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Space-Time Supersymmetry of Extended Fermionic Strings in $2 + 2$ Dimensions

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Abstract

The $N = 2$ fermionic string theory is revisited in light of its recently proposed equivalence to the non-compact $N = 4$ fermionic string model. The issues of space-time Lorentz covariance and supersymmetry for the BRST quantized $N = 2$ strings living in uncompactified $2+2$ dimensions are discussed. The equivalent local quantum supersymmetric field theory appears to be the most transparent way to represent the space-time symmetries of the extended fermionic strings and their interactions. Our considerations support the Siegel's ideas about the presence of $SO(2, 2)$ Lorentz symmetry as well as at least one self-dual space-time supersymmetry in the theory of the $N = 2(4)$ fermionic strings, though we do not have a compelling reason to argue about the necessity of the *maximal* space-time supersymmetry. The world-sheet arguments about the absence of all string massive modes in the physical spectrum, and the vanishing of all string-loop amplitudes in the Polyakov approach, are given on the basis of general consistency of the theory.

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1 Introduction

The theory of the $N = 2$ fermionic strings is a quite natural extension of the conventional NSR string theory: the latter is based on the *gauged* $N = 1$ superconformal symmetry of the string world-sheet, while the former uses the $N = 2$ extended local superconformal symmetry (see ref. [1] for a recent review). The string theory methods always look like a bridge between the two-dimensional (world-sheet) and higher-dimensional (space-time) concepts, as well as the corresponding symmetries. One of the famous relations of that type is the correspondence between the $N = 2$ *global* superconformal symmetry on the superstring world-sheet and the $N = 1$ space-time supersymmetry in the effective four-dimensional field theory resulting from the superstring compactification. Since the four-dimensional space-time supersymmetry can naturally be extended up to the $N = 4$ (if the maximal spin is 1) or up to the $N = 8$ (if the maximal spin is 2), one may ask about the existence of the *critical* string models leading to the $N = 4$ or $N = 8$ supersymmetric effective four-dimensional field theories *without* any compactification. The similar motivation has been used in the past to argue about the relevance of supermembranes, which, contrary to superstrings, have a natural room for the 11-dimensional (maximally extended!) supergravity [2].

The $N = 2$ fermionic strings, being a natural extension of the conventional superstrings, have a rather controversial status. They were first introduced and investigated in two real space-time dimensions, their space-time symmetries, if any, were always quite obscure, and the understanding of the $N = 2$ string amplitudes is still lacking. Recently, some new important developments have taken place, and they are going to change the whole status of the theory. In this paper we review some of the recent new results about the structure of the extended fermionic strings, when a special attention being paid on the status of space-time supersymmetry and its adequate description in that theories. The relevant references are cited in parallel with the discussion.

The motivation to study the $N = 2$ and $N = 4$ fermionic strings is at least two-fold. On the one side, they are just the useful polygons for analyzing the specific properties of the more complicated conventional superstrings. On the other hand, being intimately related to the self-dual field theories and, hence, to the integrable models [3], the extended fermionic strings could be used for the quantization of the latter.

Another motivation is just to relate the maximal gauged superconformal symmetry on the world-sheet with the maximal (conformal) supersymmetry in space-time.

2 World-Sheet Symmetries and Actions

The $N = 2$ superconformal algebra (SCA) in two dimensions² comprises a stress tensor $T(z)$ of (conformal) dimension 2, two real supercharges $G^i(z)$ of dimension 3/2, and an Abelian current $J(z)$ of dimension 1, with the OPE [4, 5, 6, 7]

$$\begin{aligned}
T(z)T(w) &\sim \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) , \\
T(z)G^i(w) &\sim \frac{3G^i(w)}{2(z-w)^2} + \frac{1}{z-w}\partial G^i(w) , \\
G^i(z)G^j(w) &\sim \left[\frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w} \right] \delta^{ij} + i \left[\frac{2J(w)}{(z-w)^2} + \frac{1}{z-w}\partial J(w) \right] \varepsilon^{ij} , \\
T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{1}{z-w}\partial J(w) , \\
J(z)G^i(w) &\sim i\varepsilon^{ij}\frac{G^j(w)}{z-w} , \\
J(z)J(w) &\sim \frac{c}{3(z-w)^2} ,
\end{aligned} \tag{2.1}$$

where the central charge c has been introduced, $i = 1, 2$. The internal symmetry group corresponding to the Abelian Kac-Moody (KM) current $J(z)$ can be either *compact* $U(1)$, or *non-compact* $GL(1)$.

The Abelian internal symmetry of the $N = 2$ super-Virasoro algebra (2.1) can be extended to a non-Abelian symmetry in the $N = 4$ SCA [4, 6], without introducing new “subcanonical” charges.³ The $N = 4$ SCA has a *complex doublet* of supersymmetry charges, $G_\alpha(z)$ and $\bar{G}_\alpha(z)$, transforming in the fundamental representation of the internal symmetry group G , which can also be either *compact* $SU(2)$ or *non-compact* $SU(1, 1)$.⁴ The internal symmetry generators $J^i(z)$ of the $N = 4$ SCA form the KM algebra $\widehat{G} = \widehat{SU(2)}$, or $\widehat{G} = \widehat{SU(1, 1)}$, respectively, the KM level being fixed by the central charge c :

$$J^I(z)J^J(w) \sim \frac{if^{IJK}}{z-w}J^K(w) + \frac{c}{3(z-w)^2}\delta^{IJ} , \tag{2.2}$$

²The Euclidean two dimensions are supposed to form a Riemann surface Σ parametrized by a complex variable z in local coordinate charts.

³By “subcanonical” charges [4] one means charges of dimension other than 2, 3/2 or 1. The canonical charges are just those which correspond to conformal symmetry, supersymmetry and internal symmetry, respectively.

⁴In fact, only the compact cases were the subjects of investigation in the early studies of the corresponding dual models [4, 5, 6].

where the structure constants f^{IJK} of the internal symmetry group G have been introduced, $I, J, K = 1, 2, 3$. Given the transformation properties of the $N = 4$ SCA charges, the rest of the $N = 4$ SCA OPE's is quite obvious.

All the $N > 2$ SCA's are known to contain subcanonical charges needed to close an algebra [4, 6], with just the two exceptions at $N = 4$, because of the relevant group decompositions

$$\begin{aligned} SO(4) &\cong SU(2) \otimes SU(2) , \\ SO(2, 2) &\cong SU(1, 1) \otimes SU(1, 1) . \end{aligned} \quad (2.3)$$

It is eq. (2.3) that allows to restrict the internal symmetry generators of the $N = 4$ SCA to be *self-dual* (SD) or *anti-self-dual* (ASD), and thus close the algebra, when using the minimal set of canonical charges only.

Being realized globally, both $N = 2$ and $N = 4$ SCA's arise in describing compactification of the conventional ten-dimensional superstrings down to four dimensions, when space-time supersymmetry of the effective four-dimensional theory is required [10, 12]. The unitary representations and their characters have been constructed in the past for $N = 2$ [11, 12], as well as for $N = 4$ [13].

The N -extended fermionic string models [4, 5, 6] arise when the N -extended SCA's are *locally* realized, i.e. gauged. The bridge between strings and CSA's is provided by free field realizations of the latter. In the absence of subcanonical charges in a SCA, there always exists its free field realization in terms of canonical fields to be interpreted as bosonic and fermionic coordinates [4, 5, 6].

The convenient fermionic string (free field) realization of the N -extended SCA is provided by the world-sheet covariant gauge-fixed form of the supercurrent multiplet in the (locally) N -supersymmetrized Polyakov string action [14]. It can be constructed by coupling the N -extended supergravity to the N -extended scalar multiplet in two dimensions. In the case of $N = 2$, the appropriate action (in components) has been constructed by Brink and Schwarz [15], and it takes the form

$$\begin{aligned} I_{\text{BS}} = \frac{1}{\pi} \int d^2 z e \left\{ \frac{1}{2} g^{pq} \partial_p x_\mu^* \partial_q x^\mu + \frac{i}{2} \bar{\psi}_\mu \gamma^p \overleftrightarrow{D}_p \psi^\mu \right. \\ \left. + A_p (\bar{\psi}_\mu \gamma^p \psi^\mu) + [(\partial_p x^\mu + \bar{\chi}_p \psi^\mu) (\bar{\psi}_\mu \gamma^q \gamma^p \chi_q) + \text{h.c.}] \right\} , \end{aligned} \quad (2.4)$$

where a set of D complex scalar ("matter") multiplets (x, ψ) is coupled to the $N = 2$ supergravity multiplet $(e_p^{\bar{p}}, \chi_p, A_p)$ in two dimensions; $p, q = 1, 2$; $\mu = 1, 2$. The spinors ψ and χ are both complex, whereas the bosonic zweibein $e_p^{\bar{p}}$ and the Abelian vector gauge field A_p are real. The D_p represents the standard gravitational covariant

derivative for spinors [16, 17]. The local world-sheet gauge symmetries of the BS-action (2.4) are [15, 18]:

- (i) *reparametrization invariance* ,
 - (ii) *Lorentz invariance* ,
 - (iii) *N – extended supersymmetry* ,
 - (iv) *scale invariance* ,
 - (v) *N – extended conformal supersymmetry* ,
 - (vi) *phase and chiral gauge invariance* .
- (2.5)

The symmetries (i), (ii), (iv) and (vi) are bosonic, the symmetries (iii) and (v) are fermionic.

