Duality Origami: Emergent Ensemble Symmetries in Holography and Swampland

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We discuss interrelations between several ideas in quantum gravity. One is the Swampland program, which states that a low-energy effective field theory should satisfy non-trivial constraints to have an ultraviolet (UV) completion in quantum gravity. Another is the concept of ensemble averaging in holography, where a coarse-grained description is obtained by an integral over a moduli space. To examine the relation between the two, we study ensemble averages of generalized Narain-type theories associated with a general even quadratic form and their holographic duals. We establish the emergence of global symmetries and discuss their consistency with the Swampland conjecture forbidding exact global symmetries. Out of all the zero-form symmetries, the quantum symmetries in the bulk are truly emergent, while classical symmetries are identified as vestiges of T-duality of the Narain-type theories. The latter mechanism can be formulated very generally as a "folding" of T-duality orbits via the Siegel-Weil Theorem. We also discuss the interrelations between the Swampland distance conjecture, on one hand, and ensemble averaging and spectral decompositions, on the other. The spectral decomposition also illustrates how ensemble averaging sits within the low-energy limit of certain string compactifications. Our analysis suggests fascinating links between the Swampland, the Landscape, and ensemble averaging.

arXiv:2305.10224v1 [hep-th] 17 May 2023

Introduction: Swampland versus Ensembles

In recent years two approaches to understanding quantum gravity have emerged. On one hand, there is the Swampland program [1, 2], which posits the existence of non-trivial consistency conditions for low-energy effective field theories (EFTs) to be embedded in an ultraviolet (UV) theory of quantum gravity. On the other, there is the holographic approach that leverages dualities between $(D+1)-$dimensional anti-de Sitter (AdS) spacetimes and D-dimensional conformal field theories (CFTs).

Standard holographic dualities are argued from the top-down in particular string compactifications—the quintessential example being the correspondence between Type IIB on $AdS_5 \times S^5$ and $4D$ $\mathcal{N}=4$ Super-Yang Mills theory [3]. However, there are also bottom-up holographic constructions that utilize ensemble averages of theories. Known examples of this form include Jackiw-Teitelboim (JT) gravity [4, 5] and Narain Ensembles [6, 7].

Ensemble average dualities are puzzling from both the standard holography and Swampland viewpoints. Indeed, at first glance ensemble averaging appears counter to the one-to-one holographic dualities of the standard approach. Furthermore, ensemble-averaged theories provide loopholes to several Swampland conjectures, including one of the most well-known conjectures excluding exact global symmetries in theories of quantum gravity [8–13]. Based on these considerations, there have been arguments that ensemble averaging is explicitly excluded by the Swampland program, at least in ten-dimensional string theory [14].

In this *Letter*, we address the relationship between ensembles, holography, and the Swampland by outlining the physical and mathematical framework to embed Narain ensembles into a more standard approach to holography. We first establish *emergent ensemble symmetries* that are global symmetries in the $3D$ gravity dual of generalized Narain ensembles [15] [16]. We then relate ensembles to a pre-averaged duality involving Maxwell-Chern-Simons theory and its embedding in string compactifications. This allows us to make contact with several Swampland conjectures. Finally, we utilize the theory of automorphic forms to illustrate that the ensemble average of the generalized Narain CFTs is a coarse-grained description of Maxwell-Chern-Simons and how additional fluctuations serve to break the emergent ensemble symmetries.

Global Symmetries in Chern-Simons Theories

Let us first discuss the global symmetries in the bulk theory after the ensemble average. The bulk is an Abelian $\mathrm{U}(1)^{p+q}$ Chern-Simons theory whose action is given by [17]

$$S_{\mathrm{CS}} = \frac{1}{8\pi}\int d^3x \sum_{i,j=1}^{p+q} Q_{ij} A_i dA_j . \tag{1}$$

Here Q_{ij} is an integer-valued matrix of signature (p,q) with $Q_{ii} \in 2\mathbb{Z}$. This matrix defines an even integral quadratic form $Q[\ell,\ell'] := \sum_{i,j=1}^{p+q} Q_{ij}\ell^i \ell'^j \in \mathbb{Z}$ and $Q[\ell] := Q[\ell,\ell]$.

