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Haag duality in conformal quantum field theory

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Abstract: Haag duality is established in conformal quantum field theory for observable fields on the compactified light ray S^1 and Minkowski space $S^1 x S^1$, respectively. This result provides the foundation for an algebraic approach to the classification of conformal theories. Haag duality can fail, however, for the restriction of conformal fields to the underlying non-compact spaces $\mathbb{R}$, respectively $\mathbb{R}x\mathbb{R}$. A prominent example is the stress-energy tensor with central charge $c > 1$.

1. Introduction

The classification of low dimensional conformal quantum field theories is a subject which has received much attention in recent years. In an algebraic approach to this problem, proposed in [BMT, FRS], one starts from the known structure of the observables in conformal theories [1] and constructs the fields from endomorphisms of the observables. This method is adopted from a general analysis of the superselection structure in quantum field theory by Doplicher, Haag and Roberts [DHR]. It is based on two ingredients.

The first assumption concerns the nature of the charges appearing in the theory: they have to be localizable in the sense that the total charge of a state cannot be determined in proper subregions of space. Let us illustrate this requirement by taking as observable a chiral component T(z) of the conformal stress energy tensor, which lives on a compactified light ray, viz. the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. As is well known, the corresponding positive energy (lowest weight) representations can be distinguished by two charges: the central charge c and the ground state energy (lowest weight) h [FST].

The central charge c is not localizable. Its value can be determined in any given region since c appears in the local commutation relations of T(z). For the same reason there do not exist field operators which transfer central charge and are local with respect to T(z). One therefore attributes states of different central charge to different field theories.

In contrast, the ground state energy h has the properties of a localizable charge: its value cannot be determined locally but requires integration of T(z) over all of S^1. According to the general theory [DHR] one therefore expects that there exist charged fields which are local with respect to T(z) and make transitions between states corresponding to different values of h.

For the actual construction of these fields from the observables one has to rely on a second ingredient: Haag duality [H]. Roughly speaking this condition says that the local algebras generated by the observables are maximal. Namely it has to be impossible to complement them by further local operators which are neutral, i.e. map each superselection sector into itself. If Haag duality holds then the charged fields reveal their presence through the existence of endomorphisms of the algebra of observables. Moreover, there is a systematic way to reconstruct the fields from these endomorphisms [DHR].

It is the aim of the present paper to show that Haag duality holds in conformal quantum field theory. For the proper mathematical formulation of this result we have to proceed from the underlying observable fields to local von Neumann algebras. Let us explain this step by turning again to the example of the energy density T(z).

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[1] This structure was brought to light by Schroer [Sch] and, independently, by Mack and Lüscher [ML]. We use here the review article by Furlan, Sotkov and Todorov [FST] as a convenient source for results and references in conformal quantum field theory.

We fix the central charge c and consider the smoothed-out operators T(f) in the "vacuum representation" corresponding to lowest weight h=0. Here T(f) denotes the field T(z) integrated with a testfunction f(z) on S^1. For each interval [2)] $I \subset S^1$ we define an associated von Neumann algebra $\mathfrak{A}(I)$, setting

$$\mathfrak{A}(I) = \{T(f),\ T(f)^*:\ f \text{ real, supp } f \subset I\}''. \tag{1.1}$$

As usual, a prime $'$ at a set of operators $\mathfrak{C}$ denotes the commutant of $\mathfrak{C}$, i.e. the set of bounded operators on the underlying Hilbert space which commute (strongly) with all elements of $\mathfrak{C}$. Analogously, a double prime $''$ at $\mathfrak{C}$ denotes the commutant of $\mathfrak{C}'$, etc. We note that according to the von Neumann density theorem $\mathfrak{A}(I)$ is the smallest weakly closed $*$-algebra which contains all bounded functions of the operators T(f) with supp $f \subset I$.

