Statistical properties of the radiation from SASE FEL operating in the linear regime

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Abstract

The paper presents analysis of statistical properties of the radiation from self amplified spontaneous emission (SASE) free electron laser operating in linear mode. The investigation has been performed in a one-dimensional approximation, assuming the electron pulse length to be much larger than a coherence length of the radiation. The following statistical properties of the SASE FEL radiation have been studied: field correlations, distribution of instantaneous power, distribution of the radiation energy after monochromator installed at the FEL amplifier exit and photoelectric counting statistics of SASE FEL radiation. It is shown that the radiation from SASE FEL operating in linear regime possesses all the features corresponding to completely chaotic polarized radiation.

1. Introduction

Correct design of the SASE FEL and planning of user's equipment and experiments depend strongly on knowledge of the radiation amplification process in the SASE FEL and properties of the output radiation. The process of amplification in the SASE FEL starts from the shot noise in the electron beam having stochastic nature. It means that the SASE FEL radiation is also stochastic object, that is why there exists definite problem for description of the SASE FEL process requiring development of time-dependent theory of the FEL amplifier. Some averaged output characteristics of SASE FEL have been obtained in refs. [1, 2, 3, 4, 5, 6, 7, 8]. Quantum consideration of photon statistics in SASE FEL has been performed in ref. [9, 10]. An approach for time-dependent numerical simulations of SASE FEL has been developed in ref. [8].

Nevertheless, the previous studies do not give comprehensive description of the output radiation from SASE FEL and the following statistical properties should studied in detail: field correlations, statistics of instantaneos power, statistics of the radiation energy after the monochromator installed at the exit of SASE FEL, photoelectric counting statistics of SASE FEL radiation etc. This paper gives answers on all the above mentioned problems describing statistical properties of the radiation from SASE FEL operating in linear regime. The investigation has been performed in a one-dimensional approximation. It is shown that the radiation from SASE FEL operating in the linear regime possesses all the features corresponding to completely chaotic polarized radiation. In particular, the higher order correlation functions are expressed via the first order correlation function and the probability density function of the energy after monochromator follows the gamma distribution.

2. Analysis of radiation properties in frequency domain

In the linear mode of operation the SASE FEL can be treated as a narrow band linear device which filters a wide band random input signal – shot noise. General property of such devices is that an output signal is a Gaussian random process. Since electron pulse at the entrance of SASE FEL has finite duration, we deal with nonstationary random process. Analytical study of such a process in general case is very complicated. Analysis of design parameters of VUV and X-ray SASE FELs shows that the feature of these devices is that the bunch length is much larger than the radiation coherence length. So, one can use the model

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of rectangular profile of the electron bunch and the steady-state spectral Green's function. In the frame of this model it becomes possible to describe analytically all statistical properties of the radiation from SASE FEL operating in the high gain linear regime.

Let us consider microscopic picture of the electron beam current at the entrance into the undulator. The electron beam current is constituted by moving electrons randomly arriving to the entrance of the undulator:

$$I(t) = (-e)\sum_{k=1}^{N}\delta(t-t_k) \ ,$$

where $\delta(\ldots)$ is delta-function, (-e) is the charge of the electron, N is the number of electrons in a bunch and t_k is random arrival time of the electron to the undulator entrance. The electron beam current $I(t)$ and its Fourier transform $\bar{I}(\omega)$ are connected by Fourier transformation:

$$\bar{I}(\omega) = \int\limits_{-\infty}^{\infty} e^{i\omega t} I(t) dt = (-e)\sum_{k=1}^{N} e^{i\omega t_k} \ ,$$

Let us calculate the correlation of $\bar{I}(\omega)$ and $\bar{I}(\omega')$:

$$\langle \bar{I}(\omega)\bar{I}^*(\omega')\rangle = e^2\langle \sum_{k=1}^{N}\sum_{n=1}^{N} \exp(i\omega t_k - i\omega' t_n)\rangle \ .$$

Here $\langle\ldots\rangle$ means the averaging over ensemble of bunches. The electron bunch profile is described by the profile function $F(t)$: $\langle I(t)\rangle = (-e)NF(t)$. Taking into account this relation we obtain that

$$\langle \exp(i\omega t_k)\rangle = \int\limits_{-\infty}^{\infty} F(t_k) e^{i\omega t_k} dt_k = \bar{F}(\omega) \ .$$