Introducing the oscillator representation for the real bosonic coordinates ($x^\mu = x_1^\mu + ix_2^\mu$) and the real fermionic coordinates ($\psi^\mu = \psi_1^\mu + i\psi_2^\mu$) in terms of their real modes α_{in}^μ and d_{ir}^μ respectively, the Gupta-Bleuler quantization implies, as usual in string theory, the relations [19, 5]

$$\begin{aligned} [\alpha_{in}^\mu, \alpha_{jm}^\nu] &= \delta_{ij} \eta^{\mu\nu} n \delta_{n+m,0} , \\ \{d_{ir}^\mu, d_{js}^\nu\} &= \delta_{ij} \eta^{\mu\nu} n \delta_{r+s,0} , \end{aligned} \tag{2.6}$$

where the modding for spinors can be either integer ($r, s \in \mathbf{Z}$) or half-integer ($r, s \in \mathbf{Z} + 1/2$), depending on boundary conditions.

The local gauge symmetries (2.5) can then be used to gauge away all the supergravity fields, leaving the dependence on their moduli only. The vanishing supercurrent multiplet of the BS theory (2.4) determines the constraint algebra, which is just the realization of the $N = 2$ SCA (2.1) in terms of the free oscillators (2.6):

$$\begin{aligned} T(z) &= \frac{1}{2} : P_i(z) \cdot P_i(z) : - \frac{i}{2} : \psi_i(z) \cdot \partial \psi_i(z) : , \\ G_1(z) &= \psi_i(z) \cdot P_i(z) , \quad G_2(z) = \varepsilon_{ij} \psi_i(z) \cdot P_j(z) , \\ J(z) &= \frac{i}{4} \varepsilon_{ij} \psi_i(z) \cdot \psi_j(z) , \end{aligned} \tag{2.7}$$

where the fields⁵

$$P_i^\mu(z) = i \partial x_i^\mu(z) = \sum_n \alpha_{in}^\mu z^{-n} ,$$

⁵When dealing with chiral (say, left-moving) modes of a closed string, it is just enough to consider the holomorphic dependence on z .

$$\psi_i^\mu(z) = \sum_s d_{is}^\mu z^{-s} , \quad (2.8)$$

have been introduced. The dots in eq. (2.7) mean contractions of target space vector indices with respect to a flat orthogonal metric $\eta^{\mu\nu}$ to be determined below. The $N = 2$ fermionic string theory apparently has the $SO(2) \otimes SO(D)$ or $SO(1,1) \otimes SO(D - q, q)$, $q = 0, \dots, D - 1$, as the “Lorentz” group.

The $N = 4$ counterpart to the BS action (2.4) has been constructed by Pernici and Nieuwenhuizen [20]. Their $N = 4$ fermionic string action is in fact quite similar to that of eq. (2.4), and takes the form

$$I_{\text{PN}} = \frac{1}{\pi} \int d^2z e \left\{ \frac{1}{2} g^{pq} \partial_p x_\mu^* \partial_q x^\mu + \frac{i}{2} \bar{\psi}_{\mu\mu'} \gamma^p \overleftrightarrow{D}_p \psi^{\mu\mu'} \right. \\ \left. + A_p^I \left(\bar{\psi}_{\mu\mu'} \gamma^p \sigma_I^{\mu'\nu'} \psi^{\mu\nu'} \right) + \left[\left(\partial_p x^\mu + \bar{\chi}_{p\mu'} \psi^{\mu\mu'} \right) \left(\bar{\psi}_{\mu\nu'} \gamma^q \gamma^p \chi_q^{\nu'} \right) + \text{h.c.} \right] \right\} , \quad (2.9)$$

where the fields x , ψ and χ_p are now considered to be *quaternionic*.⁶ The σ_I are the $SU(2)$ or $SU(1,1)$ generators in the fundamental representation. The $N = 4$ analogues to eqs. (2.5), (2.6), (2.7) and (2.8) are now quite obvious. In particular, one gets [6]

$$T(z) = \frac{1}{2} : P_{\alpha\mu'}(z) \cdot P^{\alpha\mu'}(z) : - \frac{i}{2} : \psi_{\mu'}(z) \cdot \partial \psi^{\mu'}(z) : , \\ G_\alpha(z) = \psi^{\mu'}(z) \cdot P_{\alpha\mu'}(z) , \\ J^I(z) = \frac{i}{2} \psi^{\mu'}(z) \cdot \sigma_{\mu'\nu'}^I \psi^{\nu'}(z) , \quad (2.10)$$

where all the fermions have been chosen to be the complex doublets with respect to $SU(2)$ or $SU(1,1)$. Notably, in eqs. (2.9) and (2.10) the index μ can be combined with the index μ' into one vector index so that, when being compared to the BS theory, the PN theory (2.9) is actually invariant under a larger “Lorentz” group $SO(2D - 2q, 2q)$, *provided* there is at least one factor of $SU(2)$ or $SU(1,1)$ in the decomposition of the $SO(2D - 2q, 2q)$ into simple factors.

Therefore, there are clearly *two* distinct extended fermionic string models both for the $N = 2$ and $N = 4$: one with the compact internal symmetry group, and another one with the non-compact group.

⁶It means that the second independent complex structure (“ j ” or “prime”) has been introduced, so that $x = x_1 + jx_2$, $\psi^{\mu'} = \psi_1 + j\psi_2$, and $x_1 = x_{11} + ix_{12}$, $x_2 = x_{21} - ix_{22}$, and similarly for fermions. The quaternionic index can equally be represented as a multi-index $(\alpha\mu')$, where $\alpha = 1, 2$ and $\mu' = 1', 2'$.

3 BRST and BFV

The Becchi-Rouet-Stora-Tyutin (BRST) quantization prescription [21, 22] implies certain conditions on an initial classical constrained system to satisfy. In light of the more general Batalin-Fradkin-Vilkovisky (BFV) quantization prescription [23, 24, 25], the constraints are to be (1) of the *first class*, and (2) *irreducible*.

The standard Dirac theorem [26] explains the way how to quantize a constrained system, first introducing the independent (physical) canonically conjugated variables arising from the solution of the constraints subject to admissible gauges, then quantizing that physical variables canonically, and finally rewriting the result back to the initial phase space. The Dirac's quantization is obviously consistent with the canonical one, unitary but non-covariant. The BRST prescription is just the Dirac's quantization in the covariant form, when a covariant (with respect to the Lorentz transformations) gauge is chosen. The covariance is maintained in the BRST quantization by extending the initial phase space by ghosts, so that the extended (fields + ghosts) system can be quantized “naively”, when using the BRST-invariant Hamiltonian and constraints, by integrating over the ghosts in the quantum generating functional. To be specific, given bosonic (B) and fermionic (F) constraints satisfying a *closed* algebra⁷

$$[B^a, B^b] = f^{ab}{}_c B^c, \quad [B^a, F^\beta] = f^{a\beta}{}_\gamma F^\gamma, \quad \{F^\alpha, F^\beta\} = f^{\alpha\beta}{}_c B^c, \quad (3.1)$$

where $B^0 \equiv H_0$ is an initial Hamiltonian, and f 's are the constraint algebra structure constants, the BRST-invariant quantities are defined by [24]

$$\mathcal{B}^a = \{\rho^a, Q\}, \quad \mathcal{F}^\alpha = [\xi^\alpha, Q]. \quad (3.2)$$

In eq. (3.2) the canonically conjugated ghosts for each constraint,

$$\text{B: } \rho^a, \eta_b, \quad \text{F: } \xi^\alpha, \lambda_\beta, \quad (3.3)$$

have been introduced. By definition, they have statistics opposite to that of the constraints and satisfy the (anti)commutation relations:

$$\{\rho^a, \eta_b\} = \delta_{ab}, \quad [\xi^\alpha, \lambda_\beta] = \delta_{\alpha\beta}. \quad (3.4)$$

The operator Q introduced in eq. (3.2) is known as the BRST charge [24]:

$$Q = B^a \eta_a + F^\alpha \lambda_\alpha - \frac{1}{2} f^{ab}{}_c \rho^c \eta_a \eta_b - f^{a\beta}{}_\gamma \xi^\gamma \eta_a \lambda_\beta - \frac{1}{2} f^{\alpha\beta}{}_c \rho^c \lambda_\alpha \lambda_\beta. \quad (3.5)$$

⁷The indices used below in eqs. (3.1)–(3.5) have the meaning different from that used in the bulk of the paper, however they could hardly be confused.

