Green's Function of Homogeneous Overmoded Waveguide with Finite Conductivity Walls

E.L. Saldin [a], E.A. Schneidmiller [a], M.V. Yurkov [b]

[a] *Deutsches Elektronen-Synchrotron (DESY), Notkestrasse 85, D-22607 Hamburg, Germany*

[b] *Joint Institute for Nuclear Research, Dubna, 141980 Moscow Region, Russia*

Abstract

We describe an approach for developing the numerical simulation codes for the FEL amplifier with the homogeneous overmode waveguide. The radiation field are calculated using Green's function method. We start with the regorous solutions for the eigenfunctions of a passive waveguide. Using these eigenfunctions, we find the Green's function. Finally, the Green's function is simplified using paraxial approximation. This algorithm of electromagnetic field calculation can be implemented in linear and nonlinear code for simulation of the waveguide FEL.

1. Formulation of the problem

To describe the FEL amplifiers operating in the millimeter or far infrared wavelenth range, one should take into account the influence of the waveguide walls on the amplification process. The most comprehensive study of the waveguide FEL is presented in the book [1]. The reader can find there extended list of the references to original papers relevant to the problem. The present study assumes a waveguide to be overmoded. Such an approximation does not reduce significantly the practical applicability of the obtained results. Indeed, the FEL amplifier has advantage against conventional vacuum tube devices only when the undulator period λ_{w} is much larger than the radiation wavelength λ. The FEL resonance condition is $\lambda_{\mathrm{w}}/v_z = \lambda/(v_{\mathrm{ph}} - v_z)$, where v_z is the longitudinal velocity of the electrons and v_{ph} is the phase velocity of the electromagnetic wave:

$$v_{\mathrm{ph}} = c\left(1 - c^2 k_{\perp}^2/\omega^2\right)^{-1/2} \ .$$

Here $k_{\perp}$ is the transverse wavenumber of the wave. It is obvious that parameter $c^2 k_{\perp}^2/\omega^2$ should be much less than unity when $\lambda_{\mathrm{w}} \gg \lambda$, i.e. the waveguide should be overmoded.

The operation of the FEL amplifier with an overmoded waveguide can be described by different methods. Our approach is based on the method of the Green's function. We start with the rigorous solutions for the eigenfunctions of a passive waveguide. Using these eigenfunction, we find the Green's function. Finally, the Green's function is simplified using paraxial approximation. The obtained expressions can be implemented in linear and nonlinear codes for simulation of the

⋆ FEL'99 Conference, Hamburg, August 23-27, 1999 Report No. Mo-P-63

TESLA FEL-Report 1999-07

FEL amplifiers. In this paper we demonstrate the application of the Green's function for solution of the initial value-problem for the FEL amplifiers with rectangular and circular waveguides.

Let us consider a helical undulator with magnetic field $H_x + H_y = H_{\rm w} \exp(-\mathrm{i}\, k_{\rm w} z)$. Electrons in the undulator move along the constrained helical trajectory in parallel to the z axis. The electron rotation angle $\theta_{\rm s} = K/\gamma$ is assumed to be small and the longitudinal electron velocity v_z is close to the velocity of light, $v_z \simeq c$. A waveguide is placed inside the undulator. The electromagnetic wave propagates in the waveguide in the same direction as the electron beam. For the electron beam with a small density perturbation, the distribution function can be written in the form

$$f = f_0(\mathcal{E}, \vec{r}_\perp) + \tilde{f}_1(z, \mathcal{E}, \vec{r}_\perp) \exp(\mathrm{i}\,\psi) + \mathrm{C.C.}\ .$$

where $\mathcal{E}$ is the energy of electron, phase $\psi = k_{\rm w} z - \omega(z/c - t)$.

We consider the initial conditions when an external electromagnetic wave and unmodulated electron beam are fed to the undulator entrance: $\tilde{f}_1|_{z=0} = 0$, $f_0 = n_0(\vec{r}_\perp)F(P)$, where $P = \mathcal{E} - \mathcal{E}_0$, $\int \mathrm{d}\, P F(P) = 1$. The evolution of the perturbation to the distribution function, $\tilde{f}_1$, is described by the Vlasov equation. Integration of the Vlasov equation gives us the relation between the longitudinal component of the beam current density, and amplitude of the wave.

