Efficient Field Calculation of 3D Bunches on General Trajectories

M.Dohlus, A.Kabel, T.Limberg

Deutsches Elektronen Synchrotron DESY, Notkestr. 85, D-22607 Hamburg

Abstract

The program TRAFIC4 calculates the effects of space charge forces and coherent synchrotron radiation on bunches moving on curved trajectories. It calculates the fields acting on the particles, as they travel along the beamline, from first principles. The bunch is modeled by small overlapping 1d, 2d or 3d continuous gaussian sub-bunches. Previously a 1d or 2d integration of the retarded source distribution was used to calculate the fields of 1d and 2d sub-bunches. A new approximation is proposed to avoid the effort of multi-dimensional integration: a 2d or 3d sub-bunch can be interpreted as the convolution of a 1d sub-bunch with a transverse density function. In the same way the field can be obtained by a convolution of the 1d-field with the transverse density function. The near field of a 1d sub-bunch can be split into a singular part which is dominated by local effects and the residual part, which depends essentially on long range interactions. As the singular part can be described by analytical functions, the convolution can be performed efficiently. The residual part depends weakly on the transverse offset. Therefore only one or few sampling points are needed for the convolution, which significantly reduces the numerical effort.

1. Introduction

Dispersive bunch compressors are used to achieve the required bunch properties for some FEL applications, e.g. ~50μm bunch length, normalized emittance of few 10^{-6} rad for the TESLA-TTF SASE FEL [1]. For this type of compressor a linear, length-correlated energy spread is superimposed by a phase shift of the accelerating RF and transformed into a length reduction by a dispersive magnet arrangement. Due to the curved trajectories and the shape variations during the compression processes transient EM fields are generated [2] which have to be taken into account for an appropriate analysis of the beam dynamic. The program TRAFIC4 [3,4,5] splits the real bunch - with time dependent shape - into a set of sub-bunches with fixed shapes, but with individual energies and paths. The fields of these sub-bunches are calculated by a retarded potential approach and considered together with the external fields in the tracking calculation for an ensemble of test particles. To obtain finite fields, the sub-bunches have to be at least two dimensional and smooth fields are generated only by three dimensional charge distributions. The numerical effort for a two or even three dimensional source integration limits the capability to simulate many arrangements under various operation conditions.

To solve this problem a new approximation is proposed which avoids multi-dimensional integration: for a particular type of two and three dimensional charge distributions, which can be described by a convolution of a 1D distribution with a transverse density function, the field quantities can be obtained by a convolution of the 1D field quantities with the density function. The near field of a 1d bunch can be split into a singular part that is dominated by local effects and the residual part, which depends essentially on long range interactions. As the singular part can be described by analytical functions, the convolution can be performed efficiently. The residual part depends weakly on the transverse offset, so that only one or few sampling points are needed for the convolution.

2. Theory

The 1d field solver of TRAFIC4 calculates the magnetic flux density $\mathbf{B}$ and the normalized Lorentz force $\mathbf{W} = \mathbf{E} + \mathbf{v}_t \times \mathbf{B}$ for a line charge density $\lambda(s,t)$ traveling along the path $\mathbf{r}_s(s)$ by a numerical integration of

$$4\pi\varepsilon_0 c_0 \mathbf{B} = \int_{-\infty}^{\infty} \mathbf{e}_s \times \mathbf{G}\beta_s ds \,, \tag{1a}$$

$$4\pi\varepsilon_0 \mathbf{W}(\mathbf{r},t) = \int_{-\infty}^{\infty} (\mathbf{e}_t - \mathbf{e}_s) \times \mathbf{e}_s \times \mathbf{G}\beta_s\beta_t ds \ + \int_{-\infty}^{\infty} \mathbf{G}\ (1-\beta_s\beta_t) ds$$
$$- \int_{-\infty}^{\infty} \frac{\lambda(s,t')}{R} \beta_s\beta_t\ \partial_s \mathbf{e}_s\ ds + \int_{-\infty}^{\infty} \frac{\dot{\lambda}(s,t')}{c_0 R} (\beta_t - \beta_s)\ \mathbf{e}_s ds \quad , \qquad (\mathbf{1}\text{b}),$$

$$\mathbf{G} = -\nabla \frac{\lambda(s,t')}{R} = \nabla R \left(\frac{\lambda(s,t')}{R^2} + \frac{\dot{\lambda}(s,t')}{c_0 R} \right) \quad , \qquad (\mathbf{1}\text{c})$$

