Coherent Effects of a Macro Bunch in an Undulator

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Abstract

The longitudinal radiative force of an electron bunch moving in an undulator has been investigated in [1] assuming an 1D density distribution. To obtain the contribution of pure curvature effects and to avoid singular fields, the 1D linear motion field was subtracted. To relate these results to the 3D case we present analytical and numerical field calculations using the field solver of TRAFIC4. The 1D and 3D cases have been calculated in the transient regime, the steady state regime and the steady state regime averaged over one undulator period for the 1GeV parameter set of the TESLA FEL.

1. Introduction

There are 2 effects caused by the macroscopic charge distribution in an undulator which increase the energy spread and the emittance of a bunch and may therefore disturb the FEL process: one is caused by the fields due to the undulator motion [1, 2, this paper], the other is related to the finite conductivity and roughness of the beam pipe [3,4,5]. The fields of the macro bunch undulator motion have been considered earlier, using the model of a one dimensional bunch excluding singular field contributions [1], and by the model of a rectangular bunch in circular motion [2]. In this paper we take into account all field contributions for one and three dimensional Gaussian charge distributions. The 1D theory considers a transverse observer offset while the 3D theory is valid for observation points without offset. It is confirmed, that the effect of the 1D singularity can be described by the superposition of the field of a 1D bunch in linear motion and a smooth non singular field [1]. The longitudinal and transverse component show a strong dependency on both the longitudinal position in the undulator and the transverse offset. In comparison to this, the field averaged over one undulator period is small and depends only weakly on the offset of the observer particle. Our analytical results agree well with the fully 3D computations using the TRAFIC4 field solver. Additionally we present calculations of the transient process of a bunch which enters the undulator. Although the local field develops its strong spatial dependency after one or few undulator periods, the one period averaged field is far from the steady state condition until the core of the *retarded* bunch is in the undulator.

2. Theory

The motion of charged particles in the magnetic field of an undulator

$$\mathbf{H}(s) = K\frac{mc_0^2}{eZ_0}\frac{2\pi}{\lambda_s}\frac{\partial s}{\partial z}\cos\left(\frac{2\pi}{\lambda_s}s\right)\mathbf{e}_x \approx K\frac{mc_0^2}{eZ_0}\frac{2\pi}{\lambda_z}\cos\left(\frac{2\pi}{\lambda_z}z\right)\mathbf{e}_x \tag{1}$$

is described by the path

$$\mathbf{r}_u(s) = \frac{u}{k_s}\left(1-\cos(k_s s)\right)\mathbf{e}_y + \left(Us + \left(\frac{u}{2}\right)^2\frac{\sin(2k_s s)}{2k_s}\right)\mathbf{e}_z + O\left(u^4\right), \tag{2}$$

with K the undulator parameter, λ_s the period length of the path, λ_z the undulator period, $u = K/\gamma\beta$ the maximal angle of the undulator path to $\mathbf{e}_z$ the undulator axis, $U = 1-(u/2)^2$ the relation between the z-period and the path period and $k_s = 2\pi/\lambda_s$. We assume a Gaussian line charge density $\lambda(s-vt)$ for one dimensional calculations with $\lambda(s) = g(s/\sigma)q/\sigma$, $v = \beta c_0$ the bunch velocity, q the bunch charge, σ

the RMS bunch length and $g(x)$ the Gaussian normal distribution. The charge density used for three dimensional calculations is

$$\rho(\mathbf{r} = \mathbf{r}_u(s) + x\mathbf{e}_x + y\mathbf{e}_y, t) = \lambda(s - vt)\frac{g(x/\sigma_r)}{\sigma_r}\frac{g(y/\sigma_r)}{\sigma_r}\frac{1}{\mathbf{e}_z \cdot \partial_s \mathbf{r}_u}, \qquad (3)$$

with σ_r the RMS bunch radius. The longitudinal electric field E is calculated for a test particle at the location $\mathbf{r}_t(S = vt) = \mathbf{r}_u(S + \delta S) + y\mathbf{e}_y$ with δS the longitudinal offset to the bunch center, y the transverse offset and $\mathbf{e}_t = \partial_S \mathbf{r}_t$ the longitudinal direction. The longitudinal electric field can be expressed by the scalar and vector potential Θ, $\mathbf{A}$ as

$$E = \mathbf{E} \cdot \mathbf{e}_t = -\left(\nabla\Theta + \dot{\mathbf{A}}\right) \cdot \mathbf{e}_t = -\frac{d\Theta}{dvt} + \frac{\partial}{\partial vt}\left(\Theta - \mathbf{A} \cdot \mathbf{v}\right). \qquad (4)$$