To simplify the consideration, we write the Vlasov equation for the case of negligibly small energy spread in the beam. The initial distribution function is the delta function in this case, $F(P) = \delta(P - P_0)$, and the Vlasov equation can be reduced to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + 2\,\mathrm{i}\,C\frac{\mathrm{d}}{\mathrm{d}\,z} + \left(\frac{4\pi e}{c\gamma_z^2\mathcal{E}_0} j_0(\vec{r}_\perp) - C^2\right)\right]\tilde{j}_1 = -\frac{\omega}{c\gamma_z^2\mathcal{E}_0} j_0(\vec{r}_\perp) U(z,\ \vec{r}_\perp)\ . \qquad (1)$$

Here $j_z = -j_0(\vec{r}_\perp) + \tilde{j}_1 \exp(\mathrm{i}\,\psi) + \mathrm{C.C.}$, is the longitudinal component of the beam current density, $-j_0(\vec{r}_\perp) \simeq -ecn_0(\vec{r}_\perp)$, $C = k_{\rm w} - \omega/(2c\gamma_z^2)$ is the detuning of the particle with nominal energy $\mathcal{E}_0$ from resonance with wave, $\gamma_z^{-2} = \gamma^{-2} + \theta_{\rm s}^2$.

In the following we assume that the transverse size of the electron beam is rather large, $r_{\rm b}^2/\gamma_z^2 \gg c^2/\omega^2$. In particular, it follows from this assumption that we can neglect the reduction of the plasma wavenumber due to the presence of the waveguide walls.

The complex amplitude of the effective potential, U, is connected with the components of the electromagnetic wave by relation

$$U = -\frac{e\theta_{\rm s}}{2\,\mathrm{i}}\left[\tilde{E}_x + \mathrm{i}\,\tilde{E}_y\right]\ . \qquad (2)$$

The transverse electric field of the wave is presented in the form:

$$\vec{E}_\perp(z,\ \vec{r}_\perp,\ t) = \Big[\vec{e}_x\tilde{E}_x(z,\ \vec{r}_\perp) + \vec{e}_y\tilde{E}_y(z,\ \vec{r}_\perp)\Big] \exp\left[\mathrm{i}\,\omega(z/c - t)\right] + \mathrm{C.C.}\ , \qquad (3)$$

To close the problem, we should solve Maxwell's equations with the boundary conditions on the waveguide walls and express the fields in terms of the first harmonic of the beam current density.

2. Green's function of homogeneous waveguide

The electromagnetic field is expressed via the vector and the scalar potential as

$$\vec{H} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\ . \qquad (4)$$

In the following we use the Coulomb gauge for the potentials,

$$\vec{\nabla} \cdot \vec{A} = 0\ . \qquad (5)$$

Substitution of (4) into the Maxwell's equations gives (at the Coulomb gauge):

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j} + \frac{1}{c}\vec{\nabla}\frac{\partial\phi}{\partial t} = -\frac{4\pi}{c}\vec{J}\ , \quad (6)$$

$$\vec{\nabla}^2\phi = -4\pi\rho_{\rm e}\ , \quad \vec{j} = \vec{v}\rho_{\rm e}\ , \qquad (7)$$

where ρ_e is the charge density. It follows from (7) and from the charge conservation law that $\vec{\nabla} \cdot \vec{J} = 0$. In the Coulomb gauge the scalar potential, ϕ, is the static Coulomb potential. The dynamical part of the field is associated with the vector potential, $\vec{A}$, only.

Let us consider a monochromatic external wave of the frequency ω. Then $\vec{A}$, ϕ and $\vec{j}$ may be written as $\vec{A} = \vec{A}_\omega e^{-i\omega t} + \mathrm{C.C.}$, $\phi = \phi_\omega e^{-i\omega t} + \mathrm{C.C.}$, $\vec{j} = \vec{j}_\omega e^{-i\omega t} + \mathrm{C.C.}$. Equation (6) for $\vec{A}_\omega$ and $\vec{J}_\omega = \vec{j}_\omega + i\omega(4\pi)^{-1}\vec{\nabla}\phi_\omega$ takes the form:

$$\vec{\nabla}^2 \vec{A}_\omega + \frac{\omega^2}{c^2}\vec{A}_\omega = -\frac{4\pi}{c}\vec{J}_\omega \ . \qquad (8)$$