with $\mathbf{R} = \mathbf{r} - \mathbf{r}_s(s)$, $R = \|\mathbf{R}\|$, $t' = t - R/c_0$, $\mathbf{v}_t = \beta_t c_0 \mathbf{e}_t$ and $\mathbf{e}_s = \partial_s \mathbf{r}_s$. The line charge density is derived from the current flow $i(s,t) := c_0 \lambda_i (t - t_c(s))$ by $\lambda(s,t) := \lambda_i (t - t_c(s))/\beta_s(s)$ with $\beta_s = (c_0 \partial_s t_c)^{-1}$. A class of two and three dimensional source distributions can be obtained by the convolution of the one dimensional density with a transverse density function $\eta(x_1, x_2)$ which is defined for an arbitrary plane $\mathbf{r}_\eta(x_1, x_2) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$ with $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\int \eta(x_1, x_2) dx_1 dx_2 = 1$. Therefore the 3d charge density is described by

$$\rho(\mathbf{r} = \mathbf{r}_s(s) + x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2, t) = \lambda(s,t) \frac{\eta(x_1, x_2)}{|(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \partial_s \mathbf{r}_s|} \quad . \qquad (2)$$

Of course this class is not sufficient to model more general shape variations as they happen in a bunch compressor, but it is adequate for an efficient sub-bunch approach. In the present version of TRAFIC4 [3] the Lorentz force and the magnetic flux density are computed by the convolution of the corresponding 1d quantities ($\mathbf{X} = \mathbf{W}$ or $\mathbf{B}$) with the transverse density:

$$\mathbf{X}_\eta(\mathbf{r},t) = \int \mathbf{X}(\mathbf{r} - \mathbf{r}_\eta(x_1, x_2), t) \eta(x_1, x_2) dx_1 dx_2 \quad . \qquad (3)$$

This convolution is very time consuming as two (three) dimensional integrations have to be carried out for the calculation of two (three) dimensional source distributions.

2.1 One Dimensional Integration with Pole Extraction

As the integrands and integrals of Eq. **1** are singular for observer positions on the path $\mathbf{r}_s(s)$ a pole extraction technique is used for the numerical integration. Therefore the path point $\mathbf{r}_0 = \mathbf{r}_s(s_0)$ with minimal distance to the observer is searched for. In the following we calculate the normalized Lorentz force for the case $\mathbf{v}_t = \mathbf{v}_0 := c_0 \beta_s(s_0) \mathbf{e}_s(s_0)$ of an observer velociy identical to the source velocity at the closest path point. A different velocity can be taken into account by adding $(\mathbf{v}_t - \mathbf{v}_0) \times \mathbf{B}$. The normalized Lorentz force and the magnetic flux density are split into a singular and a non-singular part $\mathbf{W} = \mathbf{W}_{\text{SP}} + \mathbf{W}_{\text{NS}}$, $\mathbf{B} = \mathbf{B}_{\text{SP}} + \mathbf{B}_{\text{NS}}$ where the non-singular part (index NS) can be calculated numerically for any observation point. The singular part (index SP) is analytically known:

$$4\pi\varepsilon_0 \mathbf{B}_{\text{SP}} = \frac{2\beta_0}{c_0} \left[(\mathbf{e}_0 \lambda_0) \times \mathbf{w}_1(\Delta\mathbf{r}) + \left(\frac{1}{2} \mathbf{d}_0 \times \mathbf{e}_0 \lambda_0 \right) w_2(\Delta\mathbf{r}) - (\mathbf{d}_0 \lambda'_0) \times \mathbf{w}_3(\Delta\mathbf{r}) \right] \quad , \qquad (4\text{a})$$

$$\begin{aligned} 4\pi\varepsilon_0 \mathbf{W}_{\text{SP}} &= 2\beta_0^2 \mathbf{d}_0 \times (\mathbf{e}_0 \lambda'_0 + \mathbf{d}_0 \lambda_0) \times \mathbf{w}_3(\Delta\mathbf{r}) \\ &+ \frac{2}{\gamma_0^2} \left[\lambda_0 \mathbf{w}_1(\Delta\mathbf{r}) + \left(\mathbf{e}_0 \lambda'_0 + \frac{1}{2} \mathbf{d}_0 \lambda_0 \right) w_2(\Delta\mathbf{r}) \right], \\ &+ 2\beta_0^2 \mathbf{d}_0 \lambda_0 w_2(\Delta\mathbf{r}) \end{aligned} \qquad (4\text{b})$$

with

$$\mathbf{w}_1(\Delta\mathbf{r}) = \Delta\mathbf{r} / \|\Delta\mathbf{r}\|^2 \quad , \qquad (5\text{a})$$

$$w_2(\Delta\mathbf{r}) = \ln\left(\|\Delta\mathbf{r}\|/R_{\text{ref}}\right) \quad , \tag{5b}$$

$$\mathbf{w}_3(\Delta\mathbf{r}) = \Delta\mathbf{r}\ln\left(\|\Delta\mathbf{r}\|/R_{\text{ref}}\right) \quad , \tag{5c}$$