The term $E_1 = -d\Theta/dvt$ describes the field that is needed to change the potential energy. The mean value of E_1 vanishes because the scalar potential in the undulator is a periodic function. The second term, which is not conservative, is singular for one dimensional source distributions. To obtain a finite and smooth distribution E_2 we divide the field artificially into

$$E = \underbrace{-\frac{d\Theta}{dvt}}_{E_1} + \underbrace{\frac{\partial}{\partial vt}\left(\Theta - \frac{\Theta_{LM}}{\gamma^2} - \mathbf{A} \cdot \mathbf{v}\right)}_{E_2} + \underbrace{\frac{\partial}{\partial vt}\frac{\Theta_{LM}}{\gamma^2}}_{E_3}, \qquad (5)$$

with Θ_{LM} the potential of a bunch in linear motion. Θ_{LM} is calculated in the same way as Θ, but with the source path $\mathbf{r}_{LM}(s) = s\mathbf{e}_z$ and the observer position $\mathbf{r}_{t,LM}(S) = \mathbf{r}_{LM}(S + \delta S) + y\mathbf{e}_y$. The field of a line charge density is given by the one dimensional integration of the retarded sources:

$$E_1 = \frac{-1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\partial}{\partial R}\left(\frac{\lambda(s + \delta S + \beta R)}{R}\right)\frac{\partial R}{\partial S}ds \qquad (6a)$$

$$\begin{aligned} E_2 = {} & \frac{-\beta^2}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda'(s + \delta S + \beta R)}{R_{LM}}\left(1 - \mathbf{e}(s_0) \cdot \mathbf{e}(s_1)\right)ds \\ & - \frac{1}{4\pi\varepsilon_0\gamma^2}\int_{-\infty}^{\infty}\frac{\lambda'(s + \delta S + \beta R) - \lambda'(s + \delta S + \beta R_{LM})}{R_{LM}}ds \\ & + \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\lambda'(s + \delta S + \beta R)\left(\frac{1}{R_{LM}} - \frac{1}{R}\right)\left(1 - \beta^2\mathbf{e}(s_0) \cdot \mathbf{e}(s_1)\right)ds \end{aligned} \qquad (6b)$$

$$E_3 = \frac{-1}{4\pi\varepsilon_0\gamma^2}\int_{-\infty}^{\infty}\frac{\lambda'(s + \delta S + \beta R_{LM})}{R_{LM}}ds \qquad (6c)$$

with $s_0 = S + \delta S$, $s_1 = S + \delta S + s$, $\mathbf{R} = \mathbf{r}_t(S) - \mathbf{r}_u(S + \delta S + s)$, $R = \|\mathbf{R}\|$, $\mathbf{n} = \frac{\mathbf{R}}{R}$,

$R_{LM} = \sqrt{s^2 + y^2}$, $\mathbf{e}(s) = \partial_s \mathbf{r}(s)$. The last term of Eq. 6b is of the order $qK^2\left(2 + K^2\right)/\left(\gamma^4 16\pi\varepsilon_0\sigma^2\right)$ and therefore neglected in the following. The distance R between source and observer is in first order approximated by $R_0 = \sqrt{(Us)^2 + y^2}$. For the integration of Eq. 6 for macro bunches (bunch length >> photon wavelength) we take into account the corrections $\delta R = R - R_0$ and approximate all integrands by their first order Taylor expansion with respect to δR:

$$E_1 = \frac{-1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\partial}{\partial R_0}\left(\frac{\lambda_0}{R_0}\right)\frac{\partial \delta R}{\partial S}ds\,, \tag{7a}$$