We assume the waveguide walls to be perfectly conducting. The boundary conditions for this case force the vector of the electric field be perpendicular to the waveguide wall. In addition to the Coulomb gauge condition (5), we impose the boundary condition for the scalar potential ϕ be equal to zero on the waveguide walls. The boundary conditions for the vector potential $\vec{A}$ are defined by the boundary conditions for the field:

$$\vec{n} \times \vec{A}_\omega|_S = 0 \ , \qquad (9)$$

where $\vec{n}$ is the unit vector perpendicular to the waveguide wall ($|\vec{n}| = 1$).

Under these boundary conditions, the solution of the inhomogeneous Helmholtz equation (8) has the form:

$$A^\alpha_\omega(\vec{r}) = \sum_\beta \int G^{\alpha\beta}_\omega(\vec{r},\ \vec{r}')J^\beta_\omega(\vec{r}')\,d\vec{r}' \ , \qquad (10)$$

where $G^{\alpha\beta}_\omega(\vec{r},\ \vec{r}')$ is the tensor Green's function of the waveguide and $\vec{r}$ and $\vec{r}'$ are the coordinates of the observation and the source point, respectively. It can be shown that the condition $\phi_\omega|_S = 0$ results in the zero value of the integral [2]:

$$\sum_\beta \int G^{\alpha\beta}_\omega(\vec{r},\ \vec{r}')(\vec{\nabla}\phi_\omega(\vec{r}'))^\beta\,d\vec{r}' = 0 \ . \qquad (11)$$

Hence, (10) takes the form:

$$A^\alpha_\omega(\vec{r}) = \sum_\beta \int G^{\alpha\beta}_\omega(\vec{r},\ \vec{r}')j^\beta_\omega(\vec{r}')\,d\vec{r}' \ . \qquad (12)$$

Using (12), we write the following expression for the electric field of the radiated wave:

$$E^\alpha_\omega(\vec{r}) = i\frac{\omega}{c}\sum_\beta \int G^{\alpha\beta}_\omega(\vec{r},\ \vec{r}')j^\beta_\omega(\vec{r}')\,d\vec{r}' \ , \qquad (13)$$

where tensor Green's function, $G^{\alpha\beta}_\omega$, in the paraxial approximation is given by [1]:

$$\begin{aligned} G_\omega = {} & \frac{2\pi i}{\omega}\sum_\mu \exp\left\{ i\left[\frac{\omega}{c} - \frac{c(k^{\mathrm{TE}}_\perp)^2_\mu}{2\omega}\right]|z - z'|\right\} \\ & \times \left[\vec{e}_x \frac{\partial\psi^{\mathrm{TE}}_\mu(\vec{r}_\perp)}{\partial y} - \vec{e}_y\frac{\partial\psi^{\mathrm{TE}}_\mu(\vec{r}_\perp)}{\partial x}\right] \\ & \otimes \left[\vec{e}_x \frac{\partial\psi^{\mathrm{TE}}_\mu(\vec{r}'_\perp)}{\partial y'} - \vec{e}_y\frac{\partial\psi^{\mathrm{TE}}_\mu(\vec{r}'_\perp)}{\partial x'}\right] \\ & + \frac{2\pi i}{\omega}\sum_\nu \exp\left\{ i\left[\frac{\omega}{c} - \frac{c(k^{\mathrm{TM}}_\perp)^2_\nu}{2\omega}\right]|z - z'|\right\} \\ & \times \left[\vec{e}_x \frac{\partial\psi^{\mathrm{TM}}_\nu(\vec{r}_\perp)}{\partial x} + \vec{e}_y\frac{\partial\psi^{\mathrm{TM}}_\nu(\vec{r}_\perp)}{\partial y}\right] \\ & \otimes \left[\vec{e}_x \frac{\partial\psi^{\mathrm{TM}}_\nu(\vec{r}'_\perp)}{\partial x'} + \vec{e}_y\frac{\partial\psi^{\mathrm{TM}}_\nu(\vec{r}'_\perp)}{\partial y'}\right] \ , \end{aligned} \qquad (14)$$