$\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0$, $\mathbf{e}_0 = \mathbf{e}_s(s_0)$, $\mathbf{d}_0 = \partial_s\mathbf{e}_s(s_0)$, $\beta_0 = \beta_s(s_0)$, $\lambda_0 = \lambda(s_0,t)$, $\lambda_0' = \partial_s\lambda(s_0,t)$ and R_{ref} an arbitrary normalisation length.

2.2 Convolution Method

For the calculation of the fields $\mathbf{W}_\eta(\mathbf{r},t)$ and $\mathbf{B}_\eta(\mathbf{r},t)$ of a bunch with the transverse density function $\eta(x_1,x_2)$ the convolution integral Eq. 3 has to be solved. As the non singular parts of $\mathbf{W}$ and $\mathbf{B}$ are smooth functions of $\mathbf{r}$ and the singular parts are analytically given, the numerical effort for the convolution can be significantly reduced. This is especially true if the 2d plane $\mathbf{r}_\eta(x_1,x_2)$ of the transverse density function is perpendicular to the path at the point $\mathbf{r}_0 = \mathbf{r}_s(s_0)$. For this case, and for transverse bunch dimensions which are of the same order or smaller than the longitudinal dimension, the non singular parts are almost sampled by the convolution: $\mathbf{B}_{\text{NS},\eta} \approx \mathbf{B}_{\text{NS}}$, $\mathbf{W}_{\text{NS},\eta} \approx \mathbf{W}_{\text{NS}}$. For bunches with larger transverse dimensions only few points of $\mathbf{B}_{\text{NS}}$ and $\mathbf{W}_{\text{NS}}$ have to be evaluated and fitted to compute the convolution. The condition that the 2d plane $\mathbf{r}_\eta(x_1,x_2)$ is perpendicular to the path at the closest point $\mathbf{r}_0$ has some further advantages: the observation point lies in the plane $\mathbf{r}_0 - \mathbf{r}_\eta(x_1,x_2)$ and all other observation points in this plane have the same closest path point $\mathbf{r}_0 = \mathbf{r}_s(s_0)$. Therefore the coefficients s_0, $\mathbf{e}_0$, $\mathbf{d}_0$, β_0, λ_0, λ_0' in Eq. 4 are constant for all arguments $(\mathbf{r} - \mathbf{r}_\eta(x_1,x_2),t)$ of $\mathbf{B}_{\text{SP}}$ and $\mathbf{W}_{\text{SP}}$ in the convolution integral Eq. 3. The convoluted singular parts $\mathbf{B}_{\text{SP}\eta}$, $\mathbf{W}_{\text{SP}\eta}$ are obtained from Eq. 4 by replacing the functions $\mathbf{w}_1, w_2, \mathbf{w}_3$ by

$$\mathbf{w}_{1,\eta}(\Delta\mathbf{r}) = \int \mathbf{w}_1(\Delta\mathbf{r} - \mathbf{r}_\eta(x_1,x_2))\eta(x_1,x_2)dx_1dx_2 \,, \tag{6a}$$

$$w_{2,\eta}(\Delta\mathbf{r}) = \int w_2(\Delta\mathbf{r} - \mathbf{r}_\eta(x_1,x_2))\eta(x_1,x_2)dx_1dx_2 \,, \tag{6b}$$

$$\mathbf{w}_{3,\eta}(\Delta\mathbf{r}) = \int \mathbf{w}_3(\Delta\mathbf{r} - \mathbf{r}_\eta(x_1,x_2))\eta(x_1,x_2)dx_1dx_2 \,. \tag{6c}$$

a) 2D Bunches

For 2d bunches the transverse density function $\eta(x_1)$ depends only on one parameter and and the convolution integrals are reduced to a 1d integration over x_1 with $\mathbf{r}_\eta(x_1) = x_1\mathbf{e}_1$. Therefore only few scalar integrals have to be computed to obtain the functions

$$\mathbf{w}_{1,\eta}(\Delta\mathbf{r}) = \frac{\mathbf{a}}{a}I_1 + \mathbf{e}_1 I_2 \,,\; w_{2,\eta}(\Delta\mathbf{r}) = \frac{1}{2a}I_3 \,,\; \mathbf{w}_{3,\eta}(\Delta\mathbf{r}) = \frac{1}{2}\frac{\mathbf{a}}{a}I_3 + \frac{1}{2}\mathbf{e}_1 I_4 \tag{7}$$

