CHAPTER 5

The energy momentum tensor

In classical field theory any symmetry of the Lagrangian generates a conserved current by the Noether procedure. If the Lagrangian is not invariant but only shifts by a divergence the same procedure still applies. The current which is associated to a translation of the fields is the energy momentum tensor (EMT). If we have a localized interaction translation invariance is obviously broken and the EMT is conserved only where the localization function is constant. But this is already enough in view of our interacting observable algebras.

The Lagrangian possesses a further symmetry, namely scale invariance, if no dimensionful couplings are present. Callan, Coleman and Jackiw have shown [CCJ70] that in this situation an *improved* EMT can be defined by addition of a conserved improvement tensor. This improved tensor is also traceless. Contraction of this tensor with x defines the conserved *dilatation* current reflecting scale invariance of the Lagrangian.

We pursue another way of defining the improved tensor on the example of the massless scalar field theory: The equations of motion do not fix the Lagrangian unambiguously. We find the improved tensor as the EMT of an improved Lagrangian. A similar derivation was also given by Kasper [Kas81]. Since the improvement tensor is strictly conserved, the improved EMT is only conserved up to the breaking term of the canonical one, related to the localization of the interaction. This term also causes a breaking of the dilatation current. We discuss this classical situation in section 1.

With the classical preliminaries we study the quantum theory. In section 2 we analyze the canonical EMT for a family of theories, where the free field equation is at least of second order and the interaction contains no derivated fields. We find that exactly the same conservation equation like for the classical fields can be fulfiled if we impose a further normalization condition called *Ward identity*. The Ward identity requires a suitable normalization of T-products involving the canonical free energy momentum tensor (as a symbol $\in \mathfrak{B}$). We show that the Ward identity can always be satisfied in section 3 by using the inductive method of Dütsch and Fredenhagen in [DF99]. Our result coincides with a similar result that was derived for the energy momentum tensor in the framework of Zimmermann's normal product quantization [Zim73a] by Lowenstein [Low71] and also by Zimmermann [Zim84]. We show that the interacting momentum operator (as the corresponding charge) implements the right commutation relation with the interacting fields.

In their paper [CJ71] Coleman and Jackiw argued that the trace of the improved energy momentum tensor generates an anomaly in the perturbative interacting quantum theory. Lowenstein has verified this statement for Zimmermann's normal products in [Low71]. Later, Zimmermann has given a derivation of this anomaly in [Zim84]. He verified a conjecture by Minkowski [Min76] that it can be normalized to be proportional to the β-function of the Callan-Symanzik equation. This statement already contains a result of Schroer [Sch71] that the anomaly vanishes if the coupling is a zero of the β-function. A more comprehensive result, also covering possible conformal anomalies, was given by Kraus and Sibold [KS92, KS93] using the framework of algebraic renormalization.

In accordance to these results we show in section 5 that a conserved (up to the expected ∂g breaking) improved EMT inevitably leads to the trace anomaly in φ^4-theory. The definition of the improvement tensor requires a new relation between interacting fields induced by a corresponding Ward identity which is proved in section 6. The simultaneous validity of both Ward identities forces the trace anomaly to be present. The EMT and its improved counterpart both define the same momentum operator. The breaking of dilatations given by the trace leads to anomalous contributions to the dimension of the interacting fields described in section 7. We mention that dilatations were also quite recently studied in local perturbation theory by Grigore [Gri00]. But his main focus is on the S-matrix whereas we focus on the interacting fields.

1. The energy momentum tensor in classical field theory

We discuss the EMT in a classical field theory. The Lagrangian depends on the fields $\phi_j^{\text{class}}, j = 1, \ldots, r$ and their first and second derivatives. We assume that it also depends on x explicitely via a coupling term $-g\mathscr{L}_{\text{int}}^{\text{class}}$ which is assumed to contain no derivated fields: $\mathscr{L}^{\text{class}} = \mathscr{L}^{\text{class}}(\phi_l^{\text{class}}, \phi_{l,\mu}^{\text{class}}, \phi_{l,\mu\nu}^{\text{class}}, x)$. Then the Euler-Lagrange equations read:

$$\partial_\mu \partial_\nu \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_{l,\mu\nu}^{\text{class}}} - \partial_\mu \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_{l,\mu}^{\text{class}}} + \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_l^{\text{class}}} = 0. \tag{5.1}$$

The EMT is the current associated to a spacetime translation of the fields: $\phi^{\text{class}}(x) \to \phi^{\text{class}}(x+a)$. By the Noether procedure we find the EMT to be:

$$\Theta^{\text{class}\,\mu\nu} = \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_{l,\mu}^{\text{class}}} \partial^\nu \phi_l^{\text{class}} - \left(\partial_\rho \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_{l,\mu\rho}^{\text{class}}} \right) \partial^\nu \phi_l^{\text{class}} + \\ + \frac{\partial \mathscr{L}^{\text{class}}}{\partial \phi_{l,\mu\rho}^{\text{class}}} \partial_\rho \partial^\nu \phi_l^{\text{class}} - \eta^{\mu\nu} \mathscr{L}^{\text{class}}. \tag{5.2}$$

In this situation the "conservation" equation reads:

$$\partial_\mu \Theta^{\text{class}\,\mu\nu} = \partial^\nu g \mathscr{L}_{\text{int}}^{\text{class}}. \tag{5.3}$$

In the following we investigate two specific models:

1.1. The general first order model. In this subsection we restrict ourselves to the case that there are no twice derivated fields present. The free Lagrangian is quadratic in the fields $\phi_j^{\text{class}}, \phi_{j,\mu}^{\text{class}}, j = 1, \ldots, r$. The interaction is given by the term above (containing no derivated fields).

$$\mathscr{L}^{\text{class}} = \mathscr{L}_0^{\text{class}} - g\mathscr{L}_{\text{int}}^{\text{class}} \tag{5.4}$$

$$\mathscr{L}_0^{\text{class}} = \frac{1}{2}K_{jl}^{\mu\nu}\partial_\mu\phi_j^{\text{class}}\partial_\nu\phi_l^{\text{class}} + \frac{1}{2}G_{jl}^{\mu}\partial_\mu\phi_j^{\text{class}}\phi_l^{\text{class}} - \frac{1}{2}M_{jl}\phi_j^{\text{class}}\phi_l^{\text{class}} \tag{5.5}$$

with $K_{jl}^{\mu\nu}, G_{jl}^{\mu}, M_{lj} \in \mathbb{C}$. The K, G and M are supposed to possess the following symmetries: $K_{jl}^{\mu\nu} = K_{lj}^{\nu\mu} = K_{lj}^{\mu\nu}, G_{jl}^{\mu} = -G_{lj}^{\mu}$ and $M_{jl} = M_{lj}$. The Euler-Lagrange equations read

$$(K_{jl}^{\mu\nu}\partial_\mu\partial_\nu + G_{jl}^{\mu}\partial_\mu + M_{jl})\phi_l^{\text{class}} \doteq D_{jl}\phi_l^{\text{class}} = -g\frac{\partial\mathscr{L}_{\text{int}}^{\text{class}}}{\partial\phi_j^{\text{class}}}. \tag{5.6}$$

The EMT defined by (5.2) is called the *canonical* EMT. It is given by

$$\Theta_{\text{can}}^{\text{class}\,\mu\nu} = \Theta_{0\,\text{can}}^{\text{class}\,\mu\nu} - \eta^{\mu\nu}g\mathscr{L}_{\text{int}}^{\text{class}} \tag{5.7}$$

$$\begin{aligned}\Theta_{0\,\text{can}}^{\text{class}\,\mu\nu} &= K_{lk}^{\mu\rho}\partial_\rho\phi_l^{\text{class}}\partial^\nu\phi_k^{\text{class}} + \frac{1}{2}G_{lk}^{\mu}\partial^\nu\phi_l^{\text{class}}\phi_k^{\text{class}} + \\ &\quad - \frac{1}{2}\eta^{\mu\nu}\left(K_{lk}^{\rho\sigma}\partial_\rho\phi_l^{\text{class}}\partial_\sigma\phi_k^{\text{class}} + G_{lk}^{\rho}\partial_\rho\phi_l^{\text{class}}\phi_k^{\text{class}} - M_{lk}\phi_l^{\text{class}}\phi_k^{\text{class}}\right).\end{aligned} \tag{5.8}$$

In section 2 we show that the "conservation" equation (5.3) can also be fulfiled in the interacting quantum field theory.

1.2. The massless $(\phi^{\text{class}})^4$-theory. If no dimensionful couplings are present, it is always possible to construct a conserved and traceless EMT. This tensor is called the *improved* EMT and was first introduced by Callan, et. al. in [CCJ70]. We derive this tensor in $(\phi^{\text{class}})^4$-theory as the EMT of an improved Lagrangian making use of the ambiguity in the definition of the Lagrangian. A derivation of this kind was already performed by Kasper [Kas81].

The equations of motion read:

$$\Box\phi^{\text{class}} = -g\frac{\partial\mathscr{L}_{\text{int}}^{\text{class}}}{\partial\phi^{\text{class}}}. \tag{5.9}$$

Since a total derivative in the Lagrangian does not change the equations of motion they originate from both following expressions:

$$\mathscr{L}_{\text{can}}^{\text{class}} = \frac{1}{2}\partial_\rho\phi^{\text{class}}\partial^\rho\phi^{\text{class}} - g\mathscr{L}_{\text{int}}^{\text{class}}, \tag{5.10}$$

$$\mathscr{L}_{\text{imp}}^{\text{class}} = \frac{1}{6}\partial_\rho\phi^{\text{class}}\partial^\rho\phi^{\text{class}} - \frac{1}{3}\phi^{\text{class}}\Box\phi^{\text{class}} - g\mathscr{L}_{\text{int}}^{\text{class}}. \tag{5.11}$$

The corresponding EMT's are called the canonical and the improved one, respectively. The first one already follows from the last subsection:

$$\Theta_{\rm can}^{{\rm class}\,\mu\nu} = \partial^\mu\phi^{\rm class}\partial^\nu\phi^{\rm class} - \frac{1}{2}\eta^{\mu\nu}\partial_\rho\phi^{\rm class}\partial^\rho\phi^{\rm class} + g\eta^{\mu\nu}\mathscr{L}_{\rm int}^{\rm class}, \tag{5.12}$$

$$\Theta_{\rm imp}^{{\rm class}\,\mu\nu} = \frac{2}{3}\partial^\mu\phi^{\rm class}\partial^\nu\phi^{\rm class} - \frac{1}{3}\phi^{\rm class}\partial^\mu\partial^\nu\phi^{\rm class} + \tag{5.13}$$

$$- \frac{1}{6}\eta^{\mu\nu}\partial_\rho\phi^{\rm class}\partial^\rho\phi^{\rm class} + \frac{1}{12}\eta^{\mu\nu}\phi^{\rm class}\Box\phi^{\rm class} \tag{5.14}$$

$$= \Theta_{\rm can}^{{\rm class}\,\mu\nu} - \frac{1}{3}I^{{\rm class}\,\mu\nu}, \tag{5.15}$$

where we have introduced the conserved improvement tensor

$$I^{{\rm class}\,\mu\nu} = \partial^\mu\phi^{\rm class}\partial^\nu\phi^{\rm class} + \phi^{\rm class}\partial^\mu\partial^\nu\phi^{\rm class} +$$

$$- \eta^{\mu\nu}\partial_\rho\phi^{\rm class}\partial^\rho\phi^{\rm class} - \eta^{\mu\nu}\phi^{\rm class}\Box\phi^{\rm class} \tag{5.16}$$

$$= \frac{1}{2}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box\right)(\phi^{\rm class})^2. \tag{5.17}$$

