CHAPTER 6

Operator product expansions

In [Wil69] Wilson suggested that a product of interacting field operators on separated points could be expanded into a sum of local operators if the separation goes to zero. Such an expansion is called *operator product expansion* (OPE).

Zimmermann has introduced his notion of perturbative normal products in [Zim71, Zim73a]. In his approach (interacting) operators are always defined via their Green's functions, namely vacuum expectation values of time ordered products with an arbitrary number of fields. The Green's functions are renormalized by BPHZ- or in the massless case by BPHZL subtraction.

In [Zim71, Zim73b] he gave a generalization of these local products to multi local ones. They admit a restriction of all coordinates to one yielding the local normal product. By relating a bilocal product to a time ordered one he derived an OPE verifying Wilson's hypothesis perturbatively. He found explicit formulas for the expansion coefficients in the form of Green's functions.

Since in the framework of Bogoliubov and Epstein-Glaser the operators are defined directly we try to mimick Zimmermann's procedure. We define a new time ordered product containing a bilocal expression which allows for setting its coordinates to the same value (in the sense of a restriction of a distribution). We are concerned with scalar fields only and our bilocal T-product has only the basic generators in the bilocal insertion.

The definition of a bilocal T-product gives rise to a corresponding bilocal interacting field. Following Zimmermann's notation we also call this object a normal product. The transition from the T-products to the interacting fields automatically generates an OPE for the time ordered product of two interacting fields. The coefficients depend on the coupling only locally. In φ^4-theory two coefficients appear. One consist of graphs that contribute to wave function and mass renormalization only. The other collects graphs contributing only to coupling constant renormalization.

With the normal product defined we investigate the first step towards the definition of a state on the local algebra. We find the corresponding two point function to be positive in an appropriate sense as a formal power series.

1. Bilocal time ordered products

The word *bilocal time ordered product* means a usual time ordered product where only *one* entry is a bilocal expression. We derive an explicit formula that defines these products for two scalar fields. Let us mention that our expression is only explicit up to normalization terms which restore broken Lorentz covariance (cf. chapter 3, section 5). We state the problem first:

Consider the case of the interacting scalar fields $\varphi_{g\mathscr{L}}$. We aim at the definition of a *normal product* $:\varphi,\varphi:_{g\mathscr{L}}(x_1,x_2)$ with the property

$$\lim_{\xi\to 0} :\varphi,\varphi:_{g\mathscr{L}}(x+\xi,x-\xi) = \left(\varphi^2\right)_{g\mathscr{L}}(x), \tag{6.1}$$

where $x = \frac{x_1+x_2}{2}$ denotes the central coordinate and $\xi = \frac{x_1-x_2}{2}$ the difference coordinate. Taking the definition of interacting fields into account (chapter 4), we notice that it suffices to define the corresponding T-products. This is illustrated in

1.1. The easiest example. We consider scalar $\frac{\varphi^4}{4!}$-theory. The task is to determine $T(\mathscr{L},:\varphi,\varphi:)(y,x_1,x_2)$. We proceed in the following way: First we define ${}^0T(\mathscr{L},:\varphi,\varphi:)(y,x_1,x_2)$ which consists of the usual T-product with x_1,x_2-contractions omitted. Then we subtract a suitable term with support on $y=x$ that allows for the restriction. Obviously we have

$$\begin{aligned}
{}^0T(\mathscr{L},:\varphi,\varphi:)(y,x_1,x_2) =&: \frac{\varphi(y)^4}{4!}\varphi(x_1)\varphi(x_2): + & (6.2)\\
&+ i\Delta_F(y-x_1) : \frac{\varphi(y)^3}{3!}\varphi(x_2): & (6.3)\\
&+ i\Delta_F(y-x_2) : \frac{\varphi(y)^3}{3!}\varphi(x_1): + & (6.4)\\
&+ i\Delta_F(y-x_1)i\Delta_F(y-x_2) : \frac{\varphi(y)^2}{2!}: . & (6.5)
\end{aligned}$$

The problem for the definition of the restriction of ${}^0T$ emerges on the last line, where after setting $x_1 = x_2\ (=x)$ we obtain a $\Delta_F(y-x)^2$-term that is not well defined. Graphically this procedure produces a loop, see figure 1. But we

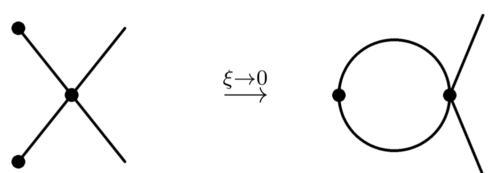

FIGURE 1. *A loop is generated if $\xi\to 0$.*

can find a term which subtraction allows for putting $x_1 = x_2 = x$, yielding $T(\mathscr{L},\varphi^2)(y,x)$ with its respective normalization. We claim that with

$$e^{(2)}(\xi) \doteq \int \mathrm{d}z\, i\Delta^F(z-\xi)i\Delta^F(z+\xi)w(z) \tag{6.6}$$

the subtracted distribution

$$i\Delta^F(y-x_1)i\Delta^F(y-x_2)\Big|_R = i\Delta^F(y-x_1)i\Delta^F(y-x_2) - \delta(x-y)e^{(2)}(\xi) \tag{6.7}$$

allows for the coincidence $x_1 = x_2 \Leftrightarrow \xi = 0$ after smearing with a test function. We calculate

$$\int \mathrm{d}y \, i\Delta_F(y-x_1) i\Delta_F(y-x_2)\Big|_R f(y) = \\ = \int \mathrm{d}y \, i\Delta_F(y-x_1) i\Delta_F(y-x_2)(f(y) - w(y-x)f(x)), \quad (6.8)$$

and this implies

$$\lim_{\xi\to 0} \left\langle i\Delta^F(\cdot+\xi) i\Delta^F(\cdot-\xi)\Big|_R, f_x\right\rangle = \left\langle i^2(\Delta^F)^2, W_{(0;w)} f_x\right\rangle, \quad (6.9)$$

with $f_x(y) = f(y+x)$, which proves our claim. Note, that the coefficient in front of the δ term is given by

$$e^{(2)}(\xi) = \int \mathrm{d}z \, \omega_0 \left({}^0T(\mathscr{L}^{(2)}, :\varphi,\varphi:)(z;\xi,-\xi)\right) w(z), \quad (6.10)$$

where the ${}^{(2)}$ again denotes twice differentiation with respect to φ. Now we return to the complete T-product. Collecting everything into one expression we find

$$T(\mathscr{L}, :\varphi,\varphi:)(y,x_1,x_2) = {}^0T(\mathscr{L}, :\varphi,\varphi:)(y,x_1,x_2) - \delta(y-x) e^{(2)}(\xi) \frac{:\varphi(x)^2:}{2}. \quad (6.11)$$

Because of the δ distribution we have set the coordinate of the last Wick monomial to x.