Here symbol $\otimes$ denotes the direct product of vectors. The waveguide functions, ψ^{TE} and ψ^{TM}, are the solutions of the Helmholtz equation:

$$\vec{\nabla}^2_\perp\psi + k^2_\perp\psi = 0 \ , \qquad (15)$$

with the boundary conditions

$$\vec{n}\cdot\vec{\nabla}\psi^{\mathrm{TE}}|_S = 0 \ , \quad \psi^{\mathrm{TM}}|_S = 0 \ , \qquad (16)$$

and the normalization condition

$$\int |\vec{\nabla}\psi|^2\,d\vec{r}_\perp = 1 \ . \qquad (17)$$

The β th component of the transverse beam current density, j^β_ω, is connected with the complex

[1] Expression (14) contains only transverse components of $G^{\alpha\beta}_\omega$. In the case of an overmoded waveguide it provides sufficient accuracy for the calculation of the transverse component of the vector potential.

amplitude, $\tilde{j}_1$, as

$$j^{\beta}_{\omega} = \frac{1}{c} v^{\beta}(z) \tilde{j}_1(z,\ \vec{r}_{\perp}) \exp\left(\mathrm{i}\, k_{\mathrm{w}} z + \mathrm{i}\, \omega \frac{z}{c}\right) \ . \quad (18)$$

Using relation $\tilde{E}_{x,y} \exp\left(\mathrm{i}\, \omega z / c\right) = E^{x,y}_{\omega}$, and equations (2), (14), (13) and (18), we write the expression for the effective potential of the interaction between the particle and the radiated electromagnetic wave:

$$U_{\mathrm{i}} = -\frac{\pi \mathrm{i}}{2c} e \theta^2_{\mathrm{s}} \int\limits_0^z \mathrm{d}\, z' \int \mathrm{d}\, \vec{r}'_{\perp} \tilde{j}_1(z',\ \vec{r}'_{\perp})$$
$$\times\ \mathcal{D}\mathcal{D}' \left\{ \sum_{\mu} \psi^{\mathrm{TE}}_{\mu}(\vec{r}_{\perp}) \psi^{\mathrm{TE}}_{\mu}(\vec{r}'_{\perp}) \right.$$
$$\times \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TE}}_{\perp})^2_{\mu}}{2\omega} (z - z') \right]$$
$$+ \sum_{\nu} \psi^{\mathrm{TM}}_{\nu}(\vec{r}_{\perp}) \psi^{\mathrm{TM}}_{\nu}(\vec{r}'_{\perp})$$
$$\left. \times \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TM}}_{\perp})^2_{\nu}}{2\omega} (z - z') \right] \right\} \quad (19)$$

where $\mathcal{D}\mathcal{D}'$ denotes

$$\mathcal{D}\mathcal{D}' = \left(\frac{\partial}{\partial x} + \mathrm{i}\, \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x'} - \mathrm{i}\, \frac{\partial}{\partial y'} \right) \ .$$

The external electromagnetic wave can be expressed in terms of the eigenmodes of the passive waveguide:

$$[\vec{e}_x(\tilde{E}_x) + \vec{e}_y(\tilde{E}_y)]_{\mathrm{ext}} = \sum_{\mu} C^{\mathrm{TE}}_{\mu} \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TE}}_{\perp})^2_{\mu} z}{2\omega} \right]$$
$$\times \left(\vec{e}_x \frac{\partial \psi^{\mathrm{TE}}_{\mu}(\vec{r}_{\perp})}{\partial y} - \vec{e}_y \frac{\partial \psi^{\mathrm{TE}}_{\mu}(\vec{r}_{\perp})}{\partial x} \right)$$
$$+ \sum_{\nu} C^{\mathrm{TM}}_{\nu} \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TM}}_{\perp})^2_{\nu} z}{2\omega} \right]$$
$$\times \left(\vec{e}_x \frac{\partial \psi^{\mathrm{TM}}_{\nu}(\vec{r}_{\perp})}{\partial x} + \vec{e}_y \frac{\partial \psi^{\mathrm{TM}}_{\nu}(\vec{r}_{\perp})}{\partial y} \right) \ , \quad (20)$$