Contracting the indices, we find $\eta_{\mu\nu}\Theta_{\rm imp}^{{\rm class}\,\mu\nu} = 0$. The improved tensor gives rise to the dilatation current:

$$D^{{\rm class}\,\mu} \doteq x_\nu\Theta_{\rm imp}^{{\rm class}\,\mu\nu}. \tag{5.18}$$

Its conservation equation reads:

$$\partial_\mu D^{{\rm class}\,\mu} = \eta_{\mu\nu}\Theta_{\rm imp}^{{\rm class}\,\mu\nu} + x_\nu\partial_\mu\Theta_{\rm imp}^{{\rm class}\,\mu\nu} = x^\mu\partial_\mu g\mathscr{L}_{\rm int}^{\rm class}. \tag{5.19}$$

The dilatation current is the Noether current corresponding to the scaling $\phi^{\rm class}(x) \to e^{d\alpha}\phi^{\rm class}(e^\alpha x)$ with $d = 1$ of the improved Lagrangian (5.11). On the other hand we can derive the dilatations from the canonical Lagrangian (5.10). In this case we find:

$$\widetilde{D}^{{\rm class}\,\mu} = x_\nu\Theta_{\rm can}^{{\rm class}\,\mu\nu} + \phi^{\rm class}\partial^\mu\phi^{\rm class}, \tag{5.20}$$

with the same conservation equation. The zero component of the difference is a divergence w.r.t. the space coordinates and therefore does not contribute to the charge:

$$\widetilde{D}^{{\rm class}\,0} - D^{{\rm class}\,0} = \frac{1}{3}\partial_j(x^{[j}\partial^{0]})(\phi^{\rm class})^2. \tag{5.21}$$

In section 5 the corresponding QFT is considered.

2. The canonical quantum energy momentum tensor

We now discuss the perturbative quantum fields which are associated to the canonical EMT. While for the free theory all equations from classical field theory can be carried over due to the Wick ordering procedure, the interacting quantum fields require a special normalization which is implied by a Ward identity. This identity enables to conserve the classical structure also in the quantum theory.

2.1. Free quantum theory. The classical fields from the last section may now serve as the symbols from our auxiliary variable algebra $\mathfrak{B}$. For our corresponding quantum fields φ_j we assume the same equations of motion to be satisfied

$$D_{jl}T(\varphi_l) = 0. \tag{5.22}$$

By investigation of the K, G and M we find that this covers a lot of equations like Klein-Gordon and Dirac equation. The commutator

$$[T(\varphi_j)(x), T(\varphi_k)(y)] = i\Delta_{jk}(x-y), \tag{5.23}$$

may therefore be an anticommutator in the case of Fermi fields. Now we regard $\Theta^{\mu\nu}_{0\,\mathrm{can}} \in \mathfrak{B}$ from (5.8) as a symbol ($\phi^{\mathrm{class}} \to \varphi$):

$$\Theta^{\mu\nu}_{0\,\mathrm{can}} = K^{\mu\rho}_{lk}\partial_\rho\varphi_l\partial^\nu\varphi_k + \frac{1}{2}G^{\mu}_{lk}\partial^\nu\varphi_l\varphi_k + \\ - \frac{1}{2}\eta^{\mu\nu}\left(K^{\rho\sigma}_{lk}\partial_\rho\varphi_l\partial_\sigma\varphi_k + G^{\rho}_{lk}\partial_\rho\varphi_l\varphi_k - M_{lk}\varphi_l\varphi_k\right). \tag{5.24}$$

Then $T(\Theta^{\mu\nu}_{0\,\mathrm{can}}) =: \Theta^{\mu\nu}_{0\,\mathrm{can}}:$ defines the free canonical quantum EMT. Since the equations of motion hold inside the Wick ordering we maintain the conservation

$$\partial_\mu :\Theta^{\mu\nu}_{0\,\mathrm{can}}:= 0. \tag{5.25}$$

2.2. Interacting quantum theory. Applying the framework of perturbative interacting fields introduced in the last chapter we investigate the consequences of switching on an interaction. We assume that the interaction $\mathscr{L}$ contains no derivated fields. Corresponding to the free canonical EMT we construct the interacting counterpart. According to (4.10) we have:

$$\Theta^{\mu\nu}_{0\,\mathrm{can} g\mathscr{L}}(x) \doteq \sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}y_1\ldots\mathrm{d}y_n R(g\mathscr{L},\ldots,g\mathscr{L};\Theta^{\mu\nu}_{0\,\mathrm{can}})(y_1,\ldots,y_n;x). \tag{5.26}$$

But this is only the part corresponding to the free fields. The total tensor $\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}}$ receives another contribution from the interaction (cp. (5.7)):

$$\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}} \doteq \Theta^{\mu\nu}_{0\,\mathrm{can} g\mathscr{L}} + \eta^{\mu\nu} g\mathscr{L}_{g\mathscr{L}}. \tag{5.27}$$

Since g is of compact support global translation invariance is broken. Hence we expect the conservation equation to be satisfied that takes account of the non-invariance of the coupling function (cp. (5.3)):

$$\partial_\mu\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}} = \partial^\nu g\,\mathscr{L}_{g\mathscr{L}}. \tag{5.28}$$

If this equation is true, we have local conservation on $\mathcal{O}$:[1]

$$\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}}(\partial_\mu f) = 0,\ \forall f \text{ with } \operatorname{supp} f \subset \mathcal{O}. \tag{5.29}$$

Equation (5.28) is the main statement. We now give a formulation in terms of the perturbative contributions. It comes out that the conservation can be completely discussed on the level of T-products. The corresponding equation is a Ward identity involving the free canonical EMT.

[1]We have assumed $g\restriction_{\mathcal{O}} = const \Rightarrow \partial_\mu g\restriction_{\mathcal{O}} = 0$.

Inserting the definition of $\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}}$ from (5.27) we see that (5.28) is equivalent to

$$\partial_\mu \Theta^{\mu\nu}_{0\,\mathrm{can} g\mathscr{L}} = -g\,\partial^\nu \mathscr{L}_{g\mathscr{L}}. \tag{5.30}$$

We expand this into the formal power series in the coupling. The RHS becomes

$$\begin{aligned}
&- g\partial^\nu \mathscr{L}_{g\mathscr{L}}(x) = \\
&= -g(x) \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_n \, \partial^\nu_x R\,(\mathscr{L},\ldots,\mathscr{L};\mathscr{L})\,(y_1,\ldots,y_n;x)\,g(y_1)\ldots g(y_n) \\
&= i \sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_{n+1}\, g(y_1)\ldots g(y_{n+1}) \times \\
&\quad \times \frac{1}{n+1} \sum_{k=1}^{n+1} \partial^\nu_x R\,(\mathscr{L},\ldots,\not{k},\ldots,\mathscr{L};\mathscr{L})\,(y_1,\ldots,\not{k},\ldots,y_{n+1};x)\delta(y_k - x) \\
&= i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_n\, g(y_1)\ldots g(y_n) \times \\
&\quad \times \sum_{k=1}^{n} \partial^\nu_x R\,(\mathscr{L},\ldots,\not{k},\ldots,\mathscr{L};\mathscr{L})\,(y_1,\ldots,\not{k},\ldots,y_n;x)\delta(y_k - x).
\end{aligned} \tag{5.31}$$

The expansion of $\Theta^{\mu\nu}_{0\,\mathrm{can} g\mathscr{L}}$ was already given at the beginning. Then (5.30) is fulfiled if

$$\begin{aligned}
&\partial^x_\mu R\,(\mathscr{L},\ldots,\mathscr{L};\Theta^{\mu\nu}_{0\,\mathrm{can}})\,(y_1,\ldots,y_n;x) = \\
&\qquad = i \sum_{k=1}^{n} \partial^\nu_x R\,(\mathscr{L},\ldots,\not{k},\ldots,\mathscr{L};\mathscr{L})\,(y_1,\ldots,\not{k},\ldots,y_n;x)\delta(y_k - x)
\end{aligned} \tag{5.32}$$

is satisfied to all orders. The R-products are completely determined in terms of the T-products (4.12). We show that it is sufficient to prove the following Ward identity:[2]

$$\begin{aligned}
&\partial^x_\mu T\,(W_1,\ldots,W_n,\Theta^{\mu\nu}_{0\,\mathrm{can}})\,(y_1,\ldots,y_n,x) = \\
&\qquad = i \sum_{k=1}^{n} \delta(y_k - x)\partial^\nu_k T\,(W_1,\ldots,W_n)\,(y_1,\ldots,y_n),
\end{aligned} \tag{\textbf{WI 1}}$$

for all $W_i \in \mathfrak{B}$ which are (not necessarily proper) sub monomials of the coupling $\mathscr{L}$. We therefore prove the statement for all $W_i \in \mathfrak{B}$ that contain no derivated fields and have $\dim \leq 4$.

[2]The index on the derivative on the RHS refers to the respective y-coordinate.

The Ward identity **WI 1** can be integrated to the functional equation:

$$\partial_\mu^x \frac{\delta}{i\delta f_{\mu\nu}(x)} S\Big(g\mathscr{L} + f\cdot\Theta_{0\,\mathrm{can}} + \sum_{j=1}^{s} h_j W_j\Big)\Big|_{f=0} =$$
$$= -\Big(g(x)\partial_x^\nu \frac{\delta}{i\delta g(x)} + \sum_{j=1}^{s} h_j(x)\partial_x^\nu \frac{\delta}{i\delta h_j(x)}\Big) S\Big(g\mathscr{L} + \sum_{j=1}^{s} h_j W_j\Big). \quad (5.33)$$

Multiplying with $S(g\mathscr{L})^{-1}$ from the left and expanding in powers of the coupling yields (5.32). Using

$$\frac{\delta}{i\delta f} S(g\mathscr{L} + f\cdot\Theta_{0\,\mathrm{can}})^{-1}\Big|_{f=0} = -S(g\mathscr{L})^{-1}\frac{\delta}{i\delta f} S(g\mathscr{L} + f\cdot\Theta_{0\,\mathrm{can}})\Big|_{f=0} S(g\mathscr{L})^{-1} \quad (5.34)$$

we find that (5.33) also holds for the inverse functional. This implies the corresponding Ward identity for the $\overline{T}$-products to have a minus sign. A simple calculation also shows that (5.33) implies

$$\partial_\mu^x \frac{\delta}{i\delta f_{\mu\nu}(x)} S_{g\mathscr{L}+f\cdot\Theta_{0\,\mathrm{can}}}(hW)\Big|_{f=0} =$$
$$= -\Big(g(x)\partial_x^\nu \frac{\delta}{i\delta g(x)} + h(x)\partial_x^\nu \frac{\delta}{i\delta h(x)}\Big) S_{g\mathscr{L}}(hW). \quad (5.35)$$

Expanding this equation in n'th order g and first order h results in the following identity for the R-products:

$$\partial_\mu^{x_1} R\,(N, \Theta_{0\,\mathrm{can}}^{\mu\nu}; W)\,(y_N, x_1; x_2) = i\sum_{k\in N}\delta(y_k - x_1)\partial_k^\nu R(N;W)\,(y_N;x_2) +$$
$$+\, i\delta(x_2 - x_1)\partial_{x_2}^\nu R(N;W)\,(y_N;x_2)\,. \quad (5.36)$$

The next section gives a proof of **WI 1**.