This theory is an Abelian topological quantum field theory (TQFT), which is completely determined by the anyon data: the anyon fusion algebra, the topological spin of the anyons (and the chiral central charge, which we have chosen to be $p-q$) [18–20]. [21]

For the TQFT defined by Eq. (1), the set of anyons is labeled by an element of the the discriminant group $\mathscr{D} := \Lambda^*/\Lambda$, where Λ is an even integral lattice $\Lambda := \{\ell \in \mathbb{Z}^{p+q} |\ Q[\ell] \in 2\mathbb{Z}\}$ and Λ^* denotes the dual lattice of Λ.

The anyon fusion algebra is simply defined by a natural addition of two elements of $\mathscr{D}$. The discriminant group is a finite Abelian group of order $|\mathscr{D}| = |\Lambda^*/\Lambda| = |\det(Q)|$.

The topological spin θ of an anyon $\alpha \in \mathscr{D}$ is given by

$$\theta(\alpha) = \exp\left(i\pi Q[\alpha]\right). \tag{2}$$

The other anyon data can be derived from $\mathscr{D}$ and θ: the braiding phase of two anyons is

$$\mathcal{B}(\alpha,\beta) = \frac{\theta(\alpha+\beta)}{\theta(\alpha)\theta(\beta)} = \exp\left(2i\pi Q[\alpha,\beta]\right), \tag{3}$$

and the modular S- and T-transformations by

$$S_{\alpha\beta} = \frac{\mathcal{B}(\alpha,\beta)}{\sqrt{|\mathcal{D}_Q|}}, \quad T_{\alpha\beta} = e^{-\frac{\pi i(p-q)}{12}}\theta(\alpha)\delta_{\alpha\beta}. \tag{4}$$

An Abelian Chern-Simons theory has global one-form and zero-form symmetries [22] which arise from automorphisms of the TQFT data [23] [24]. The one-form symmetry group is nothing but the discriminant $\mathscr{D}$ (cf. [25]). The zero-form symmetry group is defined via a permutation of the anyons (i.e., elements of $\mathscr{D}$) preserving the anyon data $(\mathscr{D},\theta)$: [26]

$$\mathrm{Aut}(\mathscr{D},\theta) := \{\sigma \in \mathrm{Aut}(\mathscr{D}) \,|\, \theta(\alpha) = \theta(\sigma\cdot\alpha)\, \forall \alpha \in \Lambda\}. \tag{5}$$

Some classical symmetries arise from the symmetries of the Lagrangian—if $(\mathscr{D},\theta)$ arises from a quadratic form Q, the classical symmetry group is the symmetry of the Lagrangian in Eq. (1)

$$\mathrm{O}_Q(p,q;\mathbb{Z}) := \{\Sigma \in \mathrm{GL}(p+q,\mathbb{Z}) \;\mid\; \Sigma^T Q \Sigma = Q\}, \tag{6}$$

which naturally induces an element of $\mathrm{Aut}(\mathscr{D},\theta)$. Otherwise, a symmetry is a quantum symmetry when it is not manifest in the Lagrangian. Notice that different quadratic forms Q, Q' can lead to the same anyon data $(\mathscr{D},\theta)$, in which case the two Lagrangians associated with Q, Q' are quantum-equivalent [18–20] [27]. In general, whether a symmetry is classical or not depends on the choice of the classical Lagrangian (i.e., the duality frame).

Holography of General Narain Ensembles

To discuss the emergence of global symmetries, let us next discuss the ensemble averaging in the CFT duals. We follow a general discussion in Ref. [15], which generalized the previous discussion for even self-dual lattices in Refs. [6, 7] (see also Refs. [28–33] for variations).

