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TOWARDS FINITENESS WITHOUT SUPERSYMMETRY

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ABSTRACT

Some aspects of finite quantum field theories in 3+1 dimensions are discussed. A model with non-supersymmetric particle content and vanishing one- and two-loop beta functions for the gauge coupling and one-loop beta functions for Yukawa-couplings is presented.

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1 Introduction

Despite the remarkable success of the standard model of electroweak and strong interactions, it is clear that it is not the ultimate theory. It lacks a description of gravity, and, more or less for aesthetic reasons, one would prefer to have a simple gauge group uniting all interactions except gravity. Another unsatisfactory point is the large number of free parameters in the model. Yet there are indications that the standard model might in fact be quite close in its structure to some theory residing at an energy level very far beyond today's experimental accessibility. One hint pointing in this direction is the renormalizability of the standard model, which would seem extremely unnatural if it were just an effective theory of some other model residing only a few orders of magnitude away; the other hint is the fact that the different coupling constants seem to come very close to each other at an energy scale which is not too far away from the Planck scale. Let us assume that there is indeed a theory at the grand unified scale which can be described in such a way that the standard model could be derived from it without too much effort (as opposed to, for example, the difficulty of predicting nuclear physics from quantum chromodynamics). Then we have a problem of explaining why the observed mass scale is extremely small compared to the Planck scale, which provides a natural cutoff for the divergences occurring in field theory.

One possible solution is supersymmetry, where the cancellation of bosonic and fermionic loops can lead to finiteness of the theory, thus providing independence of the observed mass scale from the cutoff. Indeed, $N=4$ supersymmetric field theory and many types of $N=2$ supersymmetric theories were found to be finite to all orders in perturbation theory [1]. Any one-loop finite supersymmetric theory is automatically two-loop finite; these theories have been classified [2]. There are known criteria for such a theory to be finite to all orders [3]. Supersymmetry, in fact, faces only one problem: Up to now for none of the known particles its supersymmetric partner has been discovered. It would therefore be extremely attractive to have theories sharing the finiteness properties of supersymmetry without the strict one-to-one correspondence of fermions and bosons.

Although it seems to be widely believed that only supersymmetric theories can be finite, no proof for this assumption is known. A natural approach to this question consists in considering explicitly, order by order in perturbation theory, the divergences of a general renormalizable gauge theory. Such a theory is described in terms of gauge fields, fermions and scalars. In the long run this approach should either lead to a proof that finiteness requires supersymmetry (by showing that at some order in perturbation theory the finiteness conditions can only be solved by supersymmetric theories), or, which would be more interesting, to the discovery of some non-supersymmetric finite theories. As a starting point in this direction, Böhm and Denner [4] have considered the conditions for the vanishing of all one-loop divergences and the two-loop divergence of the gauge coupling constant of a general theory with a simple gauge group. One of their results is that finiteness implies that a theory necessarily contains all three types of fields. Böhm and Denner introduce a distinction between finiteness, defined as the vanishing of the divergences associated with coupling constants, and complete finiteness, which also includes the wave functions. I prefer to use the following definition of finiteness: A theory is finite if the Lagrangian can be formulated in terms of finite bare parameters in such a way that physical quantities, calculated with the use of some regularization scheme (involving,

for example, a cutoff), have a smooth finite limit when the regularization is turned off (the cutoff goes to infinity). In this way we avoid not only the gauge dependence that plagues the concept of complete finiteness, but also complications due to the fact that beta functions can become gauge dependent at higher orders in perturbation theory. Of course, up to the orders we shall consider, vanishing of the beta functions is a necessary condition for finiteness. We still have a dependence on the regularization scheme, however, because the quadratic divergences manifest in cutoff regularization [5] are absent in dimensional regularization.

Substantial progress in the analysis of the one- and two-loop finiteness conditions was made in refs. [6, 7]. It was shown there that for any finite theory a certain quantity F , which depends only on the gauge group and the matter content of the theory, fulfils the (extremely restrictive) inequality $F \leq 1$. Furthermore, at $F = 1$ the vanishing of the one- and two-loop beta functions for the gauge coupling and the one-loop beta functions for the Yukawa couplings are equivalent to a system of highly symmetric equations, called the $F = 1$ system, for the bare Yukawa couplings of the theory. Both supersymmetric and (if they exist) completely finite theories obey $F = 1$, and there is overwhelming evidence (although no proof) that any finite theory must obey $F = 1$. This raises the question whether any solution of the $F = 1$ system has to consist of supersymmetric Yukawa couplings. The answer to this question will be given in the present paper by constructing solutions to these equations for a particle content which is definitely not supersymmetric.

In the following section we will briefly review the approach of [4] and the results of [6, 7]. In the third section we will study in detail the $F = 1$ system for a specific non-supersymmetric $F = 1$ particle content. Finally we will give a discussion of our results and an outlook to open questions.

2 Finiteness conditions and $F = 1$

We consider the general renormalizable quantum field theory defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + i\bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i + \frac{1}{2}D_\mu \phi_\alpha D^\mu \phi_\alpha - \frac{1}{2}\psi_i \psi_j Y_{ij}^\alpha \phi_\alpha - \frac{1}{2}\bar{\psi}_i \bar{\psi}_j Y_{ij}^{\alpha*} \phi_\alpha - \frac{1}{24}V^{\alpha\beta\gamma\delta} \phi_\alpha \phi_\beta \phi_\gamma \phi_\delta, \quad (1)$$

where we have omitted gauge fixing, ghost, mass and ϕ^3 terms. We assume that the gauge group G is simple. The Weyl fermions ψ_i carry indices i, j, \dots of some (generically reducible) representation R_F of G ; the bosons ϕ_α correspond to a representation R_B which may be chosen real. The covariant derivative D_μ acting on the Weyl fermions ψ_i or real bosons ϕ_α is given by $D_\mu = \partial_\mu - igA_\mu^a T^a$, where the hermitian matrices T^a are the generators T_F of R_F or T_B of R_B , respectively. They fulfill

$$[T^a, T^b] = if^{abc}T^c. \quad (2)$$

The Yukawa couplings Y_{ij}^α are symmetric under the exchange of i and j , $Y_{ij}^\alpha = Y_{ji}^\alpha$, and the scalar couplings $V_{\alpha\beta\gamma\delta}$ are symmetric under any permutation of their indices. Gauge invariance of the action implies

$$Y_{kj}^\alpha (T_F)_{ki}^a + Y_{ik}^\alpha (T_F)_{kj}^a + Y_{ij}^\beta (T_B)^{a\beta\alpha} = 0 \quad (3)$$

and

$$V^{\varepsilon\beta\gamma\delta}(T_B)^{a\varepsilon\alpha} + V^{\alpha\varepsilon\gamma\delta}(T_B)^{a\varepsilon\beta} + V^{\alpha\beta\varepsilon\delta}(T_B)^{a\varepsilon\gamma} + V^{\alpha\beta\gamma\varepsilon}(T_B)^{a\varepsilon\delta} = 0. \quad (4)$$

