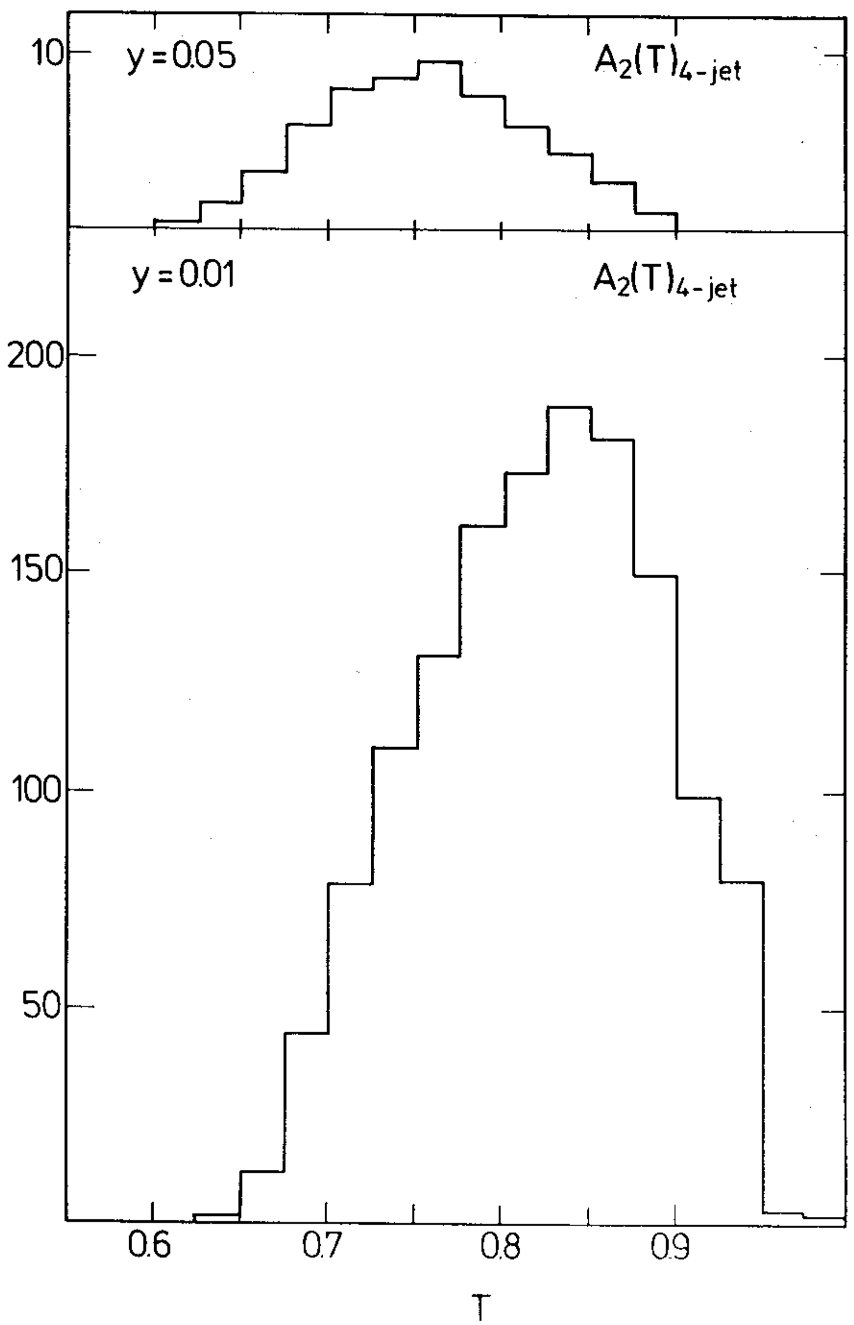
y = 0.01
A2(T)3-jet
100
-100
-200
-300
-400
-500
-600
-700
-800
-900
-1000
-1100
0.6
0.7
0.8
0.9
T