Classically, one has $\{Q, Q\}_{\text{PB}} = 0$ with respect to the Poisson bracket. Quantum mechanically, the consistent quantization requires the BRST operator to be nilpotent, $Q^2 = 0$.

Given the *reducible* constraints, the BRST rules have to be modified [25]. The BFV-prescription introduces the new ghosts beyond those needed in the BRST framework, when dealing with the reducible constraints. The derivation of the generalized BFV-BRST rules also goes back and forth: first one chooses an irreducible subset of constraints, applies BRST rules, and then rewrites the quantized theory in terms of the initial variables and constraints by using “ghosts for ghosts” [25].

The BRST techniques have been applied to the quantization of the $N = 2$ fermionic string in refs. [27, 28, 29] (see ref. [28], as for the BRST quantization of the $SU(2)$ fermionic string of Ademollo et al [6]). It is easy to check the irreducibility of the $N = 2$ first-class constraints (2.7), which justifies the applicability of the BRST rules to that case. In the oscillator representation, the BRST ghosts are

$$\begin{aligned} T(z) &\rightarrow T_n \rightarrow \rho_n, \eta_m : \quad \{\rho_n, \eta_m\} = \delta_{n+m,0} , \\ G^i(z) &\rightarrow G_r^i \rightarrow \xi_r^i, \lambda_s^j : \quad [\xi_r^i, \lambda_s^j] = \delta^{ij} \delta_{r+s,0} , \\ J(z) &\rightarrow J_n \rightarrow \omega_n, \phi_m : \quad \{\omega_n, \phi_m\} = \delta_{n+m,0} , \end{aligned} \quad (3.6)$$

in accordance to eqs. (3.3) and (3.4). The BRST charge reads [27]:

$$\begin{aligned} Q = \frac{1}{2\pi i} \oint_0 \frac{dz}{z} : & \left\{ T(z)\eta(z) + G^i(z)\lambda_i(z) + J(z)\phi(z) + z\rho(z)\eta(z)\eta'(z) \right. \\ & - \frac{1}{2}z\xi_i(z)\eta'(z)\lambda_i(z) + z\xi_i(z)\eta(z)\lambda'_i(z) - \rho(z)\lambda_i(z)\lambda_i(z) + z\omega(z)\eta(z)\phi'(z) \\ & \left. - \frac{i}{2}\varepsilon^{ij}\xi_j(z)\phi(z)\lambda_i(z) + 2i\varepsilon^{ij}z\omega(z)\lambda_i(z)\lambda'_j(z) - \alpha_0\eta(z) \right\} : , \end{aligned} \quad (3.7)$$

where the ghost fields $\eta(z) = \sum_n \eta_n z^{-n}$ etc, and the “intercept” (normal ordering ambiguity) constant α_0 have been introduced. It is now straightforward to calculate that [27]

$$\begin{aligned} Q^2 = \frac{1}{4\pi i} \oint_0 dz : & \left\{ \frac{1}{4}(D-2)(z\eta)'''z\eta + (D-2)z\lambda''_i\lambda_i + \frac{1}{4}(D-2)\phi'\phi \right. \\ & \left. + 2\alpha_0\eta'\eta + \left[\frac{1}{4}(2-D) + 2\alpha_0 \right] \lambda_i\lambda_i/z \right\} : , \end{aligned} \quad (3.8)$$

where the half-integer modding for the matter spinors ψ 's has been used. For the integer modding, the result is quite similar to that of eq. (3.8): one has only to exchange the numerical coefficients between the last two terms in the curved brackets [27]. Hence, the critical $N = 2$ fermionic strings have [18, 27, 28]

$$D = 2 , \quad \alpha_0 = 0 . \quad (3.9)$$

This critical dimension also follows from the conformal anomaly counting [18]:

$$\begin{array}{ccccccc} (\rho, \eta) & & (\xi^i, \lambda^i) & & (\phi, \omega) & & (x^i, \psi^i)^\mu \\ -26 & + & 2 \cdot 11 & + & -2 & + & 2 \cdot D \cdot (1 + \frac{1}{2}) , \end{array} \quad (3.10)$$

whose vanishing also yields $D = 2$.

The fact that the $N = 2$ fermionic strings live in 2 *complex* dimensions has been known for a long time [5, 18], however it was not known until recently how to introduce interactions in 2 complex or 4 real dimensions. That's why the early studies of the $N = 2$ fermionic string amplitudes were confined to the 2 *real* dimensions [5, 30]. The choice of two real dimensions was also suggested by the apparently 2-dimensional ‘‘Lorentz’’ or ‘‘space-time’’ group $SO(2) \otimes SO(2)$ or $SO(1, 1) \otimes SO(1, 1)$. The four-dimensional space-time interpretation of the $N = 2$ fermionic string theory was first suggested by D’Adda and Lizzi [31], who showed how to rewrite its defining constraints in the $SO(2, 2)$ covariant way.

The strange ‘‘hidden’’ four-dimensional Lorentz symmetry of the $N = 2$ theory (2.4) can be understood, when turning to the quantization of the $N = 4$ fermionic string theory (2.9). The analysis of the $N = 4$ constraints (2.10) reveals an unexpected result: they are irreducible in the compact $SU(2)$ case, but become *reducible* in the non-compact $SU(1, 1)$ case, as has been noticed recently by Siegel [32]. Being a subset of the $N = 4$ SCA constraints (2.10), the $N = 2$ SCA constraints (2.7) already eliminate all the excited string states but the ground states, so that the rest of the $N = 4$ constraints in eq. (2.10) becomes redundant. The BRST analysis of the spectrum of the $N = 2$ theory shows that all the physical states are the highest weight states of the $N = 4$ SCA, the other being either non-physical or pure gauge with respect to the $N = 2$ SCA [29]. Therefore, the $N = 2$ non-compact fermionic string theory and the $N = 4$ non-compact fermionic string theory are in fact equivalent, the latter being the covariant form of the former with respect to the ‘‘Lorentz’’ group $SO(2, 2)$. In its turn, the non-compact $N = 2$ theory can be interpreted as a ‘‘partially gauged’’ non-compact $N = 4$ theory [32]. The absence of the massive string modes in this theory also follows from a calculation of the partition function⁸ [33, 34]:

$$Z_{AAAA} = \text{sTr} \left[q^{T_0} \bar{q}^{\bar{T}_0} t_L^{J_0} t_R^{\bar{J}_0} \right] = \int d^4 p (q \bar{q})^{p^2/2} , \quad (3.11)$$

where $q = \exp(2\pi i \tau)$, $t_{L,R} = \exp(2\pi i \theta_{L,R})$. Stated differently, the non-zero modes of (x, ψ) cancel with those of ghosts.

⁸For definiteness, the case of closed $N = 2$ strings with anti-periodic (for fermions) boundary conditions has been chosen.

The very general relation between the mass m_0 of a ground state and a critical dimension D ,

$$\alpha' m_0^2 = -\frac{1}{24}(D-2) \left(\#_B + \frac{1}{2} \#_F \right) , \quad (3.12)$$

derived by Brink and Nielsen [35] from their analysis about zero-point fluctuations in dual string models, forces the ground state in the $N = 2$ string model to be always massless, $p^2 = 0$. In particular, the ground state of the Euclidean $SO(4)$ theory is therefore an identity with $p = 0$ and no dynamics, unless complex values of momenta are allowed, but then it would be just the “doubly Wick-rotated” $SO(2, 2)$ theory. The necessity of a “complex” time in the $N = 2$ fermionic string theory was also argued by Ooguri and Vafa [34] from various viewpoints.