The β -functions for this theory have been calculated in R_ξ gauge with dimensional regularization up to two loops [8]. The vanishing of the gauge coupling β function to first and second order in the loop expansion requires

$$22c_g - 4S_F - S_B = 0 \quad (5)$$

and

$$3 \operatorname{Tr}(C_F Y^{\beta\dagger} Y^\beta) - g^2 d_g (6Q_F + 6Q_B + c_g(10S_F + S_B - 34c_g)) = 0. \quad (6)$$

The Dynkin indices S_F and S_B and the quadratic Casimir operators C_F and C_B are defined by

$$\operatorname{Tr} T^a T^b = \delta^{ab} S \quad \text{and} \quad C = T^a T^a, \quad (7)$$

implying $S = \frac{1}{d_g} \operatorname{Tr} C$, where d_g is the dimension of the group G . If the group representation is irreducible, C is proportional to the unit matrix, $C = c\mathbf{1}$. c_g is the Casimir eigenvalue of the adjoint representation. By Q_F and Q_B we denote expressions $Q = \frac{1}{d_g} \operatorname{Tr} C^2$. S and Q are additive with respect to composition of representations, $S = \sum_I S_I$ and $Q = \sum_I Q_I$ for $R = \oplus R_I$. The finiteness condition coming from the Yukawa couplings at first loop order is

$$4Y^\beta Y^{\alpha\dagger} Y^\beta + Y^\alpha Y^{\beta\dagger} Y^\beta + Y^\beta Y^{\beta\dagger} Y^\alpha + Y^\beta \operatorname{Tr}(Y^{\alpha\dagger} Y^\beta + Y^{\beta\dagger} Y^\alpha) - 6g^2(Y^\alpha C_F + C_F^T Y^\alpha) = 0. \quad (8)$$

One-loop finiteness of the quartic scalar coupling requires

$$\begin{aligned} & V^{\alpha\beta\lambda\varepsilon} V^{\gamma\delta\lambda\varepsilon} + V^{\alpha\gamma\lambda\varepsilon} V^{\beta\delta\lambda\varepsilon} + V^{\alpha\delta\lambda\varepsilon} V^{\beta\gamma\lambda\varepsilon} \\ & - 3g^2(C_B^{\alpha\lambda} V^{\lambda\beta\gamma\delta} + C_B^{\beta\lambda} V^{\alpha\lambda\gamma\delta} + C_B^{\gamma\lambda} V^{\alpha\beta\lambda\delta} + C_B^{\delta\lambda} V^{\alpha\beta\gamma\lambda}) \\ & + \frac{1}{2}[\operatorname{Tr}(Y^{\alpha\dagger} Y^\lambda + Y^{\lambda\dagger} Y^\alpha) V^{\lambda\beta\gamma\delta} + \operatorname{Tr}(Y^{\beta\dagger} Y^\lambda + Y^{\lambda\dagger} Y^\beta) V^{\alpha\lambda\gamma\delta} \\ & + \operatorname{Tr}(Y^{\gamma\dagger} Y^\lambda + Y^{\lambda\dagger} Y^\gamma) V^{\alpha\beta\lambda\delta} + \operatorname{Tr}(Y^{\delta\dagger} Y^\lambda + Y^{\lambda\dagger} Y^\delta) V^{\alpha\beta\gamma\lambda}] \\ & + 3g^4(\{T_B^a, T_B^b\}^{\alpha\beta} \{T_B^a, T_B^b\}^{\gamma\delta} + \{T_B^a, T_B^b\}^{\alpha\gamma} \{T_B^a, T_B^b\}^{\beta\delta} + \{T_B^a, T_B^b\}^{\alpha\delta} \{T_B^a, T_B^b\}^{\beta\gamma}) \\ & - 2 \operatorname{Tr}[(Y^{\alpha\dagger} Y^\beta + Y^{\beta\dagger} Y^\alpha)(Y^{\gamma\dagger} Y^\delta + Y^{\delta\dagger} Y^\gamma) + (Y^{\alpha\dagger} Y^\gamma + Y^{\gamma\dagger} Y^\alpha)(Y^{\beta\dagger} Y^\delta + Y^{\delta\dagger} Y^\beta) \\ & + (Y^{\alpha\dagger} Y^\delta + Y^{\delta\dagger} Y^\alpha)(Y^{\beta\dagger} Y^\gamma + Y^{\gamma\dagger} Y^\beta)] = 0. \end{aligned} \quad (9)$$

The one-loop finiteness conditions for the scalar and fermion masses and for the ϕ^3 -coupling are proportional to the bare values of these quantities and can therefore be solved by setting them to zero. Denoting by d_R the dimension of a representation R , the finiteness conditions are invariant under an $O(d_B) \times U(d_F)$ symmetry, which is however broken by the invariance conditions (3) and (4).

In dimensional regularization quadratic divergences vanish automatically. Using a cutoff regularization, one finds the finiteness conditions

$$2c_g - 2S_F + S_B = 0 \quad (10)$$

and

$$V^{\alpha\beta\lambda\lambda} + 6g^2 C_B^{\alpha\beta} - 2 \operatorname{Tr}(Y^{\alpha\dagger} Y^\beta + Y^{\beta\dagger} Y^\alpha) = 0 \quad (11)$$

for the vanishing of the quadratic divergences of vector and scalar masses, respectively [5].

It was shown in refs. [6, 7] that eqs. 5, 6 and 8 imply

$$F := \sqrt{\frac{Q_F + Q_B + c_g(S_F - 2c_g)}{3Q_F}} \leq 1 \quad (12)$$

and that for $F = 1$, i.e.

$$Q_B - 2Q_F + c_g(S_F - 2c_g) = 0, \quad (13)$$

eqs. 6 and 8 are equivalent to

$$Y^{\alpha\dagger}Y^\alpha = 6g^2C_F, \quad (14)$$

$$\text{Im Tr } Y^{\alpha\dagger}Y^\beta = 0 \quad (15)$$

and

$$Y_{ij}^\alpha Y_{kl}^\alpha + Y_{ik}^\alpha Y_{jl}^\alpha + Y_{il}^\alpha Y_{jk}^\alpha = 0. \quad (16)$$

These equations are fulfilled by all one-loop finite supersymmetric theories, and it was conjectured that all finite theories might share the property $F = 1$. Whether or not this is true, eqs. 5 and 13 to 16 are a good starting point for a search for finite models. Only when we have a solution to these equations, we can start to consider the finiteness condition for the scalar quartic couplings (9).