where coefficients C^{TE}_{μ} and C^{TM}_{ν} are given by

$$C^{\mathrm{TE}}_{\mu} = \int \mathrm{d}\, \vec{r}_{\perp} \left(\tilde{E}_x(0,\ \vec{r}_{\perp}) \frac{\partial \psi^{\mathrm{TE}}_{\mu}}{\partial y} - \tilde{E}_y(0,\ \vec{r}_{\perp}) \frac{\partial \psi^{\mathrm{TE}}_{\mu}}{\partial x} \right) ,$$

$$C^{\mathrm{TM}}_{\nu} = \int \mathrm{d}\, \vec{r}_{\perp} \left(\tilde{E}_x(0,\ \vec{r}_{\perp}) \frac{\partial \psi^{\mathrm{TM}}_{\nu}}{\partial x} + \tilde{E}_y(0,\ \vec{r}_{\perp}) \frac{\partial \psi^{\mathrm{TM}}_{\nu}}{\partial y} \right) \ .$$

Total power of the input radiation is expressed in terms of coefficients C^{TE}_{μ} and C^{TM}_{ν} as

$$W_{\mathrm{ext}} = \frac{c}{2\pi} \left(\sum_{\mu} |C^{\mathrm{TE}}_{\mu}|^2 + \sum_{\nu} |C^{\mathrm{TM}}_{\nu}|^2 \right) \ . \quad (21)$$

In the case of a single-mode input radiation, we can let the value of the amplitude coefficient C^{TE}_{μ} (or, C^{TM}_{ν}) be real and positive constant, and write the expression for U_{ext} as
for TE-mode:

$$U^{\mathrm{TE}}_{\mathrm{ext}} = \mathrm{i}\, e \theta_{\mathrm{s}} \left(\frac{\pi W_{\mathrm{ext}}}{2c} \right)^{1/2} \left(\frac{\partial \psi^{\mathrm{TE}}_{\mu}}{\partial y} - \mathrm{i}\, \frac{\partial \psi^{\mathrm{TE}}_{\mu}}{\partial x} \right)$$
$$\times \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TE}}_{\perp})^2_{\mu} z}{2\omega} \right] \ , \quad (22)$$

for TM-mode:

$$U^{\mathrm{TM}}_{\mathrm{ext}} = \mathrm{i}\, e \theta_{\mathrm{s}} \left(\frac{\pi W_{\mathrm{ext}}}{2c} \right)^{1/2} \left(\frac{\partial \psi^{\mathrm{TM}}_{\nu}}{\partial x} + \mathrm{i}\, \frac{\partial \psi^{\mathrm{TM}}_{\nu}}{\partial y} \right)$$
$$\times \exp\left[-\mathrm{i}\, \frac{c (k^{\mathrm{TM}}_{\perp})^2_{\nu} z}{2\omega} \right] \ . \quad (23)$$

Thus, we have obtained the expressions for effective potential of particle interaction with the radiated wave, U_{i}, and with the external wave, U_{ext}. We are interested in the sum of these two contributions:

$$U = U_{\mathrm{i}} + U_{\mathrm{ext}} \ . \quad (24)$$

Substituting (24), (19) and (1) into (1), we obtain the integro-differential equation for the first harmonic of the beam current density, $\tilde{j}_1(z, \vec{r}_{\perp})$ which can be solved numerically using computer code.

3. Wall resistance effects

All the results, obtained above, refer to the case of perfictly conducting waveguide walls having the conductivity $\sigma \to \infty$. In reality the conductivity has always finit value. The problem of the exitation of the waveguide having finit conductivity can be solved in the same way as for the perfectly conducting waveguide. The vector and scalar potential are connected with the field components according to (4). Taking to account the Coulomb gauge for the vector potential (5) we find from Maxwell's equations that the vector potential is subjected to the equation (6). When the waveguide is excited at frequency ω, we have equation (7). The boundary condition for the scalar potential on the waveguide walls is $\phi_\omega = 0$. The boundary conditions for the vector potential, $\vec{A}_\omega$, must provide the boundary conditions for the field. Leontovich,s boundary condition for the field may be written in the form:

$$(\vec{n} \times \vec{A}_\omega)|_{\mathrm{S}} = \frac{\mathrm{i}\, c}{\omega n'} \vec{n} \times (\vec{n} \times (\vec{\nabla} \times \vec{A}_\omega))|_{\mathrm{S}} \ . \qquad (25)$$

where $\vec{n}$ is the unit vector ($|\vec{n}|^2 = 1$), perpendicular to the surface, and directed inside the waveguide, $n' \simeq \sqrt{\mathrm{i}\, 4\pi\sigma/\omega}$ is the index of refraction of a metal. We suppose the value of n' to be large complex number, i.e. $|n'|^2 \gg 1$. Finally, we come to the expression for the Green's function which is identical to (10) obtained for the case of perfectly conducting waveguide walls. The only difference is that the eigenvalues and eigenfunctions become to be complex values for the case of waveguide walls with finit conductivity. Scalar functions ψ^{TE} and ψ^{TM}, are the solutions of the Helholtz equation

$$\vec{\nabla}_\perp^2 \psi + k_\perp^2 \psi = 0 \ , \qquad (26)$$

The eigenfunctions are ortogonal and normalized as

$$\int \vec{\nabla}\psi_\mu \cdot \vec{\nabla}\psi_\nu \,\mathrm{d}\,\vec{r}_\perp = \delta_{\mu\nu} \ . \qquad (27)$$

It is relevant to mention that the orthogonality and normalization conditions are formulated without complex conjugation.

The boundary conditions for ψ^{TE} and ψ^{TM}, written down in paraxial approximation, are as follows:

$$[\vec{n} \cdot \vec{\nabla}\psi^{\mathrm{TE}} + (\vec{e}_z \times \vec{n}) \cdot \vec{\nabla}\psi^{\mathrm{TM}}]|_{\mathrm{S}} = -\frac{\mathrm{i}\, c}{\omega n'} (k_\perp^{\mathrm{TE}})^2 \psi^{\mathrm{TE}}|_{\mathrm{S}} \ ,$$

$$(k_\perp^{\mathrm{TM}})^2 \psi^{\mathrm{TM}}|_{\mathrm{S}} = -\frac{\mathrm{i}\, \omega}{c n'} [\vec{n} \cdot \vec{\nabla}\psi^{\mathrm{TM}} - (\vec{e}_z \times \vec{n}) \cdot \vec{\nabla}\psi^{\mathrm{TE}}]|_{\mathrm{S}} \ .$$

When $\omega/(|n' k_\perp| c) \ll 1$, these boundary conditions can be written in the form [2]:

$$\vec{n} \cdot \vec{\nabla}\psi^{\mathrm{TE}}|_{\mathrm{S}} = -\frac{\mathrm{i}\, c (k_\perp^{\mathrm{TE}})^2 \psi^{\mathrm{TE}}|_{\mathrm{S}}}{\omega n'} + \frac{\mathrm{i}\, \omega [(\vec{e}_z \times \vec{n}) \cdot \vec{\nabla}]^2 \psi^{\mathrm{TE}}|_{\mathrm{S}}}{c n' (k_\perp^{\mathrm{TE}})^2} \ ,$$

$$(k_\perp^{\mathrm{TM}})^2 \psi^{\mathrm{TM}}|_{\mathrm{S}} = \frac{\mathrm{i}\, \omega}{c n'} \vec{n} \cdot \vec{\nabla}\psi^{\mathrm{TM}}|_{\mathrm{S}} \ .$$

In conclusion we should like to stress that described approach for the problem of waveguide exitation is valid for a metallic, or dielectric overmoded waveguide. The only requirement is that of a large value of the refractive index of the waveguide walls which reveals an opportunity to use Leontovich's boundary conditions.

References

[1] H.P. Freund and T.M. Antonsen "Principles of Free electron Lasers" (Capman Hall, New York, 1992)

[2] P.M. Morse and Feshbach, "Methods of Theoretical Physics", Part I (McGraw Hill, New York, 1953)

[3] E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov, Opt.Commun.85(1991)117

[2] It should be stressed that the parameter of the perturbation theory is $\omega/(|n' k_\perp| c)$. The requirement $|n'| \gg 1$ for the overmoded waveguide is necessary, but not sufficient.