From the fact that the commutator [T(f), T(g)] vanishes if f, g have disjoint supports one is inclined to conclude that the algebras $\mathfrak{A}(I)$ satisfy the locality condition

$$\mathfrak{A}(I) \subset \mathfrak{A}(I^C)', \tag{1.2}$$

where $I^C = S^1 \backslash I$. Yet in view of the unboundedness of the opertors T(f) some care is needed at this point. We deal with this problem in Sect. 2 and derive energy bounds for chiral currents and the energy density which suffice to establish locality in the sense of relation (1.2).

Our ultimate goal, the proof of Haag duality, will be given in Sect. 3. It says that the inclusion in (1.2) can be replaced by an equality sign, i.e.

$$\mathfrak{A}(I) = \mathfrak{A}(I^C)'. \tag{1.3}$$

It is then apparent that the local algebras $\mathfrak{A}(I)$ are maximal and cannot be complemented by further operators without coming into conflict with the locality requirement (1.2).

For the sake of a better understanding of the significance of Haag duality let us make two remarks. First we note that the maximality condition (1.3) is intimately related to the transitivity of locality (Borchers property [Bo]) of quantum fields. To exhibit this fact we consider for each interval $I \subset S^1$ the algebra

$$\mathfrak{B}(I) = \{T(f),\ T(f)^*:\ f \text{ real, supp } f \subset I^C\}' = \mathfrak{A}(I^C)'. \tag{1.4}$$

A heuristic argument, based on Borchers' transitivity theorem, leads to the conclusion that the algebras $\mathfrak{B}(I)$ should also be local [La], i.e.

$$\mathfrak{B}(I) \subset \mathfrak{B}(I^C)'. \tag{1.5}$$

[2)] An interval (segment) I of S^1 is a contractable subset whose complement has an open interior.

Combining relations (1.2), (1.4) and (1.5) we then have

$$\mathfrak{A}(I) \subset \mathfrak{A}(I^C)' = \mathfrak{B}(I) \subset \mathfrak{B}(I^C)' = \mathfrak{A}(I^{CC})'', \tag{1.6}$$

where the last equality follows from the fact that $I^C = S^1 \backslash I$ is still an interval, so that definition (1.4) can be applied to $\mathfrak{B}(I^C)$. Now $I^{CC} = S^1 \backslash I^C = I$ and $\mathfrak{A}(I)'' = \mathfrak{A}(I)$ since $\mathfrak{A}(I)$ is a von Neumann algebra. Hence relation (1.6) shows that the condition (1.3) of Haag duality follows from the Borchers property (1.5). The converse statement is obvious.

This equivalence depends crucially on the fact that the underlying configuration space is compact. There is no analogous result in Minkowski space quantum field theory. In the latter case the Borchers property was established in the algebraic setting by Bisognano and Wichmann [B W], but Haag duality can fail, cf. [La] and the subsequent remarks. As was shown by Roberts [Ro], this happens whenever there is a spontaneously broken global symmetry in the theory. In a scale invariant theory internal symmetries are never spontaneously broken [Ro2]. Hence the equivalence of the Borchers property and Haag duality in conformal theories may be attributed to their high symmetry.

Our second remark concerns the relation between Haag duality and the nature of the charges appearing in a theory. It was shown by Doplicher, Haag and Roberts [DHR] that Haag duality holds in a given superselection sector if and only if the charge carried by the corresponding states arises from an abelian symmetry group. Here we establish Haag duality only for the vacuum sector. But for the important example of the energy density all lowest weight representations are explicitly known. It may therefore be possible to decide, by checking Haag duality in other sectors, which values of h and c correspond to an abelian charge, respectively to a charge arising from a non-abelian symmetry or "quantum group".