REMARK. As well as the center coordinate x we could have chosen any other point on the straight line between x_1 and x_2 (or also in the causally completed region spanned by these two points) for subtraction. But our choice is inspired by Zimmermann's work, moreover yielding a symmetrical solution.

Now we generalize the idea of subtracting a local term that compensates the overall divergence. Thereby we make use of the method of Epstein-Glaser where all lower order divergencies are appropriately handled by an

1.2. Inductive causal construction. We begin with a brief overview of the construction. Motivated by our example above we define the bilocal T-products in the following way: Denoting the sub manifold

$$\mathrm{Diag}_n^x \doteq \left\{(y_1,\dots,y_n,x_1,x_2) \in \mathbb{M}^{n+2} | y_1 = \dots = y_n = \frac{x_1+x_2}{2}\right\}, \quad (6.12)$$

we require the bilocal T-product of order n (where n is the number of the coordinates not including the two bilocal points) to be given by all bilocal T-products of lower and all local T-products of lower and same order on $\mathbb{M}^{n+2} \setminus \mathrm{Diag}_n^x$. This provides for a ${}^0T$-product which yields the local ${}^0T$-product in the limit $\xi \to 0$. Then we subtract a term with support on Diag_n^x such that the limit $\xi \to 0$ exists and yields the corresponding T-product. Hence in every order the difference between a local T-product (with x_1, x_2-contractions omitted) and a bilocal one only consists of these local terms.

In zero'th order we define: $T(:\varphi,\varphi:)(x_1,x_2) \doteq :\varphi(x_1)\varphi(x_2):$. Using our shorthand notation for the arguments (N can be any set of Wick monomials) we require the following causal factorization properties:

$$
\begin{aligned}
&T(N:\varphi,\varphi:)(y_N,x_1,x_2) = \\
&= \begin{cases}
T(I)(y_I)T\left(N\setminus I,:\varphi,\varphi:\right)\left(y_{N\setminus I},x_1,x_2\right), & \text{if } I \gtrsim N\setminus I, x_1, x_2, I\neq\emptyset, \\
T\left(I,:\varphi,\varphi:\right)(y_I,x_1,x_2)T(N\setminus I)\left(y_{N\setminus I}\right), & \text{if } I, x_1, x_2, \gtrsim N\setminus I, I\neq N, \\
T(N,\varphi,\varphi)\left(y_N,x_1,x_2\right)+ & \\
+T(I)(y_I)\big[T\left(N\setminus I,:\varphi,\varphi:\right)\left(y_{N\setminus I},x_1,x_2\right)+ & \\
-T(N\setminus I,\varphi,\varphi)\left(y_{N\setminus I},x_1,x_2\right)\big], & \text{if } I, x_1 \gtrsim N\setminus I, x, x_2, I\neq\emptyset, \\
T(N,\varphi,\varphi)\left(y_N,x_1,x_2\right)+ & \\
+\big[T(I,:\varphi,\varphi:)(y_I,x_1,x_2)+ & \\
-T(I,\varphi,\varphi)(y_I,x_1,x_2)\big]T\left(N\setminus I\right)\left(y_{N\setminus I}\right), & \text{if } I, x_1, x, \gtrsim N\setminus I, x_2, I\neq N, \\
\text{the last two expressions with } x_1\leftrightarrow x_2. &
\end{cases}
\end{aligned}
\tag{6.13}
$$

We convince ourselves that this a reasonable causal factorization. If x_1, x_2 are contracted to a point, then the first two equations obviously give the right causal decompositions. We investigate the last term on the third line, where $I, x_1 \gtrsim N\setminus I, x, x_2$. If $\xi\to 0$ we find

$$
\begin{aligned}
&T(I)(y_I)T(N\setminus I,\varphi,\varphi)\left(y_{N\setminus I},x_1,x_2\right) = \\
&\qquad = T(I)(y_I)\varphi(x_1)T(N\setminus I,\varphi)\left(y_{N\setminus I},x_2\right) \\
&\qquad \overset{\xi\to 0}{=} T(I,\varphi)(y_I,x_1)T(N\setminus I,\varphi)\left(y_{N\setminus I},x_2\right) \\
&\qquad = T(N,\varphi,\varphi)\left(y_N,x_1,x_2\right),
\end{aligned}
\tag{6.14}
$$

since x_1 becomes earlier than all y_I. This term cancels the first term of the third line in (6.13) leaving the first line of (6.13). A similar consideration leads to the same conclusion also for the last line of (6.13).

