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Gluon Plasma Frequency – the Next-to-Leading Order Term

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ABSTRACT

The longitudinal-electric oscillations of the hot gluon system are studied beyond the well known leading order term at high temperature T and small coupling g . The coefficient η in $\omega^2 = m^2(1 + \eta g\sqrt{N})$ is calculated, where $\omega \equiv \omega(\vec{q} = 0)$ is the long-wavelength limit of the frequency spectrum, N the number of colours and $m^2 = g^2 NT^2/9$. In the course of this, for the real part of the gluon self-energy, the Braaten-Pisarski resummation programme is found to work well in all details. The coefficient η is explicitly seen to be gauge independent within the class of covariant gauges. Infrared singularities cancel as well as collinear singularities in the two-loop diagrams with both inner momenta hard. However, as it turns out, none of these two-loop contributions reaches the relative order $O(g)$ under study. The minus sign in our numerical result $\eta = -.18$ is in accord with the intuitive picture that the studied mode might soften with increasing coupling (lower temperature) until a phase transition is reached at zero-frequency. The minus sign thus exhibits the 'glue' effect for the first time in a dynamical quantity of hot QCD.

1. Introduction

Mostly, our understanding of a complex physical problem profits from its known solution at the end of some parameter axis. In the case of QCD the large- N limit so far failed in 3+1 dimensions: the 'master field' is not known [1]. But for QCD in contact with a thermal bath there is indeed such a parameter and is called temperature. It appears that during the last years the essential problems with the high temperature limit of QCD have been overcome. The coupling g is weak there, perturbation theory is applicable, and the limiting form of several quantities (as e.g. the two-gluon Greens function at ingoing momentum $\sim gT$) can be written down explicitly in this limit. The above prospect is one of the reasons for the current high interest in hot QCD. The more immediate reasons are the relevance to heavy ion collisions and to the early universe.

Several difficulties specific to thermal gauge theory [2] long hampered the above understanding of temperature as a useful tool. Especially for the damping rate of the gluon plasma oscillations various numbers (mostly negative, hence unphysical) were produced, roughly one for each gauge used [3, 22, 4]. In contrast, the gauge independence of plasma parameters was demonstrated nonperturbatively [5], ranking them among the measurable physical quantities. It turned out that resummation is inevitable. However, even if gauge independence is restored on-shell by resummation [6], the latter may be incomplete.

The breakthrough came with a few papers of Pisarski [7, 8] and Braaten and Pisarski [9, 10] around 1990. In their basic paper (which is [9] and henceforth referred to as BP) it was shown that, in order to obtain soft amplitudes consistently, hard thermal loops must be added to the tree-level vertices and summed up in the gluon propagator. A momentum $\sim T$ is 'hard', but if $\sim gT$ it is 'soft' ($0 < g \ll 1$). Gauge independence of this setup was proved [9, 11] and the (gauge independent and positive) gluon damping rate was obtained [12]. The development culminated in giving this new "true zeroth order" the form of a Lagrangian [13, 14], which generates the leading terms of all soft amplitudes and can be even rewritten in a manifestly gauge-invariant form [15]. Applications cover the soft dilepton production [16], quark damping [17], screening [18], energy loss [19], kinetic equations [20] or even star matter [21]. There are current questions concerning the existence of a 'magnetic mass' [22, 23], the measurability of the damping [24] and the regulator, which prevents revived gauge dependence of the damping. But through the present work we were not forced into the former problems. With regard to the latter, all gauge dependences are cancelled algebraically on the plasmon mass-shell. We assume that potential mass-shell singularities [25] are regularized in the manner of ref. [26].

The system considered in this paper consists of only gluons in thermal equilibrium (no quarks). We concentrate on the real part of the frequency of the plasmon mode and take the first step beyond 'zeroth order'. There are three possible origins of contributions to the relative order $O(g)$ ('relative' means up to the prefactor $m^2 = g^2 T^2 N/9$). These origins form the section headings of §§ 3, 4 and 5. Their classification is due to BP. Thus, the best introduction to the present paper is the subsection 4.3 in BP. We shall not summarize this paragraph here. The predictions of BP concern the possible maximum contribution to each subset. Explicit calculation may well give something below $O(g)$ (a) for kinematical reasons, (b) by 'accidental' cancellation of prefactors and (c) by compensation among ranges of the integrals. BP give an example for case (a) in discussing the imaginary part of hard one- and two-loop diagrams. Sections 3 and 4 give examples for case (c).

As we restrict ourselves to the long-wavelength limit $\vec{q} \rightarrow 0$ there remains one single number to be calculated. This number is the prefactor η in (1.1) below. It is related to the real part of that quantity $\gamma = \gamma_r + i\gamma_i$ whose imaginary part is the damping rate:

$$\omega^2 = m^2 \left(1 + \eta g \sqrt{N}\right) \equiv m^2 - 2m\gamma_r \quad , \quad \gamma = a \frac{g^2 N T}{24\pi} \quad \Rightarrow \quad a_r = -4\pi\eta \quad . \quad (1.1)$$

Hence, our result $\eta = -.18$ (already disclosed in the abstract) means $a_r = 2.3$. The real part is thus somewhat smaller than the imaginary part $a_i = 6.635$ [12]. The second

digit in $-.18$ is not quite certain. The work leading to this number is laborious. Our motivations were:

- (i) Is QCD physically simple? If so, our expectation on the behaviour of e.g. the gluon system should be confirmed immediately in the first term of the 'true' perturbation expansion. We expect that, with increasing coupling, the 'glue' reduces the frequency of the plasmon mode below its zeroth-order value m . Moreover, this frequency could play the role of an indicator, reaching zero at the onset of glue ball formation. In figure 1, an increasing coupling might be associated with decreasing temperature [27]. Remember that often (especially in asymptotic series) the first term of a perturbation expansion gives qualitatively the full answer.
- (ii) The high-temperature limit as a perturbative starting point needs examples. We should like to give one more. The first example was given already in 1979 as Kapusta [28] calculated the pressure $p \sim T^4 (1 - 5g^2 N/16\pi^2)$. Note the minus sign.
- (iii) BP at work. While filling §4.3 of BP with detail, we will test the resummation programme independently. The test concerns the separate gauge independent sets within $O(g)$, but also the absence of infrared singularities, UV-convergence and (last not least) the physics, which here is in the minus sign in question.
- (iv) Working with the BP resummation we shall reformulate it in our Minkowski notation. This is a matter of language only, we do not claim for preferences. BP use 'English', say, and here is the translation into 'Dutch' [29].

The paper is organized as follows. It starts with the details of the 'zeroth approximation' (section 2). In the next three sections the $O(g)$ contributions are calculated. We follow the BP classification in reverse order. The two-loop diagrams (section 3) could be suspected to be outside of the realm of feasibility. But they are not. Both, the two loop diagrams and the 1-loop hard diagrams (section 4) do not (yet) contribute to $O(g)$. The main part is section 5 on the one-loop soft diagrams. The formal result of the soft analysis is summarized in section 6. Here we go until to the end of the analytical treatment. In section 7 the figure 3 gives a rough view into the numerical procedure.

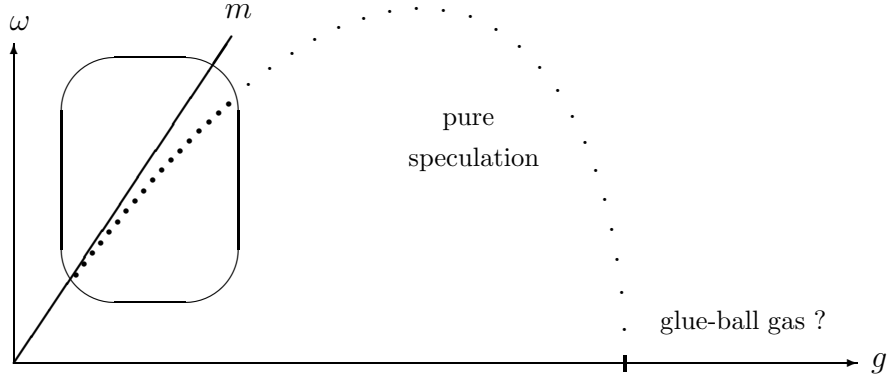


Figure 1: Longitudinal plasma frequency versus coupling (schematically). The straight line corresponds to the leading order $\omega = m$. The curve inside the window is the subject of this paper. By the dots outside, the function $g\sqrt{1-g}$ is formally followed up to stimulate speculations.

2. The frequency of the plasmon mode

In this short section we specify the subject and introduce notations. For simplicity, we allow for only gluons, activated thermally and of $N^2 - 1$ kinds. To get rid of quarks in a physical manner, they would have to be given masses much larger than the temperature. Then, the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 + \text{ghost term} \quad . \quad (2.1)$$

We use the Matsubara contour and Minkowski metric $+- --$ [29]. Hence a four vector reads $P = (i\omega_n, \vec{p})$ with $\omega_n = 2\pi nT$, $P^2 = (i\omega_n)^2 - p^2$. Let Q be the argument of the polarization function (its 'outer momentum'). We shall keep writing Q_0 even if it is already continued into the complex plane. By now the term ' Q soft' applies to Q_0 as well.

Although the terms 'gluon self energy' and 'polarization function' have identical meaning, we prefer the latter to emphasize the view of a medium having dielectric properties. The longitudinal plasmon mode (which lives on degrees of freedom not activated at zero temperature [30]) is detected as a zero of the dielectric constant or, equivalently, as a pole of the longitudinal part (index ℓ) of the gluon propagator. To 'zeroth order', i.e. when dressed with hard thermal loops, and within covariant gauges this propagator reads

$$G_{\mu\nu}(P) = A_{\mu\nu}(P)\Delta_t(P) + B_{\mu\nu}(P)\Delta_\ell(P) + D_{\mu\nu}(P)\Delta_\alpha \quad (2.2)$$

$$\text{where} \quad \Delta_\alpha = \alpha\Delta_0 \quad , \quad \Delta_0 = \frac{1}{P^2} \quad , \quad \Delta_{t,\ell} = \frac{1}{P^2 - \Pi_{t,\ell}(P)} \quad . \quad (2.3)$$

We emphasize that the above object (at soft momentum P) is much more than a certain perturbative outcome with uncertain meaning. It is, as BP have demonstrated, the exact asymptotically leading term of the propagator in the limit of high temperature ($gT \ll T$). The ghost propagator is Δ_0 and remains undressed to zeroth order. The Lorentz-matrices in (2.2) belong to the matrix-basis [22, 3, 29, 31]:

$$A = g - B - D \quad , \quad B = \frac{V \circ V}{V^2} \quad , \quad C = \frac{Q \circ V + V \circ Q}{\sqrt{2} Q^2 q} \quad , \quad D = \frac{Q \circ Q}{Q^2} \quad (2.4)$$

$$\text{with } V = Q^2 U - (U \cdot Q) Q = (-q^2, -Q_0 \vec{q}) \quad , \quad (2.5)$$

where $U = (1, \vec{0})$ is the four-velocity of the thermal bath at rest. The form (2.2) derives from $G = G^0 + G^0 \Pi G$ by using (2.4) and the polarization function $\Pi^{\mu\nu}$ at one-loop order (leading term as given in (2.8) below).

At this point it is clear how the position of the pole in the B -term of the propagator is obtained to any desired higher order in the coupling: consider the corresponding 1PI diagrams, which form $\Pi^{\mu\nu}$, but formulate them with dressed lines (2.2) and dressed vertices (see below) and consider counter terms, see §4.3 of BP, which here, however, are not yet needed. Also n -vertices with $n > 4$ do not yet occur. Once $\Pi^{\mu\nu}$ is obtained that way, one forms $\Pi_\ell = \text{Tr } B \Pi$. If one is interested in the limit $\vec{q} \rightarrow 0$ only, one obtains $\omega \equiv \omega(\vec{q} = 0)$ as $\omega = \Re \Omega$ by solving

$$\Omega^2 = \Pi_\ell(\Omega, \vec{q} = 0) \quad (2.6)$$

for the complex number Ω . At next-to-leading order in g this equation reduces to $\omega^2 = \Re \Pi_\ell(m, 0)$ with $m = g\sqrt{N}T/3$. On dimensional grounds ω is m times some function f of only g . Thus, the only explicit temperature dependence of ω is the trivial one in the prefactor m ; the other T -dependence is implicit in the running of the coupling g . At small g the function $f(g)$ need not be a pure power series in g . Possibly the asymptotics of f looks as follows,

$$\omega^2 = m^2 \left(1 + \eta g\sqrt{N} + \bar{\eta} g^2 \ln(g) + \bar{\bar{\eta}} g^2 + \dots \right) \quad , \quad (2.7)$$

because such logarithmic terms will appear in section 4. Since ω is a measurable quantity, each term of its asymptotics must be gauge independent. If not, the calculation is wrong.

We continue listing further details on the 'zeroth order'. The leading term [10, 14] of $\Pi^{\mu\nu}$ is

$$\Pi^{\mu\nu}(Q) = 3m^2 (U^\mu U^\nu - \langle (U \cdot Q) Y^\mu Y^\nu / (Y \cdot Q) \rangle) \quad (2.8)$$

$$= \frac{3}{2}m^2 g^{\mu\nu} + 4g^2 N \sum \frac{K^\mu K^\nu}{K^2(K-Q)^2} , \quad (2.9)$$

where $Y \equiv (1, \vec{e})$, $Y^2 = 0$. $\langle \dots \rangle$ is the average over the directions of the unit vector \vec{e} , and the blank summation symbol means

$$\sum_K \equiv \int_K \sum_{K_0} = \int_K T \sum_n = \left(\frac{1}{2\pi}\right)^3 \int d^3k T \sum_n . \quad (2.10)$$

We write $n(k) = 1/(e^{\beta k} - 1)$ for the Bose function and $q^* = T\sqrt{g}$ for the threshold between hard and soft momenta [18]. Since the functions $\Pi_t = \text{Tr } A\Pi/2$ and $\Pi_\ell = \text{Tr } B\Pi$ in (2.3) are related by

$$\Pi_\ell(Q) + 2\Pi_t(Q) = 3m^2 \quad (2.11)$$

(use (2.8) together with (2.4)), we have to record only

$$\Pi_\ell(Q) = 4g^2 N \sum \frac{1}{K^2(K-Q)^2} \left[k^2 - \frac{(\vec{k} \cdot \vec{q})^2}{q^2} \right] . \quad (2.12)$$

For this sum evaluated see Appendix B. There also the spectral densities of the propagators Δ_t and Δ_ℓ are detailed. For the definition of spectral densities and the general spectral representation see (5.5) below. Often differences of two propagators do occur:

$$\Delta_{\ell t} \equiv \Delta_\ell - \Delta_t \quad , \quad \Delta_{\alpha\ell} \equiv \alpha\Delta_0 - \Delta_\ell . \quad (2.13)$$

Last not least, if the outer momenta are soft (as in section 5), the 3- and 4-vertices [9] are to be dressed by one hard thermal loop each. After the colour sums are done, the remaining parts of the vertices read as follows (cf. e.g. (3.2) and (3.28) in BP, the different sign is due to notation):

$$\Gamma^{123} = (Q_1|Q_2|Q_3)^{123} + \delta\Gamma^{123} , \quad (2.14)$$

$$\begin{aligned} \text{where} \quad \delta\Gamma^{123} &= -8g^2 N \sum_K \frac{K^1 K^2 K^3}{K^2(K+Q_1)^2(K+Q_1+Q_2)^2} \\ \text{and} \quad (Q_1|Q_2|Q_3)^{123} &= (Q_1 - Q_2)^3 g^{12} + \text{cyclic} ; \end{aligned} \quad (2.15)$$

$$\Gamma^{1234} = g^{14} g^{23} + g^{13} g^{24} - 2g^{12} g^{34} + \delta\Gamma^{1234} , \quad (2.16)$$

$$\begin{aligned} \text{where} \quad \delta\Gamma^{1234} &= 16g^2 N \sum_K K^1 K^2 K^3 K^4 \left(\frac{2}{N_{1234}} + \frac{2}{N_{2134}} + \frac{1}{N_{4231}} \right) \\ \text{and} \quad N_{1234} &= K^2(K+Q_1)^2(K+Q_1+Q_2)^2(K-Q_4)^2 . \end{aligned}$$

In both expressions, (2.14) and (2.16), the sum of the Q_i must vanish. The dressed-vertex Ward identities, cf. (3.31) and (3.33) of BP, are

$$(Q_3)_3 \delta \Gamma^{123} = \Pi^{12}(Q_1) - \Pi^{12}(Q_2) \quad (2.17)$$

$$(Q_4)_4 \delta \Gamma^{1234} = -\delta \Gamma^{123}(Q_1 + Q_4, Q_2, Q_3) + \delta \Gamma^{123}(Q_1, Q_2 + Q_4, Q_3) . \quad (2.18)$$

This completes our listing of known 'zeroth order' results as we shall need them.