Let us discuss next how the information that Haag duality holds for the chiral components of observables on S^1 leads to a corresponding statement for the observables on the two-dimensional compactified Minkowski space $S^1 \times S^1$. The local algebras corresponding to "double cones" $I_l \times I_r \subset S^1 \times S^1$ are defined by

$$\mathfrak{A}(I_l \times I_r) = \mathfrak{A}_l(I_l) \otimes \mathfrak{A}_r(I_r), \tag{1.7}$$

where on the right hand side appears the tensor product of the "left" and "right" chiral von Neumann algebras associated with the intervals $I_l, I_r \subset S^1$, respectively. The causal complement of $I_l \times I_r$ (not to be confused with the spacelike complement in some Minkowski space) is again a double cone which is given by $(I_l \times I_r)^C := I_l{}^C \times I_r{}^C$. Now if Haag duality holds for the chiral algebras one gets by an application of Tomita's commutation theorem of tensor products [Sa]

$$\begin{aligned} \mathfrak{A}((I_l \times I_r)^C)' &= (\mathfrak{A}_l(I_l{}^C) \otimes \mathfrak{A}_r(I_r{}^C))' = \\ &= \mathfrak{A}_l(I_l{}^C)' \otimes \mathfrak{A}_r(I_r{}^C)' = \mathfrak{A}_l(I_l) \otimes \mathfrak{A}_r(I_r) = \mathfrak{A}(I_l \times I_r). \end{aligned} \tag{1.8}$$

The equality of the first and last member in this relation is thus the proper statement of Haag duality in conformal theories on the compactified Minkowski space $S^1 \times S^1$.

We emphasize that Haag duality can be violated for the restriction of a conformal theory to the underlying non-compact manifolds $\mathbb{R}$, respectively $\mathbb{R} \times \mathbb{R}$. To understand the origin of this feature let us consider the one dimensional situation and remove from the circle S^1 a point. The complement I' of an interval I on the punctured circle then consists of two disjoint pieces $I_<$, $I_>$ whose union is equal to I^C, apart from the point removed. Haag duality on the punctured circle would imply that the von Neumann algebra generated by $\mathfrak{A}(I_<)$ and $\mathfrak{A}(I_>)$ coincides with the commutant of $\mathfrak{A}(I)$. In view of relation (1.3) this is only possible if

$$\mathfrak{A}(I_<) \vee \mathfrak{A}(I_>) = \mathfrak{A}(I^C). \tag{1.9}$$

This strong form of "additivity" can fail, however, because of ultraviolet problems. (Note that a test function with support in I^C can in general not be splitted into the sum of two test functions which have support in $I_<$ and $I_>$, respectively.) We show in Sect. 4 that this problem arises already in the case of the energy density T(z).

Following Bisognano and Wichmann [BW] one could extend the local algebras $\mathfrak{A}(I)$ on the punctured circle, setting

$$\mathfrak{E}(I) = (\mathfrak{A}(I_<) \vee \mathfrak{A}(I_>))'. \tag{1.10}$$

This net does satisfy Haag duality. But, as is shown in Sect. 4, local conformal invariance gets lost. Only the subgroup of the conformal group which leaves the distinguished point on S^1 fixed remains as a symmetry of the extended net.

These observations vindicate the view [BMT] that conformal theories should be studied on their genuine configuration space. To restrict them to the underlying Minkowski space obscures their specific features.

2. Quantum fields and local algebras

We discuss in this section the relation between the conventional setting of conformal quantum field theory, based on point like quantum fields, and the algebraic setting, based on local von Neumann algebras.

To begin with we list the relevant assumptions and establish our notation. As already mentioned, it suffices for our problem to consider the chiral components of the given observables (such as the stress energy tensor, currents, or other chiral fields with integer spin). The underlying compactified configuration space is

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}. \tag{2.1}$$

On S^1 we have a natural action of the restricted conformal group SU(1,1) whose elements γ are represented by complex two-by-two matrices of the form

$$\gamma = \begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix} \text{ with } |a|^2 - |b|^2 = 1. \tag{2.2}$$

The action of γ on S^1 is given by

$$\gamma \cdot z = \frac{az + b}{\bar{a} + \bar{b}z}, \quad z \in S^1. \tag{2.3}$$