This shows that the bilocal T-product is completely determined by (6.13) up to the sub manifold Diag_n^x. In contrast to the definition of a local T-product where one has to perform an extension of the numerical distributions involved, the bilocal product can be defined by a suitable subtraction. This is due to the fact that all terms are well defined distributions in $n+2$ variables. We state the solution:

$$
\begin{aligned}
&T\left(N,:\varphi,\varphi:\right)\left(y_N,x_1,x_2\right) = \\
&\qquad = T(N,\varphi,\varphi)(y_N,x_1,x_2) - \omega_0\left(T(\varphi,\varphi)(x_1,x_2)\right)T(N)(y_N) + \\
&\qquad - \sum_{\substack{I\subset N\\ I\neq\emptyset}}\sum_{|\gamma|\leq\omega_I}\sum_{|\alpha|\leq\omega_I-|\gamma|}\sum_{\mu+\nu=\alpha}\frac{(-)^{|\mu|}}{\mu!\nu!\gamma!}\partial^\mu\delta(I-x)\times \\
&\qquad \times\left(e_I^{\alpha(\gamma)}(\xi)-a_I^{\alpha(\gamma)}(\xi)\right)T\left(N\setminus I,\partial^\nu\varphi^\gamma\right)\left(y_{N\setminus I},x\right),
\end{aligned}
\tag{6.15}
$$

where $\gamma \in \mathbb{N}^{|I|}$ is a multi index and $\mu, \nu \in \mathbb{N}^{4|I|}$ are multi quadri indices. Therefore the term $\partial^\mu \delta(I - x) = \prod_{i\in I} \partial^{\mu_i} \delta(y_i - x)$ (and μ_1 can be $\mu\nu\rho$ for example, with μ, ν, ρ usual Lorentz indices). The number ω_I refers to the singular order of the numerical distribution $\omega_0 \left(T(I, \varphi^2)(y_I, x)\right)$ and is given by $\omega_I = \sum_{i\in I} \dim W_i + 2 \dim \varphi - 4|I|$. Hence the sum over I only runs over subsets for which the coincidence of x_1, x_2 produces distributions with non negative singular order. These are ordered into subgraphs by the sum over γ. Only graphs for which their corresponding order, namely $\omega_I - |\gamma|$, is non negative (γ is a derivative w.r.t. φ and therefore the singular order decreases with increasing $|\gamma|$) are taken into account. The sum over α refers to the usual subtraction procedure (running from 0 up to the order of singularity of the corresponding distribution). As a matter of fact it has to be split into an action on δ and on $T(\dots, \varphi^\gamma)$ since we changed the coordinate of $\varphi^{\gamma_i}(y_i)$ into $\varphi^{\gamma_i}(x)$ according to the δ function (see also the example at the beginning).

To explain the coefficients, we have to introduce the corresponding ${}^0T$-product which is the same expression like (6.15) up to the last sum which does not contain the term $I = N$. So

$$\begin{aligned}
&{}^0T\,(N, {:}\varphi, \varphi{:})\,(y_N, x_1, x_2) = \\
&\qquad = T(N, \varphi, \varphi)(y_N, x_1, x_2) - \omega_0\left(T(\varphi, \varphi)(x_1, x_2)\right) T(N)(y_N) + \\
&\qquad\quad - \sum_{\substack{I\subset N\\ I\neq\emptyset\\ I\neq N}} \sum_{|\gamma|\leq\omega_I} \sum_{|\alpha|\leq\omega_I-|\gamma|} \sum_{\mu+\nu=\alpha} \frac{(-)^{|\mu|}}{\mu!\nu!\gamma!} \partial^\mu \delta(I - x) \times \\
&\qquad\quad \times \left(e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi)\right) T\left(N \setminus I, \partial^\nu \varphi^\gamma\right) \left(y_{N\setminus I}, x\right).
\end{aligned} \tag{6.16}$$

Hence T and ${}^0T$ only differ by a term with support on Diag_n^x. The coefficients $e_I^{\alpha(\gamma)}$ are given by the expression

$$e_I^{\alpha(\gamma)}(\xi) = \int \mathrm{d}z_I\, z^\alpha \omega_0 \left({}^0T\left(I^{(\gamma)}, {:}\varphi, \varphi{:}\right)(y_I, \xi, -\xi)\right) w_{I^{(\gamma)}}(z_I), \tag{6.17}$$

where again $I^{(\gamma)} = \{W_i^{(\gamma_i)}, i \in I\}$ and the exponent ${}^{(\gamma_i)}$ means γ_i fold differentiation with respect to φ in $\mathfrak{B}$. The coefficients $a_I^{\alpha(\gamma)}$ are chosen in such a way, that Lorentz covariance is conserved. The function $w_{I^{(\gamma)}}$ is the auxiliary function used in the extension process of the distribution $\omega_0\left(T\left(I^{(\gamma)}, \varphi^2\right)(y_I, x)\right)$ of the $|I|$ difference coordinates $y_1 - x, \dots, y_{|I|} - x$.[1]

We require our bilocal products to fulfil normalization condition **N3** in the corresponding form, namely

$$\begin{aligned}
&\left[T\,(N, {:}\varphi, \varphi{:})\,(y_N, x_1, x_2), \varphi(z)\right] = \\
&\qquad = i \sum_{k=1}^{n} T\left(N^{(e_k)}, {:}\varphi, \varphi{:}\right)(y_N, x_1, x_2)\Delta(y_k - z) + \\
&\qquad\quad + iT\,(N, \varphi)\,(y_N, x_1)\Delta(x_2 - z) + iT\,(N, \varphi)\,(y_N, x_2)\Delta(x_1 - z),
\end{aligned} \tag{\textbf{N3'}}$$

[1] Without loss of generality $I = \{1, \dots, |I|\}$.

and $e_k = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^n$ with the 1 in the k'th position.

1.3. Proof of the restriction property. We show that (6.15) yields the right T-product by restricting $x_1 = x_2$. Our proof requires the validity of **N3'** which is proven in the next subsection. Unfortunately our proof still lacks an existence statement for the distributions $a_I(\xi)$, necessary for the conservation of Poincaré covariance. Hence we have to assume that they exist.