The solutions to eqs. 5 and 13, which determine the particle content of a potentially finite theory, have been classified [6]. We will refer to these solutions as F=1 particle contents. The straightforward way to find a finite theory is to pick one of these models and try to solve eqs. 14 to 16 for the specific group representations occurring in the model. This is what we will do in the next section.

3 Solving the $F = 1$ system

The simplest non-supersymmetric model in the list of [6] corresponds to a gauge group $SU(n)$. It contains six scalars in the adjoint representation, four fermions in the antisymmetric representation and four fermions in the symmetric representation of the gauge group,

$$R_B = 6R_{Ad}, \quad R_F = 4R_A + 4R_S \quad (17)$$

(we do not yet distinguish a group representation and its complex conjugate here). With

$$S_A = n - 2, \quad S_S = n + 2, \quad S_{Ad} = c_g = 2n, \quad (18)$$

$$Q_A = 2(n - 2)^2(n + 1)/n, \quad Q_S = 2(n + 2)^2(n - 1)/n \quad \text{and} \quad Q_{Ad} = 4n^2 \quad (19)$$

one can easily check that the conditions for an $F = 1$ particle content, eqs. 5 and 13, are indeed fulfilled. Surprisingly, even eq. 10 for the vanishing of the quadratic divergence of the photon mass is fulfilled. The reason is the following: N=4 supersymmetric Yang-Mills theory contains six scalars and four fermions which are all in the adjoint representation. By replacing each of the four adjoint fermions by a fermion in the antisymmetric and a fermion

in the symmetric representation, we obtain the present model. Eqs. 5, 13 and 10 are linear in the Dynkin indices S and the quantities Q . Since $S_A + S_S = S_{Ad}$ and $Q_A + Q_S = Q_{Ad}$, the fact that N=4 supersymmetry fulfils these equations implies that our model also fulfils them. In fact, taking the particle content of any one-loop finite supersymmetric theory and replacing adjoint representations by symmetric and antisymmetric representations or vice versa will yield $F = 1$ particle contents which also satisfy the criterion (10) for the vanishing of the quadratic divergence of the photon mass.

When we try to solve the F=1-system (14) – (16), we have to destroy the symmetric form of these equations by decomposing the fermionic and the bosonic representation into their irreducible components [6, 9]. We replace the fermionic index i by (I, \tilde{i}) where the new index I runs through the set of irreducible representations and $\tilde{i} = \tilde{i}(I)$ takes the values $1, \dots, d_I$. In the same way we decompose the bosonic representation, $\alpha \rightarrow (A, \tilde{\alpha})$. Rewriting our fermions, bosons and Yukawa couplings yields

$$\psi_i \rightarrow \psi_i^I, \quad \phi_\alpha \rightarrow \phi_\alpha^A \quad \text{and} \quad Y_{ij}^\alpha \rightarrow Y_{A\tilde{i}\tilde{j}}^{\alpha IJ} = \sum_{k=1}^{N(A,I,J)} (Z^{(k)})_{IJ}^A (\Lambda^{(k)})_{\tilde{i}\tilde{j}}^{\tilde{\alpha}}, \quad (20)$$

where $N(A, I, J)$ is the number of independent invariant tensors $(\Lambda^{(k)})_{\tilde{i}\tilde{j}}^{\tilde{\alpha}}$ satisfying an analogue of eq. 3 with indices in the representations A, I and J . In the case of our model (17), A runs from 1 to 6, $\tilde{\alpha} = a$ is an index in the adjoint representation, and the set of I 's decomposes as $\{I\} = \{I_A, I_{\bar{A}}, I_S, I_{\bar{S}}\}$. Luckily, we have some obvious candidates for invariant tensors. Due to eq. 2, $(T_A)_{\mu\nu}^a$ is an invariant coupling between a scalar in the adjoint representation and fermions in the antisymmetric representation and its complex conjugate; T_A is of type (Ad, \bar{A}, A) . Of course, we also have $(T_{\bar{A}})_{\mu\nu}^a = -(T_A^T)_{\mu\nu}^a$ (of type (Ad, A, \bar{A})) and the analogous constructions for the symmetric representation and its complex conjugate. In the usual tensorial construction of $SU(n)$ representations from the fundamental representation, we can write T_A as a trace over the fundamental representation

$$(T_A)_{\mu\nu}^a = \text{Tr}(A^\mu T^a A^\nu), \quad (21)$$

where T^a is the generator of the fundamental representation and A_{ij}^μ is the tensor relating the antisymmetric representation with the fundamental one,

$$A_{ij}^\mu = -A_{ji}^\mu, \quad A^{\mu\dagger} = A^\mu, \quad A_{ij}^\mu A_{kl}^\mu = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}, \quad \text{Tr}(A^\mu A^\nu) = 2\delta^{\mu\nu}. \quad (22)$$

The same construction works for the symmetric representation with A^μ replaced by S^μ ,

$$S_{ij}^\rho = S_{ji}^\rho, \quad S^{\mu\dagger} = S^\mu, \quad S_{ij}^\rho S_{kl}^\rho = \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}, \quad \text{Tr}(S^\mu S^\nu) = 2\delta^{\mu\nu}. \quad (23)$$

Other obvious possibilities are

$$\Lambda_{\mu\rho}^a = \text{Tr}(A^\mu T^a S^\rho), \quad (24)$$

of type (Ad, \bar{A}, S) , and its transposed and complex and hermitian conjugate. It is well known that there are as many invariant tensors relating k irreducible representations of $SU(n)$ as there are singlets in the decomposition of the direct product of these representations; this is the same as the number of occurrences of the complex conjugate of the k^{th} representation in the product of the first $k - 1$ representations. In our case this means that there are as

many invariant tensors as there are adjoint representations in the product of the fermionic representations. This allows us to check that the invariants found above are indeed all that can occur. If we choose our basic invariant tensors such that a tensor of type (Ad, R_1, \bar{R}_2) is the transposed of the corresponding tensor of type (Ad, \bar{R}_2, R_1) (without extra factors), then the symmetry of the Yukawa couplings is equivalent to

$$Z_{IJ}^A = Z_{JI}^A, \quad \text{i.e.} \quad Z_{I_A J_{\bar{A}}}^A = Z_{J_{\bar{A}} I_A}^A \quad \text{etc.} \quad (25)$$