The plasmon mode is a 'longitudinal-electric' wave. To appreciate this term (used in the abstract) note that $V^\mu A_\mu^a(Q) = -i\vec{q} \cdot \vec{E}^a(Q) \equiv -iqE_{\text{long}}^a(Q)$, where $\vec{E}^a(Q)$ is the Fourier transform of $-\partial_0 \vec{A}^a(x) - \nabla A_0^a(x)$. Herewith the B -term of the full gluon propagator may be written as

$$\frac{1}{Q^2} \langle E_{\text{long}}^a(Q) E_{\text{long}}^b(-Q) \rangle = \frac{\delta^{ab}}{Q^2 - \overline{\Pi}_\ell(Q)} \quad (2.19)$$

with $\overline{\Pi}_\ell$ the exact longitudinal polarization function and $\langle \dots \rangle$ the thermal average. The strongly correlated fields near the pole are indeed longitudinal electric ones.

3. Two-loop diagrams with hard inner momenta

In this section the complete set of 2-loop diagrams is analysed with respect to its possible g^3 -contribution to the real part of the polarisation function $\Pi^{\mu\nu}(Q)$. Here, and only here, we shall restrict ourselves to Feynman gauge ($\alpha = 1$). The outer momentum reads $Q = (Q_0, \vec{q})$, and the limit $q \rightarrow 0$ is taken as early as possible. We emphasize these restrictions, although we do not expect them to be crucial for the somewhat unexpected results, namely, that the 2-loop contributions turn out to remain below the relative order $O(g)$. Because of the latter, readers who are only interested in the relevant terms might skip this section right now.

The set of 2-loop diagrams is shown in figure 2. They are numbered from $i=1$ to $i=13$. Correspondingly, there is an i -th contribution to Π , and each has an individual numerator $n^{\mu\nu}$ and denominator d under a double sum over the hard inner momenta P and K :

$$\Pi^{\mu\nu} = \frac{1}{2} g^4 N^2 \sum \sum \frac{n^{\mu\nu}}{d} , \quad \Pi_\ell = \text{Tr}(B\Pi) \equiv \frac{1}{2} g^4 N^2 H , \quad H = \sum \sum \frac{n_\ell}{d} . \quad (3.1)$$

To be specific, the denominators d are

$$\begin{aligned}
& (K - Q)^2(P - K)^2P^2 \quad (i = 1) \quad , \quad K^4(K - Q)^2(P - K)^2P^2 \quad (i = 7, 8, 9) \\
& K^2(K - Q)^2P^2(P - Q)^2 \quad (i = 2) \quad , \quad K^4(K - Q)^2P^2 \quad (i = 10) \\
& K^2(K - Q)^2(P - K)^2P^2 \quad (i = 3) \quad , \quad K^4(P - K)^2P^2 \quad (i = 11, 12) \\
& K^2(K - Q)^2(P - K)^2P^2(P - Q)^2 \quad (i = 4, 5, 6) \quad , \quad K^4P^2 \quad (i = 13) \quad .
\end{aligned} \tag{3.2}$$

In three denominators (nos. 2, 10 and 13) the factor $(P - K)^2$ is absent. In these cases the two sums are easily decoupled. The symmetry factors are given in the figure caption. They are included in $n^{\mu\nu}$ and $n_\ell \equiv \text{Tr}(Bn)$.

To exhibit the typical steps in treating any of the more complicated diagrams we shall work out one example in detail. The results for the 12 others will then be listed only. Consider the loop with an inserted ghost loop: number 8. The symmetry factor is 1. The structure constants at the ghost vertices combine via $f^{\circ\bullet a}f^{\circ\bullet b} = N\delta^{ab}$, and the Kronecker helps to treat the remaining two f 's in the same manner. Using the notation (2.15):

$$n^{\mu\nu} = -2(Q|K - Q| - K)^{\mu\lambda\rho}(P - K)_\rho(Q|K - Q| - K)^\nu{}_\lambda{}^\sigma P_\sigma \quad . \tag{3.3}$$

At this point it is convenient to leave the algebra to a little REDUCE program. Nevertheless, all calculations have been checked by hand.

The denominator $d = K^4(K - Q)^2(P - K)^2P^2$ does not change under the substitution $P \rightarrow K - P$. Thus, the numerator $n^{\mu\nu}$ may be replaced by its symmetric part under this transformation. Once symmetric, it can be expressed by invariants \mathcal{I} or by pairs of 'odd-invariants' \mathcal{O} (which change sign under $P \rightarrow K - P$). Such invariants are

$$\begin{aligned}
\mathcal{I} &= 2P^2 - 2PK \quad , \quad \mathcal{I}^{\mu\nu} = 2P^\mu P^\nu - P^\mu K^\nu - K^\mu P^\nu \quad , \\
\mathcal{O}_K &= 2PK - K^2 \quad , \quad \mathcal{O}_Q = 2PQ - KQ \quad , \quad \mathcal{O}^{\mu\nu} = P^\mu K^\nu + K^\mu P^\nu - K^\mu K^\nu \quad .
\end{aligned} \tag{3.4}$$

The result for $n^{\mu\nu}$ then reads

$$\begin{aligned}
n^{\mu\nu} &= -\frac{1}{2}g^{\mu\nu}(\mathcal{O}_K - 2\mathcal{O}_Q)^2 + \frac{1}{2}g^{\mu\nu}(K^2 - 2KQ)^2 - K^\mu K^\nu(3K^2 + 4\mathcal{I}) \\
&\quad - \mathcal{I}^{\mu\nu}(K + Q)^2 + 3\mathcal{O}^{\mu\nu}\mathcal{O}_K \quad + \quad \text{terms containing } Q^\mu \text{ or/and } Q^\nu \quad .
\end{aligned} \tag{3.5}$$

The terms not made explicit in (3.5) vanish due to $V \cdot Q = 0$ under the $\text{Tr } B \dots$ operation. Conveniently, when taking this trace (with $\mathcal{I}^{\mu\nu}$, say) we also exploit $\vec{q} \rightarrow 0$, which

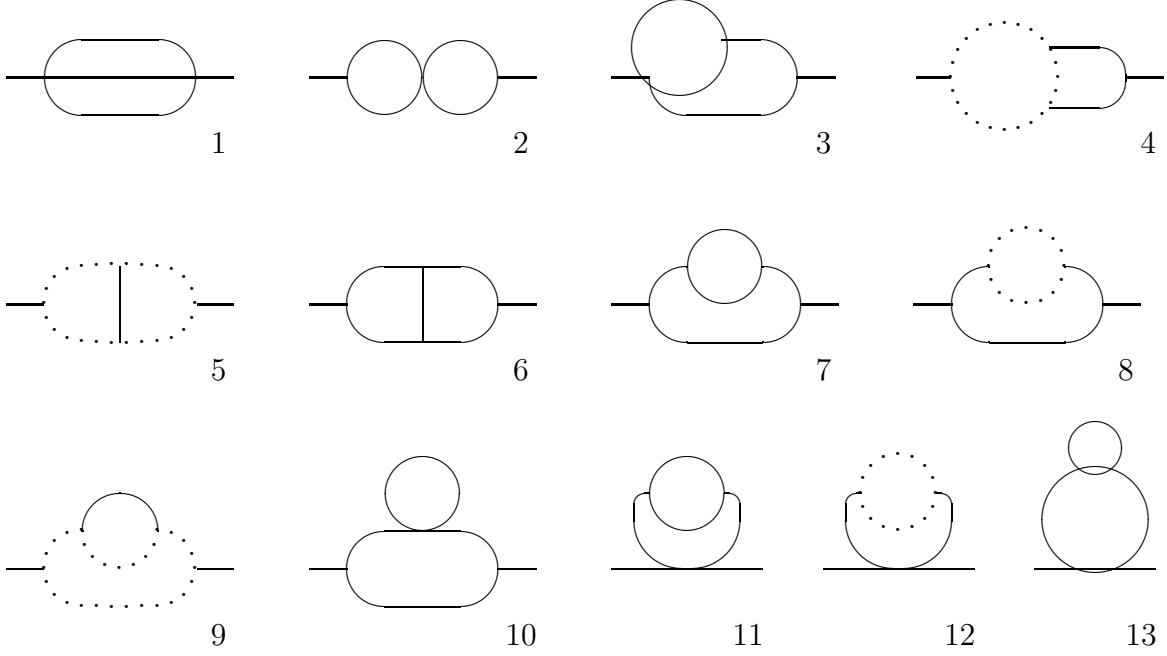


Figure 2: 2-loop diagrams. The dotted lines represent ghost propagators. Normal lines refer to hard and therefore bare gluons. The right-left mirror images of nos. 3 and 4 are included by doubling the corresponding symmetry factors. These are $1/6$ (diagram 1), $1/4$ (diagrams 2, 11, 13), $1/2$ (diagrams 6, 7, 10, 12), 1 (diagrams 3, 5, 8) and 2 (diagrams 4, 9).

amounts to $B \rightarrow -(0, \vec{q}) \circ (0, \vec{q})/q^2$. The two angular integrations in $\sum \sum$ now permit the replacement $\mathcal{I}^{\mu\nu} \rightarrow -\frac{2}{3} (p^2 - \vec{p} \cdot \vec{k})$. We obtain:

$$\begin{aligned}
n_\ell = & -\frac{1}{2}\mathcal{O}_K^2 + 2\mathcal{O}_K\mathcal{O}_Q - 2\mathcal{O}_Q^2 + \frac{1}{2}(K^2 - 2KQ)^2 + k^2K^2 \\
& + \frac{4}{3}k^2\mathcal{I} + \frac{2}{3}(p^2 - \vec{p} \cdot \vec{k})(K + Q)^2 - (2\vec{p} \cdot \vec{k} - k^2)\mathcal{O}_k .
\end{aligned} \tag{3.6}$$

Note, that here and anywhere in the following $\vec{q} = \vec{0}$ and hence $Q = (Q_0, \vec{0})$. Next we try and rewrite n_ℓ as a linear combination of factors which occur in the denominator:

$$\begin{aligned}
\mathcal{I} &= P^2 + (P - K)^2 - K^2 \rightarrow 2(P - K)^2 - K^2 \\
\mathcal{O}_K &= P^2 - (P - K)^2 \rightarrow -2(P - K)^2 \\
-\frac{1}{2}\mathcal{O}_K^2 &\rightarrow (P - K)^2(2PK - K^2) \rightarrow -(P - K)^2K^2 \\
-2\mathcal{O}_Q^2 &= -2Q_0^2(2P_0 - K_0)^2 \rightarrow -2Q^2(4(P - K)^2 - K^2 + 4p^2 - k^2) .
\end{aligned} \tag{3.7}$$

The rightharrows in (3.7) indicate allowed substitutions in (3.6). The symmetry is now abandoned in favour of cancellations. Note that if a factor $(P - K)^2$ is cancelled with that in the denominator there is another symmetry, namely $P \rightarrow -P$, which allows for the last step in the third line of (3.7). If $(K - Q)^2$ is cancelled, the new symmetry is $K, P \rightarrow -K, -P$. Through such steps we arrive at

$$\begin{aligned}
n_\ell &= (P - K)^2 \left(K^2 - 2(K - Q)^2 + \frac{2}{3}k^2 \right) + (K - Q)^2 \left(\frac{1}{2}K^2 + \frac{1}{3}k^2 - \frac{2}{3}p^2 \right) \\
&+ K^2 \left(\frac{4}{3}p^2 - k^2 \right) + n_1 + n_2 + Q^2 \left(2K^2 - \frac{1}{2}(K - Q)^2 + \frac{1}{2}Q^2 - 6(P - K)^2 \right) \quad (3.8) \\
&\text{with } n_1 = \frac{4}{3}Q^2k^2 \quad \text{and} \quad n_2 = -\frac{20}{3}Q^2p^2 .
\end{aligned}$$

This is not the last version. So far we have done nothing towards the fact that both inner momenta may be taken hard. Clearly, the last lengthy term in (3.8) is two g -orders smaller than e.g. the first one. Hence, we neglect it. It is tempting to do so with the term n_1 as well. This, however, is not allowed:

$$\begin{aligned}
4Q^2k^2 &= 4(KQ)^2 - 4Q^2K^2 \\
&= K^4 - K^2(K - Q)^2 - 2Q^2K^2 + Q^4 - Q^2(K - Q)^2 - (K - Q)^2 2KQ \\
&\approx K^4 - K^2(K - Q)^2 , \quad (3.9)
\end{aligned}$$

where the last term in the second line was omitted due to $K, P \rightarrow -K, -P$. In the case of n_2 this last step does not work. In fact, n_2 remains of the relative order g^2 and is to be neglected. The hard-hard result for number 8 is now obtained:

$$H_8 = \frac{4}{3}Z_2 - \frac{2}{3}Z'_2 - Z_1 + \frac{1}{3}Z'_1 + \frac{1}{3}Z_0 + \frac{1}{6}Z'_0 + I_0 \left(\frac{2}{3}J_1 + J_0 - 2J'_0 \right) , \quad (3.10)$$

where the objects Z, I, J are sums out of the following collection:

$$\begin{aligned}
Z_0 &= \sum \sum \frac{1}{(K - Q)^2(P - K)^2P^2} , \quad Z_{1,2} = \sum \sum \frac{(\vec{k}^2, \vec{p}^2)}{K^2(K - Q)^2(P - K)^2P^2} , \\
Z'_0 &= \sum \sum \frac{1}{K^2(P - K)^2P^2} , \quad Z'_{1,2} = \sum \sum \frac{(\vec{k}^2, \vec{p}^2)}{K^4(P - K)^2P^2} , \\
I_0 &= \sum \frac{1}{K^2} = -\frac{T^2}{12} , \quad I_1 = \sum \frac{\vec{k}^2}{K^2(K - Q)^2} = \frac{T^2}{24} , \\
J_0 &= \sum \frac{1}{K^2(K - Q)^2} , \quad J_1 = \sum \frac{\vec{k}^2}{K^4(K - Q)^2} , \quad J'_0 = \sum \frac{1}{K^4} . \quad (3.11)
\end{aligned}$$

Something enervating happened in the last steps leading to (3.10). Certain terms with a prefactor Q^2 were neglected but others not. Consider again the term n_1 in (3.8). It had the effect of adding a term $(Z_0 - Z'_0)$ to H_8 (moreover, $Z'_0 = 0$, see (3.14) below). With regard to (3.11) the two sums only differ by the kind of pole prescription. Furthermore, if we replace n_ℓ in (3.1) by n_1 , such an expression could well be among the 3-loop contributions (read Q^2 as $\omega^2 \sim g^2 T^2$ and the T^2 as e.g. I_0 , i.e. as a loop that factorizes off). To summarize, we learn that the correct definition of 2-loops requires 3-loops. Here we do what we can and evaluate the 2-loop terms as they stand. Fortunately, as it will turn out shortly, the 2-loop terms remain below the relevant order $O(g)$.