Assume that up to order $n-1$ the bilocal $T(N, :\varphi, \varphi:)(y_N, x_1, x_2)$ exists and its restriction $x_1 = x_2$ is given by $T(N, \varphi^2)(y_N, x)$. We show that T has the right causal factorization (6.13). We write (6.15) as

$$T(N, :\varphi, \varphi:)(y_N, x_1, x_2) = T(N, \varphi, \varphi)(y_N, x_1, x_2) + \\ - \sum_{I \subset N} \sum_{|\gamma| \leq \omega_I} \sum_{|\mu|+|\nu| \leq \omega_I - |\gamma|} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) T(N \setminus I, \partial^\nu \varphi^\gamma)\left(y_{N \setminus I}, x\right). \tag{6.18}$$

The $E^{\mu\nu(\gamma)}$ are given by

$$E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) = \frac{(-)^\mu}{\mu!\nu!\gamma!} \partial^\mu \delta(I - x)\big(e_I^{\mu+\nu(\gamma)}(\xi) - a_I^{\mu+\nu(\gamma)}(\xi)\big), \tag{6.19}$$

$$\text{with } E^{\mu\nu(\gamma)}(\emptyset, \varphi, \varphi)(x, \xi) = \delta_0^\mu \delta_0^\nu \delta_0^\gamma \omega_0\left(T(\varphi, \varphi)(x_1, x_2)\right)\big|_{x_1 - x_2 = 2\xi} \tag{6.20}$$

$$\text{and } \operatorname{supp} E^{\mu\nu(\gamma)}(N, \varphi, \varphi)(y_N, x, \xi) \subset \mathrm{Diag}_n^x, N \neq \emptyset. \tag{6.21}$$

If $L \gtrsim N \setminus L, x, x_1, x_2$, we have from (6.18):

$$\begin{aligned} T(N, :\varphi, \varphi:)(y_N, x_1, x_2) = \\ &= T(L)(y_L) T(N \setminus L, \varphi, \varphi)\left(y_{N \setminus L}, x_1, x_2\right) + \\ &\quad - T(L)(y_L) E(\emptyset, \varphi, \varphi)(x, \xi) T(N \setminus L)\left(y_{N \setminus L}\right) + \\ &\quad - \sum_{\substack{I \subset N \setminus L \\ I \neq \emptyset}} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) T(N \setminus I, \partial^\nu \varphi^\gamma)\left(y_{N \setminus I}, x\right) \\ &= T(L)(y_L) T(N \setminus L, \varphi, \varphi)(y_N, x_1, x_2) + \\ &\quad - T(L)(y_L) \sum_{I \subset N \setminus L} \sum_{\gamma, \mu, \nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) \times \\ &\qquad \times T((N \setminus L) \setminus I, \partial^\nu \varphi^\gamma)\left(y_{(N \setminus L) \setminus I}, x\right) \\ &= T(L)(y_L) T(N \setminus L, :\varphi, \varphi:)\left(y_{N \setminus L}, x_1, x_2\right). \end{aligned} \tag{6.22}$$

If $L, x_1 \gtrsim N \setminus L, x_2, x$, with $x_1 \gtrsim x_2$ we find:

$$
\begin{aligned}
&T(N, :\varphi, \varphi:)(y_N, x_1, x_2) = \\
&\quad = T(L,\varphi)(y_L, x_1) T(N \setminus L, \varphi)\left(y_{N\setminus L}, x_2\right) + \\
&\qquad - T(L)(y_L) E(\emptyset, \varphi, \varphi)(x, \xi) T(N \setminus L)\left(y_{N\setminus L}\right) + \\
&\qquad - \sum_{\substack{I \subset N\setminus L \\ I \neq \emptyset}} \sum_{\gamma,\mu,\nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) T(N \setminus I, \partial^\nu \varphi^\gamma)\left(y_{N\setminus I}, x\right) + \\
&\quad = T(L,\varphi)(y_L, x_1) T(N \setminus L, \varphi)\left(y_{N\setminus L}, x_2\right) + \\
&\qquad - T(L)(y_L) \sum_{I \subset N\setminus L} \sum_{\gamma,\mu,\nu} E^{\mu\nu(\gamma)}(I, \varphi, \varphi)(y_I, x, \xi) \times \\
&\qquad\quad \times T((N \setminus L) \setminus I, \partial^\nu \varphi^\gamma)\left(y_{(N\setminus L)\setminus I}, x\right) \\
&\quad = T(L,\varphi)(y_L, x_1) T(N \setminus L, \varphi)\left(y_{N\setminus L}, x_2\right) - T(L)(y_L) \times \\
&\qquad \times \left(T(N \setminus L, \varphi, \varphi)\left(y_{N\setminus L}, x_1, x_2\right) - T(N \setminus L, :\varphi, \varphi:)\left(y_{N\setminus L}, x_1, x_2\right)\right).
\end{aligned} \tag{6.23}
$$

A similar calculation also shows the right causality decomposition, if x is in the later set.

Because of this causal factorization property and the inductive assumption we immediately have

$$
{}^0T(N, :\varphi, \varphi:)(y_N, x, x) = {}^0T(N, \varphi^2)(y_N, x), \quad (y_N, x) \in \mathbb{M}^{n+1} \setminus \mathrm{Diag}_{n+1}. \tag{6.24}
$$

We show that the same equation also holds for the T-products. Because of **N3'** it is sufficient to consider vacuum expectation values only. Inserting the definition (6.15) we find:

$$
\begin{aligned}
&\omega_0\left(T(N, :\varphi, \varphi:)(y_N, x_1, x_2)\right) - \omega_0\left({}^0T(N, :\varphi, \varphi:)(y_N, x_1, x_2)\right) = \\
&\qquad\qquad = - \sum_{|\alpha| \leq \omega_N} \frac{(-)^{|\alpha|}}{\alpha!} \partial^\alpha \delta(y_N - x)\left(e_N^{\alpha(0)}(\xi) - a_N^{\alpha(0)}(\xi)\right).
\end{aligned} \tag{6.25}
$$

Since the vacuum expectation values are translation invariant we use the coordinates $z_i = y_i - x$ and $z = (z_1, \ldots, z_n)$. We set

$$
t(z, \xi) \doteq \omega_0\left(T(N, :\varphi, \varphi:)(z, \xi, -\xi)\right), \tag{6.26}
$$

$$
{}^0t(z, \xi) \doteq \omega_0\left({}^0T(N, :\varphi, \varphi:)(z, \xi, -\xi)\right). \tag{6.27}
$$