$$\begin{aligned} E_2 = &\frac{-1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\left\{\beta^2\frac{\lambda_0'}{R_{LM}}\left(1 - \mathbf{e}(s_0)\cdot\mathbf{e}(s_1)\right) + \frac{\lambda_0' - \lambda_{LM}'}{\gamma^2 R_{LM}}\right\}ds \\ &- \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\lambda_0''\beta\delta R}{R_{LM}}\left(1 - \beta^2\mathbf{e}(s_0)\cdot\mathbf{e}(s_1)\right)ds \end{aligned} \tag{7b}$$

with $\lambda_0 = \lambda(s + \delta S + \beta R_0)$, $\lambda_{LM} = \lambda(s + \delta S + \beta R_{LM})$ and

$$\begin{aligned} \delta R = &\frac{\cos(k_s s_1) - \cos(k_s s_0)}{R_0 k_s}uy - \frac{1}{2R_0}\left[\frac{\cos(k_s s_1) - \cos(k_s s_0)}{R_0 k_s}uy\right]^2 \\ &+ \left[\frac{\left(\cos(k_s s_1) - \cos(k_s s_0)\right)^2 + \frac{1}{4}Uk_s s\left(\sin(2k_s s_1) - \sin(2k_s s_0)\right)}{R_0 k_s}\right]\frac{u^2}{2k_s} + O\left(u^3\right) \end{aligned} \tag{8a}$$

$$1 - \mathbf{e}(s_0)\cdot\mathbf{e}(s_1) = \tfrac{1}{2}u^2\left(\sin\left(k_s s_1\right) - \sin\left(k_s s_0\right)\right)^2 + O(u^4)\,. \tag{8b}$$

2.1 Mid Offset Range Approximation

The mid offset range approximation estimates δR by the first term of Eq. 8a and neglects second order terms of y. Therefore it is valid for $u/k_s << |y| << \sqrt{\sigma\lambda_u}$. Although the mean value of E_1 (with respect to time) vanishes, it causes a large oscillating contribution for particles with an offset of the order of the bunch length:

$$E_1 = \frac{uy}{4\pi\varepsilon_0}\left[f_{1s}(\delta S, y)\sin\left(k_s(S + \delta S)\right) + f_{1c}(\delta S, y)\cos\left(k_s(S + \delta S)\right)\right] \tag{9a}$$

$$f_{1s}(\delta S, y) = \int_{-\infty}^{\infty}\frac{\partial}{\partial R_0}\left(\frac{\lambda_0}{R_0}\right)\frac{\cos(k_s s) - 1}{R_0}ds \tag{9b}$$

$$f_{1c}(\delta S, y) = \int_{-\infty}^{\infty}\frac{\partial}{\partial R_0}\left(\frac{\lambda_0}{R_0}\right)\frac{\sin(k_s s)}{R_0}ds \tag{9c}$$

The offset dependency of E_2 is almost negligible in comparison to E_1. Therefore we specify here only the offset independent part of this term and neglect the positive part of the integration range (which is of the order of $qK^2/(\gamma^4 16\pi\varepsilon_0\sigma^2)$):

$$E_2 = \frac{qK^2}{4\pi\varepsilon_0\gamma^2\sigma^2}\big[g_1(\delta S) + g_{1s}(\delta S)\sin(2k_s(S+\delta S)) + g_{1c}(\delta S)\cos(2k_s(S+\delta S)) + g_2(\delta S)\big] \qquad (10a)$$

$$g_1(\delta S) = \int_{-\infty}^{0} \frac{g'(x+\delta S/\sigma)}{x}\sin^2\left(\frac{\omega x}{2}\right)dx\,, \qquad (10b)$$

$$g_{1s}(\delta S) = \int_{-\infty}^{0} \frac{-g'(x+\delta S/\sigma)}{x}\sin^2\left(\frac{\omega x}{2}\right)\sin(\omega x)\,dx\,, \qquad (10c)$$