The Narain-type theory [34, 35] is defined by the same data as the bulk theory, namely a quadratic form Q. The moduli space of the CFT is pq-dimensional and is given by a double coset

$$\mathcal{M}_Q := \mathrm{O}_Q(p,q;\mathbb{Z})\backslash\left(\mathrm{O}(p,q;\mathbb{R})/(\mathrm{O}(p;\mathbb{R})\times\mathrm{O}(q;\mathbb{R})\right). \tag{7}$$

There is a family of CFT partition functions associated with an element α of the discriminant group:

$$Z^Q_\alpha(\tau,\overline{\tau};m) = \frac{\vartheta^Q_\alpha(\tau,\overline{\tau};m)}{\eta^p(\tau)\overline{\eta}^q(\overline{\tau})}. \tag{8}$$

Here the theta function ϑ^Λ_α is given by

$$\vartheta^Q_\alpha(\tau,\overline{\tau};m) := \sum_{\ell\in\Lambda+\alpha} e^{i\pi\tau_1 Q[\ell] - i\pi\tau_2 H[\ell]}, \tag{9}$$

where the moduli dependence enters through the Hamiltonian H (see Ref. [15] for more details).

Under the generators of $\mathrm{SL}(2,\mathbb{Z})$, the theta functions transform as

$$\begin{aligned}\vartheta^Q_\alpha(\tau+1;m) &= e^{\frac{\pi i(p-q)}{12}} \sum_{\beta\in\mathscr{D}} T_{\alpha\beta}\vartheta^Q_\beta(\tau;m),\\ \vartheta^Q_\alpha\left(-\frac{1}{\tau};m\right) &= e^{-\frac{\pi i}{4}(p-q)}\tau^{\frac{p}{2}}\overline{\tau}^{\frac{q}{2}} \sum_{\beta\in\mathscr{D}} S_{\alpha\beta}\vartheta^Q_\beta(\tau;m),\end{aligned} \tag{10}$$

where S, T are the modular matrices (4) for the anyons.

We now consider the ensemble average of the general Narain CFTs. Since the label α transforms non-trivially under a general element of $\mathrm{O}_Q(p,q;\mathbb{Z})$, the theta function ϑ_α in Eq. (9) is defined over the moduli space

$$\mathcal{M}_{Q,\alpha} := \mathrm{O}_{Q,\alpha}(p,q;\mathbb{Z})\backslash\left(\mathrm{O}(p,q;\mathbb{R})/(\mathrm{O}(p;\mathbb{R})\times\mathrm{O}(q;\mathbb{R})\right), \tag{11}$$

where $\mathrm{O}_{Q,\alpha}(p,q;\mathbb{Z})$ is a subgroup of $\mathrm{O}_Q(p,q;\mathbb{Z})$ preserving α. Note that this moduli space is a cover of the moduli space $\mathcal{M}_Q = \mathcal{M}_{Q,\alpha=0}$.

We can now define the ensemble average of the partition functions in Eq. (8) via

$$\langle\vartheta^Q_\alpha\rangle(\tau) := \frac{1}{\mathrm{Vol}(\mathcal{M}_{Q,\alpha})}\int_{\mathcal{M}_{Q,\alpha}} [dm]\,\vartheta^Q_\alpha(\tau;m), \tag{12}$$

where $[dm]$ denotes the canonical measure associated with the Zamolodchikov metric of the moduli space. For $p+q>4$, this average can be evaluated to be given by Siegel-Eisenstein series [7, 15, 36, 37]

$$E^Q_\alpha(\tau,\bar{\tau}) := \delta_{\alpha\in\Lambda} + \sum_{(c,d)=1,\,c>0} \frac{\gamma^Q_\alpha(c,d)}{(c\tau+d)^{\frac{p}{2}}(c\overline{\tau}+d)^{\frac{q}{2}}}, \tag{13}$$

where δ_α is equal to unity for $\alpha=0$ and zero otherwise. The summation Eq. (13) can be interpreted as a sum over geometries in the gravitational theory [6, 7]: the sum over (c,d) is a sum over the "$\mathrm{SL}(2,\mathbb{Z})$ black holes" $M_{(c,d)}$ [38, 39], where $M_{(0,1)}$ and $M_{(1,0)}$ correspond to thermal AdS and the BTZ black hole [40], respectively.