3. Proof of the Ward identity

Dütsch and Fredenhagen have presented a very general framework for proving a Ward identity of this kind in [DF99]. It was generalized by Boas [Boa99] in the presence of derivated fields. This is our situation here and we apply their methods. The strategy is as follows: A possible violation of the Ward identity is called an *anomaly*. Since all T-products are supposed to fulfil the normalization conditions **N0** – **N4** we perform a double induction, one over n and one over the degree (number of generators) of the Wick monomials. We assume that the anomaly in both lower orders is zero.

- **Step 1.** The commutator of the anomaly with the free fields vanishes. Therefore it can appear only in the vacuum sector.
- **Step 2.** If one W_i is a generator (which is the lowest degree sub monomial), the anomaly vanishes due to **N4**.
- **Step 3.** Because of **N1** and the induction on n, the anomaly is a Poincaré covariant $\mathbb{C}$-number distribution with support on the total diagonal. We

show that it vanishes by an appropriate normalization, i.e. adding a δ-polynomial with the right symmetry properties (**N0**).[3]

To save some space we use the short hand notations from above. We define the anomaly a by:

$$a^{\nu}(x, y_N) \doteq \partial_{\mu}^{x} T\left(\Theta_{0\,\mathrm{can}}^{\mu\nu}, N\right)(x, y_N) - i \sum_{k=1}^{n} \delta(y_k - x) \partial_k^{\nu} T(N)(y_N). \tag{5.37}$$

Step 1. We commute the anomaly with the free fields.[4] We use a double induction, one on n and the other on the degree of the Wick sub monomials. Using (**N3**) we need the sub monomials of $\Theta_{0\,\mathrm{can}}^{\mu\nu}$:

$$\frac{\partial \Theta_{0\,\mathrm{can}}^{\mu\nu}}{\partial \varphi_j} = -\frac{1}{2} G_{jl}^{\mu} \partial^{\nu} \varphi_l + \frac{1}{2} \eta^{\mu\nu} G_{jl}^{\rho} \partial_{\rho} \varphi_l + \eta^{\mu\nu} M_{jl} \varphi_l \tag{5.38}$$

$$\frac{\partial \Theta_{0\,\mathrm{can}}^{\mu\nu}}{\partial \varphi_{j,\rho}}, = K_{jl}^{\mu\rho} \partial^{\nu} \varphi_l + 2\eta^{\nu(\rho} K_{jl}^{\mu)\sigma} \partial_{\sigma} \varphi_l + \eta^{\nu(\rho} G_{jl}^{\mu)} \varphi_l. \tag{5.39}$$

We explicitely distinguished between the basic generators and the first order ones. Here and in the following the sums only run over the basic generators therefore. Equations (5.38),(5.39) are linear in the fields. We demand T-products containing once derivated basic generators to fulfil the following normalization:[5]

$$\partial_{\mu}^{x} T(\varphi_j, N)(x, y_N) = T\left(\partial_{\mu} \varphi_j, N\right)(x, y_N). \tag{5.40}$$

This translates into

$$\partial_{\mu} \varphi_{j\, g\mathscr{L}}(x) = \left(\partial_{\mu} \varphi_j\right)_{g\mathscr{L}} \tag{5.41}$$

---

[3] If the current itself is a sub monomial of the coupling, this may lead to non trivial conditions like in [DF99].

[4] We use the symbols $\varphi_{l,\mu}$ and $\partial_{\mu}\varphi_l$ synonymously.

[5] Because of (3.60) this means $\omega_0(T(\varphi_{j,\mu}, \varphi_k)(x,y)) = \partial_{\mu}^{x} \omega_0(T(\varphi_j, \varphi_k)(x,y))$. If the fields φ_j, φ_k are bosonic with mass dimension 1 this is automatically fulfiled because of the negative singular order. In the case of two Fermi fields with mass dimension $\frac{3}{2}$ and non vanishing anticommutator, e.g. $\psi, \overline{\psi}$ we have: $\omega_0(T(\psi_{,\mu}, \overline{\psi})(x,y)) = i\partial_{\mu} S^F(x-y) + c\gamma_{\mu}\delta(x-y)$. The normalization (5.40) requires $c = 0$.

for the interacting fields. We calculate the following relevant term:

$$
\begin{aligned}
&\partial_\mu^x\Bigg\{T\left(\frac{\partial\Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial\varphi_j},N\right)(x,y_N)\,\Delta_{ji}(x-z)+T\left(\frac{\partial\Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial\varphi_{j,\rho}},N\right)(x,y_N)\,\partial_\rho\Delta_{ji}(x-z)\Bigg\} =\\
&\quad= \partial_\mu^x\Bigg\{\Delta_{ji}(x-z)\left[-\frac{1}{2}G^\mu_{jl}\partial^\nu_x+\frac{1}{2}\eta^{\mu\nu}G^\rho_{jl}\partial^x_\rho+\eta^{\mu\nu}M_{jl}\right]+\\
&\qquad+\partial_\rho\Delta_{ji}(x-z)\Bigg[K^{\mu\rho}_{jl}\partial^\nu_x+\eta^{\nu\rho}K^{\mu\sigma}_{jl}\partial^x_\sigma-\eta^{\mu\nu}K^{\rho\sigma}_{jl}\partial^x_\sigma+\\
&\qquad+\frac{1}{2}\eta^{\nu\rho}G^\mu_{jl}-\frac{1}{2}\eta^{\mu\nu}G^\rho_{jl}\Bigg]\Bigg\}T(\varphi_l,N)\,(x,y_N)\\
&\quad=\Big(-\partial_\mu\Delta_{ji}(x-z)G^\mu_{jl}\partial^\nu_x+M_{jl}\Delta_{ji}(x-z)\partial^\nu_x+\\
&\qquad+\partial_\mu\partial_\rho\Delta_{ji}(x-z)K^{\mu\rho}_{jl}\partial^\nu_x+\partial^\nu\Delta_{ji}(x-z)M_{jl}+\\
&\qquad+\partial^\nu\Delta_{ji}(x-z)G^\mu_{jl}\partial^x_\mu+\partial^\nu\Delta_{ji}(x-z)K^{\mu\sigma}_{jl}\partial^x_\mu\partial^x_\sigma\Big)T(\varphi_l,N)\,(x,y_N)\\
&\quad=\left(D_{lj}\Delta_{ji}(x-z)\partial^\nu_x+\partial^\nu\Delta_{ji}(x-z)D^x_{jl}\right)T(\varphi_l,N)\,(x,y_N)\\
&\quad=\partial^\nu\Delta_{ji}(x-z)D^x_{jl}T(\varphi_l,N)\,(x,y_N)\,,
\end{aligned}
\tag{5.42}
$$

since Δ is a solution of the free field equation. Now we commute the anomaly with the free basic fields and use **N3**:

$$
\begin{aligned}
&[a^\nu\,(x,y_N)\,,\varphi_i(z)]=\\
&\quad=\partial^x_\mu\left[T(\Theta^{\mu\nu}_{0\,\mathrm{can}},N)\,(x,y_N)\,,\varphi_i(z)\right]+\\
&\qquad-i\sum_{k=1}^n\delta(y_k-x)\partial^\nu_k\left[T(N)\,(y_N)\,,\varphi_i(z)\right]\\
&\quad=i\partial^x_\mu\Bigg\{T\left(\frac{\partial\Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial\varphi_j},N\right)(x,y_N)\,\Delta_{ji}(x-z)+\\
&\qquad+T\left(\frac{\partial\Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial\varphi_{j,\rho}},N\right)(x,y_N)\,\partial_\rho\Delta_{ji}(x-z)\Bigg\}+\\
&\qquad+i\sum_{l=1}^n\partial^x_\mu T\left(\Theta^{\mu\nu}_{0\,\mathrm{can}},W_1\ldots,\frac{\partial W_l}{\partial\varphi_j},\ldots,W_n\right)(x,y_N)\,\Delta_{ji}(y_l-z)+\\
&\qquad+\sum_{k=1}^n\delta(y_k-x)\partial^\nu_k\left\{\sum_{l=1}^n T\left(W_1,\ldots,\frac{\partial W_l}{\partial\varphi_j},\ldots,W_n\right)(y_N)\,\Delta_{ji}(y_l-z)\right\}.
\end{aligned}
\tag{5.43}
$$

If the derivative on the last line acts on T it cancels the third line by the induction hypothesis. Only the term with $l=k$ and the derivative on Δ remains.

The δ function allows to put $y_k = x$ and by inserting (5.42) we end up with

$$
\begin{aligned}
&= \partial^\nu \Delta_{ji}(x-z) \Big[i D^x_{jl} T(\varphi_l, N)\,(x, y_N) + && (5.44)\\
&\quad + \sum_{k=1}^{n} \delta(y_k - x) T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right)\Big] && (5.45)\\
&= 0, && (5.46)
\end{aligned}
$$

because of (**N4**). Since the Ward identity commutes with all free fields the anomaly is a $\mathbb{C}$-number. Because of the causal Wick expansion (3.55) it can only appear in the vacuum sector:

$$
a^\nu(x, y_N) = \langle \Omega, a^\nu(x, y_N)\Omega \rangle\,. \tag{5.47}
$$

Step 2. We prove that the Ward identity is compatible with the normalization condition (**N4**). Using the equivalent formulation (3.60)

$$
\begin{aligned}
&\langle \Omega, T(N, \varphi_i)\,(y_N, z)\,\Omega \rangle = \\
&\quad = i \sum_{k=1}^{n} \Delta^F_{ij}(z - y_k) \left\langle \Omega, T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_j}, \dots, W_n\right)(y_N)\,\Omega \right\rangle + \\
&\quad - i \sum_{k=1}^{n} \partial_\rho \Delta^F_{ij}(z - y_k) \left\langle \Omega, T\left(W_1, \dots, \frac{\partial W_k}{\partial \varphi_{j,\rho}}, \dots, W_n\right)(y_N)\,\Omega \right\rangle .
\end{aligned} \tag{5.48}
$$

with $\Delta^F_{ij}(y-x) = \langle \Omega, \varphi_i(y)\varphi_j(x)\Omega \rangle$. Then a calculation along the lines of (5.42) shows