Fig. 13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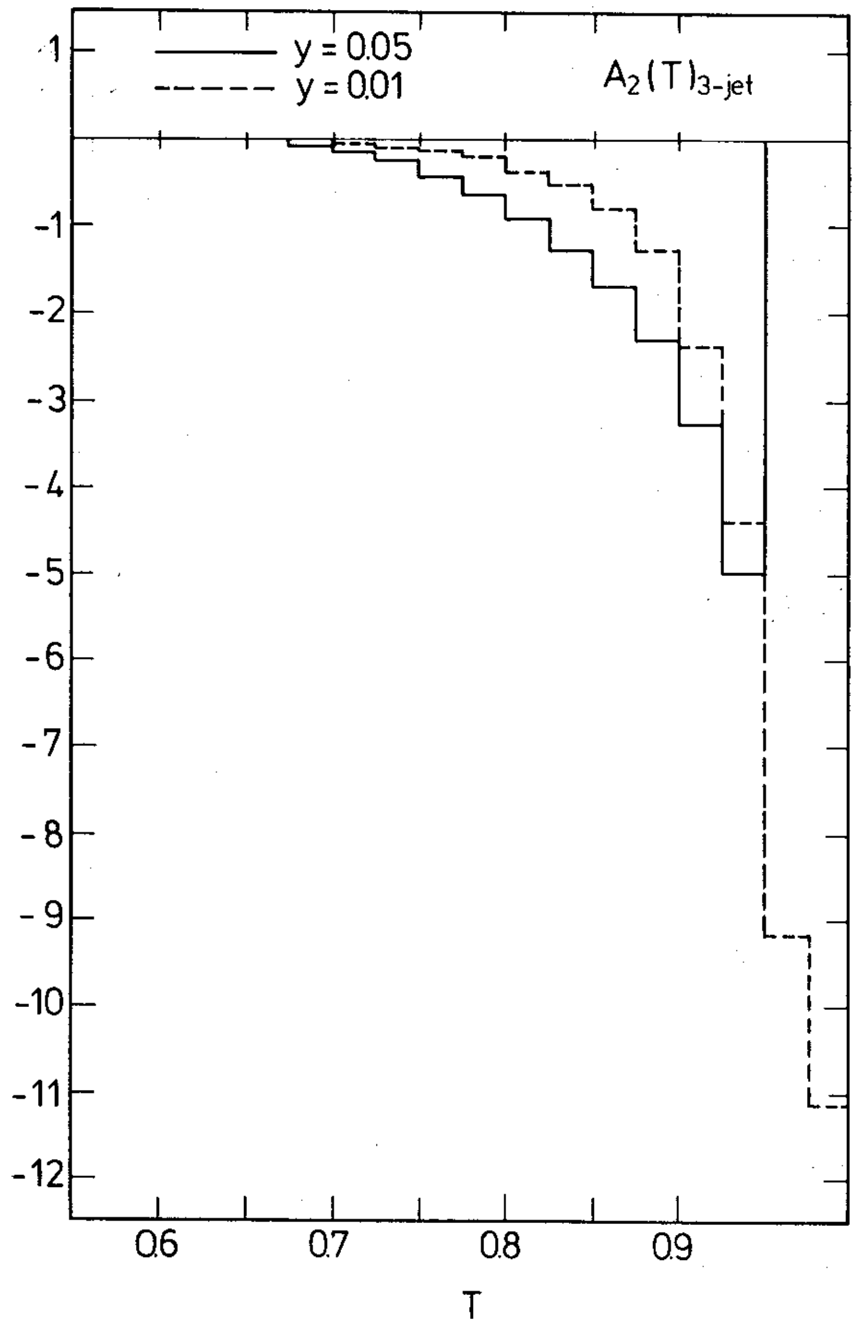