Of particular importance in our investigation are the one parameter subgroups of SU(1,1) inducing on S^1 "rotations",

$$\rho(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{2.4}$$

respectively "dilations",

$$\delta(\lambda) = \begin{pmatrix} \mathrm{ch}\,\lambda & \mathrm{sh}\,\lambda \\ \mathrm{sh}\,\lambda & \mathrm{ch}\,\lambda \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{2.5}$$

The vacuum representation of a hermitian conformal quantum field φ which lives on S^1 and has integer spin (dimension) s can now be described as follows [FST]. (For the sake of notational simplicity we restrict our attention to a single field. The extension of our results to an arbitrary number of fields is straightforward.)

First, there is a Hilbert space $\mathfrak{H}$, carrying a continuous unitary representation W of SU(1,1), and an up to a phase factor unique unit vector $\Omega \in \mathfrak{H}$ which is invariant under the action of $W(\gamma)$ for any $\gamma \in SU(1,1)$. In order to simplify notation we write

$$U(t) = W(\rho(t)) \text{ and } V(\lambda) = W(\delta(\lambda)). \tag{2.6}$$

The generator of U (the conformal Hamiltonian) is denoted by T_o and has non-negative spectrum. The vector Ω is thus the ground state of T_o in $\mathfrak{H}$ and we therefore refer to it as vacuum.

Second, there is an operator valued distribution $\varphi: C^\infty(S^1) \to \mathfrak{P}(\mathfrak{D})$ where $C^\infty(S^1)$ is the space of infinitely often differentiable functions on S^1 and $\mathfrak{P}(\mathfrak{D})$ a $*$-algebra of (unbounded) linear operators which is defined on a dense domain $\mathfrak{D} \subset \mathfrak{H}$. The domain $\mathfrak{D}$ is stable under the action of W, and $\Omega \in \mathfrak{D}$. We write formally

$$\varphi(f) = \oint \frac{dz}{2\pi i z} f(z) \varphi(z), \qquad f \in C^\infty(S^1) \tag{2.7}$$

where the integral extends over the oriented circle S^1. (Note that we use a real measure on S^1.) The closure of $\varphi(f)$, which will be denoted by the same symbol, and the adjoint $\varphi(f)^*$ are always with respect to $\mathfrak{D}$. By applying all polynomials in the smoothed-out fields φ to the vacuum one obtains a domain $\mathfrak{D}_o \subset \mathfrak{D}$ which is a core for each operator $\varphi(f)$.

The hermiticity, covariance and locality properties of φ are summarized in

$$\varphi(z)^* = \varphi(z) \tag{2.8a}$$

$$W(\gamma)\varphi(z)W(\gamma)^{-1} = D_s(z;\gamma) \cdot \varphi(\gamma \cdot z) \tag{2.8b}$$

(where $s \in \mathbb{N}$ is the spin of φ and $D_s(z;\gamma) = |az + b|^{-2s}$ with γ as in (2.2)), and

$$[\varphi(z), \varphi(z')] = 0 \text{ for } z \neq z'. \tag{2.8c}$$

These somewhat formal equations can easily be converted into rigorous statements about the smoothed-out field φ with the help of relation (2.7). We note that the field $\varphi(z)$ differs from the standard z-picture fields by a factor z^{s-1}. This definition has the advantage that the hermiticity condition (2.8a) becomes particularly simple.

The Fourier components of fields and of test functions are defined by

$$F_n = \oint \frac{dz}{2\pi i z} z^n \cdot F(z), \ n \in \mathbb{Z} \tag{2.9}$$

where F stands for φ or $f \in C^\infty(S^1)$, respectively. The hermiticity condition (2.8a) implies that

$$\varphi_n^* = \varphi_{-n} \tag{2.10}$$

on $\mathfrak{D}_o$. From the transformation law (2.8b) of φ under rotations it follows that

$$T_o \, \varphi_n = \varphi_n \, (T_o - n). \tag{2.11}$$

We also note that

$$\varphi(f) = \sum_n f_{-n} \cdot \varphi_n = \sum_n f_n \cdot \varphi_{-n} \tag{2.12}$$

where the sums converge absolutely on every vector in the domain $\mathfrak{D}_o$.