The variables for which we have to solve are therefore $Z_{I_A J_{\bar{A}}}^A, Z_{I_A J_{\bar{S}}}^A, Z_{I_S J_{\bar{A}}}^A$ and $Z_{I_S J_{\bar{S}}}^A$. By inserting the decomposition (20) into eqs. 14 to 16, we now derive equations for the Z 's. For instance,

$$\begin{aligned} (Y_A^{\dagger a} Y_B^b)_{\mu\nu}^{I_{\bar{A}} J_{\bar{A}}} &= Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^B \text{Tr}(A^\mu T^a A^\lambda) \text{Tr}(A^\lambda T^b A^\nu) + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^B \text{Tr}(A^\mu T^a S^\rho) \text{Tr}(S^\rho T^b A^\nu) \\ &= Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^B A_{ij}^\mu T_{jk}^a A_{ki}^\lambda A_{i'k'}^\lambda T_{k'j'}^b A_{j'i'}^\nu + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^B A_{ij}^\mu T_{jk}^a S_{ki}^\rho S_{i'k'}^\rho T_{k'j'}^b A_{j'i'}^\nu \\ &= Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^B A_{ij}^\mu T_{jk}^a (\delta_{kk'} \delta_{ii'} - \delta_{ki'} \delta_{ik'}) T_{k'j'}^b A_{j'i'}^\nu \\ &\quad + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^B A_{ij}^\mu T_{jk}^a (\delta_{kk'} \delta_{ii'} + \delta_{ki'} \delta_{ik'}) T_{k'j'}^b A_{j'i'}^\nu \\ &= (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^B + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^B) \text{Tr}(A^\nu A^\mu T^a T^b) \\ &\quad - (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^B - Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^B) \text{Tr}(A^\mu T^a (A^T)^\nu (T^T)^b). \end{aligned} \quad (26)$$

Summing over $a = b$ and $A = B$ and making use of

$$T_{ij}^a T_{kl}^a = \delta_{il} \delta_{jk} - \frac{1}{n} \delta_{ij} \delta_{kl} \quad (27)$$

and (22), we get

$$\begin{aligned} (Y^{\dagger \alpha} Y^\alpha)_{\mu\nu}^{I_{\bar{A}} J_{\bar{A}}} &= (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A) \text{Tr}(A^\nu A^\mu T^a T^a) \\ &\quad - (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A - Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A) \text{Tr}(A^\mu T^a (A^T)^\nu (T^T)^a) \\ &= (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A + Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A) 2(n-1/n) \delta^{\mu\nu} \\ &\quad - (Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A - Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A) 2(1+1/n) \delta^{\mu\nu} \\ &= [(2n-2-4/n) Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A + (2n+2) Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A] \delta^{\mu\nu}. \end{aligned} \quad (28)$$

Repeating these calculations for the other irreducible components of the fermionic representation and collecting the coefficients of the Kronecker deltas carrying indices of the irreducible representations, we get the following set of equations from eq. 14:

$$(2n-2-4/n) Z_{I_A K_{\bar{A}}}^{*A} Z_{J_A K_{\bar{A}}}^A + (2n+2) Z_{I_A K_{\bar{S}}}^{*A} Z_{J_A K_{\bar{S}}}^A = 6g^2 c_A \delta_{I_A J_A}, \quad (29)$$

$$(2n-2-4/n) Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A J_{\bar{A}}}^A + (2n+2) Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S J_{\bar{A}}}^A = 6g^2 c_A \delta_{I_{\bar{A}} J_{\bar{A}}}, \quad (30)$$

$$(2n-2) Z_{I_S K_{\bar{A}}}^{*A} Z_{J_S K_{\bar{A}}}^A + (2n+2-4/n) Z_{I_S K_{\bar{S}}}^{*A} Z_{J_S K_{\bar{S}}}^A = 6g^2 c_S \delta_{I_S J_S}, \quad (31)$$

$$(2n-2) Z_{K_A I_{\bar{S}}}^{*A} Z_{K_A J_{\bar{S}}}^A + (2n+2-4/n) Z_{K_S I_{\bar{S}}}^{*A} Z_{K_S J_{\bar{S}}}^A = 6g^2 c_S \delta_{I_{\bar{S}} J_{\bar{S}}}. \quad (32)$$

By taking traces over the fermionic indices in eq. 26 and its analogues and putting the different pieces together, we can translate eq. 15 to

$$\text{Im}[(2n-4) Z_{K_A I_{\bar{A}}}^{*A} Z_{K_A I_{\bar{A}}}^B + 2n Z_{K_A I_{\bar{S}}}^{*A} Z_{K_A I_{\bar{S}}}^B + 2n Z_{K_S I_{\bar{A}}}^{*A} Z_{K_S I_{\bar{A}}}^B + (2n+4) Z_{K_S I_{\bar{S}}}^{*A} Z_{K_S I_{\bar{S}}}^B] = 0. \quad (33)$$

In a similar fashion eq. 16 gives

$$Z_{I_A K_{\bar{A}}}^A Z_{J_A L_{\bar{A}}}^A = 0, \quad (34)$$

$$Z_{I_S K_{\bar{S}}}^A Z_{J_S L_{\bar{S}}}^A = 0, \quad (35)$$

$$Z_{I_A K_{\bar{A}}}^A Z_{J_S L_{\bar{S}}}^A = 0, \quad (36)$$

$$Z_{I_A K_{\bar{S}}}^A Z_{J_S L_{\bar{A}}}^A = 0, \quad (37)$$

$$Z_{I_A K_{\bar{S}}}^A Z_{J_A L_{\bar{S}}}^A + Z_{I_A L_{\bar{S}}}^A Z_{J_A K_{\bar{S}}}^A = 0, \quad (38)$$

$$Z_{I_S K_{\bar{A}}}^A Z_{J_S L_{\bar{A}}}^A + Z_{I_S L_{\bar{A}}}^A Z_{J_S K_{\bar{A}}}^A = 0, \quad (39)$$

$$Z_{I_A K_{\bar{A}}}^A Z_{J_A L_{\bar{S}}}^A - Z_{I_A L_{\bar{S}}}^A Z_{J_A K_{\bar{A}}}^A = 0, \quad (40)$$

$$Z_{I_A K_{\bar{A}}}^A Z_{J_S L_{\bar{A}}}^A - Z_{I_A L_{\bar{A}}}^A Z_{J_S K_{\bar{A}}}^A = 0, \quad (41)$$

$$Z_{I_A K_{\bar{S}}}^A Z_{J_S L_{\bar{S}}}^A - Z_{I_A L_{\bar{S}}}^A Z_{J_S K_{\bar{S}}}^A = 0, \quad (42)$$

$$Z_{I_S K_{\bar{A}}}^A Z_{J_S L_{\bar{S}}}^A - Z_{I_S L_{\bar{S}}}^A Z_{J_S K_{\bar{A}}}^A = 0. \quad (43)$$