In the case when $N \mid \bar{F}(\omega) \mid^2 \ll 1$, we can write the following expressions:

$$\langle \bar{I}(\omega)\bar{I}^*(\omega')\rangle = e^2 N \bar{F}(\omega-\omega') \ , \qquad (1)$$

$$\langle \mid \bar{I}(\omega)\mid^2 \mid \bar{I}(\omega')\mid^2\rangle = \langle \mid \bar{I}(\omega)\mid^2\rangle\langle \mid \bar{I}(\omega')\mid^2\rangle + \mid \langle \bar{I}(\omega)\bar{I}^*(\omega')\rangle \mid^2 \ . \qquad (2)$$

In the following we will analyze in detail the case of rectangular profile of the electron bunch. Transversely coherent fraction of the input shot noise signal is defined by the total beam current, so Fourier amplitude of the electric field at the exit of SASE FEL operating in the linear high-gain regime can be written as follows: $\bar{E}(\omega) = H_A(\omega-\omega_0)\bar{I}(\omega)$, where $H_A(\omega-\omega_0)$ is the Green's function and ω_0 is the resonance frequency. For many applications and for diagnostic measurements of SASE FEL radiation a monochromator with transmission function H_m will be installed at the FEL amplifier exit. In this case the expression for $\bar{E}(\omega)$ takes the form: $\bar{E}(\omega) = H_m(\omega-\omega_0)H_A(\omega-\omega_0)\bar{I}(\omega)$. Using eqs. (1) and (2) we obtain

$$g_1(\omega,\omega') = \frac{\langle \bar{E}(\omega)\bar{E}^*(\omega')\rangle}{\left[\langle \mid \bar{E}(\omega)\mid^2\rangle\langle \mid \bar{E}(\omega')\mid^2\rangle\right]^{1/2}} = \bar{F}(\omega-\omega') \ , \qquad (3)$$

$$g_2(\omega,\omega') = \frac{\langle \mid \bar{E}(\omega)\mid^2 \mid \bar{E}(\omega')\mid^2\rangle}{\langle \mid \bar{E}(\omega)\mid^2\rangle\langle \mid \bar{E}(\omega')\mid^2\rangle} = 1 + \mid g_1(\omega,\omega')\mid^2 \ . \qquad (4)$$

We should note that eq. (4) is a property of a completely chaotic polarized radiation. It follows from this conclusion that $\mid \bar{E}(\omega)\mid^2$ is distributed in accordance with the negative exponential probability density function (see ref. [11] for more detail):

$$p(\mid \bar{E}(\omega)\mid^2) = \frac{1}{\langle \mid \bar{E}(\omega)\mid^2\rangle}\exp\left(-\frac{\mid \bar{E}(\omega)\mid^2}{\langle \mid \bar{E}(\omega)\mid^2\rangle}\right) \ . \qquad (5)$$

The next problem is description of the fluctuations of the radiation bunch energy W at a detector installed after a monochromator. From the expression for Pointing's vector and Parseval's theorem we can write the expression for the average energy measured by the detector:

$$\langle W\rangle = \frac{cS}{4\pi^2}\int\limits_0^{\infty}\langle \mid \bar{E}(\omega)\mid^2\rangle d\omega = \frac{ce^2SN}{4\pi^2}\int\limits_0^{\infty} \mid H_m(\omega-\omega_0)\mid^2 \mid H_A(\omega-\omega_0)\mid^2 d\omega \qquad (6)$$

where S is the transverse area of the detector. Taking into account relation (4) we can write the expression for normalized dispersion:

$$\sigma_W^2 = \frac{\langle (W-\langle W\rangle)^2\rangle}{\langle W\rangle^2} = \frac{\int\limits_0^{\infty} d\omega \int\limits_0^{\infty} d\omega' \langle \mid \bar{E}(\omega)\mid^2\rangle\langle \mid \bar{E}(\omega')\mid^2\rangle \mid g_1(\omega',\omega')\mid^2}{\int\limits_0^{\infty} d\omega\langle \mid \bar{E}(\omega)\mid^2\rangle \int\limits_0^{\infty} d\omega'\langle \mid \bar{E}(\omega')\mid^2\rangle} \qquad (7)$$