The $N = 4$ fermionic string with the *compact* internal symmetry group [6] still has the irreducible constraints (2.10). Hence, it can be quantized along the standard BRST lines, which yield the negative critical dimension $D = -2$ [28]. It has been known for a long time that this theory suffers from inevitable ghosts both in its spectrum [6], and in its amplitudes [8]. This conclusion also follows from the conformal anomaly counting in the $SU(2)$ fermionic string:

$$\begin{array}{ccccccc} (\rho, \eta) & & (\xi_i^a, \lambda_a^i) & & (\phi^I, \omega_I) & & (x_{i\mu'}, \psi_{i\mu'})^\mu \\ -26 & + & 2 \cdot 2 \cdot 11 & + & -3 \cdot 2 & + & 2 \cdot 2 \cdot D \cdot (1 + \frac{1}{2}) , \end{array} \quad (3.13)$$

whose cancellation implies $D = -2$.

The reducibility of the $N = 4$ constraints means a linear dependence between the generators of the gauged $SU(1, 1)$ superconformal algebra. It can be understood in part by considering the gauge transformation laws of the gravitino and the $SU(1, 1)$ vector fields in the PN theory, which are themselves invariant under the Abelian complex transformations of the parameters of superconformal and internal symmetry. The $N = 4$ harmonic superspace with harmonic coordinates parametrizing the coset $SU(1, 1)/GL(1)$ should therefore be quite appropriate for a covariant description of the $N = 4$ theory, and it makes indeed the gauge symmetry generator dependence to be explicit [32]. The BFM quantization implies the appearance of the “ghosts for ghosts” according to the rule [24, 25]:

$$\begin{array}{ccccccc} & & G^a & & & & J^I \\ & \xi^a & & \lambda_a & & \phi^I & \omega_I \\ \epsilon^{(1)} & & \xi^{(1)} & & \lambda_{(1)} & & \zeta^{(1)} & & \phi^{(1)} & & \omega_{(1)} \\ \dots & \dots & & \dots & \dots & & \dots & \dots & & \dots & \dots \end{array} \quad (3.14)$$

The *complex* Abelian fermionic and bosonic ghosts, $(\xi_i^{(n)}, \lambda_{(n)}^i)$ and $(\phi_i^{(n)}, \omega_{(n)}^i)$ respectively, contribute to the conformal anomaly. Therefore, the conformal anomaly

counting (3.13) gets to be modified, presumably as ⁹

$$\begin{aligned}
& (\rho, \eta) & (\xi_i^a, \lambda_a^i) & (\phi^I, \omega_I) & (x_{i\mu'}, \psi_{i\mu'})^\mu \\
& -26 & + & 2 \cdot 2 \cdot 11 & + & -3 \cdot 2 & + & 2 \cdot 2 \cdot D \cdot (1 + \frac{1}{2}) \\
& & (\xi_i^{(1)}, \lambda_{(1)}^i) & (\phi_i^{(1)}, \omega_{(1)}^i) & & & & \\
& & + & -2 \cdot 11 & + & 2 \cdot 2 & &
\end{aligned} \tag{3.15}$$

Its cancellation now implies one quaternionic ($D = 1$) or four real dimensions.

Since the critical non-compact $N = 2$ (or $N = 4$) fermionic string implies the four-dimensional target (“space-time”) of the signature $(+, +, -, -)$ and the “Lorentz” group $SO(2, 2)$ for its consistent propagation, and it does not have any massive modes in its spectrum, this string theory should be equivalent to a local four-dimensional quantum field theory, which would describe the $N = 2(4)$ string ground states and their interactions. The spectrum of the equivalent effective field theory and its vertices have to reproduce the $N = 2$ string S -matrix. In order to identify the effective field theory of the $N = 2$ fermionic strings in $2+2$ space-time dimensions, the $N = 2$ string ground states and their scattering amplitudes should therefore be considered. Being restricted to the on-shell quantities, the $N = 2$ (non-covariant) string formalism is clearly more convenient for those purposes than the $N = 4$ (covariant) one. We are thus going to discuss first the non-covariant constraints on the effective field theory, and then propose its covariant (in the $(2+2)$ -dimensional “space-time”) action to be fixed by the symmetries and spectrum of the $N = 2(4)$ fermionic string theory.

4 Spectrum and Tree Scattering Amplitudes

We now analyze the spectrum of the physical states in the $N = 2(4)$ fermionic string theory. To be specific, let’s consider the left-moving (analytic) modes of the closed string propagating in $2 + 2$ space-time dimensions.

The physical states are all the ground states in the $N = 2$ theory, which can equally be represented in an $SO(2, 2)$ covariant way as the ground states of the $N = 4$ theory. All four real world-sheet fermions should then be considered on equal footing, when their boundary conditions are analyzed. A real world-sheet fermion $\psi(z)$ can be either *periodic* (P) or *anti-periodic* (A):

$$\psi^{\underline{a}}(2\pi) = \pm \psi^{\underline{a}}(0) , \quad \underline{a} = (a, \bar{a}) = 1, 2, 3, 4 , \tag{4.1}$$

⁹The way of treating here the anomalies of the “ghosts for ghosts” is quite similar to that used in the BFM quantization of the Green-Schwarz superstring [36]. In particular, one uses $1-1+1-1+\dots = 1/2$.

which implies 16 different sectors in the theory:

$$\begin{aligned}
& AAAA \\
& PAAA \quad APAA \quad AAPA \quad AAAP \\
& PPAA \quad PAPA \quad PAAP \quad APPA \quad APAP \quad AAPP \\
& PPPA \quad PPAP \quad PAPP \quad APPP \\
& PPPP ,
\end{aligned} \tag{4.2}$$

and, hence, $1 + 4 + 6 + 4 + 1 = 16$ different physical ground states. The periodic boundary condition implies integer modding, whereas the anti-periodic one yields only half-integer modes in the oscillator (mode) expansion of $\psi(z)$.

The sector $AAAA$ has only half-integer world-sheet fermionic modes and has no zero fermionic modes. It is therefore quite similar to the NS-sector of the conventional superstring [19], and comprises just one *space-time bosonic* state $|\text{phys}\rangle = |0, p\rangle$ with $p^2 = 0$, which follows as a solution to the physical state condition

$$Q |\text{phys}\rangle = 0 , \tag{4.3}$$

subject to the usual (Siegel's) gauge conditions

$$\rho_0 |\text{phys}\rangle = \omega_0 |\text{phys}\rangle = 0 , \tag{4.4}$$

in that sector. Each of the four sectors in the second line of eq. (4.2) contains only *one* real fermionic zero mode, which obeys the Clifford algebra as a consequence of eqs. (2.6) and (2.8). It leads to a *space-time fermionic* ground state of the form $u_\epsilon |1/2, p\rangle$, characterized by the momentum p and the spinor wave function u_ϵ with a polarization ϵ . Being the only physical state in this sector, it satisfies the on-shell conditions

$$\gamma^{\underline{a}} p_{\underline{a}} u_\epsilon = p^2 = 0 , \tag{4.5}$$

which also follow from eq. (4.3). Hence, those four sectors are all of the Ramond-type and represent space-time fermions in the theory.

The fifth line of eq. (4.2) yields the sector in which all the world-sheet fermions have integer modes and, in particular, anticommuting integer zero modes. The physical state condition (4.3) should then be supplemented by the gauge conditions

$$\xi_0 |\text{phys}\rangle = d^{\bar{a}} |\text{phys}\rangle = 0 , \quad \bar{a} = \bar{1}, \bar{2} , \tag{4.6}$$

in addition to that of eq. (4.4). Given a formal vacuum state $|0, p\rangle$ in the extended (by ghosts) Fock space for that sector, one can construct more states at the same level, when acting on that vacuum by the operators d_0^a , $a = 1, 2$:

$$|0, p\rangle, \quad d_0^a |0, p\rangle, \quad \varepsilon_{ab} d_0^a d_0^b |0, p\rangle. \quad (4.7)$$

The physical states are distinguished by eq. (4.3) which, being applied to a linear combination of all the states in eq. (4.7), picks up the longitudinal vector

$$\bar{p}_a d_0^a |0, p\rangle, \quad p^2 = 0, \quad (4.8)$$

as the only physical solution (*cf* [37]). This state obviously has vanishing ghost and excitation numbers and, therefore, represents one *space-time bosonic* physical state in the *PPPP* sector.

Similarly, six ground states corresponding to the third line of eq. (4.2) turn out to be space-time *bosons*, whereas four ground states from the fourth line of that equation are all space-time *fermions*.