With the definitions

$$\tilde{Z}_{I_A J_{\bar{A}}}^A = \sqrt{(n-2)/6g^2} Z_{I_A J_{\bar{A}}}^A, \quad \tilde{Z}_{I_A J_{\bar{S}}}^A = \sqrt{n/6g^2} Z_{I_A J_{\bar{S}}}^A, \quad (44)$$

$$\tilde{Z}_{I_S J_{\bar{A}}}^A = \sqrt{n/6g^2} Z_{I_S J_{\bar{A}}}^A, \quad \tilde{Z}_{I_S J_{\bar{S}}}^A = \sqrt{(n+2)/6g^2} Z_{I_S J_{\bar{S}}}^A, \quad (45)$$

$$\tilde{c}_A = \frac{nc_A}{2(n+1)} = n-2 \quad \text{and} \quad \tilde{c}_S = \frac{nc_S}{2(n-1)} = n+2, \quad (46)$$

eqs. 29 to 33 can be rewritten as

$$\tilde{Z}_{I_A K_{\bar{A}}}^{*A} \tilde{Z}_{J_A K_{\bar{A}}}^A + \tilde{Z}_{I_A K_{\bar{S}}}^{*A} \tilde{Z}_{J_A K_{\bar{S}}}^A = \tilde{c}_A \delta_{I_A J_A}, \quad (47)$$

$$\tilde{Z}_{K_A I_{\bar{A}}}^{*A} \tilde{Z}_{K_A J_{\bar{A}}}^A + \tilde{Z}_{K_S I_{\bar{A}}}^{*A} \tilde{Z}_{K_S J_{\bar{A}}}^A = \tilde{c}_A \delta_{I_{\bar{A}} J_{\bar{A}}}, \quad (48)$$

$$\tilde{Z}_{I_S K_{\bar{A}}}^{*A} \tilde{Z}_{J_S K_{\bar{A}}}^A + \tilde{Z}_{I_S K_{\bar{S}}}^{*A} \tilde{Z}_{J_S K_{\bar{S}}}^A = \tilde{c}_S \delta_{I_S J_S}, \quad (49)$$

$$\tilde{Z}_{K_A I_{\bar{S}}}^{*A} \tilde{Z}_{K_A J_{\bar{S}}}^A + \tilde{Z}_{K_S I_{\bar{S}}}^{*A} \tilde{Z}_{K_S J_{\bar{S}}}^A = \tilde{c}_S \delta_{I_{\bar{S}} J_{\bar{S}}}, \quad (50)$$

$$\text{Im}[\tilde{Z}_{K_A I_{\bar{A}}}^{*A} \tilde{Z}_{K_A I_{\bar{A}}}^B + \tilde{Z}_{K_A I_{\bar{S}}}^{*A} \tilde{Z}_{K_A I_{\bar{S}}}^B + \tilde{Z}_{K_S I_{\bar{A}}}^{*A} \tilde{Z}_{K_S I_{\bar{A}}}^B + \tilde{Z}_{K_S I_{\bar{S}}}^{*A} \tilde{Z}_{K_S I_{\bar{S}}}^B] = 0, \quad (51)$$

whereas eqs. 34 to 43 remain the same with the Z 's replaced by \tilde{Z} 's. Our set of equations is invariant under an $O(6)$ symmetry corresponding to a mixing of scalar representations and four $U(2)$'s from the fermionic sector. Taking traces and appropriate linear combinations of eqs. 47 – 50 we derive

$$\tilde{c}_S(N_{\bar{S}} - N_S) + \tilde{c}_A(N_{\bar{A}} - N_A) = 0, \quad (52)$$

where N_R denotes the number of occurrences of a representation R . With $N_{\bar{S}} = 4 - N_S$ and $N_{\bar{A}} = 4 - N_A$ this implies

$$(n+2)(2 - N_S) + (n-2)(2 - N_A) = 0. \quad (53)$$

For $n = 6$ this is solved by $N_S = \pm 1, N_A = \mp 4$; the corresponding field theory, however, is anomalous. The only other solution, valid for any n , is

$$N_{\bar{S}} = N_S = N_{\bar{A}} = N_A = 2. \quad (54)$$

It is quite instructive to count the number of independent equations for the Z 's. Eqs. 34 to 43 are $2 \times 10 + 2 \times 16 + 2 \times 9 + 4 \times 4$ complex equations, 47 to 50 are $4 \times 4 - 1$ real equations (the -1 comes from the fact that there is a vanishing linear combination of the traces over these equations), and eq. 51 stands for 15 real equations. In addition we have the bosonic $O(6)$ and the fermionic $(U(2))^4$ symmetry which might be fixed by adding $15 + 4 \times 4$ real equations, giving a total of 233 real restrictions on the 96 complex, i.e. 192 real variables Z_{IJ}^A .

If we consider the set $\{\tilde{Z}_{I_A J_{\bar{A}}}^A, \tilde{Z}_{I_S J_{\bar{S}}}^A\}$ as a set of 6-vectors $x_{(l)}^A$, eqs. 34 to 36 imply that two (not necessarily different) of these vectors are orthogonal to each other with respect to the scalar product defined as the sum of the products of the complex components. Taking a specific non-vanishing vector, say $x_{(1)}$, we can use the $O(6)$ freedom to rotate the real parts of all of its components into the first component; then the other components are imaginary and we can use the residual $O(5)$ to rotate them into the second component: $x_{(1)} = (c, ir, 0, 0, 0, 0)$, where c is complex and r is real. $x_{(1)}^A x_{(1)}^A = 0$ implies $c = \pm r$ (we choose the $+$ sign), and $x_{(1)}^A x_{(l)}^A = 0$ implies $x_{(l)} = (c_{(l)}, ic_{(l)}, \dots)$. The same argument can be applied to the remaining components to find $x_{(2)} = (c_{(2)}, ic_{(2)}, s, is, \dots)$ and $x_{(l)} = (c_{(l)}, ic_{(l)}, d_{(l)}, id_{(l)}, \dots)$, and finally we conclude

$$x_{(l)} = (c_{(l)}, ic_{(l)}, d_{(l)}, id_{(l)}, e_{(l)}, ie_{(l)}). \quad (55)$$