$$g_{1c}(\delta S) = \int_{-\infty}^{0} \frac{g'(x+\delta S/\sigma)}{x}\sin^2\left(\frac{\omega x}{2}\right)\cos(\omega x)\,dx\,, \qquad (10d)$$

$$g_2(\delta S) = \int_{-\infty}^{0} \frac{g'(x+\delta S/\sigma) - g'(x/(1+0.5K^2)+\delta S/\sigma)}{xK^2}\,ds\,, \qquad (10e)$$

with $\omega = \dfrac{2k_s\sigma\gamma^2}{1+K^2/2}$. The validity of this approach is demonstrated in section 3 for the case of the TESLA FEL undulator.

2.2 Approximation for the Time Averaged Field of a 3D Beam

The time averaged field $\langle E\rangle$ of a 3D beam (Eq. 3) is the sum of the time averaged term $\langle E_2\rangle$ and the time independent field E_3. For the calculation of $\langle E_2\rangle$ we make the assumption that the field is independent of the transverse offset and estimate δR by the third term of equation 8a:

$$\langle E_2\rangle = \frac{qK^2}{4\pi\varepsilon_0\gamma^2\sigma^2}\left[g_1(\delta S) + g_2(\delta S) + \frac{g_3(\delta S) + K^2 g_4(\delta S)}{\sigma k_s\beta\gamma^2}\right], \qquad (11a)$$

$$g_3(\delta S) = \int_{-\infty}^{0} \frac{\cos(x\omega)-1}{2x\omega}\,\frac{g''(x+\delta S/\sigma)}{x}\,dx\,, \qquad (11b)$$

$$g_4(\delta S) = \int_{-\infty}^{0} \left\{\frac{8\cos(x\omega) - 2\cos(2x\omega) - 6}{x\omega} + \sin(2x\omega) - 2\sin(x\omega)\right\}\frac{g''(x+\delta S/\sigma)}{32x}\,ds\,. \qquad (11c)$$

This equation differs from the time averaged field calculated with the mid range approximation (10) by the terms $g_3(\delta S)$ and $g_4(\delta S)$. Therefore the assumption of a negligible offset dependency is only consistent under the condition $1+K^2 << \sigma k_s \gamma^2$ (weak influence of g_3 and g_4). The linear motion contribution E_3 for a particle without transverse offset is

$$E_3 = \frac{-q}{4\pi\varepsilon_0 \gamma \sigma_r \sigma} \int_{-\infty}^{\infty} g'(x + \delta S/\sigma) \mathrm{F}(x\gamma\sigma/\sigma_r) dx \,, \tag{12a}$$

$$F(\xi) = \sqrt{\frac{\pi}{2}} \exp\left(\frac{\xi^2}{2}\right)\left(1 - \mathrm{erf}\left(\frac{|\xi|}{\sqrt{2}}\right)\right) \approx \frac{1}{\sqrt{1+\xi^2}} \text{ for } |\xi| >> 1 \,. \tag{12b}$$