Since in any $SO(2, 2)$ -invariant theory, there should be an equal number of states with positive and negative norms at each “spin” level, the six new bosonic states should form a space with indefinite norm, the same being true for all space-time fermions: only a half of them (say, from the second line of eq. (4.2)) should have positive norms, while the others (from the fourth line of eq. (4.2)) should then be with negative norms.

Putting all together, we thus have 8 bosonic and 8 fermionic states in the theory, which is apparently space-time supersymmetric in $2 + 2$ dimensions. To fix space-time representations of all the physical states, we now consider the ground state in the *AAAA* (=NS) sector of the theory and find first its quantum numbers.

The NS sector of the closed $N = 2$ superstrings was analyzed by Ooguri and Vafa [34] in the non-covariant $N = 2$ formulation. They noticed the NS ground state to be a massless “scalar”, and argued about the absence of other sectors in the theory. However, the arguments of ref. [34] were based on the possibility to *continuously* interpolate between what they called NS and R sectors by using “Wilson line” operators associated with the Abelian charge in the $N = 2$ superconformal algebra (sect. 2). The sectors we discussed above form a *discrete* set in that sense, but they are related by space-time supersymmetry (sect. 5).

To uncover the nature of the NS “scalar”, the gauge-fixed $N = 2$ string action

$$I_{g-f} = \frac{1}{\pi} \int d^2 z d^2 \theta d^2 \bar{\theta} \eta_{a\bar{b}} X^a \bar{X}^{\bar{b}} \equiv \frac{1}{\pi} \int d^4 Z K_0(X, \bar{X}), \quad (4.9)$$

and the vertex operator describing the emission of the NS ground state with momentum $k^a = (k^a, \bar{k}^{\bar{a}})$,

$$V_c = \frac{\kappa}{\pi} : \exp[i(k \cdot \bar{X} + \bar{k} \cdot X)] : , \quad (4.10)$$

written in terms of the $N = 2$ chiral scalar superfields $X(Z, \bar{Z}) = X(z, \bar{z}, \theta, \bar{\theta})$, can be used to calculate the $N = 2$ string tree scattering amplitudes and then analyze the effective field theory [34]. The κ is the $N = 2$ closed string coupling constant.

In particular, the 3-point tree amplitude takes the form [34]

$$\begin{aligned} A_3 &\sim \left\langle V_c|_{\theta=0}(0) \cdot \int d^2\theta d^2\bar{\theta} V_c(1) \cdot V_c|_{\theta=0}(\infty) \right\rangle \\ &= \kappa (k_1 \cdot \bar{k}_2 - \bar{k}_1 \cdot k_2)^2 \equiv \kappa c_{12}^2 , \end{aligned} \quad (4.11)$$

where the super-Möbius invariance had been used to fix three points on the $N = 2$ super-Riemannian sphere. The A_3 is non-vanishing and apparently non-covariant with respect to the Lorentz group $SO(2, 2)$.

The calculation of the 4-point $N = 2$ closed string tree amplitude yields the result [34]

$$\begin{aligned} A_4 &\sim \int d^2z \left\langle V_c|_{\theta=0}(0) \cdot \int d^2\theta d^2\bar{\theta} V_c(z) \cdot \int d^2\theta d^2\bar{\theta} V_c(1) \cdot V_c|_{\theta=0}(\infty) \right\rangle \\ &= \frac{\kappa^2}{16\pi} \int d^2z \left| \frac{1}{(1-z)^2} t(t+2) + \frac{c_{12}c_{34}}{z} + \frac{c_{23}c_{41}}{1-z} \right|^2 |z|^{-s} |1-z|^{-t} \\ &= \frac{\kappa^2}{\pi} F^2 \frac{\Gamma(1-s/2)\Gamma(1-t/2)\Gamma(1-u/2)}{\Gamma(s/2)\Gamma(t/2)\Gamma(u/2)} , \end{aligned} \quad (4.12)$$

where s, t, u are the Mandelstam variables, $s = -(k_1 \cdot \bar{k}_2 + \bar{k}_1 \cdot k_2)$, etc., and

$$F \equiv 1 - \frac{c_{12}c_{34}}{su} - \frac{c_{23}c_{41}}{tu} = 0 . \quad (4.13)$$

The A_4 amplitude is vanishing because of the (non-trivial) kinematical identity (4.13), which has also been proved in ref. [34]. The vanishing kinematical factor F in eq. (4.12) is needed for the consistency of the theory, otherwise there would be massive poles in the amplitude which are absent in the spectrum [34]. It was just the reason for the general conjecture made by Ooguri and Vafa that all the trees A_n at $n \geq 4$ should actually vanish [34] in the $N = 2$ string theory.

The effective field theory which reproduces the above-mentioned $N = 2$ string tree amplitudes in the NS sector had also been constructed in ref. [34], and it has turned out to be the Plebanski theory of *self-dual gravity* (SDG) [38] in $2 + 2$ dimensions:

$$S_P = \int d^4x \left(\frac{1}{2} \eta^{a\bar{b}} \partial_a \phi \partial_{\bar{b}} \phi + \frac{\kappa}{3} \phi \partial_a \partial_{\bar{a}} \phi \varepsilon^{ab} \varepsilon^{\bar{a}\bar{b}} \partial_b \partial_{\bar{b}} \phi \right) . \quad (4.14)$$

The NS “scalar” ϕ should therefore been identified with the deformation of the flat Kähler potential: $K = K_0 + 2\kappa\phi$. The equations of motion in the Plebanski theory (4.14) can be rewritten to the form of the SDG equations for the four-dimensional curvature tensor R to be constructed from the metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$ [38],

$${}^*R = R, \quad (4.15)$$

which are obviously $SO(2, 2)$ covariant. Hence, the NS “scalar” is not just an ordinary scalar, but non-trivially transforms under the Lorentz transformations. Since the effective field theory has turned out to be the SDG in $2 + 2$ dimensions, it now becomes clear that the NS “scalar” represents the self-dual “graviton”, which has the non-vanishing helicity (+2) and “spin” 2 to be defined with respect to the “little group” $GL(1)$ and the Lorentz group $SO(2, 2)$, respectively [44].

The situation with the $N = 2$ open strings is quite similar. The corresponding tree string amplitudes are to be the “square roots” of the closed ones, due to holomorphic factorization [43]. In addition, the Chan-Paton factors $\{\Lambda^I\}$ can now be introduced, as usual in open string theory [45]. An appropriate $N = 2$ open string world-sheet is the upper-half-superplane Σ , whose boundary is $\partial\Sigma \neq 0$. The open $N = 2$ string vertex operator takes the same form (4.10), which is supposed to be restricted to $\partial\Sigma$.

In particular, the calculation of the 3-point amplitude yields [42]

$$A_3^\circ \sim \left\langle V_o|_{\theta=0}(0) \cdot \int d^2\theta V_o(1) \cdot V_o|_{\theta=0}(\infty) \right\rangle = \kappa_o c_{12}(-i f^{IJK}), \quad (4.16)$$

where the open $N = 2$ string coupling constant κ_o and the Lie algebra structure constants f^{IJK} have been introduced,

$$f^{IJK} = \text{tr}(\Lambda^I, [\Lambda^J, \Lambda^K]). \quad (4.17)$$

This 3-point function is again non-vanishing and non-covariant with respect to the Lorentz group.

Recently, the $N = 2$ open-string 4-point tree amplitude has been calculated [42]:

$$\begin{aligned} A_4^\circ &\sim \int_0^1 dx \left\langle V_o|_{\theta=0} \cdot \int d^2\theta V_o(x) \cdot \int d^2\theta V_o(1) \cdot V_o|_{\theta=0}(\infty) \right\rangle \\ &= \frac{\kappa_o^2}{4} F \frac{\Gamma(1-2s)\Gamma(1-2t)}{\Gamma(2u)}. \end{aligned} \quad (4.18)$$

It also vanishes since $F = 0$ [42].

The effective field theory describing the $N = 2$ open string trees takes the form ¹⁰

$$S_{\text{PS}} = \frac{1}{c_R} \int d^4x \left(-\frac{1}{2} \eta^{ab} \partial_a V \partial_b V + \frac{\kappa_o}{3} \varepsilon^{\bar{a}\bar{b}} V \partial_{\bar{a}} V \partial_{\bar{b}} V \right) , \quad (4.19)$$

and it is just the self-dual Yang-Mills (SDYM) action of Parkes [39]. The equations of motion in the Parkes theory (4.19) can be rewritten to the standard form of the SDYM equations:

$$*F = F , \quad (4.20)$$

which are explicitly covariant with respect to the $SO(2, 2)$ Lorentz transformations. The Parkes “scalar” V appears to be a SDYM (non-covariant) potential with non-trivial Lorentz transformation properties. The SD interpretation of the theory (4.19) suggest to identify this “scalar” with a self-dual vector particle, whose helicity is $+1$ and “spin” 1 [40]. The SDYM equations also follow from requiring the vanishing of the sigma-model beta-functions in the $N = 2$ heterotic string theory with the Yang-Mills background [41].