Let us now turn our attention to the vectors $\{y_{(m)}^A\} = \{\tilde{Z}_{I_A J_{\bar{S}}}^A, \tilde{Z}_{I_S J_{\bar{A}}}^A\}$. In addition to eqs. 37 to 39 we know from a linear combination of the traces of eqs. 47 and 48 that $\tilde{Z}_{I_A J_{\bar{S}}}^{*A} \tilde{Z}_{I_A J_{\bar{S}}}^A = \tilde{Z}_{I_S J_{\bar{A}}}^{*A} \tilde{Z}_{I_S J_{\bar{A}}}^A$, implying that all $\tilde{Z}_{I_A J_{\bar{S}}}^A$ vanish if and only if all $\tilde{Z}_{I_S J_{\bar{A}}}^A$ vanish. A careful analysis similar to the one above shows that there are two types of solutions: One is just the equivalent of (55), whereas the second type of solution has the following form:

$$\tilde{Z}_{I_A J_{\bar{S}}}^2 = i\tilde{Z}_{I_A J_{\bar{S}}}^1, \quad \tilde{Z}_{I_A J_{\bar{S}}}^3 = \tilde{Z}_{I_A J_{\bar{S}}}^4 = \tilde{Z}_{I_A J_{\bar{S}}}^5 = \tilde{Z}_{I_A J_{\bar{S}}}^6 = 0 \quad \forall I_A, J_{\bar{S}}, \quad (56)$$

$$\tilde{Z}_{1_S 1_{\bar{A}}} = (c_{11}, ic_{11}, d_{11}, id_{11}, 0, 0) \quad \text{with} \quad d_{11} \neq 0, \quad (57)$$

$$\tilde{Z}_{1_S 2_{\bar{A}}} = (c_{12}, ic_{12}, d_{12}, id_{12}, e_{12}, ie_{12}), \quad (58)$$

$$\tilde{Z}_{2_S 1_{\bar{A}}} = (c_{21}, ic_{21}, d_{21}, id_{21}, e_{21}, -ie_{21}), \quad (59)$$

$$\tilde{Z}_{2_S 2_{\bar{A}}} = (c_{22}, ic_{22}, \frac{d_{12}d_{21} - e_{12}e_{21}}{d_{11}}, i\frac{d_{12}d_{21} + e_{12}e_{21}}{d_{11}}, \frac{e_{12}d_{21} + d_{12}e_{21}}{d_{11}}, i\frac{e_{12}d_{21} - d_{12}e_{21}}{d_{11}}), \quad (60)$$

and of course the same expressions with $(A, \bar{S}) \leftrightarrow (S, \bar{A})$ also form a solution. We will restrict our attention to the first type, i.e. we assume the same form (55) for the $\{y_{(m)}^A\}$ as for the $\{x_{(l)}^A\}$ in some (possibly different) basis. Rewriting eq. 51 as

$$I^{AB} = \text{Im}[\sum_l x_{(l)}^{*A} x_{(l)}^B + \sum_m y_{(m)}^{*A} y_{(m)}^B] = 0, \quad (61)$$

we get, in the basis where the $x_{(l)}$ take the form (55),

$$0 = I^{12} + I^{34} + I^{56} = \sum_l (|c_{(l)}|^2 + |d_{(l)}|^2 + |e_{(l)}|^2) + \sum_m \sum_{k=1}^3 \text{Im}(y_{(m)}^{*2k-1} y_{(m)}^{2k}), \quad (62)$$

implying

$$\frac{1}{2} \sum_l |x_{(l)}|^2 = \sum_m \sum_{k=1}^3 -\text{Im}(y_{(m)}^{*2k-1} y_{(m)}^{2k})$$

$$\begin{aligned}
&\leq \sum_m \sum_{k=1}^3 |y_{(m)}^{*2k-1} y_{(m)}^{2k}| \\
&\leq \frac{1}{2} \sum_m \sum_{k=1}^3 (|y_{(m)}^{2k-1}|^2 + |y_{(m)}^{2k}|^2) \\
&= \frac{1}{2} \sum_m |y_{(m)}|^2.
\end{aligned} \tag{63}$$

If we repeat the same argument in the basis where the $y_{(m)}$ take the form (55), we get

$$\frac{1}{2} \sum_l |x_{(l)}|^2 \geq \frac{1}{2} \sum_m |y_{(m)}|^2. \tag{64}$$

Therefore we must have equality in each of the steps of (63), i.e. $-\text{Im}(y_{(m)}^{*2k-1} y_{(m)}^{2k}) = |y_{(m)}^{*2k-1} y_{(m)}^{2k}|$ and $|y_{(m)}^{2k-1}| = |y_{(m)}^{2k}|$. Thus $y_{(m)}^{2k} = -i y_{(m)}^{2k-1}$ in the basis in which $x_{(l)}^{2k} = i x_{(l)}^{2k-1}$, i.e. the $O(6)$ matrix which relates the two different bases we considered is $\text{diag}(1, -1, 1, -1, 1, -1)$. It is worth noting that this is the same form of the couplings as one gets from the reality condition on scalars which are in a representation whose irreducible components are not real [6, 9]. Resubstituting this result into (51), we find

$$\text{Im}(\sum_l x_{(l)}^{*2k-1} x_{(l)}^{2k'-1} + \sum_m y_{(m)}^{*2k-1} y_{(m)}^{2k'-1}) = \text{Im}(\sum_l i x_{(l)}^{*2k-1} x_{(l)}^{2k'-1} - \sum_m i y_{(m)}^{*2k-1} y_{(m)}^{2k'-1}) = 0, \tag{65}$$

implying

$$\sum_l x_{(l)}^{*2k-1} x_{(l)}^{2k'-1} = \sum_m y_{(m)}^{2k-1} y_{(m)}^{*2k'-1}. \tag{66}$$

The remaining system of equations 40 to 43, 47 to 50 and 66 is invariant under an $SU(3)$ symmetry under which the $x_{(l)}$ transform in the fundamental and the $y_{(m)}$ transform in its complex conjugate representation. In terms of our Lagrangian this corresponds to complex bosons with couplings of the types $\psi_A \psi_{\bar{A}} \phi$, $\psi_S \psi_{\bar{S}} \phi$, $\psi_A \psi_{\bar{S}} \phi^*$ and $\psi_S \psi_{\bar{A}} \phi^*$. The Lagrangian itself is not invariant under the symmetry. Its form, however, is changed under a transformation in such a way that finiteness is preserved. Let us repeat the counting of equations here. We have 16 complex equations coming from (40) – (43), 15 (independent) real equations from (47) – (50) and 9 real equations from (66). Together with $8 + 4 \times 4$ equations that we can impose due to the $SU(3) \times U(2)^4$ symmetry, we have 80 real restrictions on the $3 \times 16 = 48$ complex quantities $Z_{IJ}^1, Z_{IJ}^3, Z_{IJ}^5$.