With the above mentioned reservation in mind we return to the full set of all 13 diagrams and list the results:

$$\begin{aligned}
H_1 &= 9Z_0 \quad , \quad H_2 = 0 \quad , \quad H_3 = 15Z_1 - \frac{27}{2}Z_0 + \frac{9}{2}Z'_0 \quad , \\
H_4 &= \frac{2}{3}Z_2 - \frac{1}{3}Z_1 - \frac{1}{6}Z_0 + \frac{1}{6}Z'_0 - \frac{2}{3}I_1J_0 \quad , \quad H_5 = \frac{1}{6}Z_1 + \frac{1}{12}Z_0 - \frac{1}{12}Z'_0 \quad , \\
H_6 &= 6Z_2 - \frac{41}{6}Z_1 + \frac{43}{12}Z_0 - \frac{19}{12}Z'_0 - 6I_1J_0 \quad , \\
H_7 &= -\frac{20}{3}Z_2 + \frac{10}{3}Z'_2 - \frac{25}{3}Z_1 - \frac{7}{3}Z'_1 + \frac{8}{3}Z_0 + \frac{5}{6}Z'_0 - 10I_0J_1 - 3I_0J_0 + 12I_0J'_0 \quad , \\
H_8 &\text{ see (3.10) } \quad , \quad H_9 = \frac{2}{3}Z_1 \quad , \\
H_{10} &= 20I_0J_1 - 6I_0J_0 - 6I_0J'_0 \quad , \quad H_{11} = -\frac{10}{3}Z'_2 + \frac{7}{3}Z'_1 - 5Z'_0 - 16I_0J'_0 \quad , \\
H_{12} &= \frac{2}{3}Z'_2 - \frac{1}{3}Z'_1 - Z'_0 + 2I_0J'_0 \quad , \quad H_{13} = 18I_0J'_0 \quad .
\end{aligned} \tag{3.12}$$

In the first line, the result for H_2 actually was $-27Q^2J_0^2/4$, which however had to be neglected in the hard-hard sense. To deal with all contributions, the collection (3.11) was sufficient. But one of these sums diverges, namely Z'_2 (see below). Appeasingly enough, the Z'_2 -terms cancel each other when adding H_7 to H_{11} , or H_8 to H_{12} . Summing up the 13 contributions H_i one obtains

$$\Pi_\ell^{\text{2-loop hh}} = g^4 N^2 \left(\frac{2}{3}Z_2 - \frac{1}{3}Z_1 + Z_0 - Z'_0 - 4I_0(J_0 - J'_0) + \frac{16}{3}I_0J_1 - \frac{10}{3}I_1J_0 \right) \quad . \tag{3.13}$$

It remains to evaluate the sums (3.11). This is done in Appendix A. There the term independent of temperature, which is contained in each frequency sum (see A.2)), is neglected from the outset. After renormalization, and if we may apply an argument of

BP (§2) in the present case, there remain only terms which are down by two powers of g . Hence, all of the following integrals are UV-controlled by Bose functions:

$$\begin{aligned} Z_0 &= -2Z_1 \ , \quad Z_1 = \frac{1}{16\pi^4} \int_{q^*}^{\infty} dp n(p) \int_{q^*}^{\infty} dk n(k) \ln \left(\frac{p+k}{p-k} \right) \ , \\ Z_2 &= -I_0 J_0 - \frac{1}{2} Z_1 + \frac{1}{32\pi^4} \int_{q^*}^{\infty} dp n(p) \int_{q^*}^{\infty} dk n(k) \frac{p}{k} \ln \left(\frac{k^2}{p^2 - k^2} \right) \ , \\ Z'_0 &= Z'_1 = 0 \ , \end{aligned} \tag{3.14}$$

where the logarithm is understood to take the absolute value of its argument. For the delicate object Z'_2 , which had cancelled in (3.13), we state the singular parts here:

$$Z'_2{}^{\text{sing}} = \frac{1}{16\pi^4} \int_{q^*}^{\infty} dp n(p) \int_{q^*}^{\infty} dk \left(p n'(k) \int_{-1}^1 du \frac{1}{1-u} - \frac{2p}{k} n(k) \int_{-1}^1 du \left[\frac{1}{1-u} \right]^2 \right) \ , \tag{3.15}$$

where $u = \cos(\vartheta)$ and ϑ the angle between \vec{k} and \vec{p} . The u -integrals diverge when the three-momenta become parallel. If we had worked with a corresponding cutoff λ , most probably, $\ln(\lambda)$ and $1/\lambda$ would have appeared in place of the u -integrals, respectively. We identify the above with the 'collinear singularity' studied recently [32] in order to establish the Kinoshita-Lee-Nauenberg theorem [33] in thermal field theory or even in hot QCD [34]. To justify our identification note that the above singularity (a) occurs in a separate factor, (b) stems from loop self-energy insertions (diagrams 7, 8, 11, 12), (c) has nothing to do with IR or UV, (d) cancels among different contributions and, once more, (e) occurs when \vec{p} and \vec{k} (or $-\vec{k}$) become parallel.

It remains to list the results of Appendix A for the single sums:

$$\begin{aligned} J_0 &= \frac{1}{\pi^2} \int_{q^*}^{\infty} dk n(k) \frac{k}{4k^2 - Q_0^2} \ , \quad J_1 = -\frac{3}{4} J_0 \ , \\ I_1 &= \frac{T^2}{24} + \frac{Q_0^2}{4} J_0 \ , \quad J'_0 = J_0^{Q_0=0} + \frac{n(q^*)}{4\pi^2} \ . \end{aligned} \tag{3.16}$$

These relations still contain the soft Q_0 , since they allow for shifting q^* down to zero.

The relative order of the 2-loop contributions is $O(g^2)$ instead of the $O(g)$ in search. To see this, we return to (3.13) and take q^* of order T in magnitude (thus allowing only for really hard inner momenta). By substituting $p = Tp'$, $k = Tk'$ the sums $Z_{1,2,3}$ become T^2 times a dimensionless number, while $J_{0,1}$ remain to be numbers of order 1 in magnitude. Thus, the contribution (3.13) to Π_ℓ is of order $g^4 N^2 T^2$ or, equivalently, $m^2 N g^2$ as stated above. If we shift q^* towards the threshold $T\sqrt{g}$, the integrands increase

and $n(k) \rightarrow T/k$. This amounts to at most $O(g^{3/2} \ln(g))$ in place of $O(g^2)$. But, instead of speculating this way, one rather should search for those other contributions which remove the toy parameter q^* at all. We shall leave this point as an open question. In the present paper we are not forced to answer it, since for the only true $O(g)$ contributions (1-loop soft) we shall see the independence of q^* explicitly.

One might ask for any deeper reason for the null result of this section. In general the real part of Π is an even function of ω . In Appendix A, especially, odd ω -powers are removed by the operation S_ω . Naturally, Π should be considered as a function of ω^2 . On the other hand, any sum Z must have the form $T^2 f(\omega^2/T^2, T/m)$ with a dimensionless function f . For hard-hard terms the second argument is absent (but it is present at 1-loop soft). Thus, the only way to get $O(g)$ is that $f(\omega^2/T^2)$ develops a root-singular dependence of its argument. This is not very probable. And it did not happen indeed.

4. One-loop diagrams with hard inner momentum

As is well known, the 1-loop contributions constitute the leading (or 'zeroth') order $\omega^{(0)} = m$ of the plasma frequency. But upon subtracting those 'strictly hard' contributions, which are really used to build up m (and which are given an upper index zero in the sequel), contributions of the relative order $O(g)$ might remain. In this section we thus concentrate on that possible origin of $O(g)$ which is second in the list of next-to-leading contributions given by BP (§4.3).

The contributions to m are hard as well as soft. Those of $O(g)$, if soft, are to be calculated with dressed propagators and vertices. Thus, in sorting contributions, one is faced with an eight-fold variety of indices: hard/soft, bare/dressed, with/without upper index 0. We will help ourselves by a proper definition of the term "1-loop hard" and thereby separating it from the term "1-loop soft".

There are three 1-loop diagrams: the loop (l), the tadpole (t) and the ghost-loop (g). Let "ltg" stand for the sum of these diagrams. A lower index "dressed" requires to use both effective propagators and vertices. Diagrammatically, our classification is:

$$\text{1-loop hard} \equiv \text{ltg}(\text{all } K)_{\text{bare}} - [\text{ltg}(\text{all } K)_{\text{bare}}]^0, \quad (4.1)$$

$$\text{1-loop soft} \equiv \text{ltg}(\text{soft } K)_{\text{dressed}} - \text{ltg}(\text{soft } K)_{\text{bare}}. \quad (4.2)$$

We now read (4.1), (4.2) as the specification of the contributions $\Delta\Pi_\ell^{1\text{-loop hard}}$ and $\Delta\Pi_\ell^{1\text{-loop soft}}$ to Π_ℓ . The prefix Δ always indicates that something is subtracted which was already counted. Consider the second line first. If we relax the restriction to soft K , (4.2) remains still valid because, with increasing K , the dressed vertices and propagators turn into bare ones automatically. Independence of any threshold q^* is thus inherent in the definition. This independence (or 'UV-convergence') can be used to test intermediate results. If we omit the K -specifications and add (4.1) to (4.2), only one subtraction is left, namely that of the zeroth order. In passing, the ghost-loop drops out in (4.2) because it is not dressed.

We return to the line (4.1) and consider the first term with respect to Π_ℓ in covariant gauges. With reference to the three diagrams l,t,g and with the notations $\Delta_0 = 1/P^2$, $\Delta_0^- = 1/(P-Q)^2$, the result reads:

$$\begin{aligned} \Pi_{\ell, \text{bare}}^1 &= \frac{1}{2}g^2N \sum \left(2\Delta_0 + 2(\alpha-1)\Delta_0 + \frac{2}{3}(\alpha-1)p^2\Delta_0^2 - \frac{10}{3}p^2\Delta_0^-\Delta_0 + 4m^2\Delta_0^-\Delta_0 \right. \\ &\quad \left. + 4(\alpha-1)m^2\Delta_0^-\Delta_0 + \frac{20}{3}(\alpha-1)m^2p^2\Delta_0^-\Delta_0^2 - \frac{1}{3}(\alpha-1)^2m^4p^2(\Delta_0^-\Delta_0)^2 \right) , \end{aligned} \quad (4.3)$$

$$\Pi_{\ell, \text{bare}}^t = \frac{1}{2}g^2N \sum \left(-6\Delta_0 - 2(\alpha-1)\Delta_0 - \frac{2}{3}(\alpha-1)p^2\Delta_0^2 \right) , \quad (4.4)$$

$$\Pi_{\ell, \text{bare}}^g = \frac{1}{2}g^2N \sum \frac{2}{3}p^2\Delta_0^-\Delta_0 . \quad (4.5)$$

Writing down (4.3) we started with 'strictly hard' terms (the first three), which are not at all influenced by an upper index 0. The tadpole contribution (4.4) is made up of only such terms. It thus drops out in (4.1). The fourth term of (4.3) survives under the 0-operation, but the remaining terms (containing m^2) do not. The 0-operation amounts to $Q \rightarrow 0$ *after* the frequency sum in \sum has been performed. To indicate this we write $[\Delta_0^-]^0 = \Delta_0^\bullet$ (instead of Δ_0). With (4.3), (4.4), (4.5) the difference (4.1) reads

$$\begin{aligned} \Delta\Pi_\ell^{1\text{-loop hard}} &= \frac{1}{2}g^2N \sum \left(-\frac{8}{3}p^2(\Delta_0^-\Delta_0 - \Delta_0^\bullet\Delta_0) + 4m^2\Delta_0^-\Delta_0 \right. \\ &\quad \left. + \text{the last three terms of (4.3)} \right) . \end{aligned} \quad (4.6)$$

Let us switch to Feynman gauge for a moment. With (3.11) and with (3.16) at $Q_0^2 = m^2$ we obtain

$$\Delta\Pi_{\ell, \alpha=1}^{1\text{-loop hard}} = \frac{1}{2}g^2N \left(-\frac{3}{3} [I_0 - I_0^\bullet] + 4m^2J_0 \right) = m^2\frac{5}{3}g^2NJ_0 . \quad (4.7)$$

Hence, the magnitude of interest is hidden in

$$J_0 = \frac{1}{\pi^2} \int_0^\infty dk n(k) \frac{k}{4k^2 - m^2} = \frac{1}{4\pi^2} \mathcal{P} \int_0^\infty dx \frac{x}{x^2 - 1} \frac{1}{e^{\varepsilon x} - 1} , \quad (4.8)$$

where $\varepsilon = m/2T = g\sqrt{N}/6$, \mathcal{P} for principal value ,

and where (A.3) has been used at $q^* = 0$ and $Q_0^2 = m^2$. At first glance J_0 seems to be of order $1/g$. However, the integral with $1/\varepsilon x$ in place of the Bose function vanishes. In fact, the first two terms of the asymptotic expansion of J_0 are

$$J_0 \sim \frac{1}{8\pi^2} \ln\left(\frac{\varepsilon}{2}\right) + \frac{1}{4\pi^2} \int_0^\infty dx \frac{1}{x} \left[\frac{1}{e^x - 1} - \frac{2}{2x + x^2} \right] \quad (\varepsilon \rightarrow +0) . \quad (4.9)$$

Thus, J_0 is large only as $\ln(g)$, and $\Delta\Pi_\ell^{\text{1-loop hard}}$ only reaches the relative order $O(g^2 \ln(g))$ instead of a pure $O(g)$ in search. The terms of (4.6), which depend on α , are also of the order of J_0 (at most).

The result (4.9) comes most opportunely, for otherwise there would have been a dilemma. BP show that the 1-loop soft terms form a separate gauge independent set, and argue that consequently the set of other $O(g)$ -terms must do so aswell. Thus, after we got no $O(g)$ from 2-loop hh, the 1-loop-hard terms, if $O(g)$, would have to be gauge independent. But, according to (4.6), they do depend on α . By the smallness of J_0 this is of no concern.

Once we have learned that, within the order $O(g)$, the line (4.1) gives zero, we may proceed simplifying the subtraction term in the 1-loop-soft line, which is the last term of (4.2) (without the minus). There, the 'total hard' parts of (4.3) and (4.4) suffice. We will write down the subtraction terms for loop and tadpole separately. The gauge dependent pieces of these two terms cancel (reflecting the gauge independence of the zeroth order). But when kept separately, these terms (e.g. the last two in (4.4)) could be (and are indeed) necessary to restore q^* -independence. However, as all α 's will drop out in the sequel before UV details need be studied, we need not keep them. The subtraction terms are now prepared as

$$\Pi_{\ell, \text{bare}}^1 = g^2 N \sum \left(\Delta_0 - \frac{5}{3} \Delta_0^- \Delta_0 p^2 \right) \quad (4.10)$$

$$\Pi_{\ell, \text{bare}}^t = g^2 N \sum (-3\Delta_0) \quad (4.11)$$

for use in the following section. (4.10) and (4.11) are real. Therefore, if one studies imaginary parts [12], no subtractions are required. Gauge dependences however may remain in the dressed tadpole and the dressed loop. We shall check their cancellation.