It follows that

$$
e_N^{\alpha(0)}(\xi) = \int \mathrm{d}z_N \, {}^0t(z, \xi) z^\alpha w_{N^{(0)}}(z) \tag{6.28}
$$

in this notation. Now, by smearing with $f \in \mathcal{D}(\mathbb{M}^n)$ in the z coordinates we find:

$$
\langle t(\cdot, \xi), f \rangle = \left\langle {}^0t(\cdot, \xi), W_{(\omega_N; w_{N^{(0)}})} f \right\rangle + \sum_{|\alpha| \leq \omega_N} \frac{a_N^{\alpha(0)}(\xi)}{\alpha!} \partial^\alpha f(0). \tag{6.29}
$$

Because of the sufficient subtraction on the test function we can put $\xi = 0$. Then we have

$$t(z,0) = \omega_0 \left(T(N,\varphi^2)(z_N,0)\right), \tag{6.30}$$

where the constants $a_N^{\alpha(0)}(0)$ can be chosen to produce any normalization of the RHS.

1.4. Lorentz covariance. Due to the definition the bilocal T-products are translation covariant. Namely, they are products of translation invariant numerical distributions and operator valued Wick products because of **N3'**. Therefore we have to consider Lorentz covariance for the the numerical distributions only.

If ${}^0t$ transforms covariantly under the Lorentz group

$$\Lambda^0 t(z,\xi) = D(\Lambda^{-1})^0 t(z,\xi), \tag{6.31}$$

w.r.t. both variables we find that t transforms the same way, if[2]

$$\left(\Lambda^\alpha{}_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta\right) e^\beta(\xi) = \left(\Lambda^\alpha{}_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta\right) a^\beta(\xi) \tag{6.32}$$

w.r.t. ξ. With e^β from (6.17) this leads to

$$\int \mathrm{d}z\, {}^0t(z,\xi) z^\alpha (\Lambda w - w)(z) = \left(\Lambda^\alpha{}_\beta D(\Lambda)\Lambda - \delta^\alpha_\beta\right) a^\beta(\xi). \tag{6.33}$$

The RHS is a one coboundary, which we have to solve for a^β. We assume that there are solutions $\mathcal{D}'(\mathbb{M}) \ni a^\beta \neq e^\beta$ with the property that $a(0)$ exists. Unlike in the case of usual T-products (cf. chapter 3, section 5) we have no existence proof, so we have to impose it as an assumption. Moreover we see that $a^\beta(\xi)$ is determined by the RHS of (6.33) only up to terms $h^\beta(\xi)$, with $\Lambda^\alpha{}_\beta D(\Lambda)\Lambda h^\beta(\xi) = h^\alpha(\xi)$.

If the central solution ($w = 1$) exists a can be chosen as

$$a^\alpha(\xi) = \int \mathrm{d}z\, {}^0t(z,\xi) z^\alpha (w-1)(z), \tag{6.34}$$

which fulfils all properties. It simply replaces the subtraction with auxiliary function w by the central subtraction.

1.5. Proof of N3'. We show that **N3'** holds by evaluating both sides independently. In the calculation we use the following formula, taken from [DF00b]:

$$\frac{1}{\alpha!}\frac{\partial(\partial^\alpha V)}{\partial\varphi_r} = \sum_{\mu+\nu=\alpha} \frac{1}{\mu!\nu!}\partial^\mu \left(\frac{\partial V}{\partial\varphi}\right) \delta^\nu_r, \tag{6.35}$$

where $V \in \mathfrak{B}$ is supposed to contain no derivated fields. We first investigate the contribution to the commutator of the RHS of **N3'** arising from the sum in

[2] We suppress the indices I and γ which are fixed in this problem.

(6.15). The following term appears:

$$\begin{aligned}
&\frac{1}{\nu_1!\dots\nu_{|I|}!\gamma_1!\dots\gamma_{|I|}!}\left[T\left(N\setminus I,\partial^{\nu_1}\varphi^{\gamma_1}\cdots\partial^{\nu_{|I|}}\varphi^{\gamma_{|I|}}\right)(y_N,x),\varphi(z)\right]=\\
&\quad=\sum_{k\in N\setminus I}\frac{1}{\nu!\gamma!}T\left((N\setminus I)^{(e_k)},\partial^\nu\varphi^\gamma\right)\left(y_{N\setminus I},x\right)i\Delta(y_k-x)+\\
&\quad+\sum_{k\in I}\sum_{\sigma_k+\rho_k=\nu_k}\frac{1}{\nu_1!\gamma_1!\cdots\not k\dots\nu_{|I|}!\gamma_{|I|}!}\frac{1}{\sigma_k!\rho_k!(\gamma_k-1)!}\times\\
&\qquad\times T\left(N\setminus I,\partial^{\nu_1}\varphi^{\gamma_1}\cdots\partial^{\sigma_k}\varphi^{\gamma_k-1}\cdots\partial^{\nu_{|I|}}\varphi^{\gamma_{|I|}}\right)(y_N,x)i\partial^{\rho_k}\Delta(x-z).
\end{aligned}\tag{6.36}$$

Without loss of generality we have put $I=\{1,\dots,|I|\}$. Denote the T-product on the last line symbolically by T^{σ_k}. Note that it only appears, if $|\gamma|>0$. With the equality

$$\frac{(-)^{|\beta|}}{\beta!}\partial^\beta\delta(y-x)f(y)=\sum_{\rho+\sigma=\beta}\frac{(-)^{|\rho|}}{\rho!\sigma!}\partial^\rho\delta(y-x)\partial^\sigma f(x),\tag{6.37}$$

we have for every $k\in I$ (now I can be any subset of N) the following contribution:

$$\begin{aligned}
\sum_{\mu_k+\nu_k=\alpha_k}\frac{(-)^{|\mu_k|}}{\mu_k!}\partial^{\mu_k}\delta(y_k-x)&\sum_{\sigma_k+\rho_k=\nu_k}\frac{1}{\sigma_k!\rho_k!}T^{\sigma_k}\partial^{\rho_k}\Delta(x-z)=\\
&\sum_{\mu_k+\nu_k=\alpha_k}\frac{(-)^{|\mu_k|}}{\mu_k!\nu_k!}\partial^{\mu_k}\delta(y_k-x)T^{\nu_k}\Delta(y_k-z).
\end{aligned}\tag{6.38}$$