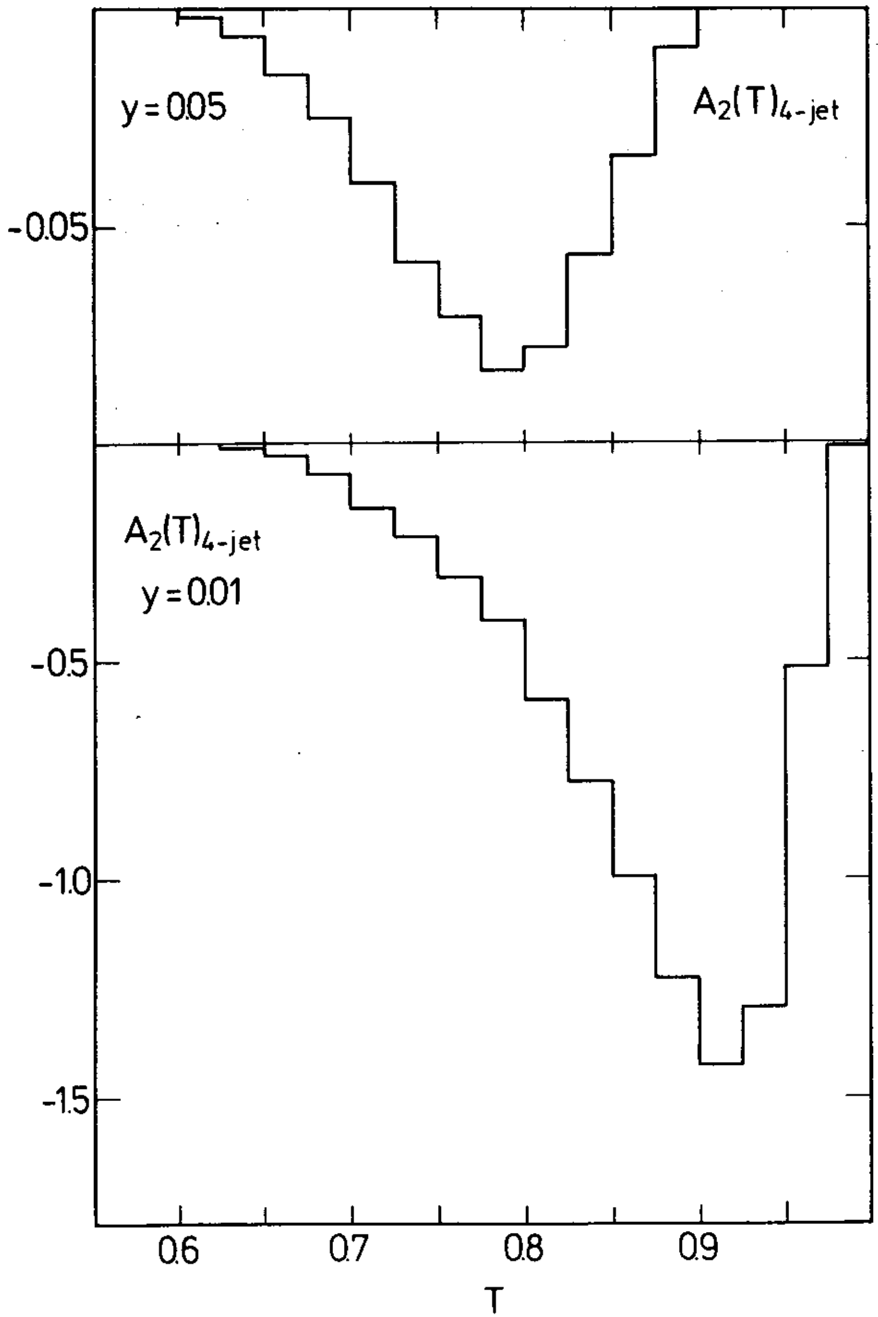
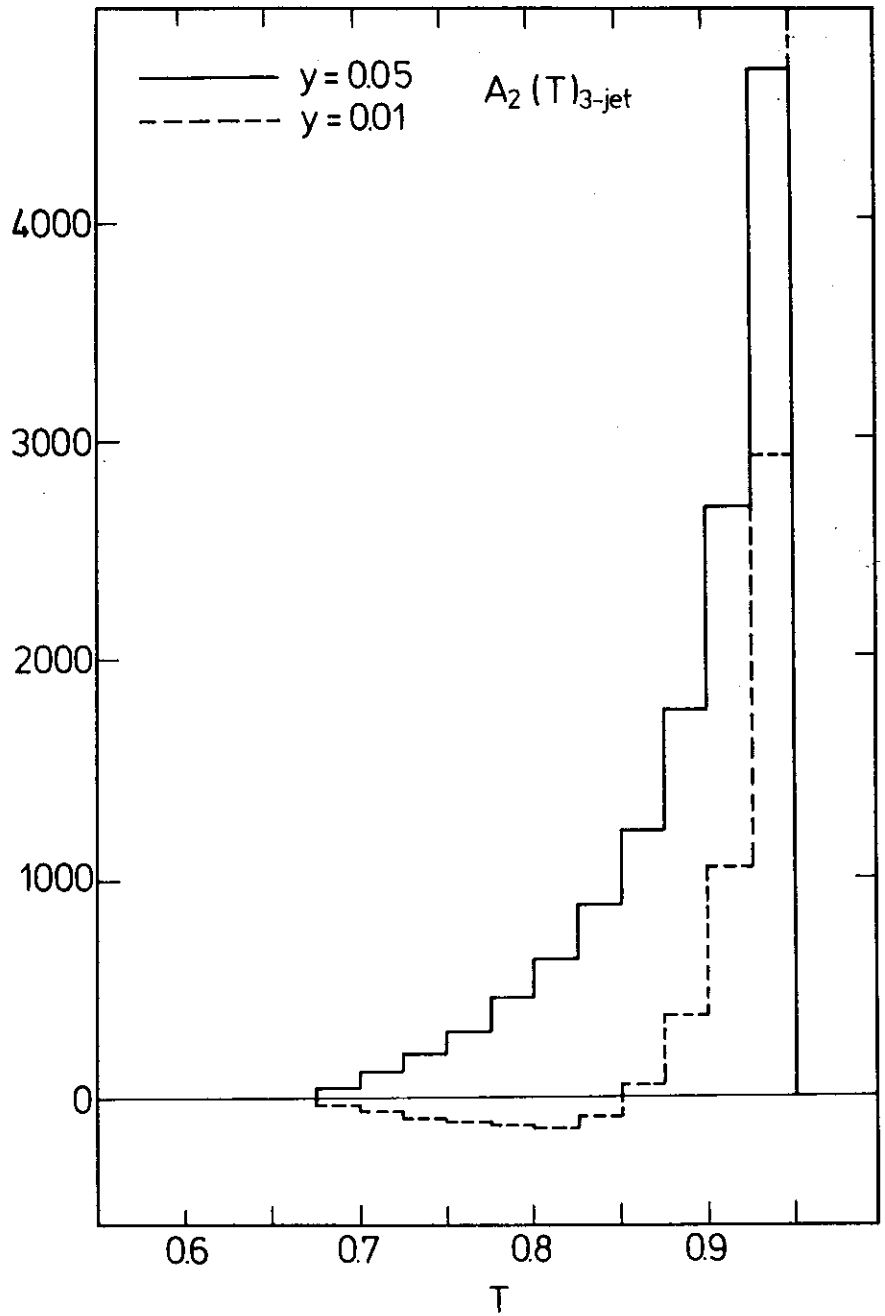
$A_2(T)_{4-jet}$
y = 0.01
$A_2(T)_{4-jet}$
200
150
100
50
0.6
0.7
0.8
0.9
T
Fig. 14