5. One-loop diagrams with soft inner momentum

5.1 THE TADPOLE DIAGRAM

As discussed in the preceding section there are precisely two diagrams, tadpole and loop, which might (and do) contribute at order $O(g)$ through (4.2). In this subsection we concentrate on the first. Using the dressed 4-vertex (2.16) and the dressed gluon (2.2) the first term in (4.2) becomes

$$\Pi_{\text{tadpole}}^{\mu\nu} = \frac{1}{2}g^2N \sum_P^{\text{soft}} G(P)_{\lambda\rho} \left([g^{\mu\rho}g^{\lambda\nu} + g^{\mu\lambda}g^{\rho\nu} - 2g^{\mu\nu}g^{\rho\lambda}] + \delta\Gamma^{\mu\nu\rho\lambda}(Q, -Q, -P, P) \right) \quad (5.1)$$

This is equation (4.23) of BP (apart from the sign, which is notational). Using the relation $A + B + D = g$, the propagator may be written as

$$G = g \Delta_\ell + D \Delta_{\alpha\ell} + A \Delta_{t\ell} \quad \text{with} \quad \Delta_{\alpha\ell} \equiv \Delta_\alpha - \Delta_\ell \text{ etc.} . \quad (5.2)$$

Turning to the longitudinal part Π_ℓ of (5.1) by taking the trace with B , i.e. by sandwiching (5.2) with vectors V and dividing by V^2 , we may decompose into three parts as follows

$$\Pi_\ell^{\text{tadpole}} = \Pi_{\text{bv}} + \Pi_{\alpha\ell} + \Pi_{t\ell} \quad (5.3)$$

The index bv refers to the bare vertex in (5.1). The other two terms combine the HTL-part of the vertex with the $\alpha\ell$ - and $t\ell$ -part of the propagator. Its first term, $g\Delta_\ell$, may be omitted, because it does not contribute to the order $O(g)$ under consideration (it leads to an integral J_1 and only reaches the order of those terms already neglected in section 3).

The quantity Π_{bv} is easily evaluated. Note that the matrices B , A , D occur at two different arguments. This amounts to

$$\text{Tr } B(Q)A(P) \rightarrow 2/3 \quad , \quad \text{Tr } B(Q)D(P) \rightarrow -p^2/3P^2 \quad (\vec{q} \rightarrow 0)$$

when the limit $\vec{q} \rightarrow 0$ is taken. In this limit we obtain

$$\Delta\Pi_{\text{bv}} = g^2N \sum_P^{\text{soft}} \left(-3\Delta_{\ell 0} - \frac{4}{3}\Delta_{t\ell} - \Delta_{\alpha\ell} - \frac{1}{3}\Delta_{\alpha\ell}\Delta_0 p^2 \right) . \quad (5.4)$$

In writing down (5.4) we have included the subtraction term (4.11), which is the second in $\Delta_{\ell 0} = \Delta_\ell - \Delta_0$. By this subtraction the first term of (5.4) becomes UV convergent.

Nevertheless, the index 'soft' on the sum in (5.4) remains necessary in order to control the two α -dependent terms.

With (5.4) we have reached a point where one can for the first time see how a true $O(g)$ -term comes about in a natural manner. So, let us stay with the $\Delta_{\ell 0}$ -term of (5.4) and evaluate. The propagators we work with (see e.g. (5.19) below) are even functions of P_0 and of p . Hence, their spectral densities $\rho(x, p)$ are odd functions of x :

$$\Delta(P) = \int dx \frac{\rho(x, p)}{P_0 - x} = \int dx x \frac{\rho(x, p)}{P_0^2 - x^2} . \quad (5.5)$$

To calculate $\sum \Delta$ we need

$$\sum_{P_0} \frac{1}{P_0^2 - x^2} = -\frac{1}{2\pi i} \int_{\bigcirc} dP_0 \frac{n(P_0)}{P_0^2 - x^2} = -\frac{1 + 2n(x)}{2x} \rightarrow -\frac{T}{x^2} , \quad (5.6)$$

where the integral surrounds the whole complex P_0 -plane counterclockwise. If p is soft also x is soft since otherwise the density ρ vanishes. Therefore, the leading term of (5.6) can be extracted as shown to the right. For any propagator Δ with such properties we thus have

$$\sum^{\text{soft}} \Delta f(p) = -T \int_P^3 f(p) \psi(p) \quad \text{with} \quad \psi(p) \equiv \int dx \frac{1}{x} \rho(x, p) , \quad (5.7)$$

where f is an arbitrary weight function. In the case at hand we have $f = 1$, and the minus-first moment $\psi_\ell(p)$ can be taken from the table (5.19) or from Appendix B:

$$-\sum^{\text{soft}} \Delta_\ell = T \int_P^3 \psi_\ell(p) = \frac{T}{2\pi^2} \int_0^{q^*} dp \frac{p^2}{3m^2 + p^2} \equiv \mathcal{L} . \quad (5.8)$$

\mathcal{L} is UV-divergent, i.e. it depends on q^* . The sum $\sum \Delta_0$, if evaluated just so, has the same sort of divergence. The difference is finite:

$$\begin{aligned} -\sum \Delta_0 &= \frac{T}{2\pi^2} \int_0^{q^*} dp , & \sum \Delta_{\ell 0} &= 3\mathcal{K} \quad \text{with} \\ \mathcal{K} &\equiv -\sum \frac{m^2}{p^2} \Delta_\ell = \frac{T}{2\pi^2} \int_0^\infty dp \frac{m^2}{3m^2 + p^2} = \frac{mT}{3} \frac{\sqrt{3}}{4\pi} . \end{aligned} \quad (5.9)$$

Thus, the considered Π -contribution is of the order $g^2 N m T = m^2 3g\sqrt{N}$ in magnitude. (5.9) shows that the 'odd power' g arises via a simple substitution thanks to the presence of the scale m . Our expressions at 2-loop hard-hard and 1-loop hard had no such scale. We learn that there are terms of order $O(g)$, indeed. Furthermore, sums over single

propagators can be evaluated (for a collection see Appendix D). But, as a rule, sums over pairs of propagators (at different arguments) need numerical evaluation.

The remaining two terms in (5.3) are more complicated. They both represent examples for the excellent utility of Ward identities. We start, using vertical vector notation, from

$$\Pi_{\{\alpha \ell\}} = \frac{1}{2} g^2 N \frac{V_\mu V_\nu}{V^2} \sum_P^{\text{soft}} \left\{ \begin{array}{l} \Delta_{\alpha\ell}(P) D_{\lambda\rho}(P) \\ \Delta_{t\ell}(P) A_{\lambda\rho}(P) \end{array} \right\} \delta\Gamma^{\mu\nu\lambda\rho}(Q, -Q, -P, P) . \quad (5.10)$$

In the upper line $D(P) = P \circ P / P^2$ brings in the momenta the Ward identities (2.18) and (2.17) are formulated with. Using both one obtains:

$$P_\lambda P_\rho \delta\Gamma^{\mu\nu\lambda\rho}(Q, -Q, -P, P) = 2\Pi^{\mu\nu}(P - Q) - 2\Pi^{\mu\nu}(Q) , \quad (5.11)$$

where use has been made of the fact that the above $\Pi^{\mu\nu}$ is an even function of its argument and so is $\Delta_{\alpha\ell}$. In the difference (5.11) we take care to do the same manipulations on both terms. Using (2.9), sandwiching with V -vectors, replacing $(VK)^2/V^2$ by $-\vec{k}^2/3$ at $\vec{q} \rightarrow 0$ and forming a common denominator (which is $K^2(K - Q)^2(K - P)^2$; the numerator is $2KP - P^2$ and the term P^2 may be neglected), a hard K -sum is obtained. It is given in Appendix C (form $\vartheta + \varphi$ there). This leads to the final form of the term studied:

$$\Pi_{\alpha\ell} = \frac{1}{3} g^2 N \sum_P^{\text{soft}} \Delta_{\alpha\ell} \Pi_\ell(P - Q) \Delta_0 \Delta_0^- p^2 \quad (5.12)$$

In the lower component of (5.10), apparently, there is no momentum in front of $\delta\Gamma$ as is needed in the Ward identity. We can produce such momenta, however, by the following exotic line which works at $\vec{q} = 0$:

$$-3 \frac{V_\mu V_\nu}{V^2} K^\mu K^\nu \rightarrow \vec{k}^2 = \frac{(KQ)^2 - K^2 Q^2}{Q^2} = \left(\frac{Q_\mu Q_\nu}{Q^2} - g_{\mu\nu} \right) K^\mu K^\nu \approx \frac{Q_\mu Q_\nu}{Q^2} K^\mu K^\nu . \quad (5.13)$$

The $g_{\mu\nu}$ -term may be neglected, if (5.13) is used with two of the K 's in $\delta\Gamma^{1234}$ (typically, such terms can be neglected when deriving the Ward identity (2.18)). Note that $\delta\Gamma$ is invariant under the interchange of Q with P . Thus, with (2.18) and (2.17), we have

$$Q_\mu Q_\nu \delta\Gamma^{\mu\nu\lambda\rho}(Q, -Q, -P, P) = Q_\mu Q_\nu \delta\Gamma^{\lambda\rho\mu\nu}(P, -P, -Q, Q) = 2\Pi^{\lambda\rho}(P - Q) - 2\Pi^{\lambda\rho}(P) . \quad (5.14)$$

Using again (2.9) together with

$$-K^\lambda A_{\lambda\rho} K^\rho = k^2 - \frac{(\vec{k} \vec{p})^2}{p^2} , \quad (5.15)$$

which is an invariant under $K \rightarrow P - K$, we end up with the sum β of Appendix C. The result is

$$\Pi_{t\ell} = g^2 N \sum_P^{\text{soft}} \Delta_{t\ell}(P) \frac{1}{3m^2} [\Pi_\ell(P - Q) - \Pi_\ell(P)] . \quad (5.16)$$

As the above derivation used the somewhat dangerous line (5.13) we like to mention, that originally $\Pi_{t\ell}$ and $\Pi_{\alpha\ell}$ were evaluated 'by hand', i.e. by first doing the frequency sums and preparing the leading terms afterwards. The results were indeed the same.

At the second term in (5.16) we encounter the possibility to cancel a self-energy in the numerator against the same contained in the propagator. We shall do so whenever possible following a hint in Ref. [12]. As this step occurs repeatedly in the next section, let us work with a short hand notation,

$$m^2 - P^2 \equiv a \quad , \quad m^2 - \Pi_\ell(P) \equiv b \quad , \quad \Delta_\ell = \frac{1}{b - a} \quad , \quad \Delta_t = \frac{-2}{b + 2a} \quad , \quad (5.17)$$

where the identity $\Pi_\ell + 2\Pi_t = 3m^2$ has been exploited. Then:

$$\Delta_{t\ell} \Pi_\ell = \left(\frac{-2}{b + 2a} - \frac{1}{b - a} \right) (m^2 - b) = m^2 \Delta_{t\ell} + 2\Gamma_t + \Gamma_\ell \quad , \quad (5.18)$$

where by $\Gamma \equiv 1 + a\Delta$ we keep terms together such that the behaviour as $1/P^2$ at large P may be associated with. Due to their neat properties we call Γ_ℓ and Γ_t 'propagators'. Let us list them together with other propagators in use:

propagator	spectral density ρ	relation	$\psi(p)$	
Δ_0	$\frac{1}{2p} [\delta(x - p) - \delta(x + p)]$	$\Delta_0 = \frac{1}{P^2}$	$\frac{1}{p^2}$	
Δ_t	ρ_t	(2.2)	$\frac{1}{p^2}$	
Δ_ℓ	ρ_ℓ	(2.2)	$\frac{1}{3m^2 + p^2}$	
Γ_t	$\rho_t (m^2 + p^2 - x^2)$	$\Gamma_t = 1 + (m^2 - P^2)\Delta_t$	$\frac{m^2}{p^2}$	(5.19)
Γ_ℓ	$\rho_\ell (m^2 + p^2 - x^2)$	$\Gamma_\ell = 1 + (m^2 - P^2)\Delta_\ell$	$\frac{-2m^2}{3m^2 + p^2}$	
Ω_t	$\rho_t (x^2 - p^2)$	$\Omega_t = m^2 \Delta_t - \Gamma_t$	0	
Ω_ℓ	$\rho_\ell (x^2 - p^2)$	$\Omega_\ell = m^2 \Delta_\ell - \Gamma_\ell$	$\frac{3m^2}{3m^2 + p^2}$	
Δ_θ	$\frac{3m^2 x}{2p} \frac{p^2 - x^2}{p^2} \theta(p^2 - x^2)$	$\Delta_\theta = m^2 - \Pi_\ell(P)$	$2m^2$	

The argument of each propagator is P . The object Δ_θ is nothing but the quantity b in (5.17). It is a 'propagator' having no pole contribution. According to (5.5), in each case the minus-first moment ψ equals minus the propagator at $P_0 = 0$.

To apply the above reformulations, the third term of the soft tadpole contribution (5.3) is given by

$$\Pi_{t\ell} = -g^2 N \frac{1}{3m^2} \sum_P^{\text{soft}} \left[2\Gamma_t + \Gamma_\ell + \Delta_{t\ell}^- \Delta_\theta \right] . \quad (5.20)$$

One may require that the expression (5.20) is UV-stable automatically and needs no cutoff. This is indeed the case. Consider for example $2\Gamma_t + \Gamma_\ell$. As the table (5.19) shows, the moment ψ of this combination behaves as p^{-4} at large p as required.

To summarize this subsection we evaluate the sums over single propagators in (5.4) and (5.20), see Appendix D, and obtain

$$\Delta \Pi_\ell^{\text{tadpole}} = g^2 N \left(-3\mathcal{K} - \frac{1}{3m^2} \sum_P \Delta_{t\ell}^- \Delta_\theta \right) + \begin{array}{l} \text{terms containing } \Delta_{\alpha\ell}, \text{ which will} \\ \text{cancel against those in the loop} \end{array} . \quad (5.21)$$

5.2 THE LOOP DIAGRAM

The loop has the symmetry factor $1/2$. It is made up of two dressed 3-vertices (2.14) and two dressed gluons. After the colour sums are done the contribution reads

$$\Pi_{\text{loop}}^{\mu\nu} = \frac{1}{2} g^2 N \sum_P^{\text{soft}} G(P-Q)_{\rho\sigma} G(P)_{\lambda\tau} \Gamma(Q, P-Q, -P)^{\mu\sigma\tau} \Gamma(Q, P-Q, -P)^{\nu\rho\lambda} . \quad (5.22)$$

Turning to the longitudinal part, each vertex is contracted with one V -vector. Since each propagator G is made up of the three terms (5.2), which we now number from 1 to 3, nine contributions can be distinguished:

$$\Pi_\ell^{\text{loop}} = \frac{1}{V^2} V_\mu \Pi_{\text{loop}}^{\mu\nu} V_\nu \equiv \sum_{i,j=1}^3 \Pi_{ij} \quad , \quad \Pi_{ij} = \frac{1}{2} g^2 N \sum_P^{\text{soft}} \mathcal{R}_{ij} . \quad (5.23)$$

An element Π_{ij} or \mathcal{R}_{ij} depends on α , if at least one index takes the value 2. Thus, if (5.23) is arranged as a 3×3 -matrix, the gauge dependent elements form a 'red cross'. To comment the next steps consider for example \mathcal{R}_{23} :

$$\mathcal{R}_{23} = \Delta_{\alpha\ell}^- \Delta_{t\ell} \frac{1}{(P-Q)^2} \frac{1}{V^2} W^\lambda A_{\lambda\tau} W^\tau \quad , \quad \text{with} \quad W^\lambda \equiv V_\mu (P-Q)_\rho \Gamma^{\mu\rho\lambda} . \quad (5.24)$$

Using the details of Γ^{123} , (2.14), one obtains

$$\begin{aligned}
WAW &= (P^2 - Q^2)^2 \left(V^2 + \frac{(VP)^2}{p^2} \right) \\
&- 16g^2 N(P^2 - Q^2) \sum \frac{1}{N_K} (KP - KQ) \left((VK)^2 - (VK)(VP) \frac{\vec{k} \vec{p}}{p^2} \right) \\
&- (8g^2 N)^2 \sum \sum \frac{(KP - KQ)(RP - RQ)}{N_K N_R} (VK)(VR) \left(\vec{k} \vec{r} - \frac{(\vec{k} \vec{p})(\vec{r} \vec{p})}{p^2} \right), \quad (5.25)
\end{aligned}$$

where $N_K = K^2(K - Q)^2(K - P)^2$. The expression simplifies slightly through $\vec{q} \rightarrow 0$ since e.g. $(VK)(VP)/V^2 \rightarrow -(\vec{q} \vec{k})(\vec{q} \vec{p})/q^2 \rightarrow -\vec{k} \vec{p}/3$, where the last step exploited the angular integrations contained. Note that $Q = (Q_0, \vec{0})$ in all what follows. The double sum, which runs over K and R , both hard, can be decoupled, see Appendix C. This decoupling is possible in all nine cases and leads to squares of various single sums (over hard K and soft outer variable), which can be evaluated towards the leading term. They are listed in Appendix C and denoted by small greek letters. It should be noted that, most probably, all we do at this stage was already worked out by BP in §4.2. But let us be obstinate in order to have an independent test.