Let us derive analytical expression for σ_W^2. In the framework of the one-dimensional model and when the effects of the space charge field and energy spread in the beam can be neglected, operation of the

FEL amplifier is described in terms of the gain parameter Γ and the efficiency parameter ρ (see, e.g. refs. [3, 12, 13]) which are connected by the relation $\rho = \lambda_w \Gamma / 4\pi$, where λ_w is the undulator period. Using expression for the solution of the initial-value problem in a high-gain limit we can write [1, 4, 6]:

$$\mid H_A \mid^2 = A \exp\left[-\frac{(\omega - \omega_0)^2}{2\sigma_A^2} \right] , \qquad \sigma_A = 3\sqrt{\frac{2}{\sqrt{3}} \frac{\rho\omega_0}{\sqrt{\hat{z}}}} . \tag{8}$$

Substituting expressions (8) into eq. (7) we obtain for the case of a Gaussian profile of the monochromator line $\mid H_m \mid^2 = \exp\left[-(\omega - \omega_0)^2 / 2\sigma_m^2 \right]$:

$$\sigma_W^2 = \frac{\sqrt{\pi}}{\hat{\sigma}^2} \int\limits_0^{\hat{\sigma}} \mathrm{erf}(x) dx , \qquad \hat{\sigma} = \frac{\hat{\sigma}_A \hat{\sigma}_m}{\sqrt{\hat{\sigma}_A^2 + \hat{\sigma}_m^2}} , \qquad \hat{\sigma}_m = \sigma_m T . \tag{9}$$

We assume pulse duration T to be large, $\sigma_A T \simeq \rho\omega_0 T \gg 1$. The value $\rho\omega_0 T$ is of the order of $10^2 \div 10^3$ for modern projects of VUV and X-ray SASE FELs, so the obtained result can be used for practical calculations.

The next practical problem is to find the probability density distribution of the radiation energy after monochromator, $p(W)$. Using arguments similar to that of ref. [11] one can show that the distribution of the radiation energy after the monochromator is described rather well by a gamma probability density function:

$$p(W) = \frac{M^M}{\Gamma(M)} \left(\frac{W}{\langle W \rangle} \right)^{M-1} \frac{1}{\langle W \rangle} \exp\left(-M \frac{W}{\langle W \rangle} \right) , \tag{10}$$

where $\Gamma(M)$ is gamma function and $M = 1/\sigma_W^2$. Parameter M can be interpreted as the average number of "degrees of freedom" or "modes" in a radiation pulse. It follows from eq. (7) that this parameter can not be less than unity. When M tends to unity, the distribution (10) tends to the negative exponential distribution (5). When $M \gg 1$, the distribution (10) tends to the Gaussian distribution.

3. Analysis of the radiation properties in time domain

Time dependence of the radiation field has the form $E(z,t) = \left[\tilde{E}(z,t) e^{-i\omega_0 (z/c - t)} + C.C. \right]$ at any position along the undulator. The first and second order time correlation functions are defined as follows:

$$g_1(t - t') = \frac{\langle \tilde{E}(t) \tilde{E}^*(t') \rangle}{\left[\langle \mid \tilde{E}(t) \mid^2 \rangle \langle \mid \tilde{E}(t') \mid^2 \rangle \right]^{1/2}} . \tag{11}$$

$$g_2(t - t') = \frac{\langle \mid \tilde{E}(t) \mid^2 \mid \tilde{E}(t') \mid^2 \rangle}{\langle \mid \tilde{E}(t) \mid^2 \rangle \langle \mid \tilde{E}(t') \mid^2 \rangle} . \tag{12}$$