The von Neumann algebras corresponding to any interval $I \subset S^1$ are defined in analogy to (1.1) by [3)]

[3)] There exist more refined definitions of the local algebras which require less regularity of the underlying fields, cp. [DSW].

$$\mathfrak{A}(I) = \{\varphi(f), \varphi(f)^* : f \text{ real, supp } f \subset I\}''. \tag{2.13}$$

There are several immediate consequences of this definition. The first one is isotony,

$$\mathfrak{A}(I_1) \subset \mathfrak{A}(I_2) \text{ if } I_1 \subset I_2, \tag{2.14}$$

and the second one covariance,

$$W(\gamma)\mathfrak{A}(I)W(\gamma)^{-1} = \mathfrak{A}(\gamma \cdot I), \quad \gamma \in SU(1,1) \tag{2.15}$$

where $\gamma \cdot I$ denotes the interval obtained by the action of γ on the points $z \in I$. As a consequence of conformal invariance one finds that the algebras $\mathfrak{A}(I)$ depend continuously on the interval I. In particular

$$\mathfrak{A}(I) = \mathfrak{A}(\bar{I}), \tag{2.16}$$

where the bar denotes closure. For the proof of this equality we make use of an argument in [LRT] and choose a sequence $\gamma_n \in SU(1,1)$ such that $\gamma_n \cdot \bar{I} \subset I$ and $\gamma_n \to e$ (e being the unit of SU(1,1)). It follows from relations (2.15) and (2.14) that $W(\gamma_n)\mathfrak{A}(\bar{I})W(\gamma_n)^{-1} = \mathfrak{A}(\gamma_n \cdot \bar{I}) \subset \mathfrak{A}(I)$. Taking also into account that $W(\gamma_n)$ converges strongly to 1 and $\mathfrak{A}(\bar{I})$ is weakly closed it is then clear that $\mathfrak{A}(\bar{I}) \subset \mathfrak{A}(I)$. The opposite inclusion follows from isotony.

As already pointed out, the locality properties of the field φ may not necessarily imply that the corresponding von Neumann algebras $\mathfrak{A}(I)$ are local. We therefore add this requirement as an extra assumption,

$$\mathfrak{A}(I) \subset \mathfrak{A}(I^c)'. \tag{2.17}$$

Relations (2.14) to (2.17) comprise the relevant structure of local conformal von Neumann algebras which we anticipate in our discussion of Haag duality in the subsequent section.

In the remainder of the present section we show that for currents J(z) and the energy density T(z) the locality condition (2.17) follows from the respective Lie field structure. The essential step in the proof is the demonstration that the fields satisfy so-called linear energy-bounds. We can then rely on a result by Driessler and Fröhlich [DF] showing that the algebras (2.13) are local.

Let us consider first the energy density T(z). By the Lüscher–Mack theorem [FSR] the Fourier components T_n of this field generate the Virasoro algebra

$$\left[T_n, T_m\right] = (n-m)\,T_{n+m} + \frac{c}{12}\, n(n^2-1)\delta_{n+m,\,0} \tag{2.18}$$

where $c > 0$ is the central charge. We need bounds on the norms $\|T_n \Psi_N\|$, where Ψ_N is any normalized eigenvector of T_o corresponding to the eigenvalue $N \in \mathbb{N}_o$. (Note that all vectors Ψ_N are elements of the domain $\mathfrak{D}_o$.)