1
y = 0.05
y = 0.01
$A_2(T)_{3-jet}$
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
0.6
0.7
0.8
0.9
T
Fig. 15



y = 0.05
$A_2(T)_{4-jet}$
-0.05
$A_2(T)_{4-jet}$
y = 0.01
-0.5
-1.0
-1.5
0.6
0.7
0.8
0.9
T

Fig. 16



y = 0.05
y = 0.01
$A_2(T)_{3-jet}$
4000
3000
2000
1000
0
0.6
0.7
0.8
0.9
T

Fig. 17

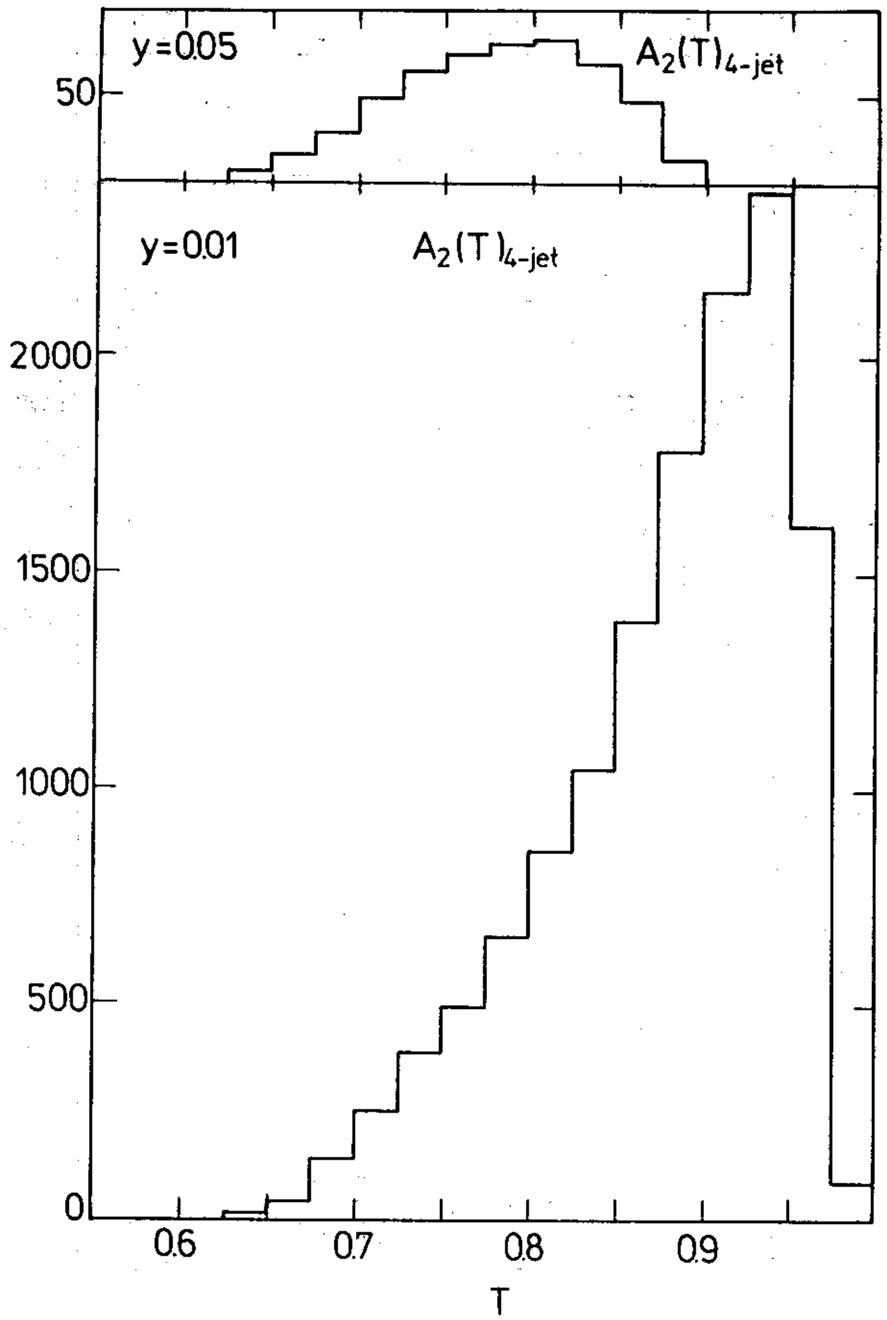

y=0.05
$A_2(T)_{4\text{-jet}}$
50
y=0.01
$A_2(T)_{4\text{-jet}}$
2000
1500
1000
500
0
0.6
0.7
0.8
0.9
T