Now we insert this result into the last term of (6.15), commuted with $\varphi(z)$:

$$\begin{aligned}
&[\text{last term of (6.15)},\varphi(z)]=\\
&\quad=\sum_{\substack{I\subset N\\ I\neq\emptyset}}\sum_{|\gamma|\le\omega_I}\sum_{|\alpha|\le\omega_I-|\gamma|}\sum_{\mu+\nu=\alpha}\frac{(-)^{|\mu|}}{\mu!\nu!\gamma!}\partial^\mu\delta(I-x)\times\\
&\qquad\times\left(e_I^{\alpha(\gamma)}(\xi)-a_I^{\alpha(\gamma)}(\xi)\right)\sum_{k\in N\setminus I}T\left(N^{(e_k)}\setminus I,\partial^\nu\varphi^\gamma\right)\left(y_{N\setminus I},x\right)i\Delta(y_k-x)+\\
&\qquad+\sum_{\substack{I\subset N\\ I\neq\emptyset}}\sum_{1\le|\gamma|\le\omega_I}\sum_{k\in I}\sum_{|\alpha|\le\omega_I-|\gamma|}\sum_{\mu+\nu=\alpha}\frac{(-)^{|\mu|}}{\mu!\nu!(\gamma-e_k)!}\partial^\mu\delta(I-x)\times\\
&\qquad\times\left(e_I^{\alpha(\gamma)}(\xi)-a_I^{\alpha(\gamma)}(\xi)\right)T\left(N\setminus I,\partial^\nu\varphi^{\gamma-e_k}\right)\left(y_{N\setminus I},x\right)i\Delta(y_k-x).
\end{aligned}\tag{6.39}$$

If we shift the multi index $\gamma \to \gamma + e_k$ in the second term and note that $e_I^{\alpha(\gamma+e_k)} = e_{I^{(e_k)}}^{\alpha(\gamma)}$ by definition of $a_I^{\alpha(\gamma)}$, (6.17), we have:

$$\begin{aligned} &\text{last term of (6.39)} = \\ &= \sum_{\substack{I \subset N \\ I \neq \emptyset}} \sum_{0 \leq |\gamma| \leq \omega_I - 1} \sum_{k \in I} \sum_{|\alpha| \leq \omega_I - |\gamma| - 1} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \\ &\quad \times \left(e_{I^{(e_k)}}^{\alpha(\gamma)}(\xi) - a_{I^{(e_k)}}^{\alpha(\gamma)}(\xi) \right) T\left(N \setminus I, \partial^\nu \varphi^\gamma\right) \left(y_{N \setminus I}, x\right) i\Delta(y_k - x). \end{aligned} \tag{6.40}$$

Then we can sum up both terms and obtain:

$$\begin{aligned} &[\text{last term of (6.15)}, \varphi(z)] = \\ &= \sum_{k=1}^{n} \sum_{\substack{I \subset N^{(e_k)} \\ I \neq \emptyset}} \sum_{|\gamma| \leq \omega_I} \sum_{|\alpha| \leq \omega_I - |\gamma|} \sum_{\mu + \nu = \alpha} \frac{(-)^{|\mu|}}{\mu! \nu! \gamma!} \partial^\mu \delta(I - x) \times \\ &\quad \times \left(e_I^{\alpha(\gamma)}(\xi) - a_I^{\alpha(\gamma)}(\xi) \right) T\left(N^{(e_k)} \setminus I, \partial^\nu \varphi^\gamma\right) \left(y_{N \setminus I}, x\right) i\Delta(y_k - x). \end{aligned} \tag{6.41}$$

Note, that if $k \in I$ the singular order ω_I in (6.41) is automatically lowered by one compared to (6.40) since now $k \in N^{(e_k)}$. In (6.40) the singular order is measured with respect to the set $I \subset N$ without differentiation of the k'th symbol.

This shows the equality of the last term, if we insert (6.15) into **N3'**. The remaining two terms of (6.15) on the LHS of **N3'** commute with φ according to

$$\begin{aligned} &[T(N, \varphi, \varphi)(y_N, x_1, x_2) - \omega_0 \left(T(\varphi, \varphi)(x_1, x_2)\right) T(N)(y_N), \varphi(z)] = \\ &\qquad = \sum_{k=1}^{n} \Big[T\left(N^{(e_k)}, \varphi, \varphi\right) (y_N, x_1, x_2) + \\ &\qquad - \omega_0 \left(T(\varphi, \varphi)(x_1, x_2)\right) T\left(N^{(e_k)}\right) (y_N) \Big] i\Delta(y_k - z) + \\ &\qquad + T(N, \varphi)(y_N, x_1) i\Delta(x_2 - z) + T(N, \varphi)(y_N, x_2) i\Delta(x_1 - z), \end{aligned} \tag{6.42}$$

such that they match the missing terms in the RHS of **N3'**. This finishes the proof.

2. The operator product expansion

In order to derive the interacting normal product $(:\varphi, \varphi:)_{g\mathscr{L}}$ we only have to evaluate the R products which are given in terms of T-products according to

(4.12).

$$\begin{aligned}
&R(N;\varphi,\varphi)(y_N,x_1,x_2) - R\left(N;:\varphi,\varphi:\right)(y_N,x_1,x_2) = \\
&\quad = \sum_{I\subset N} (-)^{|I|}\overline{T}(I)(y_I)\big[T(N\setminus I,\varphi,\varphi)\left(y_{N\setminus I},x_1,x_2\right) + \\
&\quad\quad - T(N\setminus I,:\varphi,\varphi:)\left(y_{N\setminus I},x_1,x_2\right)\big] \\
&\quad = \sum_{I\sqcup J\sqcup K=N}\sum_{\mu,\nu,\gamma} E^{\mu\nu(\gamma)}(I,\varphi,\varphi)(y_I,x,\xi)(-)^{|J|}\overline{T}(J)(y_J)T(K,\partial^\nu\varphi^\gamma)(y_K,x) \\
&\quad = \sum_{I\subset N}\sum_{\mu,\nu,\gamma} E^{\mu\nu(\gamma)}(I,\varphi,\varphi)(y_I,x,\xi)R\left(N\setminus I,\partial^\nu\varphi^\gamma\right)\left(y_{N\setminus I},x\right).
\end{aligned} \tag{6.43}$$