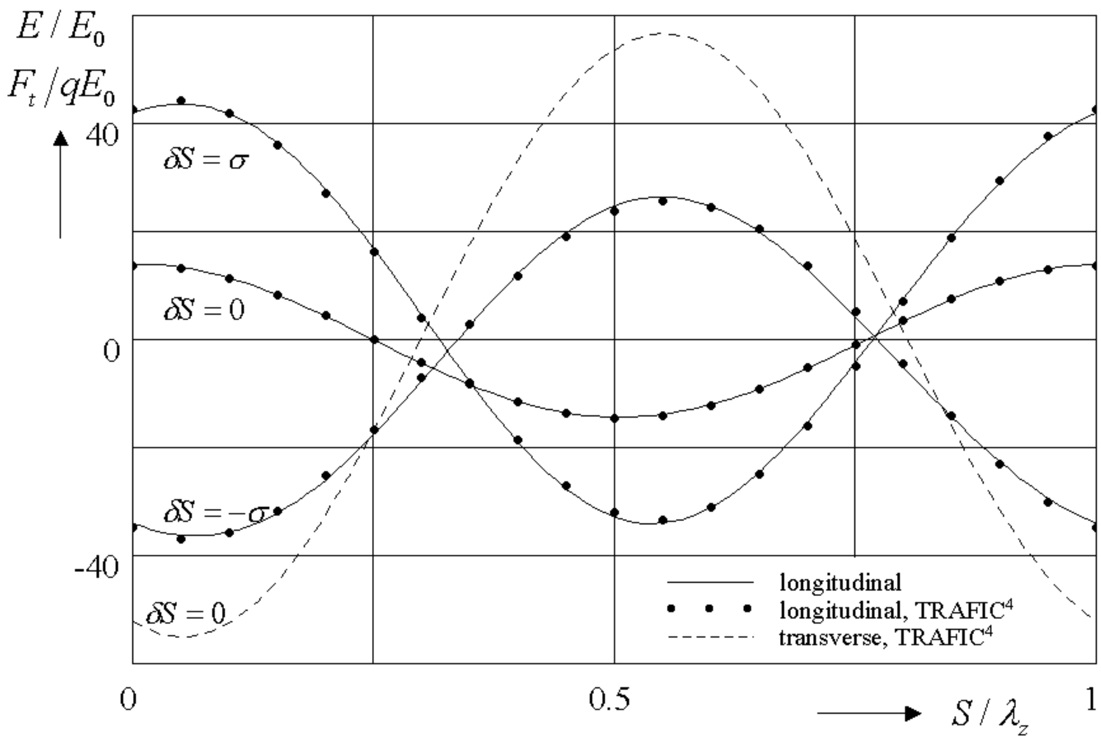

Fig. 1: Longitudinal electric field for test particles with $y = 50\mu$m and $\delta S = 0, -\sigma, +\sigma$ and transverse component ($\partial_S \mathbf{r}_t \times \mathbf{e}_x$) of the Lorentz force for a test particle with $y = 50\mu$m, $\delta S = 0$.

3. Calculations

The following calculations were carried out for the parameters of the 1GeV parameter set of the TESLA FEL ($K = 1.27$, $\lambda_z = 2.73$cm, $\gamma = 1957$, $\sigma = 50\mu$m) with the formulas described above and by the TRAFIC4 field solver [6] which integrates Eq. 4 numerically without any approximations. All fields are normalized to $E_0 = q/4\pi\varepsilon_0\gamma^2\sigma^2$ which is 939 V/m for a bunch charge of 1 nC. We distinguish between the steady state regime that assumes an infinite undulator and the transient regime.

	$eE_{\mathrm{mean}} L_u / \gamma m c_0^2$	$eE_{\mathrm{rms}} L_u / \gamma m c_0^2$
1D approach without E_3, [1]	$-5.14\cdot10^{-6}$	$3.59\cdot10^{-5}$
1D approach without E_3, Eq. 11	$-5.11\cdot10^{-6}$	$3.57\cdot10^{-5}$
round beam, TRAFIC4	$-5.16\cdot10^{-6}$	$10.0\cdot10^{-5}$

Table 1: The mean energy offset and the rms energy spread for the parameters of the 1GeV parameter set of the TESLA FEL ($K = 1.27$, $\lambda_z = 2.73$cm, $\gamma = 1957$, $\sigma = 50\mu$m, length=27m).