Using approximation $\rho\omega_0 T \gg 1$ and formulae (3), (4) we can write

$$g_1(t - t') = \frac{\int\limits_{-\infty}^{\infty} d(\Delta\omega) \mid H_A(\Delta\omega) \mid^2 \exp\left[-i\Delta\omega (t - t') \right]}{\int\limits_{-\infty}^{\infty} d(\Delta\omega) \mid H_A(\Delta\omega) \mid^2} , \tag{13}$$

$$g_2(t - t') = 1 + \mid g_1(t - t') \mid^2 . \tag{14}$$

The distribution of the instantaneous radiation power $P \propto \mid \tilde{E} \mid^2$ is the negative exponential distribution:

$$p(\mid \tilde{E}(t) \mid^2) = \frac{1}{\langle \mid \tilde{E}(t) \mid^2 \rangle} \exp\left(-\frac{\mid \tilde{E}(t) \mid^2}{\langle \mid \tilde{E}(t) \mid^2 \rangle} \right) , \tag{15}$$

and finite-time integrals of the instantaneous power follow the gamma distribution (10).
Substitution expression (8) into eq. (13) we obtain ($\tau = (t - t')$):

$$g_1(\tau) = \exp\left(-\frac{9\rho^2 \omega_0^2 \tau^2}{\sqrt{3}\hat{z}} \right) . \tag{16}$$

Following the approach of ref. [11], we define the coherence time τ_c as

$$\tau_c = \int\limits_{-\infty}^{\infty} \mid g_1(\tau) \mid^2 d\tau = \frac{\sqrt{\pi}}{\sigma_A} = \sqrt{\frac{\sqrt{3}\pi\hat{z}}{18}} \frac{1}{\rho\omega_0} . \tag{17}$$

Let us discuss the problem how to measure statistical properties of the SASE FEL radiation. Typical pulse duration of existent projects of SASE FELs is of about a fraction of picosecond. The resolution time of modern fast photoelectric detectors is much larger than this value, of about a fraction of nanosecond, which allows to measure total energy of the radiation pulse only. On the other hand, there are no such evident technical limitations for measurement of statistical properties of SASE FEL radiation in frequency domain.

4. Photoelectric Detection of SASE FEL Radiation

It has been shown above that the energy, W, in the radiation pulse reaching the photodetector is unpredictable, we can predict the probability density $p(W)$ only. In this case the probability of detection of K photons by a photodetector is given by [11]:

$$P(K) = \int\limits_0^\infty \frac{(\alpha W)^K}{K!} \exp(-\alpha W) p(W) dW \ , \qquad (18)$$

where $\alpha = \eta/\hbar\omega_0$ and η is the quantum efficiency of the photodetector. Using formula (18) we get the expressions for the mean and for the variance of K value [11]:

$$\langle K \rangle = \alpha \langle W \rangle \ , \qquad \sigma_K^2 = \frac{\langle K^2 \rangle - \langle K \rangle^2}{\langle K \rangle^2} = \frac{1}{\langle K \rangle} + \sigma_W^2 \ , \qquad (19)$$

where $\sigma_W^2 = 1/M$ is given by formula (7). The ratio of the classical variance to the "photon shot noise" variance is equal to $\delta_c = \langle K \rangle / M$. Parameter δ_c is named as the photocount degeneracy parameter.

When SASE FEL operates in the linear regime, the probability density of the energy after monochromator, $p(W)$, is the gamma distribution (10). Substituting (10) into (18) and performing integration we come to the negative binomial distribution [11]:

$$P(K) = \frac{\Gamma(K+M)}{\Gamma(K+1)\Gamma(M)} \left(1 + \frac{M}{\langle K \rangle} \right)^{-K} \times \left(1 + \frac{\langle K \rangle}{M} \right)^{-M} . \qquad (20)$$

It can be shown that the negative binomial distribution tends to the gamma distribution (10) at large values of the photocount degeneracy parameter δ_c.

The formula for calculation of the photocount degeneracy parameter δ_c is given by $\delta_c \simeq \eta R_m \Delta\omega_m N_{ph}(\omega)/M$, where R_m is integral reflection coefficient of the mirrors and dispersive element of the monochromator, N_{ph} is spectral density of photons radiated within one pulse. Peculiar feature of SASE FEL is that the degeneracy parameter is always extremely large. Typical values of δ_c will be about $10^8 \div 10^{10}$ for X-ray FELs, so we can state that classical approach is adequate for description of statistical properties of the radiation from SASE FEL.

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