We leave the above special example and notice the result for all nine terms at the same intermediate level. The element \mathcal{R}_{22} in the center of the 'red cross' is the only one containing squares of α :

$$\mathcal{R}_{22} = -\frac{1}{3} \Delta_{\alpha\ell}^- \Delta_{\alpha\ell} \Delta_0 \Delta_0^- p^2 \left[Q^2 - 8g^2 N \tau \right]^2 \quad (5.26)$$

The object τ is the hard K -sum mentioned above. Its evaluation is straightforward:

$$\tau \equiv \sum \frac{(KP)(KQ - KP)}{N_K} \frac{\vec{k} \vec{p}}{p^2}, \quad 8g^2 N \tau = m^2 + \text{non-leading terms}. \quad (5.27)$$

Using (5.27) the square bracket in (5.26) becomes $Q^2 - m^2$. Thus the whole term vanishes on mass shell. No α^2 survives. The other gauge dependent pieces stand at the four ends of the 'red cross':

$$\begin{aligned}
\mathcal{R}_{12} &= \Delta_{\ell}^- \Delta_{\alpha\ell} \Delta_0 \left([(P - Q)^2 - Q^2]^2 + \frac{p^2}{3} (P^2 - 2PQ - Q^2) \right. \\
&\quad \left. - \frac{16g^2 N}{3} [(2PQ - P^2)(\vartheta + \varphi) + \tau] - \frac{1}{3} (8g^2 N)^2 \left[\frac{\lambda^2}{p^2} - \frac{\vartheta^2}{2} - \varphi^2 \right] \right) \\
\mathcal{R}_{23} &= \frac{1}{6} \Delta_{\alpha\ell}^- \Delta_{t\ell} \Delta_0^- (2P^2 - 2Q^2 + 8g^2 N \mu)^2 \\
\mathcal{R}_{21} &= \mathcal{R}_{12}, \quad \mathcal{R}_{32} = \mathcal{R}_{23}, \quad (5.28)
\end{aligned}$$

where the hard K -sums read ϑ , φ , τ , λ , μ and are defined in Appendix C. Their leading terms are:

$$8g^2 N\varphi = m^2 - \Pi_\ell^- - p^2 \Delta_0^- \Pi_\ell^- \quad , \quad 8g^2 N\mu = \Pi_\ell - m^2 \quad , \quad \vartheta = \mu^- \quad , \quad \lambda = (P_0 - Q_0)(\varphi + \vartheta) \quad . \quad (5.29)$$

In passing, the symmetry $\mathcal{R}_{ij} = \mathcal{R}_{ji}$ arises only when the leading parts of the K -sums are taken. In general, if $i \neq j$, \mathcal{R}_{ji} differs from \mathcal{R}_{ij} in the denominators, which are $K^2(K+Q)^2(K+Q-P)^2$ in place of N_K .

It is irresistible to look back at the gauge dependent pieces of the tadpole contribution: (5.12) and the last two terms in (5.4). Both contain $\Delta_{\alpha\ell}$, and the expressions (5.28) do so as well (note that $P \rightarrow Q - P$ is allowed in any \mathcal{R} -element). To rewrite \mathcal{R}_{23} we set $Q^2 = m^2$, insert (5.29) and use the notation (5.17):

$$\mathcal{R}_{23} = -\frac{1}{3}\Delta_{\alpha\ell}^- \Delta_0^- (b+2a) - \frac{1}{6}\Delta_{\alpha\ell}^- \Delta_\ell \Delta_0^- (b+2a)^2 \quad . \quad (5.30)$$

For \mathcal{R}_{12} we observe that it is the sum of three squared brackets containing one 'greek' sum each. The next steps are $P \rightarrow Q - P$, inserting (5.29), using again the a - b -notation, but still maintaining the order of the three squared 'greeks' of (5.28):

$$\mathcal{R}_{12} = \frac{1}{3}\Delta_{\alpha\ell}^- \Delta_\ell \Delta_0^- \left((b-a)^2 \Delta_0^2 P_0^4 + \frac{1}{2}(b+2a)^2 - (b-a)^2 \Delta_0^2 P_0^2 p^2 \right) \quad . \quad (5.31)$$

The second term will cancel the last term of \mathcal{R}_{23} . and one $(b-a)$ suffices to kill Δ_ℓ . After a few rather trivial steps (including the omission of a term which is odd in P) we end up with

$$\mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{23} + \mathcal{R}_{32} = \frac{2}{3}\Delta_{\alpha\ell} \Delta_0 \left(3P^2 + p^2 - \Delta_0^- p^2 \Pi_\ell^- \right) \quad . \quad (5.32)$$

Obviously, these three terms precisely cancel the gauge dependence parts of the tadpole. BP are right. The order $O(g)$ -terms form a gauge independent set. As this result was expected (see the text below (2.7)), it merely tells us that the procedure followed so far works smoothly.

The physics is contained in the four \mathcal{R} -elements in the corners of the matrix:

$$\begin{aligned} \mathcal{R}_{11} &= \Delta_\ell^- \Delta_\ell \left(2P^2 - 2PQ + 5Q^2 - \frac{10}{3}p^2 - \frac{16g^2 N}{3} [(Q_0 - 2P_0)(\beta + \gamma) + 2\sigma] \right. \\ &\quad \left. - \frac{1}{3}(8g^2 N)^2 \left[\frac{\sigma^2}{p^2} - \beta^2 - 2\gamma^2 + \frac{3\rho^2}{2p^2} + \frac{\zeta^2}{p^2} \right] \right) \\ \mathcal{R}_{13} &= \Delta_\ell^- \Delta_{\ell\ell} \left(\frac{2}{3}(P+Q)^2 - \frac{8p^2}{3} + \frac{16g^2 N}{3} [(P_0 + Q_0)\beta + \rho] + \frac{1}{3}(8g^2 N)^2 \left[\frac{\beta^2}{2} - \frac{\rho^2}{p^2} \right] \right) \end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{31} &= \mathcal{R}_{13} \\
\mathcal{R}_{33} &= -\Delta_{t\ell}^- \Delta_{t\ell} \frac{1}{6p^2} (4p^2 - 8g^2 N \rho)^2 .
\end{aligned} \tag{5.33}$$

The hard K -sums contained here, which are again defined in Appendix C, can be traced back to those already given in (5.29):

$$Q_0 \beta = \mu^- - \mu \ , \ Q_0 \gamma = \varphi - \varphi^- \ , \ Q_0 \sigma = \lambda^- - \lambda \ , \ \rho = P_0 \beta - \vartheta \ , \ \zeta = \sigma - \rho \ . \tag{5.34}$$

In the sequel the above \mathcal{R} -elements are subject to several transformations and regroupings with the general aim of simplification. For instance, we cancel self-energies in the numerator as in (5.18) and reduce P -powers by changing from Δ - to Γ -propagators. The procedure ends up with the four standard expressions given in (6.1), (6.2) below. To illustrate the steps we concentrate on the derivation of \mathcal{M}_2 , see (6.2). This term is part of \mathcal{R}_{33} . Admittedly, this is the simplest term of (5.33). At first we insert ρ , (C.9), square out and reduce the number of terms slightly by means of $P \rightarrow Q - P$. In pure factors Q_0^2 or $1/Q_0^2$ we replace Q_0^2 by m^2 , thus anticipating the analytical continuation otherwise to be done at the end. The resulting expression

$$\mathcal{R}_{33} = -\frac{1}{3m^2 p^2} \Delta_{t\ell}^- \Delta_{t\ell} \left(8m^2 p^4 + P_0(Q_0 - P_0)bb^- - 8p^2 P_0 Q_0 b + P_0^2 b^2 \right) \tag{5.35}$$

is ready for cancellations of b as often as possible:

$$\Delta_{t\ell} b = -2\Gamma_t - \Gamma_\ell \ , \quad \Delta_{t\ell} b^2 = -3b + a(4\Gamma_t - \Gamma_\ell) \ . \tag{5.36}$$

Clearly, there remain terms of the form $\Delta^- b$, which will be collected at the end to give \mathcal{M}_1 , see (6.2). Next we keep only terms having the index t twice and denote this selection by \mathcal{R}_{33}^{tt} :

$$\frac{1}{2} \mathcal{R}_{33}^{tt} = -\frac{4}{3} \Delta_t^- \Delta_t p^2 - \frac{2P_0(Q_0 - P_0)}{3m^2 p^2} \Gamma_t^- \Gamma_t - \frac{8}{3m^2} \Delta_t^- P_0 Q_0 \Gamma_t - \frac{2}{3m^2 p^2} \Delta_t^- P_0^2 a \Gamma_t \ . \tag{5.37}$$

In the last term we use the identity

$$P_0^2 a = (3m^2 + 2P_0 Q_0 + P_0^2) a^- - (3m^2 + p^2) 2P_0 Q_0 - 3m^2 p^2 \ , \tag{5.38}$$

valid at $Q_0^2 = m^2$, in order to reduce P_0 -powers. Note that $a^- \Delta_t^- = \Gamma_t^- - 1$, where the -1 leads to terms with only one propagator. The product $2P_0 Q_0$ may be replaced by m^2 , if it occurs with factors symmetric under $P \rightarrow Q - P$. Even if it occurs with $\Delta_t^- \Gamma_t$, we may write

$$\Delta_t^- \Gamma_t 4P_0 Q_0 = 4\Delta_t^- P_0 Q_0 + \Delta_t^- \Delta_t 4P_0 Q_0 a$$

and symmetrize in the last term by

$$4P_0Q_0a \rightarrow m^2(3a + 3a^- - 3m^2 - 4p^2)$$

followed by $\Delta_t a = \Gamma_t - 1$. Note the two more origins of single-propagator terms (SPT). The procedure ends up with

$$\frac{1}{2}\mathcal{R}_{33}^{tt} = -3m^2\Delta_t^-\Delta_t - 3(m^2\Delta_t^- - \Gamma_t^-)(m^2\Delta_t - \Gamma_t) + \text{SPT} \quad (5.39)$$

with

$$\begin{aligned} \text{SPT} = & \frac{8}{3}\Gamma_t\frac{1}{p^2} - \frac{2}{3}(\Gamma_t^- - \Gamma_t)\frac{P_0}{Q_0}\frac{1}{p^2} - 4\Delta_t\frac{m^2}{p^2} + 2(\Delta_t^- - \Delta_t)\frac{P_0Q_0}{p^2} \\ & + \frac{4}{3}\Delta_t - \frac{2}{3}(\Delta_t^- - \Delta_t)\frac{P_0}{Q_0} . \end{aligned} \quad (5.40)$$

For the sum over SPT we read off from Appendix D that

$$\sum \text{SPT} = -\frac{8}{3}\mathcal{V} + \frac{2}{3}\mathcal{V} + 4\mathcal{V} - 2\mathcal{V} - 2\mathcal{K} - \frac{2}{3}\mathcal{L} , \quad (5.41)$$

where the first four terms correspond to the first four in (5.40). Each of these four sums diverges in the infrared, since

$$\mathcal{V} \equiv -\sum \Delta_t\frac{m^2}{p^2} = \frac{Tm^2}{2\pi^2} \int_0^{q^*} dp \frac{1}{p^2} , \quad (5.42)$$

but they cancel each other in (5.41). We also see how terms denoted by \mathcal{K} arise. They are collected in the first term of (6.1). The UV-singular objects \mathcal{L} either cancel or are needed in (6.2) to compensate the q^* -dependence of the first part of \mathcal{M}_1 . Clearly, the first two terms of (5.39) give \mathcal{M}_2 in (6.2), as announced. At first glance, the propagator Ω_t is introduced for a shorter notation only. Note however that in the IR-region Ω_t is less dangerous than Γ_t , as can be seen in (5.19).

The treatment of \mathcal{R}_{11} , \mathcal{R}_{13} and even of the remaining parts of \mathcal{R}_{33} along the steps just described leads into a lengthy and tedious procedure. Here we only comment on one more detail. After the b -cancellations in \mathcal{R}_{11} are done, a term $\Delta_0^-\Delta_0 p^2$ is left, which also occurs in the subtraction-term (4.10) (but the two do not cancel). The sum over it gives I_1 as defined in (3.11) and to be evaluated there at hard inner momentum. But since the result (3.16) allowed for $q^* \rightarrow 0$, we may write

$$\sum \Delta_0^-\Delta_0 p^2 = -\frac{1}{2} \sum \Delta_0 + \frac{m^2}{4} J_0 \quad (5.43)$$

and turn to soft integration momentum. Since the J_0 -term may be neglected, see (4.8) and (4.9), we learn that the replacements

$$\sum \Delta_0^- \Delta_0 p^2 \rightarrow \frac{1}{2} \mathcal{L} + \frac{3}{2} \mathcal{K} \quad \text{and} \quad m^2 \sum \Delta_0^- \Delta_0 \rightarrow 0 \quad (5.44)$$

are allowed in \mathcal{R}_{11} as well as in the subtraction term. By including the latter this subsection ends up with

$$\Delta \Pi_\ell^{\text{loop}} = g^2 N \left(\frac{11}{6} \mathcal{L} + \frac{11}{2} \mathcal{K} + \sum_P^{\text{soft}} \left[\frac{1}{2} \mathcal{R}_{11} + \mathcal{R}_{13} + \frac{1}{2} \mathcal{R}_{33} \right] \right) . \quad (5.45)$$

6. Result

The analysis of the preceeding sections may be summarized as follows. The only $O(g)$ -contributions to the real part of the polarisation function Π_ℓ arise from the soft tadpole and the soft loop, (5.21) and (5.45). Their sum is independent of the gauge parameter α and may be cast into the following form:

$$\Delta\Pi_\ell^{\text{1-loop soft}} = g^2 N \left(4\mathcal{K} + \sum_{j=1}^4 \mathcal{M}_j \right) \quad (6.1)$$

with

$$\begin{aligned} \mathcal{M}_1 &= \frac{1}{6m^2} \sum \Delta_t^- \Delta_\theta \frac{P^2}{p^2} + \frac{1}{3} \mathcal{L} \quad , \quad \mathcal{M}_2 = -3m^2 \sum \Delta_t^- \Delta_t - 3 \sum \Omega_t^- \Omega_t \frac{1}{p^2} \quad , \\ \mathcal{M}_3 &= -\frac{3}{2m^2} \sum \Omega_\ell^- \Omega_t \frac{P^2}{p^2} \quad , \quad \mathcal{M}_4 = -\frac{3}{2} \sum \Omega_\ell^- \Omega_\ell \frac{1}{p^2} \quad . \end{aligned} \quad (6.2)$$

Any non-covariant gauge should lead to (6.1) as well. The first term of (6.1), $4\mathcal{K}$, is positive. But the whole contribution is expected to be negative.

Obviously, the above four terms carry different index pairs. But there are more properties in favour of the decomposition. Each term \mathcal{M}_j

- (a) converges at large P_0 when summing over frequencies
- (b) is UV-stable, i.e. it does not depend on the cutoff q^*
- (c) is IR-stable (this forces the two terms of \mathcal{M}_2 together, see below)
- (d) contains two propagators, where one is taken at argument $Q - P$. Moreover, each \mathcal{M} either has the form $\sum \Delta^- \Delta f(p)$ or it can be cast into it (see below).