The size of the contribution from each geometry is determined by a coefficient given by a quadratic Gauss sum $\gamma^Q_\alpha(c,d) := \lambda^Q_{\alpha,0}(M^{-1})$, where

$$\begin{aligned}\lambda^Q_{\alpha,\beta}(M) :=& \frac{1}{\sqrt{|\mathcal{D}_Q|}}\, e^{-\frac{\pi i}{4}(p-q)} c^{-\frac{p+q}{2}}\\ &\times \sum_{\ell_c\in\Lambda/(c\Lambda)} e^{\frac{\pi i}{c}\left(aQ[\ell_c+\alpha] - 2Q[\ell_c+\alpha,\beta] + dQ[\beta]\right)}.\end{aligned} \tag{14}$$

This expression is known to coincide with the lens space partition function of the Chern-Simons theory [41, 42] (with insertions of Wilson lines labeled by α,β), as pointed out in [15]. It appears in the modular transformation under a general element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\vartheta^Q_\alpha(\tau_{[M]};m) = \sum_{\beta\in\mathscr{D}} \mathcal{U}^Q_{\alpha,\beta}(M,\tau)\,\vartheta^Q_\beta(\tau;m), \tag{15}$$

$$\mathcal{U}^Q_{\alpha,\beta}(M,\tau) := (c\tau+d)^{\frac{p}{2}}(c\overline{\tau}+d)^{\frac{q}{2}}\lambda^Q_{\alpha,\beta}(M), \tag{16}$$

which follows from the repeated use of Eq. (10).

Emergence of Global Symmetries in the Boundary CFT

We can now describe the emergence of a global symmetry after averaging generalized Narain CFTs. By definition, a zero-form symmetry σ preserves the anyon data $(\mathscr{D},\theta)$ up to a permutation of anyon labels. Since the anyon data determine the modular S,T-matrices (4) and hence the partition function, we find that Eisenstein series are simply permuted:

$$E^Q_{\sigma\cdot\alpha}(\tau,\overline{\tau}) = E^Q_\alpha(\tau,\overline{\tau}). \tag{17}$$

Of course, this is expected since the zero-form symmetries are also the global symmetries of the boundary theory. The question we address in this section is to discuss how this symmetry arises in the process of ensemble averaging.

When the zero-form symmetry is a quantum symmetry, the symmetry changes the functional form of the theta function, and the symmetry is not simply a permutation of the theta function. The symmetry is then truly emergent which appears only after ensemble averaging.

By contrast, for a classical zero-form symmetry $\Sigma \in \mathrm{O}_Q(p,q;\mathbb{Z})$, we know that the symmetry preserves the CFT partition function.

$$\vartheta^Q_{\Sigma\cdot\alpha}(\tau,\overline{\tau};\Sigma\cdot m) = \vartheta^Q_\alpha(\tau,\overline{\tau};m). \tag{18}$$

It is straightforward to check this statement explicitly. More physically, this essentially follows from the fact that the moduli space (7) is already quotiented by the action of the T-duality group $\mathrm{O}_Q(p,q;\mathbb{Z})$ (6). Note that before the ensemble average the T-duality group acts both on the anyon label and the moduli, while after the average it acts only on the anyon label. What this means is that the T-duality group of the original theory is "folded" into an emergent global symmetry via ensemble averaging—a process we call *duality origami*.

While we discussed the case of generalized Narain theories, we can formulate this folding in general. Let us consider the ensemble average of an observable $\mathcal{O}(m,x)$ of a theory $\mathcal{T}(m)$ (CFT or otherwise) over a moduli space $m\in\mathcal{M}$ [43]. The ensemble average of $\mathcal{O}(m,x)$ is defined as

$$\langle\mathcal{O}\rangle(x) := \frac{1}{\mathrm{Vol}(\mathcal{M})}\int_{\mathcal{M}}[dm]\,\mathcal{O}(m,x), \tag{19}$$

where $[dm]$ is an appropriate measure of the moduli space.

Let us first assume that (1) there exists a symmetry G which preserves the moduli space $\mathcal{M}$ as well as its measure: $[d(g\cdot m)] = [dm]$. Let us also assume that (2) G acts covariantly on the observable $\mathcal{O}$ as $\mathcal{O}(g\cdot m, g\cdot x) = \mathcal{O}(m,x)$. We call G to be an *ensemble symmetry* when the two conditions are satisfied. Note that an ensemble symmetry is *not* a symmetry of a boundary theory in the standard sense: an element $g\in G$ maps one theory (at m in the moduli space) to another (at $g\cdot m$), and hence it relates two different theories inside the same ensemble.