with

$$I_1(a,b) = \int_{-\infty}^{\infty} \frac{a}{a^2+x^2}\eta(b-x)dx \,,\; I_2(a,b) = \int_{-\infty}^{\infty} \frac{x}{a^2+x^2}\eta(b-x)dx \,,$$

$$I_3(a,b) = \int_{-\infty}^{\infty} a\ln\left(\frac{a^2+x^2}{R_{\text{ref}}^2}\right)\eta(b-x)dx \,,\; I_4(a,b) = \int_{-\infty}^{\infty} x\ln\left(\frac{a^2+x^2}{R_{\text{ref}}^2}\right)\eta(b-x)dx \tag{8}$$

and $b = \mathbf{e}_1 \cdot \Delta\mathbf{r}$, $\mathbf{a} = \Delta\mathbf{r} - b\mathbf{e}_1$, $a = \|\mathbf{a}\|$. The substitution of these functions into Eq. 4 leads to the contributions $\mathbf{B}_{\text{SP}\eta}$, $\mathbf{W}_{\text{SP}\eta}$ and together with $\mathbf{B}_{\text{NS},\eta} \approx \mathbf{B}_{\text{NS}}$, $\mathbf{W}_{\text{NS},\eta} \approx \mathbf{W}_{\text{NS}}$ a numerically efficient approximation is obatined.

b) Round 3d Gaussian bunches

The 2d convolutions Eq. 6 for three dimensional beams with a round Gaussian transverse density function $\eta(x_1,x_2) = g\left(\sqrt{x_1^2+x_2^2}/\sigma_r\right)/\left(\sqrt{2\pi}\sigma_r^2\right)$ can be reduced to the computation of three scalar functions of one parameter:

$$\mathbf{w}_{1,\eta}(\Delta\mathbf{r} = r_\Delta \mathbf{e}_\Delta) = \frac{\mathbf{e}_\Delta}{r_\Delta} H_1(r_\Delta/\sigma_r)\,, \tag{9a}$$

$$w_{2,\eta}(\Delta\mathbf{r} = r_\Delta \mathbf{e}_\Delta) = H_2(r_\Delta/\sigma_r) + \ln\left(\frac{r_\Delta}{R_{\mathrm{ref}}}\right), \tag{9b}$$

$$\mathbf{w}_{3,\eta}(\Delta\mathbf{r} = r_\Delta \mathbf{e}_\Delta) = \Delta\mathbf{r}\left\{H_3(r_\Delta/\sigma_r) + \ln\left(\frac{r_\Delta}{R_{\mathrm{ref}}}\right)\right\} \tag{9c}$$

with

$$H_1(x) = \sqrt{2\pi}\, x \int_0^\infty \mathrm{I}_1(\rho x)\exp(-\rho x) g(x-\rho)\, d\rho\,, \tag{10a}$$

$$H_2(x) = \sqrt{2\pi} \int_0^\infty \rho \ln\left(\frac{\rho}{x}\right) \mathrm{I}_0(\rho x)\exp(-\rho x)\, g(x-\rho) d\rho\,, \tag{10b}$$

$$H_3(x) = \frac{\sqrt{2\pi}}{x} \int_0^\infty \rho^2 \ln\left(\frac{\rho}{x}\right) \mathrm{I}_1(x)\exp(-\rho x) g(x-\rho)\, d\rho\,, \tag{10c}$$

and $g(x) = \exp(-x^2/2)/\sqrt{2\pi}$. The functions $H_1(x)$, $H_2(x)$, $H_3(x)$ have been calculated numerically and are approximated by Chebychev polinomials.

3. Calculations

To verify the presented singularity convolution method $\mathbf{W}_\eta \approx \mathbf{W}_{\mathrm{NS}} + \mathbf{W}_{\mathrm{NS},\eta}$ we compare this method with a direct multi-dimensional integration for a bunch in circular motion. Fig. 1 shows the longitudinal and radial component of the normalized Lorentz force for 2d and 3d bunches with q = 1nC, γ = 100 and the radius of curvature $R_c = 10$ m. The 2d bunch is a round Gaussian disc in the longitudinal-radial plane with σ = 50 μm, the 3d bunch is a spherical Gaussian distribution with the same σ. The ordinate in Fig. 1 is the longitudinal bunch coordinate (from head to tail for increasing s). All curves are in good agreement, while the numerical effort for the singularity convolution method is about two orders smaller than for the multidimensional integration.