We are now in a position to discuss the full space-time covariant formulation of the $N = 2$ fermionic strings in terms of the equivalent local quantum supersymmetric field theory, when all 16 sectors of the string theory being taken into account.

5 Space-Time Symmetries and Covariant Actions

As we have learned from the previous sections, the effective field theory describing $N = 2$ open string modes (or left-moving modes of the $N = 2$ closed string) in $2 + 2$ dimensions should be

- *Lorentz covariant* with respect to $SO(2, 2) \cong SL(2, \mathbf{R}) \otimes SL(2, \mathbf{R})$,
- space-time *supersymmetric*,
- *self-dual*.

In addition, it comprises just $8_{\text{B}} \oplus 8_{\text{F}}$ degrees of freedom in the open case, and $16_{\text{B}} \oplus 16_{\text{F}}$ degrees of freedom in the closed case. The space-time supersymmetry actually implies an invariance of the theory under a larger group $SL(2|N) \otimes SL(2|N)$, where the number N of space-time supersymmetries is yet to be determined.

¹⁰The c_R is the quadratic Casimir operator eigenvalue for the gauge group generators.

The covariant description of the self-dual supersymmetry and supergravity in $2 + 2$ dimensions has recently been developed in refs. [46, 47, 48]. The existence of *Majorana-Weyl* (MW) spinors in $2 + 2$ dimensions is the key point in all those constructions. In particular, in the real (Majorana) representation of the $(2 + 2)$ -dimensional Dirac 4×4 matrices Γ^a ,

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \text{diag}(-, -, +, +), \quad (5.1)$$

the Γ_5 -matrix is *real*,

$$\Gamma_5 = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4, \quad \Gamma_5^2 = 1. \quad (5.2)$$

The MW spinors transform in the fundamental (real) representation of one of the $SL(2, \mathbf{R})$ factors in the Lorentz group, and, hence, they have only $2/2 = 1$ degree of freedom on-shell [46]. The self-dual vector particle and MW spinor are naturally united into one $N = 1$ supersymmetric self-dual *vector* multiplet with $1_B \oplus 1_F$ on-shell components. The self-dual *scalar* $N = 1$ supermultiplet, comprising a *real* scalar and a MW spinor, can also be constructed [46].

In *extended* supersymmetry, there exists $N = 2$ self-dual vector multiplet [47], comprising a self-dual vector, two MW spinors of the same chirality and a real scalar ($2_B \oplus 2_F$ on-shell components). Naively, one could expect that the $N = 4$ SDYM would also follow the same pattern, and thus contain $4_B \oplus 4_F$ components in its on-shell spectrum, but it turns out *not* to be the case [48]. The $N = 4$ SDYM actually needs twice as many degrees of freedom for its definition even on-shell [48]. The $N = 4$ SDYM has the following on-shell field contents [44]

$$\left(A_{\underline{a}}^I, G_{\underline{ab}}^I, \rho^I, \tilde{\lambda}^I, S_i^I, T_i^I \right), \quad (5.3)$$

where $G_{\underline{ab}}$ is anti-symmetric and anti-self-dual, ρ and $\tilde{\lambda}$ are anti-MW and MW spinors, respectively, S_i and T_i are scalars, $\hat{i} = \hat{1}, \hat{2}, \hat{3}$; all fields being real and in the adjoint representation of a gauge group G , $I = 1, 2, \dots, \dim G$.

The situation with self-dual supergravities (SDSG's) in $2 + 2$ dimensions is quite similar [40, 48]. There exist the N -extended SDSG's up to the $N \leq 4$, which comprise $N_B \oplus N_F$ on-shell degrees of freedom in $2 + 2$ dimensions, but it is no longer true for $N > 4$ SDSG's, in which the number of the on-shell degrees of freedom is doubled: $2N_B \oplus 2N_F$ [40].

There are just $8_B \oplus 8_F$ on-shell components in the irreducible $N = 4$ SDYM multiplet, and just $16_B \oplus 16_F$ on-shell componets in the irreducible multiplet of the $N = 8$ SDSG. But in order to investigate the possibility to identify them with the $N = 2(4)$ open and closed string modes respectively, as was suggested by Siegel

[40, 44], one should still compare the interactions to be defined separately, in terms of the $N = 2$ open and closed string diagrams and in terms of the $N = 4$ SDYM or $N = 8$ SDSG Feynman graphs, respectively. First, however, one needs covariant actions for the supersymmetric self-dual field theories.

As for the *ordinary* SDYM theory, its covariant action has recently been proposed by Kalitzin and Sokatchev [49] in the *harmonic* space, which can be $M_{2+2} \otimes S^2$. Their action reads ¹¹

$$S_{\text{KS}} = \int d^4x d^2u \operatorname{tr} [\Lambda^{(-3)\alpha} \partial_\alpha^+ (e^\mathcal{V} D^{++} e^{-\mathcal{V}})] . \quad (5.4)$$

The harmonic coordinates on the sphere $S^2 = SU(2)/U(1)$ are introduced as

$$\begin{pmatrix} u^{+\alpha'} \\ u^{-\alpha'} \end{pmatrix} \in SU(2) , \quad \varepsilon_{\alpha'\beta'} u^{+\alpha'} u^{-\beta'} = 1 . \quad (5.5)$$

The relevant derivatives in eq. (5.4) are

$$D^{++} = u^{+\alpha'} \frac{\partial}{\partial u^{-\alpha'}} , \quad \partial_\alpha^+ = u^{+\beta'} \partial_{\alpha\beta'} , \quad \underline{a} = (\alpha\alpha') . \quad (5.6)$$

The Yang-Mills theory in the harmonic space is defined by the relations

$$\begin{aligned} \partial_{\alpha\beta'} &\rightarrow \mathcal{D}_{\alpha\beta'} \equiv \partial_{\alpha\beta'} + A_{\alpha\beta'}(x) , \\ \mathcal{D}_\alpha^\pm &\equiv u^{\pm\beta'} \mathcal{D}_{\alpha\beta'} , \rightarrow D^{++} A_\alpha^+(x, u) = 0 , \end{aligned} \quad (5.7)$$

which imply the constraint [49]

$$[D^{++}, \mathcal{D}_\alpha^+] = 0 . \quad (5.8)$$

The SDYM condition $F_{\alpha'\beta'} = 0$ can now be rewritten to the form [49]

$$[\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+] = 0 , \quad (5.9)$$

because of the relation $F_{\alpha\beta}^{++} = \varepsilon_{\alpha\beta} u^{+\alpha'} u^{+\beta'} F_{\alpha'\beta'}$. The zero-curvature condition (5.9) can be explicitly solved:

$$\mathcal{D}_\alpha^+(x, u) = e^{-\mathcal{V}(x, u)} \partial_\alpha^+ e^{\mathcal{V}(x, u)} , \quad (5.10)$$

and then eq. (5.8) becomes the equation of motion for the harmonic field $\mathcal{V}(x, u)$. The action which reproduces that equation is just the Kalitzin-Sokatchev action (5.4). This action does give the SDYM condition on-shell, when varying with respect to the Lagrange field $\Lambda^{(-3)}(x, u)$. On the other hand, the Lagrange multiplier itself turns

¹¹The \pm or the number indices (in parentheses) mean the $U(1)$ charge.

out *not* to be a propagating field but a pure gauge in the KS action (5.4), which is invariant under the gauge transformations [49]:

$$\Lambda^{(-3)}(x, u) \rightarrow \Lambda^{(-3)}(x, u) + \partial^{+\alpha} b^{(-4)} . \quad (5.11)$$

The substitution $\mathcal{V} \rightarrow V^{++} = e^{\mathcal{V}} D^{++} e^{-\mathcal{V}}$, which is non-local in the harmonic space, but *local* in space-time, makes the action (5.4) to be the action of a *free* theory. Hence, there is *no* scattering in the quantized KS theory, which was demonstrated to a great extent in ref. [50]. The $N = 1$ and $N = 2$ supersymmetric extensions of the KS action can also be constructed along the similar lines [51], and there should be no principal problems to write down the KS-type actions for the N -extended SDSG's of ref. [48] for $N \leq 4$. Nevertheless, they are all actually the *free* theories. The lesson we should learn is that the “purely” self-dual covariant field theories, i.e. those without any additional propagating degrees of freedom, are actually free and do not have scattering.