Taking traces and appropriate linear combinations of eqs. 47 to 50 and 66, one easily finds

$$\tilde{Z}_{IAK\bar{A}}^{*A} \tilde{Z}_{IAK\bar{A}}^A = (3\tilde{c}_A - \tilde{c}_S)/2 = n - 4, \tag{67}$$

$$\tilde{Z}_{IAK\bar{S}}^{*A} \tilde{Z}_{IAK\bar{S}}^A = (\tilde{c}_A + \tilde{c}_S)/2 = n, \tag{68}$$

$$\tilde{Z}_{ISK\bar{A}}^{*A} \tilde{Z}_{ISK\bar{A}}^A = (\tilde{c}_A + \tilde{c}_S)/2 = n, \tag{69}$$

$$\tilde{Z}_{ISK\bar{S}}^{*A} \tilde{Z}_{ISK\bar{S}}^A = (3\tilde{c}_S - \tilde{c}_A)/2 = n + 4. \tag{70}$$

We can use the $SU(2)$ symmetry among the fermions in the antisymmetric representation to diagonalise the expression $\tilde{Z}_{IAK\bar{A}}^{*A} \tilde{Z}_{JAK\bar{A}}^A$. Then eq. 47 implies that also $\tilde{Z}_{IAK\bar{S}}^{*A} \tilde{Z}_{JAK\bar{S}}^A$ is diagonal. In the same way all the expressions occurring on the l.h.s. of (47) – (50) can be assumed to

be diagonal. By making use of the $SU(3)$ symmetry and eq. 66, we can also make both sides of eq. 66 diagonal with respect to k and k' .

One way of solving our system is the following: We split the fermions into two identical groups, each of which contains one fermion of each type. Then we let the first of the three complex bosons interact only with the first group and the second boson with the second group while the third scalar remains free of Yukawa interactions. With this ansatz eqs. 40 to 43 and eq. 66 for $k \neq k'$ are fulfilled trivially. The remaining equations lead, uniquely up to phases, to

$$\tilde{Z}_{1_A 1_{\bar{A}}}^1 = -i\tilde{Z}_{1_A 1_{\bar{A}}}^2 = \tilde{Z}_{2_A 2_{\bar{A}}}^3 = -i\tilde{Z}_{2_A 2_{\bar{A}}}^4 = \sqrt{n-4}/2, \quad (71)$$

$$\tilde{Z}_{1_S 1_{\bar{S}}}^1 = -i\tilde{Z}_{1_S 1_{\bar{S}}}^2 = \tilde{Z}_{2_S 2_{\bar{S}}}^3 = -i\tilde{Z}_{2_S 2_{\bar{S}}}^4 = \sqrt{n+4}/2, \quad (72)$$

$$\tilde{Z}_{1_A 1_{\bar{S}}}^1 = +i\tilde{Z}_{1_A 1_{\bar{S}}}^2 = \tilde{Z}_{2_A 2_{\bar{S}}}^3 = +i\tilde{Z}_{2_A 2_{\bar{S}}}^4 = \sqrt{n}/2, \quad (73)$$

$$\tilde{Z}_{1_S 1_{\bar{A}}}^1 = +i\tilde{Z}_{1_S 1_{\bar{A}}}^2 = \tilde{Z}_{2_S 2_{\bar{A}}}^3 = +i\tilde{Z}_{2_S 2_{\bar{A}}}^4 = \sqrt{n}/2, \quad (74)$$

with all other \tilde{Z}_{IJ}^A vanishing. In terms of the original Z 's this means

$$Z_{1_A 1_{\bar{A}}}^1 = -iZ_{1_A 1_{\bar{A}}}^2 = Z_{2_A 2_{\bar{A}}}^3 = -iZ_{2_A 2_{\bar{A}}}^4 = \sqrt{3(n-4)/2(n-2)}g, \quad (75)$$

$$Z_{1_S 1_{\bar{S}}}^1 = -iZ_{1_S 1_{\bar{S}}}^2 = Z_{2_S 2_{\bar{S}}}^3 = -iZ_{2_S 2_{\bar{S}}}^4 = \sqrt{3(n+4)/2(n+2)}g, \quad (76)$$

$$Z_{1_A 1_{\bar{S}}}^1 = +iZ_{1_A 1_{\bar{S}}}^2 = Z_{2_A 2_{\bar{S}}}^3 = +iZ_{2_A 2_{\bar{S}}}^4 = \sqrt{3/2}g, \quad (77)$$

$$Z_{1_S 1_{\bar{A}}}^1 = +iZ_{1_S 1_{\bar{A}}}^2 = Z_{2_S 2_{\bar{A}}}^3 = +iZ_{2_S 2_{\bar{A}}}^4 = \sqrt{3/2}g. \quad (78)$$

Our next step should be the solution of the finiteness condition (9) for the quartic scalar couplings. The general way towards a solution of these equations is the same as for the finiteness conditions for the Yukawa couplings: One identifies all tensors fulfilling the invariance condition (4), expands the couplings $V^{\alpha\beta\gamma\delta}$ in terms of these tensors, extracts equations for the coefficients from (9) and solves these equations. Unfortunately, our solution (75) – (78) for the Yukawa couplings does not admit a real solution for the quartic scalar couplings. This can be seen with the following refinement of an argument by Böhm and Denner [4]: We set $\alpha = \gamma$ and $\beta = \delta$ in (9) and sum over all α and β in the last two adjoint representations, i.e. in those representations whose Yukawa couplings vanish (in the following equations summations over α, β, \dots are to be understood in this way, summations over λ, ε extend over the whole range; in order to avoid complicated notation we do not indicate this separately):

$$\begin{aligned} 0 &= 2V^{\alpha\beta\lambda\varepsilon}V^{\alpha\beta\lambda\varepsilon} + V^{\alpha\alpha\lambda\varepsilon}V^{\beta\beta\lambda\varepsilon} - 12g^2C_B^{\alpha\lambda}V^{\lambda\alpha\beta\beta} \\ &\quad + 6g^4\{T_B^a, T_B^b\}^{\alpha\beta}\{T_B^a, T_B^b\}^{\alpha\beta} + 3g^4\{T_B^a, T_B^b\}^{\alpha\alpha}\{T_B^a, T_B^b\}^{\beta\beta} \\ &= 2V^{\alpha\beta\lambda\varepsilon}V^{\alpha\beta\lambda\varepsilon} + (V^{\alpha\alpha\lambda\varepsilon}V^{\beta\beta\lambda\varepsilon} - V^{\alpha\alpha\gamma\delta}V^{\beta\beta\gamma\delta}) + (V^{\alpha\alpha\gamma\delta} - 6g^2c_g\delta^{\gamma\delta})(V^{\beta\beta\gamma\delta} - 6g^2c_g\delta^{\gamma\delta}) \\ &\quad - 36g^4c_g^2 \cdot 2d_g + 6g^4 \cdot 6c_g^2d_g + 3g^4 \cdot 16c_g^2d_g \\ &= 2V^{\alpha\beta\lambda\varepsilon}V^{\alpha\beta\lambda\varepsilon} + V^{\alpha\alpha\mu\nu}V^{\beta\beta\mu\nu} + (V^{\alpha\alpha\gamma\delta} - 6g^2c_g\delta^{\gamma\delta})(V^{\beta\beta\gamma\delta} - 6g^2c_g\delta^{\gamma\delta}) + 12g^4c_g^2d_g, \end{aligned} \quad (79)$$

where the summation over μ and ν extends over the first four representations (those with non-vanishing Yukawa couplings). Since the last expression is always positive, this equation cannot be fulfilled.