To begin with we assume that $n \geq 1$. Then $T_n \Psi_N$ is either 0 or an eigenvector of T_o corresponding to the eigenvalue $N - n \geq 0$. In the latter case we obtain, bearing in mind that Ψ_N is normalized (cp. the proof of Lemma 2.1 in [Bu]),

$$\begin{aligned} \|T_n \Psi_N\|^4 &\leq (T_n \Psi_N, T_n T_n^* T_n \Psi_N) \\ &= \left\{ 2n(N-n) + \frac{c}{12} n(n^2-1) + \|T_n \Psi_{N-n}\|^2 \right\} \cdot \|T_n \Psi_N\|^2 , \end{aligned} \tag{2.19}$$

where in the step from the first to the second line we made use of the hermiticity condition (2.10) and the commutation relations (2.18). After division of relation (2.19) by $\|T_n \Psi_N\|^2$ we see by induction in N that $\|T_n \Psi_N\|^2 \leq N^2 + \frac{c}{12}(n^2-1) \cdot N$. If $n \leq -1$ we can write with the help of relation (2.18) $\|T_n \Psi_N\|^2 = \|T_{-n} \Psi_N\|^2 - 2nN - \frac{c}{12} n(n^2-1)$. This leads, together with the preceding bound, to the estimate for arbitrary $n \in \mathbb{Z}$

$$\|T_n \Psi_N\| \leq N + \frac{c}{24} n^2 + |n| = \|(T_o + \frac{c}{24} n^2 + |n|) \Psi_N\|. \tag{2.20}$$

Now let $\Psi \in \mathfrak{D}_o$ be any vector. Since Ψ can be represented as a rapidly converging sum $\Psi = \sum_N c_N \Psi_N$ and since the vectors $T_n \Psi_N$ are, for fixed n, orthogonal for different values of N it follows from (2.20) that $\|T_n \Psi\| \leq \|(T_o + \frac{c}{24} n^2 + |n|) \Psi\|$. Making use of relation (2.12) and the fact that the Fourier components f_n of test functions are rapidly decreasing for large $|n|$, we conclude that for each $f \in C^\infty(S^1)$ there exists some constant c_f such that

$$\|T(f) \Psi\| \leq c_f \cdot \|(T_o + 1) \Psi\| \text{ for all } \Psi \in \mathfrak{D}_o. \tag{2.21}$$

Next, let us consider the case of currents J(z). A variant of the Lüscher-Mack theorem yields [FST] that the Fourier components J_n satisfy the commutation relations

$$[J_n, J_m] = \frac{K}{2} \cdot n \, \delta_{n+m,\, 0} \tag{2.22}$$

where $K > 0$. Hence by a similar (actually simpler) argument as before we find that

$$\|J(f) \Psi\| \leq c_f \cdot \|(T_o + 1) \Psi\| \text{ for all } \Psi \in \mathfrak{D}_o. \tag{2.23}$$

The operator T_o on the right hand side of this inequality could in fact be replaced by its square root.

The domain $\mathfrak{D}_o$ is a core for any power of the operator T_o as well as for the smoothed-out fields J(f) and T(f), respectively. Moreover, the commutator of these fields with iT_o yields $J(f')$, respectively $T(f')$, where $f'(z) = iz \frac{d}{dz} f(z)$. These facts and the bounds established above put us into the position to apply Theorem 3.1 in [DF]. According to that result the operators J(f) and T(f) are, for real f, essentially selfadjoint on $\mathfrak{D}_o$. Furthermore, if $[T(f), T(g)] \restriction \mathfrak{D}_o = 0$, then all bounded functions of T(f) and T(g) commute, and an analogous statement holds for the current. Hence the algebras $\mathfrak{A}(I)$ constructed from these fields satisfy the locality condition (2.17), as claimed.

It is apparent that this method of establishing energy-bounds and locality works in all positive energy (lowest weight) representations of currents and the energy density.

3. Modular operators and Haag duality

We turn now to the proof of Haag duality, i.e. of relation (1.3). Our strategy is similar to the one developed by Bisognano and Wichmann in [BW]. In fact we could resort to that work by restricting the field φ to some Minkowski space. But we found it worth while to give a direct argument which takes advantage of the simplifying features of the conformal Hamiltonian T_o.