The $N = 4$ SDYM theory has a quite different covariant component action, when the “extra” fields needed to complete the supermultiplet are playing the role of the Lagrange multipliers for the rest of the fields and vice versa [44, 48],

$$\begin{aligned} \mathcal{L}_{\text{SP}}^{N=4 \text{ SDYM}} = & -\frac{1}{2} G^{abI} (F_{\underline{ab}}^I - \frac{1}{2} \epsilon_{\underline{ab}}^{\underline{cd}} F_{\underline{cd}}^I) + \frac{1}{2} (\nabla_{\underline{a}} S_i^I)^2 - \frac{1}{2} (\nabla_{\underline{a}} T_i^I)^2 \\ & + i(\rho^I \sigma^{\underline{a}} D_{\underline{a}} \tilde{\lambda}^I) - i f^{IJK} \left[(\tilde{\lambda}^I \alpha_i^J \tilde{\lambda}^K) S_i^K + (\tilde{\lambda}^I \beta_i^J \tilde{\lambda}^K) T_i^K \right] , \end{aligned} \quad (5.12)$$

where $\nabla_{\underline{a}}$ and $D_{\underline{a}}$ are the gauge-covariant derivatives, $\sigma^{\underline{a}}$ represent the Dirac matrices in the 2-component notation for the $SO(2, 2)$ Lorentz group,

$$\Gamma^{\underline{a}} = \begin{pmatrix} 0 & \sigma^{\underline{a}} \\ \tilde{\sigma}^{\underline{a}} & 0 \end{pmatrix} , \quad (5.13)$$

in which the Γ_5 is supposed to be diagonal. The α and β matrices are the second independent set of the gamma matrices for the $SO(4)$ or $SO(2, 2)$ internal symmetry [48]. The similar action does also exist for the $N = 8$ SDSG [40].

It is not known how to rewrite that actions in the maximally extended superspace. However, one can construct the SP-type self-dual field theory actions at lower N , when introducing the Lagrange multipliers to the fields of the N -extended SDYM and SDSG [48]. For instance, the $N = 1$ PS-type SDYM action in the $N = 1$ superspace takes the form [48]

$$S_{\text{SP}}^{N=1 \text{ SDYM}} = \int d^4 x d^2 \theta \Lambda^{\alpha I} W_{\alpha}^I , \quad (5.14)$$

where the chiral real $N = 1$ SDYM superfield strength W_α^I and the Lagrange superfield $\Lambda^{\alpha I}$ have been introduced. In components, eq. (5.14) reads

$$S_{\text{SP}}^{N=1 \text{ SDYM}} = \int d^4x \left[-\frac{1}{2} G^{abI} (F_{\underline{ab}}^I(A) - \frac{1}{2} \epsilon_{\underline{ab}}^{cd} F_{\underline{cd}}^I(A)) + i \rho^{\alpha I} (\sigma_{\alpha\beta}^{\underline{a}} \cdot \nabla_{\underline{a}} \tilde{\lambda}^{\beta I}) \right] , \quad (5.15)$$

where all the auxiliaries have been eliminated, and the following propagating fields have been introduced:

$$\begin{aligned} \Lambda_\alpha^I| &= \rho_\alpha^I , & \nabla_\alpha \Lambda_\beta^I| &= (\sigma^{\underline{ab}})_{\alpha\beta} G_{\underline{ab}}^I + \dots , \\ W_\alpha^I| &= \lambda_\alpha^I , & \nabla_\alpha W_\beta^I| &= (\sigma^{\underline{ab}})_{\alpha\beta} F_{\underline{ab}}^I(A) + \dots . \end{aligned} \quad (5.16)$$

The action (5.14) can be combined with a similar action for the three self-dual *scalar* $N = 1$ supermultiplets [48] to be accompanied by the corresponding three scalar Lagrange multiplier $N = 1$ superfields. It will result in the $N = 1$ superspace representation of the $N = 4$ SDYM theory, which is quite convenient for analyzing quantum loops. The $N = 8$ SDSG in terms of the $N = 1$ self-dual superfields and their $N = 1$ Lagrange multipliers follows the same pattern.

The SP-type actions (5.12) and (5.14) are covariant and have non-vanishing 3-point vertices. Hence, there is a non-trivial scattering of covariant objects in that theories. In addition, their quantum loops are all vanishing, because of the non-renormalization theorem in (extended) supersymmetry [53]. It actually implies the N -extended *superconformal* invariance of those theories [40, 44]:

$$\begin{array}{ccc} SO(2,2) \cong SL(2) \otimes SL(2) & \xrightarrow{\text{susy}} & SL(2|N) \otimes SL(2|N) \\ & \downarrow \text{conf} & \downarrow \text{s-conf} \\ SO(3,3) \cong SL(4) & \xrightarrow{\text{susy}} & SL(4|N) \end{array} \quad (5.17)$$

Since the $N = 2(4)$ superstring has no massive modes and has to be equivalent to the effective local quantum field theory, there should be an exact correspondence between the string and field amplitudes. The space-time covariance, supersymmetry and self-duality apparently force the equivalent quantum field theory to be the $N = 4$ SDYM in the open case, and the $N = 8$ SDSG in the closed case, *provided* the string physical states are all belong to the one *irreducible* supermultiplet. Given the covariant equivalence, it implies the need to re-examine the status of the known $N = 2$ superstring amplitudes (see the previous section) with respect to their space-time symmetries, if any.

6 Covariance in Loops Forces Them to Vanish

The $N = 2(4)$ fermionic string theory has the same dilemma as the conventional superstrings: it has to have reducible gauge symmetry generators in order to be explicitly space-time covariant and supersymmetric, but quantization can only be efficiently performed in a non-covariant gauge. That's why our strategy is to consider the quantized $N = 2$ strings and impose covariance and supersymmetry as the consistency conditions.

In the Polyakov approach to the $N = 2$ string theory, the partition function takes the form

$$Z = \int [Dg D\chi DAdx D\psi] e^{-I_{BS}} , \quad (6.1)$$

where the action I_{BS} has been introduced in eq. (2.4). A topology of a closed $N = 2$ super-Riemannian surface is characterized by the two integers: the genus g and the first Chern class c . In the component approach we adopted, the genus g is simply related to the Euler characteristics $\chi(\Sigma)$ of the Riemann surface,

$$\chi(\Sigma) \equiv \frac{1}{2\pi} \int_{\Sigma} d^2z \sqrt{g} R = 2 - 2g , \quad (6.2)$$

whereas the first Chern class c can be identified with the “instanton” number for the Abelian gauge field Wilson lines along the cycles of the surface.

The deformations of the world-sheet metric can be orthogonally decomposed as usual [37, 54]

$$\delta g_{mn} = \{\delta\sigma g_{mn}\} \oplus \{P_1 \delta v\}_{mn} \oplus Ker P_1^\dagger , \quad (6.3)$$

with respect to the natural norm

$$||\delta g_{mn}||^2 = \int_{\Sigma} d^2z \sqrt{g} g^{lm} g^{np} \delta g_{ln} \delta g_{mp} , \quad (6.4)$$

where $(P_1 \delta v)_{mn} \equiv \nabla_m (\delta v)_n + \nabla_n (\delta v)_m - g_{mn} \nabla^p (\delta v)_p$; δv_p and $\delta\sigma$ are the infinitesimal parameters of reparametrizations and Weyl transformations, respectively. Similarly, one has [37]

$$\delta\chi_n = \{\gamma_n \delta\Lambda\} \oplus \{P_{1/2} \delta\zeta\} \oplus Ker P_{1/2}^\dagger , \quad (6.5)$$

where $(P_{1/2} \delta\zeta)_n \equiv \tilde{\nabla}_n \delta\zeta - \frac{1}{2} \gamma_n \gamma^m \tilde{\nabla}_m \delta\zeta$, $\tilde{\nabla}_n \equiv \nabla_n - iA_n$; $\delta\zeta$ and $\delta\Lambda$ are the infinitesimal superconformal and super-Weyl transformation parameters, respectively. The deformations of the Abelian gauge field follow the same pattern [37]:

$$\delta A_n = (P_0 \delta\varepsilon)_n \oplus (\hat{P}_0 \delta\theta)_n \oplus \cap Ker (P_0^\dagger, \hat{P}_0^\dagger) , \quad (6.6)$$