The statement (a) is a rather trivial consequence of the spectral representation (5.6). The latter shows that, at large P_0 , the leading term of a propagator is $1/P_0^2$ times its first moment $\int dx x \rho$.

The statement (b) is ultimately justified by evaluation. However, the special assignment of the \mathcal{L} -term can be understood immediately. At hard inner momentum the propagator Δ_t^- in \mathcal{M}_1 turns into the bare one. This is Δ_0 and cancels the extra factor P^2 . Now, the remaining sum is easily evaluated by means of (5.7) and (5.19) to give $-\mathcal{K} - \mathcal{L}/3$.

The statements (c) and (d) lead into further analysis. There is a next step in common to all four expressions (see (6.6) below), if we are able to get rid of the extra P 's in \mathcal{M}_1

and \mathcal{M}_3 . This is achieved by introducing temporarily two more propagators:

propagator	spectral density ρ	relation	$\psi(p)$	
Ω_θ	$\rho_\theta (x^2 - p^2)$	$\Omega_\theta = P^2 \Delta_\theta - \frac{2}{5} m^2 p^2$	$-\frac{8}{5} m^2 p^2$	(6.3)
Λ_t	$\rho_t (x^2 - p^2)^2$	$\Lambda_t = P^2 \Omega_t - m^2$	m^2	.

With the 'relation'-column of (6.3) and with view to Appendix D we obtain

$$\mathcal{M}_1 = \frac{4}{15} \mathcal{L} - \frac{1}{5} \mathcal{K} + \frac{1}{6m^2} \sum \Delta_t^- \Omega_\theta \frac{1}{p^2} \quad (6.4)$$

$$\mathcal{M}_3 = \frac{9}{2} \mathcal{K} - \frac{3}{2m^2} \sum \Lambda_t^- \Omega_\ell \frac{1}{p^2} . \quad (6.5)$$

Now all terms are either known (\mathcal{K} , \mathcal{L}) or have the desired form $\sum \Delta^- \Delta f$. Using the spectral representation for both propagators and doing the frequency sum, one is left with three integrations (over p and two x). The following formula reduces them to two. Let A and B be two of our propagators (not necessarily different) and ψ_A , ψ_B their minus-first moments. Then:

$$\begin{aligned} \sum A^- B f(p) &= \sum B^- A f(p) = T \int_P^3 f(p) \psi_A(p) \psi_B(p) + \\ &+ T \int_P^3 f(p) \int dx \frac{1}{x} \rho_A(x, p) \frac{Q_0}{x - Q_0} \left[B(Q_0 - x, p) + \psi_B(p) \right] . \end{aligned} \quad (6.6)$$

This formula may be derived in a straightforward manner. It is much easier, however, to go in the backward direction. Inserting the ψ -definitions (5.7) as well as the spectral representation (5.5) of B to the right of (6.6) and symmetrizing with respect to x , one arrives at the conclusion that

$$\sum_{P_0} \frac{1}{P_0^2 - x^2} \frac{1}{(P_0 - Q_0)^2 - y^2} = \frac{T}{x^2 y^2} \left(1 - \frac{1}{2} S_x \frac{Q_0 (Q_0 + x)}{(Q_0 + x)^2 - y^2} \right) \quad (6.7)$$

might have been used for the frequency sum on the left of (6.6). $S_x f(x) \equiv f(x) + f(-x)$. If x and y are soft, (6.7) is indeed correct and is valid in the same sense as (5.6). If $A \neq B$, (6.6) may be used in two versions. In such a case we favour the transversal density ρ_t to appear on the r.h.s., because it is slightly more convenient in the numerical procedure.

Using the spectral representation of B and taking (6.6) at $Q_0 = m + i\varepsilon$, the imaginary part of (6.6) is easily obtained as

$$\Im m \sum A^- B f(p) = \frac{mT}{2\pi} \int_0^\infty dp f(p) p^2 \int dx \frac{1}{x(x - m)} \rho_A(x, p) \rho_B(x - m, p) . \quad (6.8)$$

Using this formula for (6.1) one arrives at precisely the analytical expression for the damping rate. The latter is equation (25) in [12]. Note that it was obtained there in Coulomb gauge. Our longitudinal spectral density ρ_ℓ is $p^2/(p^2 - x^2)$ times that of ref. [12].

We return to the real part. The analytical continuation $Q_0 \rightarrow m + i\varepsilon$ was so far carried through in trivial expressions only. However, in (6.6) this continuation requires more care. Any propagator B can be expressed by Δ_ℓ or Δ_t . The functions Π in their denominator get an imaginary part. The real parts of Δ include this Landau damping, see (B.4) and (B.5), and are denoted in Appendix B by $\Delta^r(x, p)$. Note further that the denominator $x - Q_0$ in (6.6) is harmless because at $x = Q_0$ the square bracket vanishes too: $\psi_B(p) = -B(0, p)$. To summarize, after analytical continuation and when taking the real part of (6.6), the propagator Δ , which is B or occurs in B , has to be replaced by $\Delta^r(m - x, p)$. It includes Landau damping if $p^2 > (m - x)^2$.

Consider \mathcal{M}_2 to see the above steps at work and to verify the statement (c) as announced. Using (6.6), (B.12), the table (5.19) and the abbreviation $t \equiv x - m$ we obtain

$$\begin{aligned} \mathcal{M}_2 &= -3 \frac{m^2 T}{2\pi^2} \int_0^\infty dp \left(\mathcal{N}_\Delta + \mathcal{N}_\Omega \right) \quad \text{with} \quad \mathcal{N}_\Delta = \int dx \frac{1}{x} \rho_t \left(1 + \frac{m}{t} \left[1 + p^2 \Delta_t^r(t, p) \right] \right) \\ \text{and} \quad \mathcal{N}_\Omega &= \int dx \frac{1}{x} \rho_t \frac{p^2 - x^2}{mt} \left[1 + (p^2 - t^2) \Delta_t^r(t, p) \right] . \end{aligned} \quad (6.9)$$

The two terms \mathcal{N} still correspond to the two terms in (6.2). To realize that both \mathcal{N} are IR singular, consider p small, neglect ρ_t^{pole} and use the delta function asymptotics $\rho_t^{\text{cut}}/x \rightarrow \delta(x)/p^2$, see (B.13). With (B.6) one obtains

$$\mathcal{N}_\Delta \rightarrow \frac{5}{6p^2} \quad , \quad \mathcal{N}_\Omega \rightarrow -\frac{5}{6p^2} . \quad (6.10)$$

Clearly each of these terms would make \mathcal{M}_2 divergent in the infrared. But the two singularities cancel.

The above expression (6.9) also shows that by splitting off a factor mT of each \mathcal{M} , dimensionless quantities are obtained. One may set $m = 1$ in these quantities, whereafter the two integrations run over dimensionless variables p and x . This is the point where further evaluation is relegated to the computer.

To summarize input and output, let $\mathcal{M}_0 \equiv 4\mathcal{K}$ and $\eta_j \equiv 3\mathcal{M}_j / mT$. Then:

$$\omega^2 = m^2 \left(1 + \eta g \sqrt{N} \right) \quad \text{with} \quad \eta = \sum_{j=0}^4 \eta_j = -0.18 \quad , \quad (6.11)$$

where the result is composed of the following individual numbers:

$$\eta_0 = +0.55\,1 \quad \eta_1 = -0.13\,8 \quad \eta_2 = -0.25\,6 \quad \eta_3 = -0.12\,6 \quad \eta_4 = -0.21\,3 \quad . \quad (6.12)$$

Hence, all non-trivial contributions have the 'right' sign. And the total is negative in accordance with the intuitive picture given in the introduction. The third digits are uncertain. Consequently, even the second digit in (6.11) can not be stated with conviction.

7. On the numerical problems

This short last section is an attempt to list the pitfalls one encounters in the numerical treatment of the twofold integral in each of the four terms η_j . The structure (6.9) is typical. Both, a density ρ and some expression linear in a propagator $\Delta^r(1-x, p)$, develop their specialities over the (dimensionless) p - x -plane. Consider for example

$$\begin{aligned} \eta_4 = & -\frac{9\sqrt{3}}{16\pi} + \frac{9}{4\pi^2} \int_0^\infty dp \int dx \frac{1}{x} \rho_\ell(x, p) (x^2 - p^2) \mathcal{Q}(t, p) \\ \text{with } t \equiv x - 1, \quad \mathcal{Q}(t, p) = & \frac{p^2}{t} \left(\frac{1}{3 + p^2} + \frac{p^2 - t^2}{p^2} \Delta_\ell^r(t, p) \right) \end{aligned} \quad (7.1)$$

and with ρ_ℓ and Δ_ℓ^r from Appendix B. The delta functions in the pole contribution of ρ_ℓ lead to single integrals. In figure 3 they run along the dotted lines which start from the points $x = \pm 1$ at the x -axis. In each step (to larger p) the frequency $\omega_\ell(p)$ is determined numerically from (B.9). On the whole left dotted line the propagator Δ_ℓ^r is undamped. On the right it is damped.

The integration over the cut-part of ρ_ℓ is conveniently done with the variables $a = p + x$, $b = p - x$. It runs over an undamped pole of Δ_ℓ^r , see the dotted line inside the area $p^2 > x^2$. We imbedded this pole in the sense $x/(x^2 + \varepsilon^2)$ and worked with a suitable variable step-width. There is one more singular line at $a = 1$ which separates the damped/undamped regions of Δ_ℓ^r . With increasing p these two singular lines approach each other exponentially. To handle this speciality we separated a stripe around the $a = 1$ -line and introduced a logarithmic variable $\tau = -\ln(a - 1)$ there. In passing, there were no problems at the vertical line $x = 1$ since the "0/0" on this line can be avoided analytically.

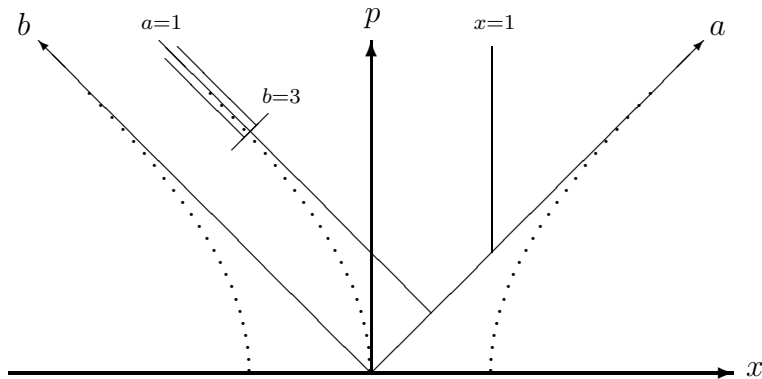


Figure 3: Integration area in the numerical evaluation of η_4 . On the dotted lines there are poles of the spectral density (outside) or of the propagator. The stripe on the line $a = 1$ had a width of $\Delta a = .2$.

Let a last warning stand at the end. In order to get all numerically relevant pieces in the p - x -area, we had to run with p up to pretty high values: not 10, not 100, but 2500 !

8. Conclusions

Braaten-Pisarski resummation is applied to calculate the real part of the gluon plasmon frequency in the next-to-leading order. Two of the three classes of contributions, which were predicted by BP to be separate gauge independent sets within the order $O(g)$ of interest, do not reach this order. But this surprise of a 'gauge independent number zero' is not in conflict with neither prediction nor any principle. The gauge dependent terms in the remaining class (soft one-loop diagrams in the effective expansion) are explicitly shown to cancel out. Within the covariant gauges (used therein) and within $O(g)$ we had no (true) IR-problem. Also, the result is independent of the soft-hard threshold q^* .

The most laborious part has been the reformulation of the loop contributions of subsection 5.2. Most probably, there is a shorter and more elegant way of evaluation. However, once one is half way inside the jungle, it is a hard decision to go back for only the belief in a better world. For each contribution, we have immediately restricted ourselves to the part $\text{Tr } BII$ and to the limit $\vec{q} \rightarrow 0$. We were thus for instance unable to check transversality [35] of the polarization function to $O(g)$.

The next-to-leading order term has been obtained with a negative coefficient. This minus sign is in accord with the intuitive picture of a system whose longitudinal-electric

mode becomes soft with decreasing temperature and increasing coupling. The details (although very special) support the general prospect of using the known high temperature limit to understand QCD perturbatively 'from above'.

Permanently, while this work grew up (and ran into every pitfall), there was a very enjoyable and helpful contact to Anton Rebhan. Thanks to Fritjof Flechsig, who checked the imaginary parts, an algebraic error could be eliminated (it invalidates the result presented in the preprint foregoing this paper). I also acknowledge encouraging discussions with Neven Bilic, Max Kreuzer, Rob Pisarski, Martin Reuter and Uwe-Jens Wiese.

Appendix A

Here the various sums (3.11) of section 3 are evaluated. The integration momenta read K and P and are considered hard in the sense $q^* < k$ with $q^* = T\sqrt{g}$. But as far as no use is made of this inequality we may play around with q^* . To start with (and to introduce notations) consider I_1 and J_0 which we combine in vector form:

$$\{J_0, I_1\} = \sum \frac{\{1, k^2\}}{K^2(K-Q)^2} = \frac{1}{2\pi^2} \int_{q^*}^{\infty} dk \{k^2, k^4\} T \sum_n \frac{1}{K^2(K-Q)^2} . \quad (\text{A.1})$$

Note that $K = (i\omega_n, \vec{k})$ but $Q = (i\omega_{n'}, \vec{0})$ with n' an outer index. To do the frequency sum we use the formula [29]

$$T \sum_n F(i\omega_n) = \frac{1}{2\pi i} \oint d\Omega F(\Omega) - \frac{1}{2\pi i} \oint d\Omega n(\Omega) S_{\Omega} F(\Omega) \equiv \oint + \oint \approx \oint , \quad (\text{A.2})$$

where S is an operator which symmetrizes: $S_{\Omega} F(\Omega) = F(\Omega) + F(-\Omega)$. The intergal with arrow runs along the imaginary axis in the Ω -plane, that with circle surrounds the right half plane counterclockwise. As is indicated to the right of (A.2) and reasoned in the main text, we omit the temperature-independent arrowed integral in this section. In (A.1) the poles surrounded in the right-half plane lie at $K_0 = k$ and at $K_0 = k + Q_0$. Note that $n(k + Q_0) = n(k)$. One obtains

$$T \sum_n \frac{1}{K^2(K-Q)^2} = -\frac{n(k)}{k} \frac{1}{Q_0} \left(\frac{1}{Q_0 - 2k} + \frac{1}{Q_0 + 2k} \right)$$

and thus

$$\{J_0, I_1\} = \frac{1}{\pi^2} \int_{q^*}^{\infty} dk k n(k) \frac{\{1, k^2\}}{4k^2 - Q_0^2} . \quad (\text{A.3})$$

For hard k the $-Q_0^2$ in the denominator can be omitted, of course. But J_0 also occurs in the 1-loop-hard analysis, where the 'strictly hard' part is to be subtracted. In the hard sense we have thus obtained $I_1 = T^2/24$. Note the relation $2I_1 = -I_0$. We may see of how this relation derives from (A.1). Compared to I_0 , the factor $\vec{k}^2/(K-Q)^2$ introduces an additional pole (factor 2). The extra propagator combines with the first as $-1/4k^2$. This gives $-1/2$. In essence, the same mechanism also applies to Z_1 and Z_0 giving $2Z_1 = -Z_0$. This way we avoid examining Z_0 here.

The sums J_1 and J'_0 contain a K^4 in the denominator. Surrounding $1/(K_0 - k)^2$ leads also to a derivative of $n(k)$, which one gets rid through integration by parts. The result is

$$J_1 = -\frac{3}{4}J_0 \quad \text{and} \quad J'_0 = J_0^{Q_0 \rightarrow 0} + \frac{1}{4\pi^2}n(q^*) \quad . \quad (\text{A.4})$$

Analytical continuation of the discrete Q_0 -values amounts to $Q_0 \rightarrow \omega + i\epsilon$. The 3/4-relation in (A.4) then holds true also for the imaginary parts.