Acknowledgement

We wish to thank E.L. Saldin, E.A. Schneidmiller and J. Rossbach for many useful discussions.

References

[1] R.Brinkmann: Conceptual Design of a 500 GeV e+e- Linear Collider with Integrated X-ray Laser Facility. DESY 1997-048, ECFA 1997-192.

[2] E.Saldin, E.Schneidmiller, M.Yurkov: On the Coherent Radiation of an Electron Bunch Moving in an Arc of a Circle. TESLA-FEL 96-14.

[3] M.Dohlus, A.Kabel, T.Limberg: Wake Fields of a Bunch on a General Trajectory due to Coherent Synchrotron Radiation. PAC97

[4] M.Dohlus, A.Kabel, T.Limberg: Uncorrelated Emittance Growth in the TTF-FEL Bunch Compression Sections Due to Coherent Synchrotron Radiation and Space Charge Effects. EPAC98.

[5] K.Rothemund, M.Dohlus, U.v.Rienen: Calculation of Coherent Fields of Charged Particle Bunches on General Trajectories. FEL99 (same Conference)

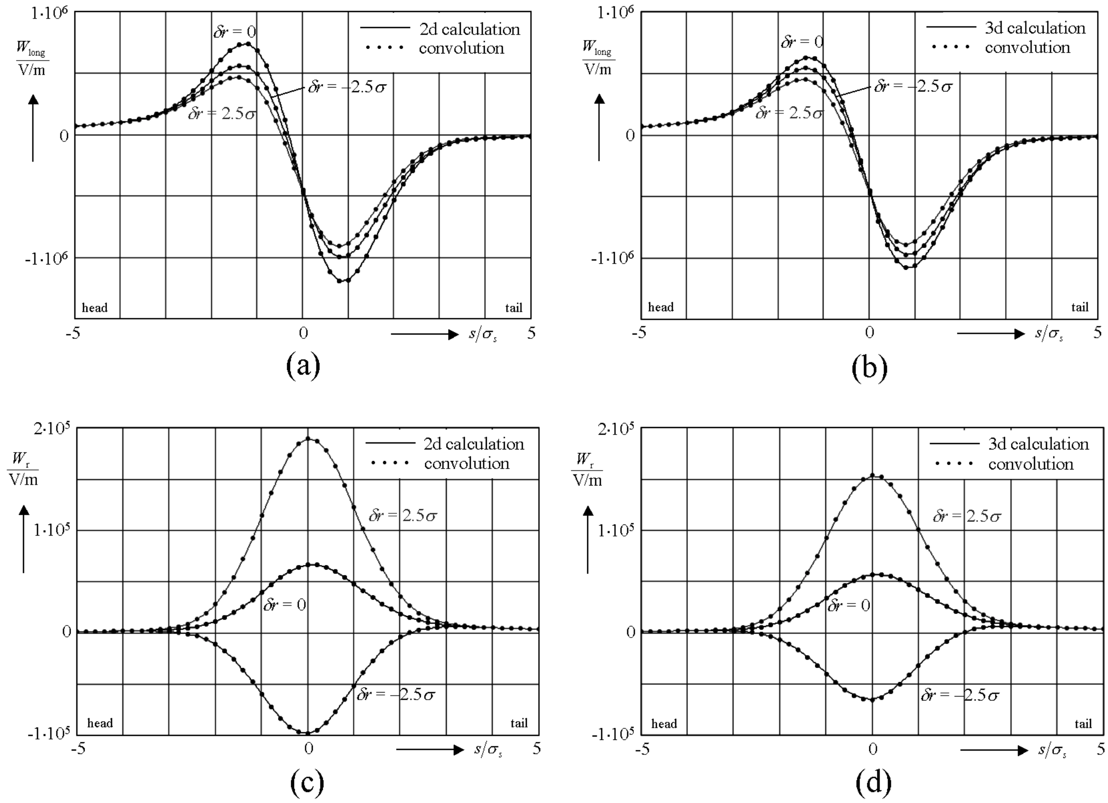

Fig. 1: Longitudinal and radial component of $\mathbf{W} = \mathbf{E} + \mathbf{v}_t \times \mathbf{B}$ for a bunch in circular motion (parameters: curvature radius 10 m, bunch charge 1 nC, bunch length σ = 50 μm, γ = 100). All figures compare the results of a direct multidimensional integration (solid lines) and the singularity convolution method (dotted lines). The fields of a 2d disc shaped bunch are shown in (a) and (c) and of a 3d spherical bunch in (b) and (d). The disc shaped bunch lies in the plane of curvature.