As a result of our particular ansatz, the matrices $\tilde{Z}_{I_A K_{\bar{A}}}^{*A} \tilde{Z}_{J_A K_{\bar{A}}}^A$ etc. turned out to be proportional to unity. Although there is no reason to believe that this has to be true for every solution, we will from now on demand this particularly symmetric form. With (67) to (70) this implies

$$\tilde{Z}_{I_A K_{\bar{A}}}^{*A} \tilde{Z}_{J_A K_{\bar{A}}}^A = \frac{n-4}{2} \delta_{I_A J_A}, \quad \dots, \quad \tilde{Z}_{K_S I_{\bar{S}}}^{*A} \tilde{Z}_{K_S J_{\bar{S}}}^A = \frac{n+4}{2} \delta_{I_{\bar{S}} J_{\bar{S}}}. \quad (80)$$

The resulting equations could obviously be fulfilled by

$$\tilde{Z}_{I_A J_{\bar{A}}}^{2k-1} = \frac{\sqrt{n-4}}{2} \Sigma_{IJ}^k, \quad \tilde{Z}_{I_A J_{\bar{S}}}^{2k-1} = \frac{\sqrt{n}}{2} \Sigma_{IJ}^k, \quad \tilde{Z}_{I_S J_{\bar{A}}}^{2k-1} = \frac{\sqrt{n}}{2} \Sigma_{IJ}^k, \quad \tilde{Z}_{K_S J_{\bar{S}}}^{2k-1} = \frac{\sqrt{n+4}}{2} \Sigma_{IJ}^k \quad (81)$$

with any set of matrices Σ^k fulfilling

$$\Sigma_{IK}^{\dagger k} \Sigma_{KJ}^k = \Sigma_{IK}^k \Sigma_{KJ}^{\dagger k} = \delta_{IJ}, \quad (82)$$

$$\text{Tr } \Sigma^{\dagger k} \Sigma^{k'} = 0 \quad \forall k \neq k' \quad (83)$$

and

$$\Sigma_{IK}^k \Sigma_{JL}^k = \Sigma_{IL}^k \Sigma_{JK}^k. \quad (84)$$

Our first solution corresponds, in this context, to

$$\Sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (85)$$

Eqs. 82 to 84 are invariant under $U(2) \times U(2)$ transformations acting on the lower indices; for eq. 84 this becomes clear if one notes that it is equivalent to

$$\varepsilon^{IJ} \varepsilon^{KL} \Sigma_{IK}^k \Sigma_{JL}^k = 0. \quad (86)$$

Regarding transformations acting on the upper index, (82) is invariant under a $U(3)$, (83) under a subgroup of $U(3)$ (generically $(U(1))^3$) depending on how many of the eigenvalues of the matrix $M^{kk'} = \text{Tr } \Sigma^{\dagger k} \Sigma^{k'}$ are equal, and (84) is invariant under an $O(3)$. With a rather tedious analysis it is possible to see that all solutions of eqs. 82 and 83, up to the invariances of these equations, have the form

$$\Sigma^1 = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma^2 = b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma^3 = c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (87)$$

with positive real numbers a, b, c satisfying $a^2 + b^2 + c^2 = 1$. The freedom in choosing a, b and c and some unitary transformation on the upper index which leaves (83) but not (84) invariant can be used to construct many solutions which also fulfill eq. 84. Examples are our first solution, which corresponds to

$$a = b = 1/\sqrt{2}, \quad c = 0 \quad (88)$$

and a rotation between $k = 1$ and $k = 2$,

$$a = 1/\sqrt{2}, \quad b = c = 1/2, \quad (89)$$

or solutions of the same form but with complex a, b and c fulfilling $|a|^2 + |b|^2 + |c|^2 = 1$ and $a^2 = b^2 + c^2$. With such a solution, e.g.

$$a = 1/\sqrt{3}, \quad b = (1 + i/\sqrt{3})/2, \quad c = (1 - i/\sqrt{3})/2, \quad (90)$$

we can even make $M^{kk'} = \text{Tr } \Sigma^{\dagger k} \Sigma^{k'}$ and therefore $\text{Tr } Y^{\dagger\alpha} Y^\beta$ proportional to unity.

4 Discussion and outlook

The main result of our investigation is the fact that an $F = 1$ particle content together with the $F = 1$ system of equations for the Yukawa couplings does not imply supersymmetry. For the model we considered, we saw that solutions of the $F = 1$ system naturally group pairs of real scalars into complex scalars, thereby fulfilling automatically a large number of equations and turning an overdetermined system into an underdetermined one. The simplest of the various explicit solutions we were able to give turned out not to admit finite ϕ^4 -couplings. Since this was due to the asymmetric way in which the scalars occurred in this solution, it is not unlikely that some other solution which is more symmetric will admit a set of ϕ^4 -couplings which will not need divergent renormalizations, at least at one loop level. The corresponding systems of equations are currently under investigation. Of course, even if we find such solutions, there still remains the question of higher loop divergences. If there exists an all-orders finite non-supersymmetric theory (whether it is the present model or any other), it is quite likely that the route to a proof of finiteness will proceed in a way similar to the case of $N = 1$ supersymmetry [3]: Starting from a one-loop finite theory, the matter couplings could be made finite order by order by viewing them as expansions in the gauge coupling [10]; for the gauge coupling we would need a suitable extension of the theorem, valid for supersymmetric theories, that n -loop finiteness implies vanishing of the gauge beta function at $n + 1$ loops [11].

Looking for a finite theory which is able to accommodate the standard model, it will be no problem to find a suitable $F = 1$ particle content, since we can take any one-loop finite supersymmetric theory and replace adjoint representations by antisymmetric and symmetric representations, thereby even guaranteeing the vanishing of the quadratic divergence of the mass of the gauge field. The resulting $F = 1$ equations, however, will be more complicated than in the model of the present work.

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