If we now insert into the power series for the interacting fields (4.11) only the $\mu = 0$ coefficients contribute, since $g\restriction_{\mathcal{O}} = const$. We omit this index on the $E^{\mu\nu(\gamma)}$-terms. The RHS of (6.43) is just the n'th order contribution of the product of two power series. So we find the expansion:

$$T\left(\varphi,\varphi\right)_{g\mathscr{L}}(x_1,x_2) = \left(:\varphi,\varphi:\right)_{g\mathscr{L}}(x_1,x_2) + \sum_{|\gamma|\le 2}\sum_{\alpha\le 2-|\gamma|} E^{\alpha(\gamma)}_{g\mathscr{L}}(\xi)\left(\partial^\alpha\varphi^\gamma\right)_{g\mathscr{L}}(x). \tag{6.44}$$

This is the *operator product expansion.* We used the fact, that the maximal singular order $\omega_0\left(T(N,\varphi^2)\right)$ is two in a renormalizable field theory. If the interaction is a sum of Wick monomials $g\cdot\mathscr{L} = \sum_{r=1}^{s} g_r\mathscr{L}_r$, the expansion coefficients read:

$$\begin{aligned}
E^{\alpha(\gamma)}_{g\mathscr{L}}(\xi) = &\sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{r_1,\dots,r_n=1}^{s} g_{r_1}(x)\dots g_{r_n}(x)\,\times \\
&\quad\times \sum_{|\gamma|\le\omega_{r_1,\dots,r_n}}\ \sum_{|\alpha|\le\omega_{r_1,\dots,r_n}-|\gamma|}\Bigg[-a^{\alpha(\gamma)}_{r_1,\dots,r_n}(\xi) + \\
&+\int \mathrm{d}z_N\, z^\alpha\omega_0\left({}^0T\left(\mathscr{L}^{(\gamma_1)}_{r_1},\dots,\mathscr{L}^{(\gamma_n)}_{r_n},:\varphi,\varphi:\right)(z_N,\xi,-\xi)\right)w^{\gamma_1\cdots\gamma_n}_{r_1\dots r_n}(z_N)\Bigg].
\end{aligned} \tag{6.45}$$

As was expected from the general theorem of the unitary equivalence of the local algebras [BF96], the expansion coefficients depend only locally on g.

The operator product expansion (6.44) has the general form:

$$\begin{aligned}
T\left(\varphi,\varphi\right)_{g\mathscr{L}}(x_1,x_2) = \left(:\varphi,\varphi:\right)_{g\mathscr{L}}(x_1,x_2) + E^{0(0)}_{g\mathscr{L}}(\xi)\mathbb{I}_{g\mathscr{L}} + E^{0(1)}_{g\mathscr{L}}(\xi)\varphi_{g\mathscr{L}}(x) + \\
+ E^{\mu(1)}_{g\mathscr{L}}(\xi)\partial_\mu\varphi_{g\mathscr{L}}(x) + E^{0(2)}_{g\mathscr{L}}(\xi)\left(\varphi^2\right)_{g\mathscr{L}}(x).
\end{aligned} \tag{6.46}$$

If we consider a pure φ^4-coupling we have $E^{0(1)}_{g\mathscr{L}} = E^{\mu(1)}_{g\mathscr{L}} = 0$, since the vacuum expectation value of an odd number of fields is zero. In that case the coefficients read

$$E^{0(0)}_{g\mathscr{L}}(\xi) = i\Delta_F(2\xi) + \sum_{n=2}^{\infty} \frac{i^n}{n!} g(x)^n \Big[-a^{0(0)}_{N^{(0)}}(\xi) + \\ + \int \mathrm{d}z_N\, \omega_0 \left({}^0T\left(\mathscr{L},\dots,\mathscr{L},{:}\varphi,\varphi{:}\right)(z_N,\xi,-\xi)\right) w_{N^{(0)}}(z_N) \Big] \quad (6.47)$$

$$E^{0(2)}_{g\mathscr{L}}(\xi) = \sum_{n=1}^{\infty} \frac{i^n}{n!} g(x)^n \sum_{|\gamma|=2} \frac{1}{\gamma_1!\dots\gamma_n!} \Big[-a^{0}_{N^{(\gamma)}}(\xi) + \\ + \int \mathrm{d}z_N\, \omega_0 \left({}^0T\left(\mathscr{L}^{(\gamma_1)},\dots,\mathscr{L}^{(\gamma_n)},{:}\varphi,\varphi{:}\right)(z_N,\xi,-\xi)\right) w_{N^{(\gamma)}}(z_N) \Big]. \quad (6.48)$$

A closer inspection of the two terms reveals that $E^{0(0)}_{g\mathscr{L}}$ contains the terms which appear in the mass and wave function renormalization. If we put $w = 1$ (which is allowed if $m > 0$) and do a resummation over one particle irreducible graphs, we would end up with the usual geometric series found in the literature for the interacting propagator (cf. [IZ85]). In that case all disconnected graphs disappear. But this is only due to that special choice of normalization. The series $E^{0(2)}_{g\mathscr{L}}$ contain the contributions to the renormalization of the coupling constant.

3. Towards the definition of a state

In this section we introduce the idea to define a state on the algebra of local observables with the help of the OPE. We remind the definition of a state ω as a linear normed positive functional in the free field theory. In that case there is also an OPE, namely Wick's theorem (cf. (3.19) – (3.21)). The vacuum state ω_0 on the algebra of observables (of the free field) was defined by $\omega_0({:}A{:}) = 0$ and $\omega_0(\mathbb{I}) = 1$. Since Wick's theorem allows to expand any observable into a series of Wick polynomials the state is uniquely defined.