Fig. 18

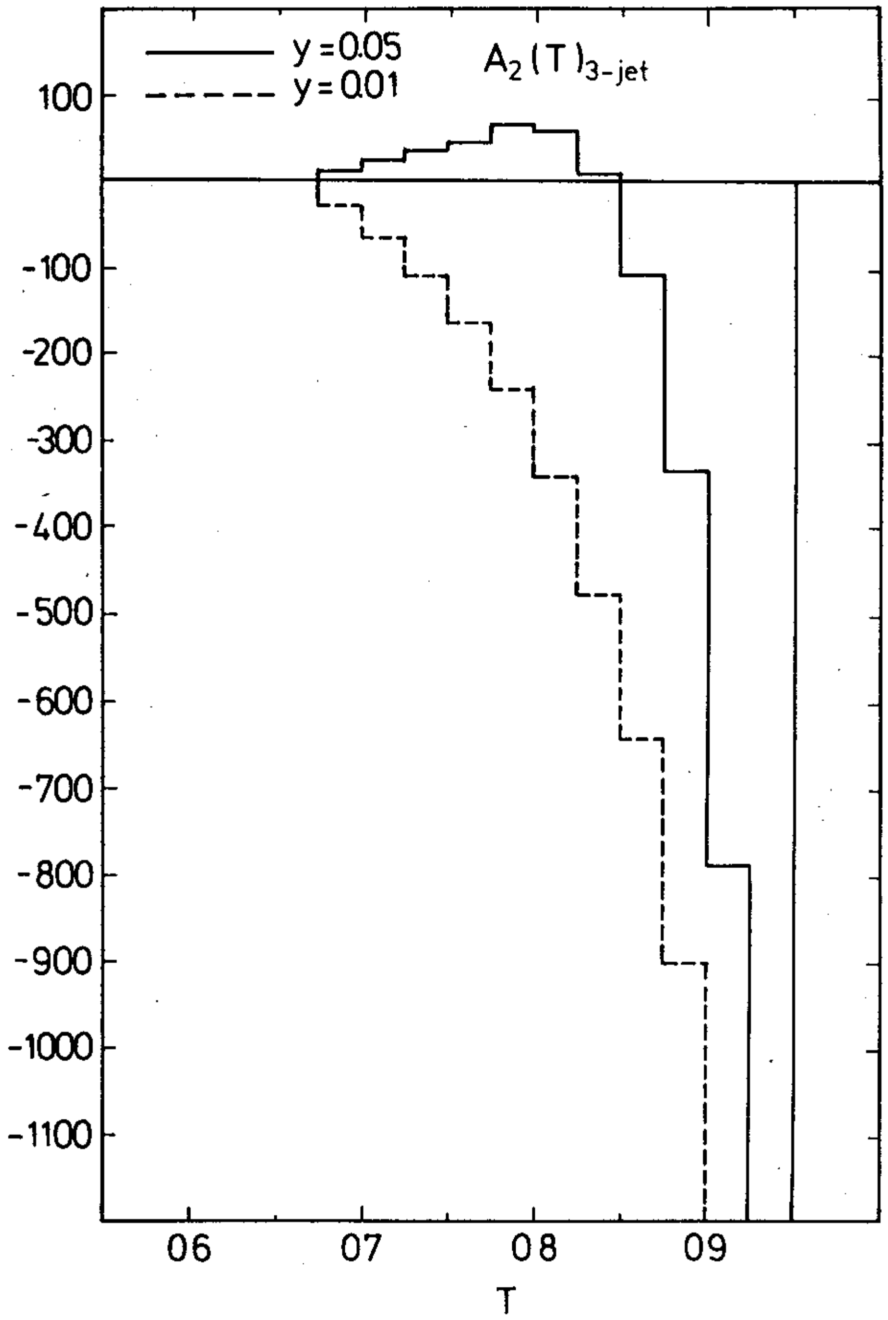

y=0.05
y=0.01
$A_2(T)_{3\text{-jet}}$
100
-100
-200
-300
-400
-500
-600
-700
-800
-900
-1000
-1100
0.6
0.7
0.8
0.9
T