$$
\begin{aligned}
&\partial^x_\mu \Bigg\{ \Delta^F_{ij}(z-x) \left\langle \Omega, T\left(\frac{\partial \Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial \varphi_j}, N\right)(x, y_N)\,\Omega \right\rangle + \\
&\quad - \partial_\rho \Delta^F_{ij}(z-x) \left\langle \Omega, T\left(\frac{\partial \Theta^{\mu\nu}_{0\,\mathrm{can}}}{\partial \varphi_{j,\rho}}, N\right)(x, y_N)\,\Omega \right\rangle \Bigg\} = \\
&\qquad = \left(D_{jl} \Delta^F_{ij}(z-x) \partial^\nu_x - \partial^\nu \Delta^F_{ij}(z-x) D^x_{jl} \right) \langle \Omega, T(\varphi_l, N)\,(x, y_N)\,\Omega \rangle \\
&\qquad = \left(\delta_{li} \delta(z-x) \partial^\nu_x - \partial^\nu \Delta^F_{ij}(z-x) D^x_{jl} \right) \langle \Omega, T(\varphi_l, N)\,(x, y_N)\,\Omega \rangle\,,
\end{aligned} \tag{5.49}
$$

since Δ^F is a Green's function of the equation of motion: $D_{jl}\Delta^F_{ij} = D_{lj}\Delta^F_{ji} = \delta_{li}\delta$. We set $a^\nu\,(x, y_N, z)$ like before with $W_{n+1} \doteq \varphi_i$ and compute its vacuum expectation value. We obtain

$$
\begin{aligned}
&a^\nu(x, y_N, z) = \\
&= i\partial_\mu^x \Big\{ \Delta_{ij}^F(z-x) \left\langle \Omega, T\left(\frac{\partial\Theta_{0\,\mathrm{can}}^{\mu\nu}}{\partial\varphi_j}, N\right)(x, y_N)\,\Omega \right\rangle + \\
&\quad - \partial_\rho \Delta_{ij}^F(z-x) \left\langle \Omega, T\left(\frac{\partial\Theta_{0\,\mathrm{can}}^{\mu\nu}}{\partial\varphi_{j,\rho}}, N\right)(x, y_N)\,\Omega \right\rangle \Big\} + \\
&\quad + i\sum_{l=1}^n \Delta_{ij}^F(z-y_l)\partial_\mu^x \left\langle \Omega, T\left(\Theta_{0\,\mathrm{can}}^{\mu\nu}, W_1, \ldots, \frac{\partial W_l}{\partial\varphi_j}, \ldots, W_n\right)(x, y_N)\,\Omega \right\rangle + \\
&\quad - i\delta(z-x)\partial_z^\nu \langle \Omega, T(\varphi_i, N)(z, y_N)\,\Omega \rangle + \\
&\quad + \sum_{k=0}^n \delta(y_k - x)\partial_k^\nu \left\{ \sum_{l=1}^n \Delta_{ij}^F(z-y_l) \left\langle \Omega, T\left(W_1, \ldots, \frac{\partial W_l}{\partial\varphi_j}, \ldots, W_n\right)(y_N)\,\Omega \right\rangle \right\}.
\end{aligned}
\tag{5.50}
$$

Again, if in the last line the derivative acts on the T-products these terms cancel the third line by the induction hypothesis. The term $l = k$ with the derivative on Δ^F remains. Inserting (5.49) gives:

$$
= -i\partial^\nu \Delta_{ij}^F(z-x) D_{jl}^x \langle \Omega, T(\varphi_l, N)(x, y_N)\,\Omega \rangle + \tag{5.51}
$$

$$
\qquad - \sum_{k=0}^n \delta(y_k - x)\partial^\nu \Delta_{ij}^F(z-y_k) \left\langle \Omega, T\left(W_1, \ldots, \frac{\partial W_k}{\partial\varphi_j}, \ldots, W_n\right)(y_N)\,\Omega \right\rangle
$$

$$
= 0, \tag{5.52}
$$

because of (**N4**).

Step 3. We show how the anomaly can be removed by an appropriate normalization. The above steps have shown that it has the following form:

$$
a^\nu(x, y_N) = \partial_\mu^x \langle \Omega, T(\Theta_{0\,\mathrm{can}}^{\mu\nu}, N)(x, y_N)\,\Omega \rangle - i\sum_{k=1}^n \delta(y_k - x)\partial_k^\nu \langle \Omega, T(N)(y_N)\,\Omega \rangle \tag{5.53}
$$

$$
= M^\nu(\partial)\delta(y_1 - x)\ldots\delta(y_n - x). \tag{5.54}
$$

Here, $\partial = (\partial_1, \ldots, \partial_n)$ and M^ν is a Lorentz vector valued polynomial of degree ≤ 5, since $\operatorname{sing\,ord} \langle \Omega, T(\Theta_{0\,\mathrm{can}}^{\mu\nu}, N), \Omega \rangle = \dim \Theta_{0\,\mathrm{can}}^{\mu\nu} + \sum_{i=1}^n \dim W_i - 4n \leq 4$ according to (3.61). If $M^\nu(\partial)$ has the form

$$
M^\nu(\partial) = \sum_{i=1}^n \partial_\mu^i M_1^{\mu\nu}(\partial), \tag{5.55}
$$

with M_1 again a polynomial, the normalization $T(\Theta_{0\,\mathrm{can}}^{\mu\nu}, N) \to T(\Theta_{0\,\mathrm{can}}^{\mu\nu}, N) + M_1^{\mu\nu}(\partial)\delta$ removes the anomaly.

We show that this is the case. We introduce the free momentum operator:[6]

$$P^\mu \doteq \int \mathrm{d}^3\mathbf{x}\, \Theta^{0\nu}_{0\,\mathrm{can}}(x). \tag{5.56}$$

It is a hermitian operator that annihilates the vacuum.

For every $(y_1, \dots, y_n)$ we take a double cone $\mathcal{O}$ with $y_i \in \mathcal{O}$ for all $i = 1, \dots, n$. Choosing a g, with $g\restriction_{\overline{\mathcal{O}}} = 1$ we decompose $\partial_\mu g = a_\mu - b_\mu$, such that $\mathrm{supp}\, a_\mu \cap (\overline{V}_- + \mathcal{O}) = \mathrm{supp}\, b_\mu \cap (\overline{V}_+ + \mathcal{O}) = \emptyset$. We smear out the first term on the r.h.s of (5.37) with this g and use the causal factorization of the T-products:

$$\int \mathrm{d}x\, \partial^x_\mu T(\Theta^{\mu\nu}_{0\,\mathrm{can}}, N)\,(x, y_N)\, g(x) = \tag{5.57}$$

$$= -\Theta^{\mu\nu}_{0\,\mathrm{can}}(a_\mu) T(N)\,(y_N) + T(N)\,(y_N)\,\Theta^{\mu\nu}_{0\,\mathrm{can}}(b_\mu) \tag{5.58}$$

$$= -\left[\Theta^{\mu\nu}_{0\,\mathrm{can}}(a_\mu), T(N)\,(y_N)\right] - T(N)\,(y_N)\,\Theta^{\mu\nu}_{0\,\mathrm{can}}(\partial_\mu g).$$

The second term vanishes because $\Theta^{\mu\nu}_{0\,\mathrm{can}}$ is a conserved current. Then, in the commutator $\Theta^{\mu\nu}_{0\,\mathrm{can}}(a_\mu)$ can be replaced by $-P^\nu$, since $T(N)$ is localized in $\mathcal{O}$:

$$= \left[P^\nu, T(N)\,(y_N)\right]. \tag{5.59}$$

Therefore the vacuum expectation value of (5.59) vanishes. Smearing the second term of (5.37) with g and taking the vacuum expectation value we find:

$$i \int \mathrm{d}x \sum_{j=1}^{n} \delta(y_j - x)\partial^\nu_j \,\langle \Omega, T(N)\,(y_N)\,\Omega\rangle\, g(x) = i \sum_{j=1}^{n} \partial^\nu_j \,\langle \Omega, T(N)\,(y_N)\,\Omega\rangle = 0, \tag{5.60}$$

because of translation invariance. Hence we get[7]

$$\int \mathrm{d}x\, a^\nu(x, y_N) = 0. \tag{5.61}$$

To prove (5.55) we work in Fourier space:

$$\widehat{a^\nu}(x, p_1, \dots, p_n) = \int \mathrm{d}y_1 \dots \mathrm{d}y_n\, a^\nu(x, y_1, \dots, y_n) e^{i(p_1 y_1 + \dots + p_n y_n)} \tag{5.62}$$

$$= M^\nu(-ip_1, \dots, -ip_n) e^{i(p_1 + \dots + p_n)x}. \tag{5.63}$$

If we integrate over x and use (5.61) we find:

$$\int \mathrm{d}^4x\, \widehat{a^\nu}(x, p_1, \dots, p_n) = (2\pi)^4 M^\nu(-ip_1, \dots, -ip_n)\delta(p_1 + \dots + p_n) = 0, \tag{5.64}$$

$$\Rightarrow M^\nu(-ip_1, \dots, -ip_n)\Big|_{p_1 + \dots + p_n = 0} = 0. \tag{5.65}$$

---

[6]One has to give some meaning to the formal integral in (5.56). We refer to the method of Requardt [Req76]. This shows that a charge like P^μ can be defined for massive theories in general and for certain massless theories, if the infrared behaviour is not "too bad". We explicitly show the existence in section 4 for the massless φ^4-model. But the same conclusion also holds if the mass dimension of $\Theta^{\mu\nu}_{0\,\mathrm{can}}$ is not less than four. This is the case here, if the fields in (5.24) contracted with K are bosonic and the ones contracted with G are fermionic, as usual.

[7]Note that the RHS of (5.54) is of compact support in the difference variables $y_i - x$.

We set $q = \sum_{i=1}^{n} p_i$, and write $\widetilde{M^\nu}(q, p_1, \ldots, p_{n-1}) = M^\nu(-ip_1, \ldots, -ip_n)$. Performing a Taylor expansion at the origin we get:

$$\widetilde{M^\nu}(q, p_1, \ldots, p_{n-1}) = \sum_{k=1}^{\mathrm{degree}\widetilde{M^\nu}} \sum_{|\alpha|+|\beta|=k} \frac{q^\alpha p^\beta}{\alpha!\beta!} \partial_\alpha^q \partial_\beta^p \widetilde{M^\nu}(0, \ldots, 0), \tag{5.66}$$

$p = (p_1, \ldots, p_{n-1})$. Because of (5.65) there are only terms with $|\alpha| \geq 1$ in the second sum. If we Fourier transform back into coordinate space this implies (5.55).