In the interacting field theory the fields are (operator valued distributional) formal power series in g. Denote by $\mathbb{C}_g$ the formal power series in g with complex coefficients. Following [DF99] a state $\omega_{g\mathscr{L}}$ on $\mathfrak{A}_{g\mathscr{L}}(\mathcal{O})$ is a mapping:

$$\omega_{g\mathscr{L}} : \mathfrak{A}_{g\mathscr{L}}(\mathcal{O}) \mapsto \mathbb{C}_g, \quad (6.49)$$

$$\omega_{g\mathscr{L}}\left(a_g A_{g\mathscr{L}} + B_{g\mathscr{L}}\right) = a_g \omega_{g\mathscr{L}}(A_{g\mathscr{L}}) + \omega_{g\mathscr{L}}(B_{g\mathscr{L}}), \quad a_g \in \mathbb{C}_g \quad (6.50)$$

$$\omega_{g\mathscr{L}}\left(A^*_{g\mathscr{L}}\right) = \overline{\omega_{g\mathscr{L}}\left(A_{g\mathscr{L}}\right)} \quad (6.51)$$

$$\omega_{g\mathscr{L}}(\mathbb{I}_{g\mathscr{L}}) = 1, \quad (6.52)$$

$$\omega_{g\mathscr{L}}\left(A^*_{g\mathscr{L}} A_{g\mathscr{L}}\right) \geq 0, \quad (6.53)$$

$A_{g\mathscr{L}}, B_{g\mathscr{L}} \in \mathfrak{A}_{g\mathscr{L}}(\mathcal{O})$. Inspired by the definition of the vacuum state for the free field algebra we set

$$\omega_{g\mathscr{L}}(({:}A{:})_{g\mathscr{L}}) = 0, \quad (6.54)$$

$$\omega_{g\mathscr{L}}(\mathbb{I}_{g\mathscr{L}}) = 1. \quad (6.55)$$

Unfortunately we do not have an OPE for the general product of interacting fields and time ordered products of them. But we can already check, if the above criteria hold in our case.

We consider pure φ^4-interaction. Since $(:\varphi,\varphi:)^*_{g\mathscr{L}} = (:\varphi,\varphi:)_{g\mathscr{L}}$ on $\mathcal{D}$ the adjoint of (6.46) is:

$$\overline{T}\,(\varphi,\varphi)_{g\mathscr{L}}\,(x_1,x_2) = (:\varphi,\varphi:)_{g\mathscr{L}}\,(x_1,x_2) + \overline{E^{0(0)}_{g\mathscr{L}}(\xi)}\mathbb{I}_{g\mathscr{L}} + \overline{E^{0(2)}_{g\mathscr{L}}(\xi)}\,\left(\varphi^2\right)_{g\mathscr{L}}(x). \tag{6.56}$$

The singular order of $E^{0(0)}$ is 2 and of $E^{0(2)}$ is 0. Therefore we can form

$$\varphi_{g\mathscr{L}}(x_1)\varphi_{g\mathscr{L}}(x_2) = \\ = \theta(x_1^0 - x_2^0)T\,(\varphi,\varphi)_{g\mathscr{L}}\,(x_1,x_2) + \theta(x_2^0 - x_1^0)\overline{T}\,(\varphi,\varphi)_{g\mathscr{L}}\,(x_2,x_1). \tag{6.57}$$

Then the interacting two point function is given by

$$\omega_{2g\mathscr{L}}(x_1,x_2) \doteq \omega_{g\mathscr{L}}(\varphi_{g\mathscr{L}}(x_1)\varphi_{g\mathscr{L}}(x_2)) = \theta(\xi^0)E^{0(0)}_{g\mathscr{L}}(\xi) + \theta(-\xi^0)\overline{E^{0(0)}_{g\mathscr{L}}(\xi)}. \tag{6.58}$$

For the notion of positivity we refer to the work [DF99]. A formal power series $\mathbb{C}_g \ni b_g = \sum_n b_n g^n$ is defined to be positive if it can be written as the square of another power series $b_g = \overline{c_g}c_g$. This is equivalent to the conditions: $b_n \in \mathbb{R}, \forall n \in \mathbb{N}_0$ and for the first non vanishing b_l it is required that $b_l > 0$ and l is even. So for $f \in \mathcal{D}(\mathbb{M})$ we have:

$$\begin{aligned}\omega_{2,g\mathscr{L}}(\overline{f},f) &= \\ &= \int \mathrm{d}x\,\mathrm{d}y \left(\theta(x^0-y^0)E^{0(0)}_{g\mathscr{L}}(x-y) + \theta(y^0-x^0)\overline{E^{0(0)}_{g\mathscr{L}}(x-y)}\right)\overline{f(x)}f(y) \\ &= 2\int \mathrm{d}x\,\mathrm{d}y\ \mathrm{Re}\ \theta(x^0-y^0)E^{0(0)}_{g\mathscr{L}}(x-y)\overline{f(x)}f(y),\end{aligned} \tag{6.59}$$

where we exchanged $x \leftrightarrow y$ in the second integral and used the fact that $E^{0(0)}_{g\mathscr{L}}$ is even. The first non vanishing contribution comes from the free two point function which is positive, cf. (2.22). Under the above criterion our two point function is positive.

4. Remarks

Our OPE is a consequence of the definition of our normal product of two fields. The definition for other but scalar fields can be derived straight forward by the applied methods. A generalization to composed fields requires a modification of the terms arising by x_1, x_2-contractions. Unfortunately, we have not succeeded in finding a definition for higher order (> bi) normal products.

On the other hand the bi-OPE already contains all contributions which are necessary for the definition of a mass-, wavefunction- and coupling normalization, which up to now always requires the adiabatic limit [Sch95], [EG73]. For the first two of them this should be possible by a suitable condition on the measure that one can derive from the Jost-Lehmann-Dyson representation of our (time ordered) two point function.