3.1 Steady State

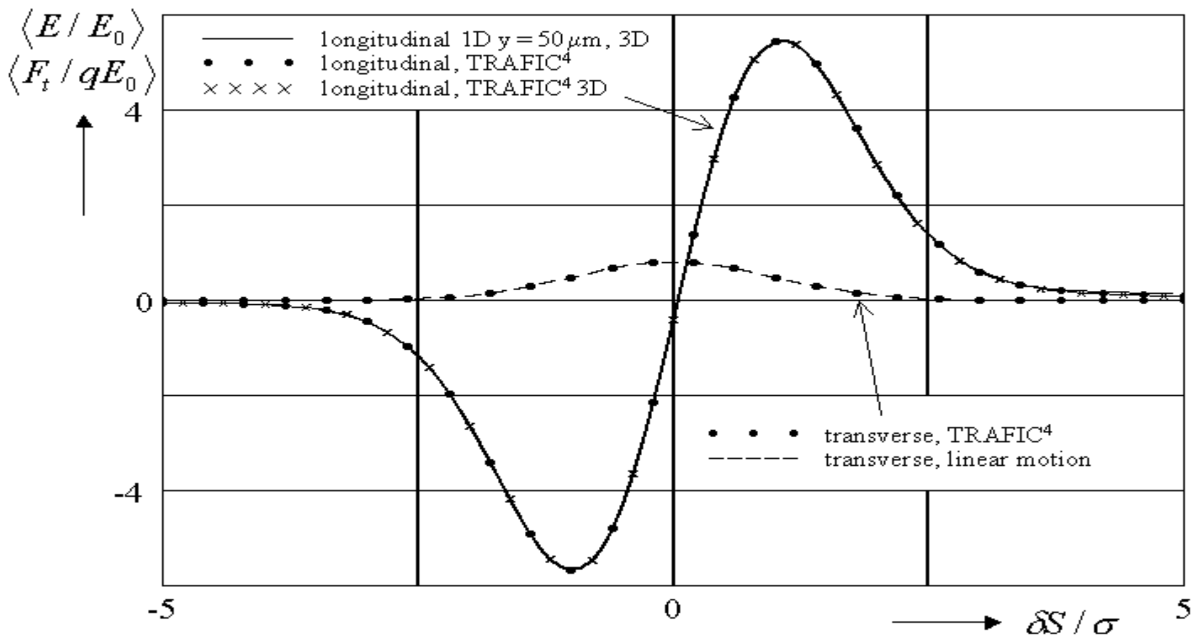

Fig. 2: Longitudinal electric field and transverse component ($\partial_S \mathbf{r}_t \times \mathbf{e}_x$) of the Lorentz force averaged over one undulator period. The comparison is between the 1D longitudinal field for a test particle with y = 50μm (calculated by mid range approach and TRAFIC4) and the 3D longitudinal field (calculated by Eq. 11&12 and TRAFIC4). Further the averaged transverse field (1D, y = 50μm) is compared with the linear motion field.

The time dependent field of particles with a transverse offset y = 50μm and $\delta S = 0, -\sigma, +\sigma$ is shown in Fig. 1. The slight deviation between the mid range approximation and the TRAFIC4 points is essentially caused by neglecting the third term of Eq. (8a) in the factor $\partial_S \delta R$ which is needed for the calculation of E_1 by Eq. (7b). The major contribution to the strong oscillation of $E(S, \delta S = \text{const})$ comes from the variation of the potential energy $-d\Theta/dS$. It was proposed in [2] to estimate this oscillation by a circular motion model with the local curvature radius. In the given case this would overestimate the amplitude of the $+\sigma$ particle by a factor of 3/2. The transverse component (perpendicular to the magnetic field) of the Lorentz force is the space charge force of a beam in linear motion superimposed by the centrifugal space charge force. Averaged over one undulator period the strong oscillations cancel (see Fig. **2**). The averaged longitudinal fields for an observer offset y = 50μm and a round beam σ_r = 50 μm without offset calculated by this theory are plotted on the same line. This result is also in good agreement with the fully three dimensional calculation of TRAFIC4. The deviation between $\langle E_2 \rangle$ calculated by Eq. 11 and $\langle E - E_3 \rangle$ calculated by TRAFIC4 (3D) is less then $0.05 \cdot E_0$ for all positions in the bunch ($|\delta S| < 5\sigma$, $|y| < 5\sigma_r$). This confirms that $\langle E_2 \rangle$ is really insensitive to the transverse position of the test particle. The averaged transverse component of the Lorentz force is almost identical to the transverse field of a bunch in longitudinal motion. The mean energy offset and the rms energy spread are listed in Tab. 1 (undulator length L_u = 27 m).