Once we have an ensemble symmetry, one can easily derive

$$\langle\mathcal{O}\rangle(g\cdot x) = \langle\mathcal{O}\rangle(x) \quad \text{for} \quad g\in G \tag{20}$$

from the two conditions of ensemble symmetries. Since the dependence of the moduli m is now removed, we find a global symmetry acting on a single theory—this global symmetry is an emergent symmetry after the ensemble average.

We can formulate this as a general lesson in ensemble averages: a symmetry connecting different theories in an ensemble can be turned into an emergent global symmetry of a single theory after averaging. This is an interesting loophole to the holographic argument [12, 13] that there are no global symmetries in theories of quantum gravity.

Emergence of Global Symmetries in the Bulk

Let us next discuss the emergence of global symmetries in the holographic bulk. Prior to the ensemble average, the bulk is related to the Maxwell-Chern-Simons theory [15, 44]

$$S_{\mathrm{MCS}} = \frac{1}{16\pi^2}\sum_{i,j=1}^{p+q}\int_M\left(-\frac{1}{2e^2}\lambda^{-1}_{ij}dA^i\wedge *dA^j\right) + S_{\mathrm{CS}}, \tag{21}$$

where e^2 is the coupling that has dimensions of mass, and λ^{-1} is a dimensionless, symmetric, positive definite matrix with a determinant one. Given that e^2 is dimensionful, the Maxwell term is irrelevant and therefore the Chern-Simons term is expected to dominate in the IR. This corresponds to the topological limit $e^2\to\infty$, which leaves only the Chern-Simons term. The effect of the Maxwell term, however, still remains, since the quantization conditions for the gauge fields in the topological limit depend on the parameters λ, and hence on a point of the corresponding moduli space $\mathcal{M}_Q$.

As pointed out in [15], with standard boundary conditions imposed, the wavefunction of Maxwell-Chern-Simons theory matches the partition function of an irrational CFT with dependence on Narain moduli space (up to an overall constant) [44], thereby realizing the holography duality prior to averaging. The same T-duality symmetries described earlier still exist for the Maxwell-Chern-Simons theory, where the mapping between different points in moduli space corresponds to a mapping between different points in theory space for the Maxwell-Chern-Simons theory.

Maxwell-Chern-Simons theory of Eq. (21) is believed to describe the long-distance limit of string theory on $AdS_3 \times K_7$, where K_7 is a compact 7-manifold [44] [45]. Although the gauge fields of Maxwell-Chern-Simons theory couple to other degrees of freedom in the corresponding low energy supergravity description, it was conjectured in [44], based on the decoupling of topological modes at long distance, that the complete partition of string theory on $AdS_3 \times K_7$ is described by a linear combination of the partition functions ϑ^Q_α, which are now regarded as the wavefunction of Maxwell-Chern-Simons theory in the topological limit. Our discussion of the holography is therefore relevant not only for quantum gravities in three spacetime dimensions, but also for the ten-dimensional string theory.

Emergent Global Symmetries and Swampland Distance Conjecture

Once we are in string theory, there is an alternative method to obtain global symmetries: going to the infinity of the moduli space. Indeed, the Swampland distance conjecture [2] states that there is a tower of states in the infinite distance limit, where we often expect an emergence of a global symmetry [46, 47].

In our situation, the natural limit for the modulus τ is to choose $\tau \to i\infty$, or its $SL(2,\mathbb{Z})$ images; these are the cusps for the fundamental region of the torus (as in Eq. (13)), and are in the infinite distance limit of the moduli space. Geometrically, each cusp corresponds to a geometry [15], and in the infinite distance limit to the cusp, one of the cycles of the associated geometries shrinks to zero size (so that three-dimensional gravity reduces to two-dimensional gravity). As we have discussed before, this means that the leading divergence at the cusp is given by the lens space partition function, which is the quantity associated with the post-averaged bulk theory.