The normalization term $M_1^{\mu\nu}$ cannot be added to the first two terms of $T(\Theta_{0\,\mathrm{can}}^{\mu\nu})$ only. The next two terms (conf. (5.24)) are multiples of the traces of the first ones. This symmetry has to be preserved since we demand linearity of the T-products:

$$\eta_{\mu\nu} T(\partial^\mu \varphi_j \partial^\nu \varphi_l, N) = T(\partial^\mu \varphi_j \partial_\mu \varphi_l, N) \tag{5.67}$$

$$\Rightarrow \eta_{\mu\nu} (\partial^\mu \varphi_j \partial^\nu \varphi_l)_{g\mathscr{L}} = (\partial^\mu \varphi_j \partial_\mu \varphi_l)_{g\mathscr{L}}. \tag{5.68}$$

Therefore we add the normalization terms according to

$$K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial^\nu \varphi_k, N) \to K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial^\nu \varphi_k, N) + \left(M_1^{\mu\nu} - \frac{1}{6} \eta^{\mu\nu} M_{1\ \rho}^{\rho} \right) (\partial)\delta, \tag{5.69}$$

$$\Rightarrow K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial_\mu \varphi_k, N) \to K_{lk}^{\mu\rho} T(\partial_\rho \varphi_l \partial_\mu \varphi_k, N) + \frac{1}{3} M_{1\ \rho}^{\rho}(\partial)\delta. \tag{5.70}$$

If the normalization has to be performed on the other terms we put:

$$G_{lk}^{\mu} T(\partial^\nu \varphi_l \varphi_k, N) \to G_{lk}^{\mu} T(\partial^\nu \varphi_l \varphi_k, N) + 2 \left(M_1^{\mu\nu} - \frac{1}{3} \eta^{\mu\nu} M_{1\ \rho}^{\rho} \right) (\partial)\delta, \tag{5.71}$$

$$\Rightarrow G_{lk}^{\mu} T(\partial_\mu \varphi_l \varphi_k, N) \to G_{lk}^{\mu} T(\partial_\mu \varphi_l \varphi_k, N) - \frac{2}{3} M_{1\ \rho}^{\rho}(\partial)\delta. \tag{5.72}$$

These normalizations remove the anomaly.

4. The interacting momentum operator

Now we investigate the interacting charge generated by the conserved energy momentum tensor. It defines the interacting momentum operator since its commutator implements the infinitesimal action on $\mathfrak{A}_{g\mathscr{L}}(\mathcal{O})$ according to:

$$\left[\Theta_{\mathrm{can}\,g\mathscr{L}}^{0\nu}(f), W_{g\mathscr{L}}(x)\right] = i\partial^\nu W_{g\mathscr{L}}(x) \tag{5.73}$$

for all $W \in \mathfrak{B}$ containing no derivated fields. The test function $f \in \mathcal{D}(\mathbb{M})$ is supposed to be $f(y) = h(y^0)$ for all $y = (y^0, \mathbf{y}) \in \mathbb{M}$ in a neighbourhood of $x + (\overline{V}_+ \cup \overline{V}_-)$ and $\int \mathrm{d}y^0\, h(y^0) = 1$.

4.1. Proof of (5.73). We follow the idea of [DF99]. We use the abbreviation $\mathrm{d}y_N\, g(y_N) \doteq \prod_{i\in N} \mathrm{d}y_i g(y_i)$. With the definition of the commutator (4.20) the support properties of the R-products and the choice of f from above we have:

$$\begin{aligned}\left[\Theta^{0\nu}_{0\,\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\right] &= \int \mathrm{d}y\, h(y^0)\, \left[\Theta^{0\nu}_{0\,\mathrm{can}\,g\mathscr{L}}(y), W_{g\mathscr{L}}(x)\right] = \\ &= -\sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N\, \mathrm{d}y\, g(y_N) \times \\ &\quad \times \Big[(h(y^0) - h(y^0-a) + h(y^0-a))R\left(N, \Theta^{0\nu}_{0\,\mathrm{can}}; W\right)(y_N, y; x) + \\ &\quad - (h(y^0) - h(y^0-b) + h(y^0-b))R\left(N, W; \Theta^{0\nu}_{0\,\mathrm{can}}\right)(y_N, x; y)\Big],\end{aligned} \tag{5.74}$$

where $W_i = \mathscr{L}, i = 1, \ldots, n$. Since h has compact support we can choose $a > 0$ and $b < 0$ big enough that the contributions of $h(y^0 - a)$ and $h(y^0 - b)$ vanish due to the support properties of the R-products, see figure 1. If we define the

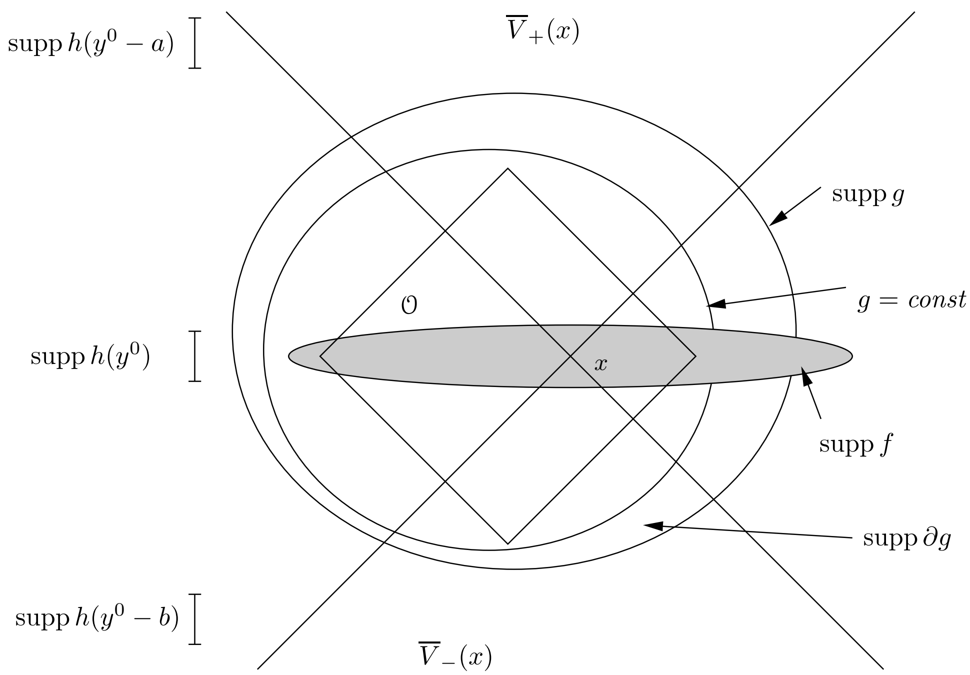

FIGURE 1. Supports of $f, h, g, \partial g$ and R-products.

following two functions

$$k(y) \doteq k(y^0) = \int_{-\infty}^{y^0} \mathrm{d}z\, (h(z) - h(z-a)), \tag{5.75}$$

$$\widetilde{k}(y) \doteq \widetilde{k}(y^0) = \int_{y^0}^{\infty} \mathrm{d}z\, (h(z) - h(z-b)), \tag{5.76}$$

the commutator becomes

$$
\begin{aligned}
&\left[\Theta^{0\nu}_{0\,\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\right] = \\
&\quad = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N \, \mathrm{d}y \, g(y_N) \times \\
&\qquad \times \Big[k(y)\partial^y_\mu R\left(N, \Theta^{\mu\nu}_{0\,\mathrm{can}}; W\right)(y_N, y; x) + \widetilde{k}(y)\partial^y_\mu R\left(N, W; \Theta^{\mu\nu}_{0\,\mathrm{can}}\right)(y_N, x; y)\Big] \\
&\quad = i(k(x) + \widetilde{k}(x))\partial^\nu W_{g\mathscr{L}}(x) + \\
&\qquad + i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N \, g(y_N) \times \\
&\qquad \times \sum_{j=1}^{n} \Big[k(y_j)\partial^\nu_j R\left(N; W\right)(y_N; x) + \widetilde{k}(y_j)\partial^\nu_j R\left(N \setminus j, W; j\right)\left(y_{N\setminus j}, x; y_j\right)\Big] \\
&\quad = i\partial^\nu W_{g\mathscr{L}}(x) + \\
&\qquad - i\eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N \, g(y_N) \times \\
&\qquad \times \sum_{j=1}^{n} \Big[(h(y^0_j) - h(y^0_j - a))R\left(N; W\right)(y_N; x) + \\
&\qquad\quad - (h(y^0_j) - h(y^0_j - b))R\left(N \setminus j, W; j\right)\left(y_{N\setminus j}, x; y_j\right)\Big] + \\
&\qquad - i \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N \sum_{j=1}^{n} g(y_1) \ldots \partial^\nu g(y_j) \ldots g(y_n) \times \\
&\qquad \times \Big[k(y_j)R\left(N; W\right)(y_N; x) + \widetilde{k}(y_j)R\left(N \setminus j, W; j\right)\left(y_{N\setminus j}, x; y_j\right), \Big]
\end{aligned}
\tag{5.77}
$$

where we have inserted the Ward identities (5.32), (5.36) and used the fact that

$$
k(x) + \widetilde{k}(x) = \int \mathrm{d}z \, h(z) - \int_{-\infty}^{x^0} \mathrm{d}z \, h(z-a) - \int_{x^0}^{\infty} \mathrm{d}z \, h(z-b) = 1. \tag{5.78}
$$

The support of f in the time direction and therefore of h can be made sufficiently small, see figure 1. Then the term $\partial^\nu g(y_j)k(y_j)R(N;W)(y_N;x)$ vanishes due to the support of properties. The same is true for the second term in the last integrand of (5.77). In the first integrand the $h(y^0_j - a)$ and $h(y^0_j - b)$ can be omitted, again. We obtain

$$
\begin{aligned}
\big[\Theta^{0\nu}_{0\,\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\big] &= \\
&= i\partial^\nu W_{g\mathscr{L}}(x) + \\
&\quad - i\eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \mathrm{d}y_N\, g(y_N) \times \\
&\quad \times \sum_{j=1}^{n} h(y_j^0) \Big[R\,(N;W)\,(y_N;x) - R\,(N\setminus j, W; j)\,\big(y_{N\setminus j}, x; y_j\big)\Big] \\
&= i\partial^\nu W_{g\mathscr{L}}(x) + \\
&\quad + \eta^{0\nu} \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} \int \mathrm{d}y_N \sum_{j=1}^{n} g(y_1)\dots g(y_j) h(y_j^0)\dots g(y_n) \times \\
&\quad \times \frac{1}{n}\Big[R\,(N\setminus j, j; W)\,\big(y_{N\setminus j}, y_j; x\big) - R\,(N\setminus j, W; j)\,\big(y_{N\setminus j}, x; y_j\big)\Big] \\
&= i\partial^\nu W_{g\mathscr{L}}(x) - \eta^{0\nu}\left[\mathscr{L}_{g\mathscr{L}}(gf), W_{g\mathscr{L}}(x)\right],
\end{aligned}
\tag{5.79}
$$

since all terms in the sum of the last integrand are equal because of the symmetry. Hence we find:

$$
\begin{aligned}
\big[\Theta^{0\nu}_{\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\big] &= \big[\Theta^{0\nu}_{0\,\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\big] + \eta^{0\nu}\left[\mathscr{L}_{g\mathscr{L}}(gf), W_{g\mathscr{L}}(x)\right] \\
&= i\partial^\nu W_{g\mathscr{L}}(x).
\end{aligned}
\tag{5.80}
$$