Fig. 19

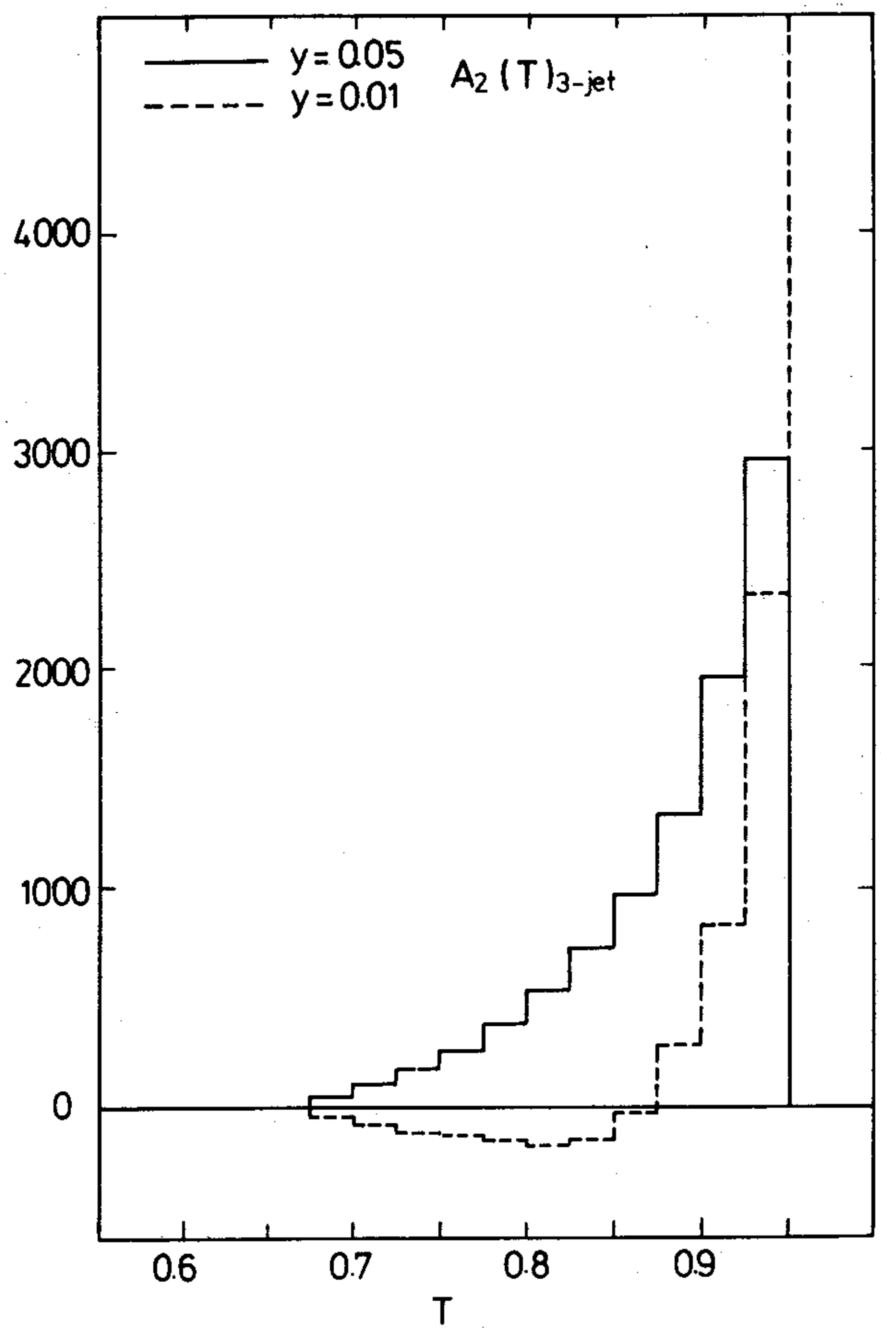


Fig. 20