3.2 Transient

The transient behavior of the longitudinal E-field of a 1D bunch that enters an undulator has been calculated by TRAFIC4. Therefore it is assumed that the magnetic field (Eq.1) vanishes for z < 0 and the path (Eq. 2) is linear for negative path coordinates: $\mathbf{r}_u(s < 0) = -s\,\mathbf{e}_z$. The field observed by test particles with a transverse offset y = 50μm and $\delta S = 0, -\sigma, +\sigma$ is shown in Fig. **3**. The instantaneous field $E(S,\delta S)$ (left diagram) after half an undulator period looks almost like the steady state field in Fig. 1. Nevertheless the averaged field $\langle E(S,\delta S)\rangle = 1/\lambda_s \int_{S-\lambda_s/2}^{S-\lambda_s/2} E(\xi,\delta S)\, d\xi$ (right diagram) needs several hundred undulator periods to approach the steady state condition as calculated in Fig. **2**.

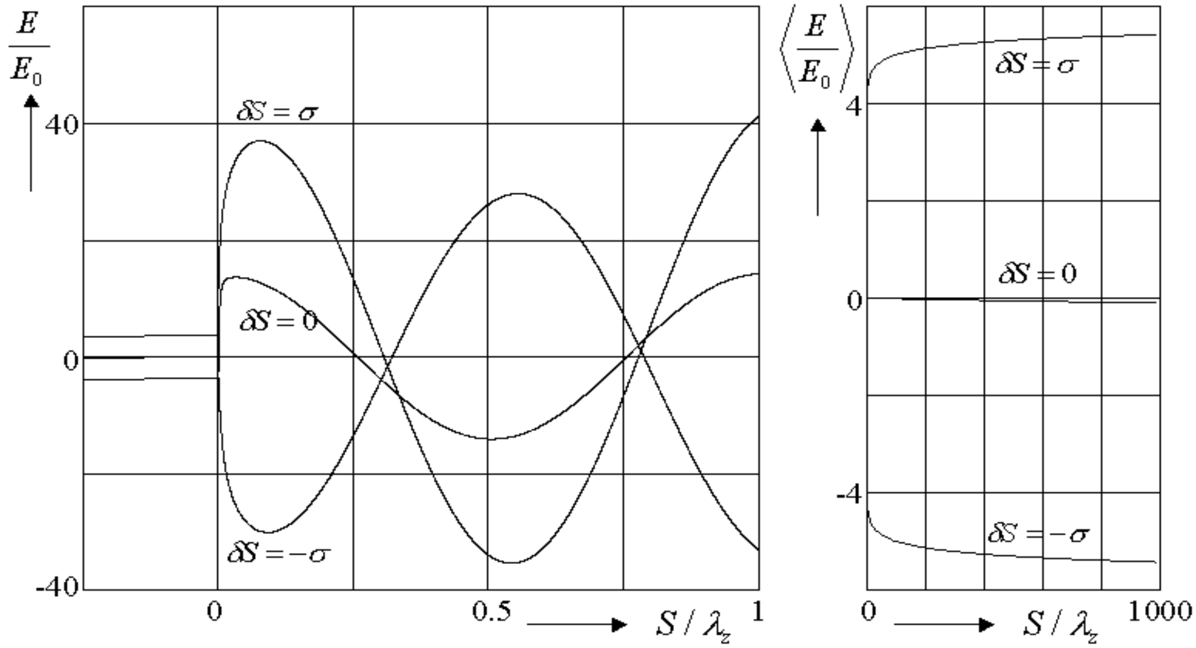

Fig. 3: Transient longitudinal electric field for test particles with y = 50μm and $\delta S = 0, -\sigma, +\sigma$. The left picture shows the field in the first undulator period (compare Fig. 1). The right picture shows the averaged field $\langle E\rangle = 1/\lambda_s \int_{S-\lambda_s/2}^{S+\lambda_s/2} E(\xi) d\xi$ for 1000 periods.

4. Conclusion

The longitudinal as well as the transverse component of the Lorentz force caused by a macrobunch in the TESLA FEL undulator at 1GeV can be one to two orders larger than the forces of the same bunch in linear motion. Averaged versus one undulator period these components are of the same order as for linear motion.

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