We show that one can define a momentum operator according to

$$
P^\nu_{g\mathscr{L}} \doteq \int \mathrm{d}^3\mathbf{x}\, \Theta^{0\nu}_{\mathrm{can}\,g\mathscr{L}}(x^0,\mathbf{x}). \tag{5.81}
$$

Due to the definition of interacting fields (4.11) we start investigating the free contribution. Following the method of Requardt [Req76] we consider the localized momentum operator:

$$
{}_\lambda P^\nu \doteq \int \mathrm{d}^4x\, k_\lambda(x^0) h_\lambda(\mathbf{x}) \Theta^{0\nu}_{0\,\mathrm{can}}(x^0,\mathbf{x}), \tag{5.82}
$$

with test functions $h \in \mathcal{D}(\mathbb{R}^3)$, $h(0) \equiv 1$ and $k \in \mathcal{D}(\mathbb{R})$ with $\int \mathrm{d}x^0 k(x^0) = 1$. We set $h_\lambda(\mathbf{x}) \doteq h(\lambda\mathbf{x})$ and $k_\lambda(x^0) \doteq \lambda k(\lambda x^0)$. Calculation of the correlation function of two EMTs results in:

$$
\begin{aligned}
&\Big\langle \Omega, \Theta^{0\mu}_{0\,\mathrm{can}}(x)\Theta^{0\nu}_{0\,\mathrm{can}}(y)\Omega \Big\rangle = \\
&= \partial^0\partial^0 D_+(x-y)\partial^\mu\partial^\nu D_+(x-y) + \partial^0\partial^\mu D_+(x-y)\partial^0\partial^\nu D_+(x-y) + \\
&\quad - 2\eta^{0(\mu}\partial^{\nu)}\partial^\rho D_+(x-y)\partial_\rho\partial^0 D_+(x-y) + \\
&\quad + \frac{1}{2}\eta^{0\mu}\eta^{0\nu}\partial_\rho\partial_\sigma D_+(x-y)\partial^\rho\partial^\sigma D_+(x-y) \\
&= \lambda^8 \Big\langle \Omega, \Theta^{0\mu}_{0\,\mathrm{can}}(\lambda x)\Theta^{0\nu}_{0\,\mathrm{can}}(\lambda y)\Omega \Big\rangle,
\end{aligned}
\tag{5.83}
$$

since the massless two-point function D_+ is homogenous of degree -2. We have

$$\langle \Omega, {}_\lambda P^\mu {}_\lambda P^\nu \Omega \rangle = \lambda^2 \int \mathrm{d}^4x\, \mathrm{d}^4y\, k(x^0)k(y^0)h(\mathbf{x})h(\mathbf{y}) \left\langle \Omega, \Theta^{0\mu}_{0\,\mathrm{can}}(x)\Theta^{0\nu}_{0\,\mathrm{can}}(y)\Omega \right\rangle, \tag{5.84}$$

and this implies $\lim_{\lambda\to 0} ||{}_\lambda P^\nu \Omega|| = 0$. For an arbitrary Wick polynomial W we additionally have

$$\lim_{\lambda\to 0} [{}_\lambda P^\nu, W(x)] = \partial^\nu W(x). \tag{5.85}$$

The domain $\mathcal{D}$ is the linear hull of all $\Phi = W_1(f_1)\dots W_r(f_r)\Omega$, with W_i Wick monomials and f_i test functions. With the derivation property of the commutator, this defines the momentum $P^\nu = \lim_{\lambda\to 0} {}_\lambda P^\nu$ in a strong limit on $\mathcal{D}$: $P\Phi\Omega = [P, \Phi]\Omega$.

In the interacting contribution of (5.81) the space integral is restricted to the hypersurface of constant x^0 intersecting $\overline{V}_+(\mathrm{supp}\, g)$ and hence compact because of the support properties of the R-products. The interacting canonical tensor further contains the interaction term (cf. (5.27)) which is localized. Hence the integral in (5.81) exists.

5. The interacting improved tensor in massless φ^4-theory

This section treats the possibilities for defining an improved EMT. It results in the unavoidable appearance of the well known trace anomaly. The definition of a suitable improvement tensor requires the validity of a further differential equation involving interacting fields. This equation is proved by a corresponding Ward identity in the next section.

We consider the free massless scalar field φ. It satisfies the wave equation and the commutation relation:

$$\Box\varphi = 0, \qquad [\varphi(x), \varphi(y)] = iD(x-y), \tag{5.86}$$

for $\varphi(x) = T(\varphi)(x)$. D is the Pauli-Jordan distribution. The corresponding Feynman propagator is denoted by D^F. The free field allows to define a conserved and traceless improved EMT by the expression from classical field theory in form of Wick products:

$$:\Theta^{\mu\nu}_{0\,\mathrm{imp}}: = :\Theta^{\mu\nu}_{0\,\mathrm{can}}: - \frac{1}{3} :I^{\mu\nu}:, \tag{5.87}$$

$$:I^{\mu\nu}: = :\partial^\mu\varphi\partial^\nu\varphi: + :\varphi\partial^\mu\partial^\nu\varphi: - \eta^{\mu\nu} :\partial_\rho\varphi\partial^\rho\varphi: \tag{5.88}$$

$$= \partial^\mu :\varphi\partial^\nu\varphi: - \eta^{\mu\nu}\partial_\rho :\varphi\partial^\rho\varphi: \tag{5.89}$$

$$= \frac{1}{2}(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box) :\varphi^2: . \tag{5.90}$$

It is well known that the dilatations can be implemented as a unitary symmetry on the invariant domain $\mathcal{D}$ of Fock space $U : \mathcal{D} \mapsto \mathcal{D}, U_\lambda \varphi(x) U_\lambda^{-1} = \lambda\varphi(\lambda x)$. The infinitesimal transformation is given by the commutator of the dilatation charge $Q^R_D = \int \mathrm{d}x^0\, \alpha(x^0) f_R(\mathbf{x}) D^0(x)$ for sufficiently large R, where $\int \mathrm{d}x^0 \alpha(x^0) = 1$ and f_R is a smooth version of the characteristic function of the ball of radius R. For large R the commutator becomes independent of α and

f_R [MR71]. The dilatation current is $D_0^\mu(x) = x_\nu :\Theta^{\mu\nu}_{0\,\mathrm{imp}}(x):$. Since the symmetry is conserved we have $\lim_{R\to\infty} \omega_0([Q_D^R, W]) = 0$, for any observable $W = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n :W_1(x_1): \ldots :W_n(x_n): f(x_1)\ldots f(x_n)$ with $W_i \in \mathfrak{B}$, $f_i \in \mathcal{D}(\mathbb{M})$.[8]

Switching on the interaction, the field equation becomes

$$\Box \varphi_{g\mathscr{L}} = -g \left(\frac{\partial \mathscr{L}}{\partial \varphi}\right)_{g\mathscr{L}}. \tag{5.91}$$

Since the model is just a special case of the general situation discussed in the last sections this leads to a locally conserved canonical EMT (5.24),(5.27),(5.28):

$$\Theta^{\mu\nu}_{\mathrm{can}\, g\mathscr{L}} = (\partial^\mu\varphi\partial^\nu\varphi)_{g\mathscr{L}} - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\varphi\partial^\rho\varphi)_{g\mathscr{L}} + g\eta^{\mu\nu}\mathscr{L}_{g\mathscr{L}}. \tag{5.92}$$

In order to define an interacting improvement tensor based on (5.89) we require the following identity for some $c \in \mathbb{R}, c \neq \frac{1}{4}$:

$$\begin{aligned}\partial^\mu(\varphi\partial^\nu\varphi)_{g\mathscr{L}} &= \\ &= (\partial^\mu\varphi\partial^\nu\varphi)_{g\mathscr{L}} + (\varphi\partial^\mu\partial^\nu\varphi)_{g\mathscr{L}} - c\eta^{\mu\nu}(\varphi\Box\varphi)_{g\mathscr{L}} - c\eta^{\mu\nu} g\left(\varphi\frac{\partial\mathscr{L}}{\partial\varphi}\right)_{g\mathscr{L}}. \end{aligned} \tag{5.93}$$

The equation is obviously symmetric in μ, ν. We show this identity to be satisfied by a corresponding Ward identity in the next section. The exclusion of the case $c = \frac{1}{4}$ is due to the fact that (5.93) has to be satisfied with **WI 1** simultaneously. Since the latter one fixes the normalization of $(\partial^\mu\varphi\partial^\nu\varphi)_{g\mathscr{L}}$ the new interacting field $(\varphi\partial^\mu\partial^\nu\varphi)_{g\mathscr{L}}$ must not appear in a traceless combination.

Now the improvement tensor is defined by

$$I^{\mu\nu}_{g\mathscr{L}} \doteq \partial^\mu(\varphi\partial^\nu\varphi)_{g\mathscr{L}} - \eta^{\mu\nu}\partial_\rho(\varphi\partial^\rho\varphi)_{g\mathscr{L}}. \tag{5.94}$$

It is conserved due to the μ, ν-symmetry of (5.93).

$$\partial_\mu I^{\mu\nu}_{g\mathscr{L}} = 0. \tag{5.95}$$

To discuss the consequences of the improvement we introduce the dimension operator d on monomials $W \in \mathfrak{B}$ according to:

$$dW \doteq \sum_r (r+1)\varphi_{,\mu_1\ldots\mu_r}\frac{\partial W}{\partial\varphi_{,\mu_1\ldots\mu_r}}. \tag{5.96}$$

The 1 in parenthesis refers to the dimension of the scalar field φ. Obviously, d has integer eigenvalues. In case of a pure $\mathscr{L} \propto \varphi^4$-interaction we have:

$$d\mathscr{L} = \varphi\frac{\partial\mathscr{L}}{\partial\varphi} = 4\mathscr{L}. \tag{5.97}$$

Now we define the interacting improved EMT according to (5.15) by:

$$\Theta^{\mu\nu}_{\mathrm{imp}\, g\mathscr{L}} \doteq \Theta^{\mu\nu}_{\mathrm{can}\, g\mathscr{L}} - \frac{1}{3} I^{\mu\nu}_{g\mathscr{L}}. \tag{5.98}$$

[8] Instead of writing $\lim_{R\to\infty}[Q_D^R, W]$ or using the better convergent definition by Requardt [Req76] we use the expression $[D^0(f), W]$ with a suitable test function f.