We now turn to the doubly sums Z_1 and Z_2 . In a first step each sum is made explicit by using (A.1) and (A.2), performing the two operations S and marking the poles in the P_0 right-half plane. After the P_0 -integration is done there remain terms involving $n(\varpi)$ where $\varpi \equiv |\vec{p} - \vec{k}|$. But under the spatial P -integral (which is $f_P^3 \equiv (2\pi)^{-3} \int d^3p$) the substitution $\vec{p} \rightarrow \vec{k} - \vec{p}$ interchanges ϖ with p . This way we obtain

$$\begin{aligned} Z_{1,2} = & \int_K^3 \int_P^3 \left\{ k^2, \frac{p^2 + \varpi^2}{2} \right\} \frac{1}{2\pi i} \oint dK_0 n(K_0) \frac{1}{K_0^2 - k^2} \left(\frac{1}{(K_0 - Q_0)^2 - k^2} \right. \\ & \left. + \frac{1}{(K_0 + Q_0)^2 - k^2} \right) \frac{n(p)}{p} \left(\frac{1}{(K_0 - p)^2 - \varpi^2} + \frac{1}{(K_0 + p)^2 - \varpi^2} \right) \quad . \quad (\text{A.5}) \end{aligned}$$

With view to the K_0 -integration, poles in the right-half plane come from the denominators $K_0 - k$, $K \pm Q_0 - k$, $K_0 - p - \varpi$, but from $K_0 - p + \varpi$ and $K_0 + p - \varpi$ only if $p > \varpi$ or $p < \varpi$, respectively. The (lengthy) result will thus involve the step functions $\theta(p - \varpi)$ and $\theta(\varpi - p)$.

All Q_0 -dependence is in various denominators, and these get $i\epsilon$ terms by analytical continuation. Pairs of the corresponding delta-functions could contribute to the real part. We find, however, that any two delta peaks either do not cross or they force k (or p) down to $O(\omega)$. Thus, and thereby explicitly referring to $q^* < k$, we may omit all the $i\epsilon$ at all. Note that this no-contribution of delta pairs can be made responsible for the non- $O(g)$ of the 2-loop diagrams.

In the following, ω stands in place of an earlier Q_0 . After the K_0 -integral is done there are again unwanted ϖ in some Bose function arguments. This time we exploit the interchange of ϖ with k under the corresponding substitution in the spatial K -integral. To write down the next Z -version we need appropriate notations,

$$\begin{aligned} N_{\pm} &= \frac{1}{(p \pm k)^2 - \varpi^2} = \frac{1}{2pk} \frac{1}{u \pm 1} \quad , \quad u \equiv \cos(\vartheta) \\ R_{\pm} &= \frac{1}{(\omega \pm p + k)^2 - \varpi^2} \quad , \quad R_- = R_-^{\omega \rightarrow -\omega} \\ R_1 &= \frac{1}{\omega^2 + 2k\omega + 2p(k-p)(1+u)} \quad , \quad R_2 = R_1^{p \rightarrow -p} \quad , \end{aligned} \quad (\text{A.6})$$

where ϑ is the angle between \vec{k} and \vec{p} . We may then write $Z_{1,2} = S_{\omega} Z_{1,2}^+$ with

$$\begin{aligned} Z_{1,2}^+ &= \int_K \int_P \left\{ k^2, \frac{p^2 + \varpi^2}{2} \right\} \frac{n(p)n(k)}{2pk} \frac{1}{\omega(\omega + 2k)} (N_+ + N_- + R_+ + R_-) \\ &+ \int_K \int_P \left\{ \varpi^2, \frac{p^2 + k^2}{2} \right\} \frac{n(p)}{p} \left[\frac{n(p+k)}{2k} N_+ R_+ \right. \\ &\left. - \theta(p-k) \frac{n(p-k)}{2k} N_- R_- + \theta(k-p) \frac{n(k-p)}{2k} N_- R_- \right] \quad . \end{aligned} \quad (\text{A.7})$$

The unwanted step functions can be recombined. To show this we restrict to the relevant pieces of the twofold integral over the square bracket,

$$A = \int_0^{\infty} dk k^2 \int_{-1}^1 du \left\{ \varpi^2, \frac{p^2 + k^2}{2} \right\} \left[\dots \right] \equiv A_1 + A_2 + A_3 \quad , \quad (\text{A.8})$$

where A_j refers to the j -th term in the square bracket. In A_1 we substitute $k \rightarrow k - p$. This gives

$$A_1 = \int_p^{\infty} dk \int_{-1}^1 du \left\{ k^2 + 2p(p-k)(1+u), \frac{p^2 + (k-p)^2}{2} \right\} \frac{n(k)}{4p(1+u)} R_1 \quad . \quad (\text{A.9})$$

In A_2 we substitute $k \rightarrow p - k$ followed by $u \rightarrow -u$. This leads to just the above expression for A_1 except that the integration limits now are 0 and p . A_1 and A_2 thus combine to one integral from 0 to ∞ . In A_3 the step function sets a lower limit at p . But this turns to 0 as well by $k \rightarrow k + p$ (followed by $u \rightarrow -u$):

$$A_3 = - \int_0^{\infty} dk \int_{-1}^1 du \left\{ k^2 + 2p(k+p)(1+u), \frac{p^2 + (k+p)^2}{2} \right\} \frac{n(k)}{4p(1+u)} R_2 \quad . \quad (\text{A.10})$$

The step functions have gone, and wanted arguments enter the Bose functions. But we become aware that each integral A_j diverges when the angular integration runs down to

$u = -1$. We show that these singularities cancel (in the case at hand, but not in Z'_2). Let us first summarise the form of $Z_{1,2}^+$ so far reached:

$$Z_{1,2}^+ = \frac{1}{32\pi^4} \int_0^\infty dp n(p) \int_0^\infty dk n(k) \int_{-1}^1 du \left(\left\{ k^2, \frac{p^2 + \varpi^2}{2} \right\} \frac{2pk}{\omega(\omega + 2k)} [N_+ + N_- + R_+ + R_-] + \left[\left\{ \frac{k^2}{1+u} + 2p(p-k), \frac{p^2 + (k-p)^2}{2(1+u)} \right\} R_1 - \text{ditto}_{p \rightarrow -p} \right] \right) . \quad (\text{A.11})$$

If we decompose into partial fractions,

$$\frac{1}{1+u} R_1 = \frac{1}{\omega(\omega + 2k)} \left(\frac{1}{1+u} + 2p(p-k) R_1 \right) ,$$

the collinear singularity in Z_1^+ is removed immediately. Note that $N_+ + N_- \rightarrow N_+ - N_- = 0$ by $u \rightarrow -u$. In Z_2^+ the crucial terms add up as

$$\frac{-pk}{\omega(\omega + 2k)} \left(\frac{u}{1+u} - \frac{u}{1-u} + \frac{2}{1+u} \right) \rightarrow \frac{-2pk}{\omega(\omega + 2k)} .$$

The big round bracket in (A.11) now reads

$$\begin{aligned} \left(\dots \right) &= \frac{1}{\omega(\omega + 2k)} \left\{ 2k^2(Q_+ + Q_-) + 2(\omega + k)^2(Q_1 - Q_2) , -4pk \right. \\ &\quad \left. + \left([p^2 + (p+k+\omega)^2] Q_+ + [p^2 + (p-k)^2] Q_1 - \text{ditto}_{p \rightarrow -p} \right) \right\} , \quad (\text{A.12}) \\ &\text{where } Q_\pm = pkR_\pm \text{ and } Q_1 = p(p-k)R_1 , \end{aligned}$$

and is ready to be integrated over the relative angle. For the resulting logarithms we write $\ln(x)$ but mean $\ln(|x|)$. At this point we remember that evaluation at hard-hard momenta is sufficient. We thus reintroduce q^* as lower limits in (A.11) and expand in powers of the soft ω . For example:

$$\frac{1}{\omega(\omega + 2k)} = -\frac{1}{4k^2} + \frac{1}{2k\omega} + O(\omega) , \quad \frac{(\omega + k)^2}{\omega(\omega + 2k)} = \frac{3}{4} + \frac{k}{2\omega} + O(\omega)$$

and

$$\begin{aligned} 2 \int_{-1}^1 du Q_+ &= \ln \left(\frac{4pk + 2(p-k)\omega + \omega^2}{\omega[2p + 2k + \omega]} \right) \\ &= -\ln(\omega) + \ln \left(\frac{2pk}{p+k} \right) + \omega \frac{p+k}{2pk} - \omega \frac{1}{2p+2k} + O(\omega^2) . \quad (\text{A.13}) \end{aligned}$$

Since Z^+ will be operated with S_ω , only terms even in ω need be retained. Then, at the end, S_ω amounts to a factor of 2. Working this way we obtain the hard-hard part

$$\int_{-1}^1 du \left(\dots \right) = \left\{ \ln \left(\frac{p+k}{p-k} \right) , \frac{2p}{k} - \frac{1}{2} \ln \left(\frac{p+k}{p-k} \right) + \frac{p}{2k} \ln \left(\frac{k^2}{p^2 - k^2} \right) \right\} . \quad (\text{A.14})$$

The first term in the second component leads to decoupled integrals, which are easily identified with the hard parts of I_0 and J_0 . The result for $Z_{1,2}$, as given in (3.14) in the main text, is now obtained.

The sums Z'_1 and Z'_2 need not really be calculated, since they cancel between different diagrams. We like to show, however, how the collinear singularity looks like. We may start with the expression (A.5) taken at $Q_0 = 0$. Due to the square singularity $1/(K_0 - k)^2$ the further calculation is slightly different from the above. But (A.8) to (A.10) may be used again, except that now $\omega = 0$. At the level of (A.11) the result is

$$\begin{aligned}
Z'_{1,2} = & \frac{1}{16\pi^4} \int_0^\infty dp n(p) \int_0^\infty dk n(k) \int_{-1}^1 du \left(\left\{ k^2, \frac{p^2 + \varpi^2}{2} \right\} \right. \\
& \left[\left(p \frac{n'(k)}{n(k)} - \frac{p}{k} \right) (N_+ + N_-) - 2p(p+k)N_+^2 + 2p(p-k)N_-^2 \right] \\
& \left. + \left[\left\{ k^2, \frac{p^2 + (p-k)^2}{2} \right\} \frac{1}{1+u} N_1 - \text{ditto}_{p \rightarrow -p} \right] \right) , \quad (\text{A.15})
\end{aligned}$$

where $N_1 = R_1^{\omega=0} = \frac{-1}{2p(p-k)} \frac{1}{1+u}$.

The second square bracket agrees with that of (A.11) if taken at $\omega = 0$ (which allowed the second term of the first component to cancel). Consider the first component of (A.15). Making explicit all denominators containing u and using $u \rightarrow -u$, the collinear singularity can be packed up in a separate factor:

$$Z'_1 = \frac{1}{16\pi^4} \int_{-1}^1 du \left(\frac{1}{1+u} \right)^2 \int_0^\infty dp n(p) \int_0^\infty dk n(k) \frac{kp}{k^2 - p^2} = 0 . \quad (\text{A.16})$$

The whole term vanishes because the p - k -integral runs over a function which is antisymmetric under an interchange of p with k . We turn to the second component and treat it in a similar manner. Singular terms can again be localized, but now their prefactors remain non-zero:

$$\begin{aligned}
Z'_2 = & \frac{1}{16\pi^4} \int_0^\infty dp n(p) p \int_0^\infty dk \left(-2n'(k) + n'(k) \int_{-1}^1 du \frac{1}{1+u} \right. \\
& \left. + \frac{2}{k} n(k) - \frac{2}{k} n(k) \int_{-1}^1 du \left[\frac{1}{1+u} \right]^2 \right) . \quad (\text{A.17})
\end{aligned}$$

The last object to be considered is Z'_0 . After all, it is a rather simple sum and becomes zero through $N_+ + N_- \rightarrow 0$. To end up, we note that this is in accord with the relation $2Z'_1 = -Z'_0$ as both sides vanish.

Appendix B

Here properties of the propagators Δ_t and Δ_ℓ are detailed. These propagators are defined in (2.2) and related by (2.3) to the polarization functions $\Pi_t = \frac{1}{2} \text{Tr } A\Pi$ and $\Pi_\ell = \text{Tr } B\Pi$ at one-loop order. While here we merely only list the known facts on Π , Δ and their spectral densities ρ , they acquire a unique notation. From (2.8) to (2.11) and with P_0 a complex variable still already apart from the imaginary axis:

$$\Pi_t(P) = \frac{3}{2}m^2 g\left(\frac{P_0}{p}\right) \quad , \quad \Pi_\ell(P) = 3m^2 \left[1 - g\left(\frac{P_0}{p}\right)\right] \quad (\text{B.1})$$

$$\text{with } g(z) = z^2 - \frac{1}{2}z(z^2 - 1) \ln\left(\frac{z+1}{z-1}\right) \quad (\text{B.2})$$

$$g(z) = \frac{2}{3} + \frac{2}{15z^2} + \frac{2}{35z^4} + \dots \quad (z \rightarrow \infty) \quad .$$

If P_0 approaches the real x-axis, one derives from (B.1), (B.2) that

$$\Im \Pi_t(x + i\varepsilon, p) = -\frac{3\pi}{4} \xi \eta \theta(p^2 - x^2) \quad , \quad \xi \equiv \frac{x}{p} \quad , \quad \eta \equiv 1 - \xi^2 \quad , \quad (\text{B.3})$$

The corresponding real part is (B.1), (B.2) with the logarithm taken at the absolute value of its argument. For the imaginary part of Π_ℓ one may use the identity $\Pi_\ell + 2\Pi_t = 3m^2$. From (2.3) and (B.3) and by anticipating the notations of (B.8) we obtain

$$\Re \Delta_t(x + i\varepsilon, p) = \frac{4}{m^2} \frac{-D_t}{D_t^2 + \theta(p^2 - x^2) C_t^2} \equiv \Delta_t^r(x, p) \quad , \quad (\text{B.4})$$

$$\Re \Delta_\ell(x + i\varepsilon, p) = \frac{2}{m^2 \eta} \frac{-D_\ell}{D_\ell^2 + \theta(p^2 - x^2) C_\ell^2} \equiv \Delta_\ell^r(x, p) \quad . \quad (\text{B.5})$$

(B.4) and (B.5) are even functions of x . For a special purpose in section 6 we add the small- p behaviour

$$\Delta_t^r(m, p) = -\frac{5}{6p^2} + O(1) \quad . \quad (\text{B.6})$$

The spectral densities [7] ρ_t and ρ_ℓ have a common structure :

$$\rho = \rho^{\text{pole}} + \rho^{\text{cut}} \quad , \quad \rho^{\text{pole}} = r \delta(x - \omega) - r \delta(x + \omega) \quad , \quad \rho^{\text{cut}} = \theta(q^2 - x^2) \frac{1}{m^2} \frac{N}{D^2 + C^2} \quad , \quad (\text{B.7})$$

$$\text{with} \quad r_t = \frac{\omega_t (\omega_t^2 - p^2)}{3m^2 \omega_t^2 - (\omega_t^2 - p^2)^2} \quad , \quad r_\ell = \frac{\omega_\ell}{3m^2 - \omega_\ell^2 + p^2} \quad ,$$

$$\begin{aligned}
N_t &= 12\xi\eta \quad , \quad N_\ell = -6\frac{\xi}{\eta} \quad , \\
C_t &= 3\pi\xi\eta \quad , \quad C_\ell = 3\pi\xi \quad , \\
D_t &= 4\frac{p^2}{m^2}\eta + 6\xi^2 + 3\xi\eta \ln\left(\frac{1+\xi}{1-\xi}\right) \quad , \quad D_\ell = 2\frac{p^2}{m^2} + 6 - 3\xi \ln\left(\frac{1+\xi}{1-\xi}\right) \quad , \quad (\text{B.8})
\end{aligned}$$

The frequencies ω_t, ω_ℓ are the positive solutions ($\neq p$) of

$$\omega^2 = p^2 + \Pi_j(\omega, p) \quad (j = t, \ell) \quad . \quad (\text{B.9})$$

They are obtained by solving (B.9) numerically.