It is interesting to compare the two different types of emergence of global symmetries discussed above. One is obtained by taking an infinite-distance limit of a modulus and is associated with the Swampland distance conjecture. Another is obtained by ensemble averaging of the CFT moduli space. In both cases, we are led to exactly the same expression, namely the lens space partition function. We believe that the general lesson from this is that ensemble averages and Swampland conjectures are intrinsically tied together and that ensemble averages in holography play a natural role inside the Swampland program and string theory. [48]

Fluctuations around Ensemble Averages and Breaking of Emergent Symmetries

When we seriously take the correspondence between ensemble averaging and distance conjecture, one natural question is how the global symmetries are broken in honest theories of quantum gravity. In the case of the distance conjecture, this is achieved by staying at a finite distance in the moduli space. The natural counterpart in ensemble averaging is to consider the "fluctuations" away from the averaging, and this should break any emergent global symmetry in the bulk. The question is then how to formulate these ideas into a precise mathematical formalism.

Remarkably, the relevant mathematics is already known in the literature, as the Roeckle-Selberg spectral decomposition. This is a decomposition of a square-integrable modular form as in [49, 50]. This technology was applied to the partition functions of Narain CFTs defined via even, self-dual lattices in [51], where it was shown that the ensemble average arises as the moduli-independent piece of the spectral decomposition. For our generalized CFTs, we require a spectral decomposition for non-holomorphic modular forms of congruence subgroups with non-zero weight. For square-integrable forms $f(\tau,\bar\tau)$, the decomposition is [52–54]

$$\begin{aligned} f(\tau,\bar\tau) &= \sum_{i=1}^{\infty} \langle f, u_i\rangle\, u_i(\tau) + \sum_{j=1}^{N} \langle f, v_j\rangle\, v_j(\tau) \\ &+ \sum_{k=1}^{n} \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\langle f, E_{\mathfrak{a}_k}\left(\tau, \frac{1}{2}+it\right)\right\rangle E_{\mathfrak{a}_k}\left(\tau, \frac{1}{2}+it\right) dt. \end{aligned} \tag{22}$$

Here $\{u_i\}$ is the orthonormal basis of cusp forms, $\{u_j\}$ the residues of the Eisenstein series, $E_{\mathfrak{a}_k}$ are Eisenstein series labeled by the cusps $\mathfrak{a}_k$ of the relevant congruence subgroup, and $\langle -,-\rangle$ is the Petersson inner product; see [54, Theorem 6.7.1] for more information. When we apply this decomposition to the combination measuring the deviation from the ensemble average

$$f(\tau,\bar\tau) = \tau_2^{(p+q)/4}(\vartheta_{Q,h}(\tau,\bar\tau;m) - E_{Q,h}(\tau,\bar\tau)), \tag{23}$$

the terms in the decomposition are all moduli-dependent and represent deviations from the ensemble average, hence triggering the breaking of the the emergence of global symmetries. This suggests that the counterparts of the "tower of states" in the Swampland distance conjecture should be included inside the moduli-dependent terms of the spectral decomposition. We also expect that such an analysis will be related to the discussion of wormholes before averaging in [55]. It is an interesting question to study this point further.

One of the surprises of quantum gravity and string theory is that they have successfully incorporated a wide variety of ideas. It is therefore natural to expect that

Swampland and ensembles, which at first seem to be in tension with each other, are combined nicely in string theory, and that such a combination will lead us to deeper insights into the mysteries of quantum gravity.

Acknowledgments We thank A. Kidambi, C. Nazaroglu, A. Westphal, and Y. Zheng for useful discussions. M.Y. is grateful to the KITP, Santa Barbara for hospitality during the Integrable22 workshop. M.A. and M.Y. are supported in part by the JSPS Grant-in-Aid for Scientific Research (19H00689, 20H05860). J.M.L. is supported by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy - EXC 2121 "Quantum Universe". J.M.L. thanks Kavli Institute for the Physics and Mathematics of the Universe for its hospitality during the completion of a portion of this work. M.Y. is also supported in part by the JSPS Grant-in-Aid for Scientific Research (19K03820, 23H01168), and by JST, Japan (PRESTO Grant No. JPMJPR225A, Moonshot R&D Grant No. JPMJMS2061).

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