But the trace of that tensor is not zero. We find:

$$\eta_{\mu\nu}\Theta^{\mu\nu}_{\mathrm{imp}\,g\mathscr{L}} = -\left(\partial_\rho\varphi\partial^\rho\varphi\right)_{g\mathscr{L}} + 4g\mathscr{L}_{g\mathscr{L}} + \partial_\rho\left(\varphi\partial^\rho\varphi\right)_{g\mathscr{L}} \tag{5.99}$$

$$= (1-4c)\left(\left(\varphi\Box\varphi\right)_{g\mathscr{L}} + g\left(\varphi\frac{\partial\mathscr{L}}{\partial\varphi}\right)_{g\mathscr{L}}\right). \tag{5.100}$$

This is the well known trace anomaly of the EMT. We see that it is undetermined up to a multiplicative real parameter. The anomaly is zero in case that one of the factors vanishes. But this contradicts the non existence of a scale invariant renormalization.[9]

The improved EMT defines the same momentum operator:

$$\left[\Theta^{0\nu}_{\mathrm{imp}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\right] = \left[\Theta^{0\nu}_{\mathrm{can}\,g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\right] = \partial^\nu W_{g\mathscr{L}}(x), \tag{5.111}$$

---

[9] To show the contradiction we have to treat the two cases

$$(i) \quad c = \frac{1}{4} \text{ and} \tag{5.101}$$

$$(ii) \quad \left(\varphi\Box\varphi\right)_{g\mathscr{L}} = -g\left(\varphi\frac{\partial\mathscr{L}}{\partial\varphi}\right)_{g\mathscr{L}}. \tag{5.102}$$

Assume (i) is true. With

$$I_0^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + \varphi\partial^\mu\partial^\nu\varphi - \eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \frac{1}{4}\eta^{\mu\nu}\varphi\Box\varphi, \tag{5.103}$$

$$I^{\mu\nu}_{g\mathscr{L}} = I^{\mu\nu}_{0\,g\mathscr{L}} + \frac{3}{4}\eta^{\mu\nu}g\left(d\mathscr{L}\right)_{g\mathscr{L}}, \tag{5.104}$$

we could define the locally conserved dilatation current by

$$D^\mu_{g\mathscr{L}} \doteq x_\nu\Theta^{\mu\nu}_{\mathrm{imp}\,g\mathscr{L}} = x_\nu\left(\Theta^{\mu\nu}_{\mathrm{can}\,g\mathscr{L}} - \frac{1}{3}I^{\mu\nu}_{g\mathscr{L}}\right) \tag{5.105}$$

$$= x_\nu\left(\Theta^{\mu\nu}_{0\,\mathrm{can}g\mathscr{L}} - \frac{1}{3}I^{\mu\nu}_{0\,g\mathscr{L}}\right) \tag{5.106}$$

$$= \left(x_\nu\left(\Theta^{\mu\nu}_{0\,\mathrm{can}} - \frac{1}{3}I_0^{\mu\nu}\right)\right)_{g\mathscr{L}}. \tag{5.107}$$

The conservation is equivalent to the Ward identity

$$\partial^x_\mu T(D^\mu, N)(x, y_N) = i\sum_{k=0}^{n}\delta(x-y_k)(d_k + y_k\cdot\partial_k)T(N)(y_N), \tag{5.108}$$

where d_k denotes the k'th dimension (see next footnote). Passing to the integrated Ward identity by integration with a function g chosen like in step 3 of section 3 this leads to

$$[D^0(f), T(N)(y_N)] = \sum_{k=1}^{n}(d_k + y_k\cdot\partial^k)T(N)(y_N), \tag{5.109}$$

where f is a test function like in section 4. Since the dilatations are the infinitesimal symmetry transformations of the unitarily implementable scale transformations on the free field algebra, the RHS vanishes in the vacuum state ω_0. But this implies a scale invariant renormalization of $\omega_0(T(N))$ which is not possible. Therefore $c \neq \frac{1}{4}$.

If we assume (ii) to be true, the last two terms of (5.93) vanish leading to the conserved improvement tensor $I^{\mu\nu}_{g\mathscr{L}}$ with

$$I^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + \varphi\partial^\mu\partial^\nu\varphi - \eta^{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \eta^{\mu\nu}\varphi\Box\varphi. \tag{5.110}$$

The dilatations according to (5.105) are conserved and both Ward identities (5.108),(5.109) hold with D^μ replaced by $D_0^\mu = x_\nu(\Theta^{\mu\nu}_{0\,\mathrm{can}} - \frac{1}{3}I^{\mu\nu})$. Because of the same argument this is a contradiction.

because the $I^{0\nu}_{g\mathscr{L}}$-component is either a derivative (or divergence) w.r.t. to the space components and $\partial_j f(y) = 0, y \in \overline{V}_+(x) \cup \overline{V}_-(x)$:

$$I^{00}_{g\mathscr{L}} = \partial_j \left(\varphi \partial^j \varphi\right)_{g\mathscr{L}}, \tag{5.112}$$

$$I^{0j}_{g\mathscr{L}} = \partial^j \left(\varphi \partial^0 \varphi\right)_{g\mathscr{L}}. \tag{5.113}$$

The trace of the improved EMT expresses the breaking of scale invariance. If the dilatations are defined by (cp. (5.18))

$$D^{\mu}_{g\mathscr{L}} = x_\nu \Theta^{\mu\nu}_{\mathrm{imp}\, g\mathscr{L}}, \tag{5.114}$$

we obtain

$$\partial_\mu D^{\mu}_{g\mathscr{L}} = \Theta_{\mathrm{imp}\, g\mathscr{L}}{}^{\mu}{}_{\mu} + x^\mu \partial_\mu g\mathscr{L}_{g\mathscr{L}}. \tag{5.115}$$

If we define the dilatations alternatively by (cp. (5.20))

$$\widetilde{D}^{\mu}_{g\mathscr{L}} = x_\nu \Theta^{\mu\nu}_{\mathrm{can}\, g\mathscr{L}} + (\varphi \partial^\mu \varphi)_{g\mathscr{L}}, \tag{5.116}$$

we find the same breaking (5.115). The (not time independent) charge remains unchanged due to:

$$\widetilde{D}^{0}_{g\mathscr{L}} - D^{0}_{g\mathscr{L}} = \frac{2}{3} \partial_j \left(x^{[j} \left(\varphi \partial^{0]} \varphi\right)_{g\mathscr{L}} \right). \tag{5.117}$$

The next section gives a proof of (5.93).

6. Proof of the Ward identity

We prove equation (5.93) in analogy to the conservation of the canonical EMT by the validity of the following Ward identity:[10]

$$\partial^\mu_x T(\varphi\partial^\nu\varphi, N)(x, y_N) = T(\partial^\mu\varphi\partial^\nu\varphi, N)(x, y_N) + T(\varphi\partial^\mu\partial^\nu\varphi, N)(x, y_N) + \\ - c\eta^{\mu\nu} T(\varphi\Box\varphi, N)(x, y_N) + ic\eta^{\mu\nu} \sum_{k=1}^{n} \delta(y_k - x) d_k T(N)(y_N). \quad (\textbf{WI 2})$$

The proof follows the procedure in section 3. For $N = \emptyset$ **WI 2** is obviously fulfiled. Then we make a double induction over $n = |N|$ and the degree of the Wick monomials. Under the assumption that **WI 2** is fulfiled in lower orders we denote the possible anomaly by

$$\begin{aligned} a^{\mu\nu}(N)(x, y_N) \doteq\; & \partial^\mu_x T(\varphi\partial^\nu\varphi, N)(x, y_N) - T(\partial^\mu\varphi\partial^\nu\varphi, N)(x, y_N) + \\ & - T(\varphi\partial^\mu\partial^\nu\varphi, N)(x, y_N) + c\eta^{\mu\nu} T(\varphi\Box\varphi, N)(x, y_N) + \\ & - i\eta^{\mu\nu} c \sum_{k=1}^{n} \delta(y_k - x) d_k T(N)(y_N). \end{aligned} \tag{5.118}$$

We show that a normalization of the $T(\varphi\partial^\mu\partial^\nu, N)$ exists such that the anomaly vanishes if we require the following normalization condition for the twice derivated basic field:

$$T(\partial_\mu\partial_\nu\varphi, N)(x, y_N) \doteq \partial^x_\mu \partial^x_\nu T(\varphi, N)(x, y_N). \tag{5.119}$$

[10] d_k is the dimension of the k'th monomial in the T-product: $d_k T(N) = T(W_1, \ldots, dW_k, \ldots, W_n)$ and d given by (5.96).

By comparison to (3.60), the integrated form of **N4**, this is achieved by fixing the two point function: $\omega_0(T(\partial^\mu\partial^\nu\varphi,\varphi)(x,y)) = i\partial^\mu\partial^\nu D^F(x-y)$. For the corresponding interacting fields we have $(\partial^\mu\partial^\nu\varphi)_{g\mathscr{L}} = \partial^\mu\partial^\nu\varphi_{g\mathscr{L}}$. In order to condense the notation we introduce the following abbreviation:

$$N^{(e_k)} \doteq \left\{W_1,\ldots,\frac{\partial W_k}{\partial\varphi},\ldots,W_n\right\}. \tag{5.120}$$

Step 1. We commute the anomaly with the free field $\varphi(z)$:

$$\begin{aligned}
&[a^{\mu\nu}(N)(x,y_N),\varphi(z)] = \\
&\quad= \partial_x^\mu[T(\varphi\partial^\nu\varphi,N)(x,y_N),\varphi(z)] - [T(\partial^\mu\varphi\partial^\nu\varphi,N)(x,y_N),\varphi(z)] + \\
&\quad\quad - [T(\varphi\partial^\mu\partial^\nu\varphi,N)(x,y_N),\varphi(z)] + c\eta^{\mu\nu}[T(\varphi\Box\varphi,N)(x,y_N),\varphi(z)] + \\
&\quad\quad - ic\eta^{\mu\nu}\sum_{l=1}^{n}\delta(y_l-x)d_k[T(N)(y_N),\varphi(z)] \\
&\quad= \partial_x^\mu\partial_x^\nu T(\varphi,N)(x,y_N)iD(x-z) + \partial_x^\nu T(\varphi,N)(x,y_N)\partial^\mu D(x-z) + \\
&\quad\quad + \partial_x^\mu T(\varphi,N)(x,y_N)i\partial^\nu D(x-z) + T(\varphi,N)(x,y_N)i\partial^\mu\partial^\nu D(x-z) + \\
&\quad\quad - \partial_x^\nu T(\varphi,N)(x,y_N)i\partial^\mu D(x-z) - \partial_x^\mu T(\varphi,N)(x,y_N)i\partial^\nu D(x-z) + \\
&\quad\quad - T(\partial^\mu\partial^\nu\varphi,N)(x,y_N)iD(x-z) - T(\varphi,N)(x,y_N)i\partial^\mu\partial^\nu D(x-z) + \\
&\quad\quad + c\eta^{\mu\nu}T(\Box\varphi,N)(x,y_N)iD(x-z) + \\
&\quad\quad - ic\eta^{\mu\nu}\sum_{k=1}^{n}\delta(y_k-x)T(N^{(e_k)})(y_N)iD(y_k-z) + \\
&\quad\quad + \sum_{k=1}^{n}a^{\mu\nu}(N^{(e_k)})(x,y_N)iD(y_k-z) \\
&\quad= 0,
\end{aligned} \tag{5.121}$$

if we apply the induction assumption ($a^{\mu\nu}(N^{(e_k)})=0$) to the last line and and the normalization (5.119) and normalization condition **N4** to the previous lines. Therefore, the anomaly has to be a vacuum expectation value:

$$a^{\mu\nu}(N)(x,y_N) = \omega_0\left(a^{\mu\nu}(N)(x,y_N)\right). \tag{5.122}$$