The most important moments [18, 31] of the densities ρ are

$$n = 1 \quad \int dx \, x \, \rho_j(x, p) = 1 \quad (j = t, \ell) \quad , \quad (\text{B.10})$$

$$n = 3 \quad \int dx \, x^3 \, \rho_j(x, p) = m^2 + p^2 \quad (j = t, \ell) \quad , \quad (\text{B.11})$$

$$n = -1 \quad \int dx \, \frac{1}{x} \, \rho_t(x, p) = \frac{1}{p^2} \quad , \quad \int dx \, \frac{1}{x} \, \rho_\ell(x, p) = \frac{1}{3m^2 + p^2} \quad . \quad (\text{B.12})$$

All the above details of the spectral densities are particularly important in the course of the numerical evaluation of the 'magnificent seven' of section 6. Note that the two cut-parts of ρ behave quite different. Consider p fixed and let x approach the borders $\pm p$ of the interval set by the step function. Then ρ_ℓ^{cut} runs to $-\infty$ due to the prefactor $1/\eta$. Only the squared logarithm in D_ℓ keeps the integral in e.g. (B.10) finite. On the other hand, the cut part of the transversal density has no such prefactor. If $p \ll 1$, this density is concentrated around $x = 0$. With $v \equiv \frac{4}{3\pi m^2}$ one obtains

$$\frac{1}{x} \rho_t^{\text{cut}}(x, p) \rightarrow \theta(p^2 - x^2) \frac{1}{\pi} \frac{vp}{x^2 + v^2 p^6} \rightarrow \frac{1}{p^2} \delta(x) \quad (p \rightarrow 0) \quad . \quad (\text{B.13})$$

Note that this formula trivially leads to the minus-first moment (B.12). But in the other two moments (B.10), (B.11), the pole-contribution dominates at small p .

Appendix C

Here we comment upon the treatment of the hard-hard double sums, which occur in the loop diagram of subsection 5.2. Then the resulting single sums (the 'greek sums') are collected. A typical and sufficient general example is (5.25). The denominator is already factorized. Thus the only task is rewriting the numerator appropriately. As (5.25) shows, this leads to products of the following three single sums:

$$\sum_K \frac{K_0^2}{N} \vec{k} = \sigma \frac{\vec{p}}{p^2} \quad (\text{C.1})$$

$$\sum_K \frac{K_0}{N} \vec{k} \circ \vec{k} = \beta \frac{1}{2} \left(1 - \frac{\vec{p} \circ \vec{p}}{p^2} \right) + \gamma \frac{\vec{p} \circ \vec{p}}{p^2} \quad (\text{C.2})$$

$$\begin{aligned} \sum_K \frac{k_i k_j k_\ell}{N} = & \rho \frac{1}{2} \left[\left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) \frac{p_\ell}{p^2} + \left(\delta_{j\ell} - \frac{p_j p_\ell}{p^2} \right) \frac{p_i}{p^2} + \right. \\ & \left. + \left(\delta_{i\ell} - \frac{p_i p_\ell}{p^2} \right) \frac{p_j}{p^2} \right] + \zeta \frac{p_i p_j p_\ell}{p^4} , \end{aligned} \quad (\text{C.3})$$

where $N \equiv K^2(K-Q)^2(K-P)^2$ ($= N_K$ in the main text). The form of the right-hand sides is dictated by symmetry. Note that $\vec{q} = 0$. Thus, the only direction, which the K -sums 'know' of, is that of \vec{p} . The coefficients $\sigma, \beta, \gamma, \rho$ and ζ are determined by taking traces and by multiplications with \vec{p} . The results are given in (C.9) to (C.13) below.

There occur ten such 'greek sums' in the main text. Irrespective of several relations between these sums, we shall list them all as in a table of integrals. Their evaluation towards hard integration momentum K is rather familiar and not given here. Note that the two outer momenta Q and P are soft. The shorthand notations $\Pi_\ell = \Pi_\ell(P)$, $\Pi_\ell^- = \Pi_\ell(P-Q)$, $\Delta_0 = 1/P^2$ and $\Delta_0^- = 1/(P-Q)^2$ are also used in the main text. But $\overline{P}_0 \equiv Q_0 - P_0$, $[-] \equiv [k^2 - (\vec{k} \vec{p})^2/p^2]$ and $\phi \equiv 8g^2 N$ are special to the following table. Some of the results simplify by using $b \equiv m^2 - \Pi_\ell$.

definition	result (leading terms)
$\mu = \sum \frac{KP - KQ}{N} [-] \quad ,$	$\phi \mu = \Pi_\ell - m^2 = -b$
$\tau = \sum \frac{(KP)(KQ - KP)}{N} \frac{\vec{k} \vec{p}}{p^2} \quad ,$	$\phi \tau = m^2$
$\vartheta = \sum \frac{KP}{N} [-] \quad ,$	$\phi \vartheta = \Pi_\ell^- - m^2 = -b^-$

$$\varphi = \sum \frac{KP}{N} \frac{(\vec{k} \vec{p})^2}{p^2} \quad , \quad \phi \varphi = m^2 - \overline{P}_0^2 \Delta_0^- \Pi_\ell^- \quad (\text{C.7})$$

$$\lambda = \sum \frac{K_0 (KP) (\vec{k} \vec{p})}{N} \quad , \quad \phi \lambda = p^2 \overline{P}_0 \Delta_0^- \Pi_\ell^- \quad . \quad (\text{C.8})$$

The above five sums occur in the gauge dependent contributions from the soft loop. The following five sums, which determine the physical soft loop contributions, are either symmetric (first three) or antisymmetric (last two) under the shift $P \rightarrow Q - P$:

$$\rho = \sum \frac{\vec{k} \vec{p}}{N} [-] \quad , \quad \phi Q_0 \rho = Q_0 m^2 - P_0 \Pi_\ell - \overline{P}_0 \Pi_\ell^- = P_0 b + \overline{P}_0 b^- \quad (\text{C.9})$$

$$\sigma = \sum \frac{K_0^2 (\vec{k} \vec{p})}{N} \quad , \quad \phi Q_0 \sigma = p^2 \Delta_0 P_0 \Pi_\ell + p^2 \Delta_0^- \overline{P}_0 \Pi_\ell^- \quad (\text{C.10})$$

$$\zeta = \sum \frac{1}{N} \frac{(\vec{k} \vec{p})^3}{p^2} \quad , \quad \phi Q_0 \zeta = -Q_0 m^2 + \Delta_0 P_0^3 \Pi_\ell + \Delta_0^- \overline{P}_0^3 \Pi_\ell^- \quad (\text{C.11})$$

$$\beta = \sum \frac{K_0}{N} [-] \quad , \quad \phi Q_0 \beta = \Pi_\ell^- - \Pi_\ell = b - b^- \quad (\text{C.12})$$

$$\gamma = \sum \frac{K_0}{N} \frac{(\vec{k} \vec{p})^2}{p^2} \quad , \quad \phi Q_0 \gamma = \Delta_0 P_0^2 \Pi_\ell - \Delta_0^- \overline{P}_0^2 \Pi_\ell^- \quad . \quad (\text{C.13})$$

Relations between these sums are given in (5.29) and (5.34) in the main text. They can be derived directly from the definitions.

Appendix D

Here the soft sums over single propagators are collected, which occur in section 5. For their evaluation see (5.5) to (5.8) as well as the table (5.19). For the quantities \mathcal{K} , \mathcal{L} and \mathcal{V} , the results are formulated with, see also the main text at (5.8), (5.9) and (5.42).

$$\sum \Delta_\ell = \sum (\Delta_\ell^- - \Delta_\ell) \frac{P_0}{Q_0} = -\mathcal{L} \quad (\text{D.1})$$

$$\sum \Delta_\ell \frac{m^2}{p^2} = \sum (\Delta_\ell^- - \Delta_\ell) \frac{P_0}{Q_0} \frac{m^2}{p^2} = -\mathcal{K} \quad (\text{D.2})$$

$$\sum \Delta_0 = \sum \Delta_t = \sum (\Delta_t^- - \Delta_t) \frac{P_0}{Q_0} = -3\mathcal{K} - \mathcal{L} \quad (\text{D.3})$$

$$\sum \Gamma_\ell \frac{1}{p^2} = \sum (\Gamma_\ell^- - \Gamma_\ell) \frac{P_0}{Q_0} \frac{1}{p^2} = \frac{1}{m^2} \sum \Gamma_\ell^- \frac{P_0^2}{p^2} = 2\mathcal{K} \quad (\text{D.4})$$

$$\frac{1}{m^2} \sum \Gamma_\ell = 2\mathcal{L} \quad , \quad \frac{1}{m^2} \sum \Gamma_t = -3\mathcal{K} - \mathcal{L} \quad (\text{D.5})$$

$$\sum \Delta_t \frac{m^2}{p^2} = \sum (\Delta_t^- - \Delta_t) \frac{P_0}{Q_0} \frac{m^2}{p^2} = -\mathcal{V} \quad (\text{D.6})$$

$$\sum \Gamma_t \frac{1}{p^2} = \sum (\Gamma_t^- - \Gamma_t) \frac{P_0}{Q_0} \frac{1}{p^2} = -\mathcal{V} \quad (\text{D.7})$$

$$\sum \Gamma_\ell \frac{P_0^2}{p^2} = \sum \Gamma_t \frac{P_0^2}{p^2} = 0 \quad (\text{D.8})$$

$$\sum \Omega_t = 0 \quad , \quad \frac{1}{m^2} \sum \Omega_\ell = -3\mathcal{L} \quad (\text{D.9})$$

References

- [1] S. Coleman, *Aspects of symmetry* (Cambridge Univ. Press, Cambridge, 1985) .
- [2] V. V. Klimov, Sov. Phys. JETP **55** (1982) 199 ; H.A. Weldon, Phys. Rev. **D 26** (1982) 1394 ; J.I. Kapusta, *Finite-temperature field theory* (Cambridge Univ. Press, Cambridge, 1989) .
- [3] K. Kajantje, J. Kapusta, Ann. Phys. **160** (1985) 477 .
- [4] O.K. Kalashnikov and V.V. Klimov, Sov. J. Nucl. Phys. **31** (1980) 699 ; T.H. Hannson and I. Zahed, Phys. Rev. Lett. **58** (1987) 2397 ; Nucl. Phys. B **292** (1987) 725 ; U. Heinz, K. Kajantje and T. Toimela, Phys. Lett. **B 183** (1987) 96 ; Ann. Phys. (NY) **176** (1987) 218 ; H.Th. Elze, U. Heinz, K.Kajantje and T. Toimela, Z. Phys. **C 37** (1988) 305 ; R. Kobes and G. Kunstatter, Phys. Rev. Lett. **61** (1988) 392 ; Physica **A 158** (1989) 192 ; S. Nadkarni, Phys. Rev. Lett. **61** (1988) 396 ; Physica **A 158** (1989) 226 ; R. Kobes, G. Kunstatter and K.W. Mak, Phys. Lett. **B 223** (1989) 433 ; Z. Phys. **C 45** (1989) 129 ; H. Nachbagauer, Z. Phys. **C 56** (1992) 407 .
- [5] R. Kobes, G. Kunstatter and A. Rebhan, Phys. Rev. Lett. **64** (1990) 2992 ; Nucl. Phys. **B 355** (1991) 1 .
- [6] M. Kreuzer, A. Rebhan and H. Schulz, Phys. Lett. **B 244** (1990) 58 ; U. Kraemmer, M. Kreuzer, A. Rebhan and H. Schulz, *Lecture Notes in Physics* **361** (1990) 285 .

- [7] R.D. Pisarski, Physica **A 158** (1989) 146 .
- [8] R.D. Pisarski, Physica **A 158** (1989) 246 ; Phys. Rev. Lett. **63** (1989) 1129 ; Nucl. Phys. **A 498** (1989) 423c .
- [9] E. Braaten and R. D. Pisarski, Nucl. Phys. **B 337** (1990) 569 .
- [10] E. Braaten and R. D. Pisarski, Nucl. Phys. **B 339** (1990) 310 .
- [11] J. Frenkel and J.C. Taylor, Nucl. Phys. **B 334** (1990) 199 .
- [12] E. Braaten and R. D. Pisarski, Phys. Rev. **D 42** (1990) 2156 .
- [13] J.C. Taylor and S. M. Wong, Nucl. Phys. **B 346** (1990) 115 ; Frenkel and C. Taylor, Nucl. Phys. **B 374** (1992) 156 .
- [14] R.D. Pisarski, in: *From fundamental fields to nuclear phenomena*, eds. J. A. McNeil and C. E. Price (World Scientific Publ. Co., 1991) .
- [15] E. Braaten and R. D. Pisarski, Phys. Rev. **D 45** (1992) 1827 .
- [16] E. Braaten, R.D. Pisarski and T.C. Yuan, Phys. Rev. Lett. **64** (1990) 2242 ; S.M.H. Wong, Z. Phys. **C 58** (1993) 159 .
- [17] R. Kobes, G. Kunstatter and K. Mak, Phys. Rev. **D 45** (1992) 4632 ; E. Braaten and R.D. Pisarski, Phys. Rev. **D 46** (1992) 1829 .
- [18] E. Braaten and T. C. Yuan, Phys. Rev. Lett. **66** (1991) 2183 .
- [19] E. Braaten and M.H. Thoma, Phys. Rev. **D 44** (1991) 1298 ; H. Thoma and M. Gyulassy, Nucl. Phys. **B 351** (1991) 491 .
- [20] J.P. Blaizot and E. Iancu, Nucl. Phys. **B 390** (1993) 589 ; preprint SACLAY-SPHT-93-1 .
- [21] T. Altherr and U. Kraemmer, Astropart. Phys. **1** (1992) 133 .
- [22] D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. **53** (1981) 43 .
- [23] A. K. Rebhan, Bielefeld preprint BI-TP 93/42
- [24] A.V. Smilga, Bern preprint BUTP-92/39 .

- [25] R. Baier, G. Kunstatte and D. Schiff, Phys. Rev. **D 45** (1992) R4381 ; Nucl. Phys. **B 388** (1992) 287 .
- [26] A. Rebhan, Phys. Rev. **D 46** (1992) 4779 .
- [27] K. Enqvist and K. Kainulainen, Z. Phys. **C 53** (1992) 87 .
- [28] J.I. Kapusta, Nucl. Phys. **B 148** (1979) 461 .
- [29] P. Landsman and Ch. van Weert, Phys. Reports **145** (1987) 141 .
- [30] P.V. Landshoff and A. Rebhan, Wien preprint TUW-93-03 .
- [31] H. Schulz, Phys. Lett. **B 291** (1992) 448 .
- [32] M. Le Bellac and P. Reynaud, Nucl. Phys. **B 380** (1992) 423 ; and references therein .
- [33] T. Kinoshita, J. Math. Phys. **3** (1962) 650 ; T.D. Lee and M. Nauenberg, Phys. Rev. **B 133** (1964) 1549 .
- [34] R. Baier, B. Pire and D. Schiff, Phys. Rev. **D 38** (1988) 2814 ; Y. Gabellini, T. Grandou and D. Poizat, Ann. Phys. (N.Y.) **202** (1990) 436 ; T. Altherr and T. Becherroway, Nucl. Phys. **B 330** (1990) 174 .
- [35] G. Kunstatte, Winnipeg preprint WIN-92-08 .