where $(P_0\delta\varepsilon)_n = \partial_n\delta\varepsilon$, $(\hat{P}_0\delta\theta)_n = \epsilon_{nm}\partial_m\theta$; $\delta\varepsilon$ and $\delta\theta$ are the infinitesimal phase and chiral gauge transformation parameters, respectively. The spaces of moduli deformations $\text{Ker } P_1^\dagger$, $\text{Ker } P_{1/2}^\dagger$ and $\cap\text{Ker } (P_0^\dagger, \hat{P}_0^\dagger)$, as well as the conformal Killing (CK) spaces $\text{Ker } P_1$, $\text{Ker } P_{1/2}$ and $\cap\text{Ker } (P_0, \hat{P}_0)$, are all finite-dimensional, their dimensions being determined in general by g and c , in accordance to the Riemann-Roch theorem. In particular, for a surface with $g \geq 2$, one has $(3g-3)$ complex moduli for the metric, $2(2g-2)$ complex fermion moduli for the two “gravitini”, and g complex moduli for the Abelian vector gauge field. For the torus ($g=1$), the CK vectors (CKV’s), spinors (CKSp’s) and scalars (CKS’s) are to be taken into account [37]. For our purposes, it is enough to notice that the partition function can be reduced to a finite-dimensional integral over the $N=2$ supermoduli space $\mathcal{M}_{g,c}$ at the given genus g and the first Chern class c , which schematically takes the form

$$Z = \sum_{g,c=0}^{\infty} \sum_{\substack{\text{spin} \\ \text{structures}}} \int_{\mathcal{M}_{g,c}} d(\text{WP}) \frac{\{\text{zero-mode contributions}\}}{\text{Vol}(\text{CKV})\text{Vol}(\text{CKSp})\text{Vol}(\text{CKS})} \prod_{\{\alpha\}} \det' \Delta_{\{\alpha\}}, \quad (6.7)$$

where $d(\text{WP})$ denotes the Weyl-Petersson measure, $\prod_{\{\alpha\}} \det' \Delta_{\{\alpha\}}$ symbolically represents the contributions of all non-zero modes, and the zero modes are treated separately. The appearance of the Weyl-Petersson measure has to be expected if one insists on the modular invariance of the theory, whereas all the non-zero mode contributions should actually cancel altogether, as a consequence of boson-fermion supersymmetry. It can also be understood in the world-sheet terms because of the natural pairing between the coordinates and ghosts in the $N=2$ string theory, which was also responsible for the actual absence of the non-zero physical modes. The detailed one-loop calculations of ref. [37] also support that conjecture.

Being transported along closed cycles, the world-sheet fermions can either be periodic or anti-periodic, and all that information is just equivalent to assigning a spin structure [54]. The partition function is supposed to be defined as a sum over all spin structures. The integration over Wilson lines of the Abelian gauge field of the $N=2$ string changes spin structures, and it is thus equivalent to the summation over all *continuous* changes of fermionic boundary conditions. However, it does not cover the discontinuous changes in eq. (4.2), since it would then contradict the Lorentz invariance, in particular. It was always assumed that all sectors of the $N=2$ string should contribute with the same phase to the partition function, and that argument was always supported by the fact that all the contributions from various sectors are separately modular invariant (see e.g., [37]). Without any symmetry restrictions, it was indeed a good assumption to reduce an arbitrariness in the theory, since any other reasoning like factorizability or unitarity are rather doubtful in $2+2$ dimensions. But

when the $SO(2, 2)$ Lorentz symmetry and space-time supersymmetry have to be taken into account, the situation becomes quite different. It is space-time supersymmetry which dictates the relative phases between really different (discontinuous) sectors of the $N = 2$ string theory, and which forces separate equal contributions to sum to zero in the total partition function.

The self-dual space-time supersymmetry actually forces all string amplitudes to vanish, at any positive genus and “instanton” number. There is no need even for the maximal number of supersymmetries: just one space-time supersymmetry in self-dual and space-time covariant theory makes the job done. As was calculated in ref. [57], the one-loop 3-point amplitude in the NS-type sector of the $N = 2$ string theory is non-vanishing and severely IR-divergent. However, the contribution of space-time fermions from the R-type sectors of the string theory is just supposed to cancel it.

From this viewpoint, the status of the non-vanishing 3-point tree string amplitudes discussed in sect. 4 should be reconsidered. The tree amplitudes can be defined inside each sector independently of the existence of any other sectors in the theory. Therefore, those tree amplitudes cannot be “made to vanish”. However, they actually describe the scattering of the non-covariant quantities. Applying the Lorentz boost to any of the 3-point tree amplitudes of sect. 4 results in the multiplicative redefinition of the coupling constant as the only change [39]. Therefore, there is no Lorentz-invariant scattering to be described by those amplitudes. It clearly matches with the often repeated statement about the “no tree scattering in SDYM” [56]. In other words, there is no covariant amplitude which would reduce, say, to eq. (4.11) in an appropriate gauge. If we now take a look on the equivalent quantum field theory of eq. (5.12), we immediately notice only the presence of the 3-point vertices in the Lagrangian, which relate *different* sectors of the string theory. Though there is no covariant scattering inside each of the sectors in eq. (4.2), it is consistent with space-time covariance and supersymmetry to have 3-point interactions between SD fields and their Lagrange multipliers. Since the states corresponding to the Lagrange multipliers are supposed to have negative or zero norms, this result is not very optimistic for the future prospects of the theory under consideration.

7 Conclusion

The outcome of our discussion can be summarized as follows.

The critical $N = 2$ fermionic strings live in 2 complex dimensions, the Lorentz

group being $SO(2,2)$ and the signature being $(-, -, +, +,)$. They have only zero modes as the physical states. The open $N = 2$ fermionic string spectrum is space-time supersymmetric and comprises 8 bosonic and 8 fermionic states. The $N = 2$ closed string spectrum has 16 bosonic and 16 fermionic states. The spin contents of that supermultiplets exactly coincides with that of the $N = 4$ supersymmetric SDYM and the $N = 8$ SDSG, respectively, which are the equivalent local quantum space-time supersymmetric field theories for the $N = 2$ fermionic strings. Those results were first outlined by Siegel [32, 40, 44] on the ground of pure symmetry considerations. We have seen above how his ideas can be supported inside the NSR-like formulations for the $N = 2$ and $N = 4$ fermionic strings. However, our results do not give the ultimate proof of an actual presence of the *maximal* space-time supersymmetry: it is the case under the additional assumption about the irreducibility of the zero mode supermultiplets. The existence of the GS-type formulation of the $N = 2(4)$ strings, recently proposed by Siegel [58], with the fermionic degrees of freedom to be represented by the $SO(2,2)$ *spinors*, gives the alternative covariant and space-time supersymmetric description of the same theory but it does not rule out the possibility to have the space-time supersymmetry which is not maximal.

The space-time covariance and supersymmetry in the theory of $N = 2$ fermionic strings imply the absence of scattering for any covariant quantities inside of each sector of that string theory. The only allowed type of covariant scattering is the 3-point scattering between the self-dual fields and their Lagrange multipliers corresponding to the different sectors, when the specific vertices being given by the appropriate Siegel-Parkes-type action. In those supersymmetric field theory PS-type actions, the Lagrange multipliers are necessarily the propagating fields. All the string loop amplitudes vanish as the consequence of the *alternative* sum over spin structures, which is the only consistent with the space-time covariance and supersymmetry. Therefore, the $N = 2$ fermionic string theory is equivalent to the supersymmetric local quantum field theory which looks like a topological field theory.

The huge degeneracy of the $N = 2(4)$ non-compact fermionic string theory is ultimately responsible for the mutual cancellation of various non-trivial contributions to its tree and loop amplitudes. Therefore, one should expect that even slight deformation of this theory can make it to be very non-trivial and the amplitudes to be non-vanishing. One of the way has recently been suggested in ref. [59] by using the background charge to be represented by the additional term in the stress tensor along the standard lines of the Dotsenko-Fateev construction [60] in conformal field theory, or, equivalently, at the expense of the non-trivial dilaton expectation value in the sigma-model approach to strings. Those modified $N = 2$ fermionic strings turn out to

live in $1+2m$ dimensions, $m = 1, 2, \dots$, and do not have vanishing 4-point amplitudes [61]. Therefore, the theory of the extended fermionic strings still has a potential to be non-trivial not only topologically but also geometrically, i.e. with non-vanishing tree and string loop amplitudes, all having a rich geometrical structure.

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