Step 2. We show that the anomaly vanishes if one Wick monomial is a basic generator φ. Because of Step 1 we only consider vacuum expectation values and use (3.60):

$$
\begin{aligned}
\omega_0\left(a^{\mu\nu}(N,\varphi)(x,y_N,z)\right) = & \\
&= \partial_x^\mu \omega_0\left(T(\varphi\partial^\nu\varphi,N,\varphi)(x,y_N,z)\right) + \\
&\quad - \omega_0\left(T(\partial^\mu\varphi\partial^\nu\varphi,N,\varphi)(x,y_N,z)\right) + \\
&\quad - \omega_0\left(T(\varphi\partial^\mu\partial^\nu\varphi,N,\varphi)(x,y_N,z)\right) + \\
&\quad + c\eta^{\mu\nu}\omega_0\left(T(\varphi\Box\varphi,N,\varphi)(x,y_N,z)\right) + \\
&\quad - ic\eta^{\mu\nu}\sum_{l=1}^{n}\delta(y_l-x)\omega_0(d_l T(N,\varphi)(y_N,z)) + \\
&\quad - ic\eta^{\mu\nu}\delta(z-x)\omega_0(T(N,\varphi)(y_N,z)) \\
&= -i\partial^\mu D^F(z-x)\omega_0\left(T(\partial^\nu\varphi,N)(x,y_N)\right) + \\
&\quad + iD^F(z-x)\partial_x^\mu\omega_0\left(T(\partial^\nu\varphi,N)(x,y_N)\right) + \\
&\quad + i\partial^\mu\partial^\nu D^F(z-x)\omega_0\left(T(\varphi,N)(x,y_N)\right) + \\
&\quad - i\partial^\nu D^F(z-x)\partial_x^\mu\omega_0\left(T(\varphi,N)(x,y_N)\right) + \\
&\quad + i\partial^\mu D^F(z-x)\omega_0\left(T(\partial^\nu\varphi,N)(x,y_N)\right) + \\
&\quad + i\partial^\nu D^F(z-x)\omega_0\left(T(\partial^\mu\varphi,N)(x,y_N)\right) + \\
&\quad - iD^F(z-x)\omega_0\left(T(\partial^\mu\partial^\nu\varphi,N)(x,y_N)\right) + \\
&\quad - i\partial^\mu\partial^\nu D^F(z-x)\omega_0\left(T(\varphi,N)(x,y_N)\right) + \\
&\quad + ic\eta^{\mu\nu}D^F(z-x)\omega_0\left(T(\Box\varphi,N)(x,y_N)\right) + \\
&\quad + ic\eta^{\mu\nu}\Box D^F(z-x)\omega_0\left(T(\varphi,N)(x,y_N)\right) + \\
&\quad - ic\eta^{\mu\nu}\sum_{k=1}^{n}\delta(x-y_k)\omega_0(T(N^{(e_k)})(y_N))iD^F(z-y_k) + \\
&\quad - ic\eta^{\mu\nu}\delta(z-x)\sum_{k=1}^{n}iD^F(z-y_k)\omega_0(T(N^{(e_k)})(y_N)) + \\
&\quad + \sum_{k=1}^{n}\omega_0\left(a^{\mu\nu}(N^{(e_k)})(x,y_N)\right)iD^F(z-y_k) \\
&= 0.
\end{aligned}
\tag{5.123}
$$

We have used $\Box D^F = \delta$, the normalization (5.119), **N4** and the induction assumption.

Step 3. Because of the induction assumption the anomaly is a local term:

$$
\omega_0(a^{\mu\nu}(N)(x,y)) = M^{\mu\nu}(\partial)\delta(x,y_N), \tag{5.124}
$$

where $\delta(x, y_N) = \delta(x - y_1) \dots \delta(x - y_n)$ and $M^{\mu\nu}$ is a polynomial in $\partial = (\partial_1, \dots, \partial_n)$ of degree ≤ 4. Therefore we can absorb the anomaly by the following normalization:

$$\omega_0\left(T(\varphi\partial^\mu\partial^\nu\varphi, N)(x, y_N)\right) \rightarrow \omega_0\left(T(\varphi\partial^\mu\partial^\nu\varphi, N)(x, y_N)\right) + M^{\mu\nu}(\partial)\delta(x, y_N) + \\ - \frac{c}{4c-1}\eta^{\mu\nu} M^\rho{}_\rho(\partial)\delta(x, y_N), \tag{5.125}$$

$$\Rightarrow \omega_0\left(T(\varphi\Box\varphi, N)(x, y_N)\right) \rightarrow \omega_0\left(T(\varphi\Box\varphi, N)(x, y_N)\right) - \frac{1}{4c-1} M^\rho{}_\rho(\partial)\delta(x, y_N), \tag{5.126}$$

$$\Rightarrow \omega_0(a^{\mu\nu}(N)(x, y_N)) \rightarrow 0. \tag{5.127}$$

As long as the trace of the anomaly $M^\rho{}_\rho$ is non vanishing the normalization can only be done for $c \neq \frac{1}{4}$.

7. The anomalous dimension

Although the dilatation current is not conserved in the interacting theory we calculate the commutator of the corresponding charge in order to obtain a term that measures the anomalous dimension. From **WI 2** we derive the following Ward identities for the R-products:

$$\partial^y_\mu R(N, \varphi\partial^\mu\varphi; W)(y_N, y; x) = \\ = R(N, \partial_\mu\varphi\partial^\mu\varphi; W)(y_N, y; x) + (1 - 4c)R(N, \varphi\Box\varphi; W)(y_N, y; x) + \\ + 4ic\sum_{j\in N}\delta(y_j - y)d_j R(N; W)(y_N; x) + 4ic\delta(x - y)R(N; dW)(y_N; x). \tag{5.128}$$

If we set $M = \{V_1, \dots, V_m\}, V_i \in \mathfrak{B}$ we find:

$$\partial^x_\mu R(N; \varphi\partial^\mu\varphi, M)(y_N; x, x_M) = \\ = R(N, \partial_\mu\varphi\partial^\mu\varphi, M)(y_N; x, x_M) + (1 - 4c)R(N; \varphi\Box\varphi, M)(y_N; x, x_M) + \\ + 4ic\sum_{j\in N}\delta(y_j - x)d_j R(N \setminus j; j, M)\left(y_{N\setminus j}; y_j, x_M\right) + \\ + 4ic\sum_{j\in M}\delta(x_j - x)d_j R(N; M)(y_N; x_M). \tag{5.129}$$

A calculation similar to the one given in section 4 shows that the following commutation relation holds:

$$\begin{aligned}
&\left[D^0_{g\mathscr{L}}(f), W_{g\mathscr{L}}(x)\right] = \\
&\quad = i\,(dW)_{g\mathscr{L}}(x) + ix^\mu\partial_\mu W_{g\mathscr{L}}(x) + \\
&\qquad + (1-4c)\Bigg\{\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}y_N\mathrm{d}y\, g(y_N)\,\times \\
&\qquad \times\Big[\Big(R(N, gd\mathscr{L}; W)(y_N, y; x) + R(N, \varphi\Box\varphi; W)(y_N, y; x)\Big)k(y) + \\
&\qquad + \Big(R(N, W; gd\mathscr{L})(y_N, x; y) + R(N, W; \varphi\Box\varphi)(y_N, x; y)\Big)\widetilde{k}(y)\Big] + \\
&\qquad - i\,(dW)_{g\mathscr{L}}(x)\Bigg\}.
\end{aligned} \tag{5.130}$$

The terms in braces are the anomalous contributions. They are necessarily non vanishing (because of the normalization that excludes case (ii) in (5.102)). Moreover they are operator valued. We compute them for the case $W = \varphi$, where the normalization of the R-products is known due to **N4** [DF99].

$$\begin{aligned}
&\left[D^0_{g\mathscr{L}}(f), \varphi_{g\mathscr{L}}(x)\right] = \\
&\quad = i\varphi_{g\mathscr{L}}(x) + ix^\mu\partial_\mu\varphi_{g\mathscr{L}}(x) + \\
&\qquad + i(1-4c)\Bigg\{\sum_{n=0}^{\infty}\frac{i^n}{n!}\int \mathrm{d}y_N\mathrm{d}y\, g(y_N)\,\times \\
&\qquad \times\Bigg\{\sum_{l=1}^{n}\Bigg[D_{\mathrm{ret}}(x-y_l)\Bigg(g(y)R\left(N\setminus l, d\mathscr{L}; \frac{\partial\mathscr{L}}{\partial\varphi}\right)(y_{N\setminus l}, y; y_l) + \\
&\qquad\qquad + R\left(N\setminus l, \varphi\Box\varphi; \frac{\partial\mathscr{L}}{\partial\varphi}\right)(y_{N\setminus l}, y; y_l)\Bigg)k(y) + \\
&\qquad\qquad + D_{\mathrm{av}}(x-y_l)\Bigg(g(y)R\left(N^{(e_l)}; d\mathscr{L}\right)(y_N; y) + \\
&\qquad\qquad + R\left(N^{(e_l)}; \varphi\Box\varphi\right)(y_N; y)\Bigg)\widetilde{k}(y)\Bigg] + \\
&\qquad + \big(D_{\mathrm{ret}}(x-y)k(y) + D_{\mathrm{av}}(x-y)\widetilde{k}(y)\big)R\left(N; \varphi\frac{\partial^2\mathscr{L}}{\partial\varphi^2}\right)(y_N; y)k(y)\Bigg\}\Bigg\}.
\end{aligned} \tag{5.131}$$

On the other hand we can study the interacting Ward identities of the dilatations. Since time ordered products of interacting fields are already determined by time ordered products of free fields with an arbitrary number of interactions according to (4.21), we find with (5.128), (5.129):

$$\begin{aligned}\partial^x_\mu T\left(D^\mu_{g\mathscr{L}}, \varphi, \dots, \varphi\right)_{g\mathscr{L}}(x, x_1, \dots, x_m) = \\ = T\left(\Theta_{\mathrm{imp}}{}^\mu{}_\mu, \varphi, \dots, \varphi\right)_{g\mathscr{L}}(x, x_1, \dots, x_m) + \\ + x^\mu \partial_\mu g(x) T\left(\mathscr{L}, \varphi, \dots, \varphi\right)_{g\mathscr{L}}(x, x_1, \dots, x_m) + \\ + i \sum_{l=1}^m \delta(x_l - x)(4c + x^\mu_l \partial^l_\mu) T\left(\varphi, \dots, \varphi\right)_{g\mathscr{L}}(x_1, \dots, x_m).\end{aligned} \tag{5.132}$$

The anomalous terms defined above are no anomalous dimensions in the form of a formal (local) power series in the coupling that multiply the interacting fields. This is no surprise since already for the free massive field the dilatations are no symmetry and the above method of commuting with the (non time invariant) dilatation charge does not produce a number in that case. Nevertheless also the massive scalar field is given the canonical dimension one. In [CJ71] the authors suggest to define the dimension by equal time commutators. For the free field this method is well defined and produces the right result. But it is not clear that this carries over to the interacting fields since they may become more singular objects in the time coordinate as their free counterparts.