We begin by noting that it is sufficient to establish the equality

$$\mathfrak{A}(I_+) = \mathfrak{A}(I_-)', \tag{3.1}$$

where $I_\pm = \{z \in S^1 : \pm \operatorname{Im} z > 0\}$ are opposite semicircles. Making use of continuity, cp. (2.16), we can replace I_- in (3.1) by $\bar{I}_- = I_+^c$, proving Haag duality for the region I_+. Given any open interval I we choose $\gamma \in SU(1,1)$ such that $\gamma \cdot I_+ = I$. Because of covariance, cp. (2.15), we then deduce from (3.1) that

$$\mathfrak{A}(I) = W(\gamma) \mathfrak{A}(I_+) W(\gamma)^{-1} = \left(W(\gamma) \mathfrak{A}(I_+^c) W(\gamma)^{-1} \right)' = \mathfrak{A}(I^c)', \tag{3.2}$$

which proves Haag duality for arbitrary intervals.

Next we note that the vacuum Ω is cyclic and separating [4)] for the algebras $\mathfrak{A}(I_\pm)$. This is a consequence (cp. Theorem 4.1 in [DF]) of the Reeh-Schlieder theorem which holds for the underlying field φ since T_o is non-negative. We can thus define an antilinear operator S, familiar from the Tomita-Takesaki theory [BR], setting

$$S\, A\, \Omega = A^* \Omega, \quad A \in \mathfrak{A}(I_-). \tag{3.3}$$

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4) We recall that a vector Φ is cyclic for some algebra $\mathfrak{C}$ if the set of vectors $\{C\Phi : C \in \mathfrak{C}\}$ is dense in the underlying Hilbert space. It is separating if $C\Phi = 0$ for $C \in \mathfrak{C}$ is only possible if $C = 0$.

The following general properties of S are well known [BR]: S is closable, its closure (which we also denote by S) has the unique polar decomposition $S = J \cdot \Delta^{1/2}$. The operator J (the modular conjugation) is an antiunitary involution, and Δ (the modular operator) is positive and selfadjoint. Moreover, there holds the fundamental equality

$$J\mathfrak{A}(I_-)J^{-1} = \mathfrak{A}(I_-)'. \tag{3.4}$$

It is crucial for our argument that we can identify J and Δ more explicitly. In analogy to the results in [BW] we will find that J coincides with the PCT operator Θ, which is fixed by its action on the underlying field φ,

$$\Theta\varphi(z)\Theta^{-1} = (-1)^s\varphi(z^{-1})^*, \tag{3.5}$$

and $\Theta\Omega = \Omega$. (The existence of Θ is a byproduct of our analysis.) The modular operator Δ will turn out to be equal to $V(i\pi)$, where V is the representation of the dilations defined in (2.6). Thus we will have to show

<u>Lemma:</u> Let S be the antilinear operator defined in (3.2). Then S has the polar decomposition

$$S = \Theta \cdot V(i\pi/2).$$

Before embarking on the somewhat technical proof of this lemma let us show that equation (3.1) follows from it. Making use of the geometric action of Θ on the underlying field φ we see on the other hand from the definition (2.13) of the local algebras that $\mathfrak{A}(I_+) = \Theta\mathfrak{A}(I_-)\Theta^{-1}$. Knowing on the other hand that Θ is the modular conjugation we infer from (3.4) that $\Theta\mathfrak{A}(I_-)\Theta^{-1} = \mathfrak{A}(I_-)'$. The combination of these facts yields equation (3.1), as claimed. Hence, apart from the proof of the lemma, we have established

<u>Theorem:</u> Let $\mathfrak{A}(I)$, $I \subset S^1$ be a local net of von Neumann algebras which is generated by a conformal field φ, as discussed in Sect. 2. Then there holds for all intervals I the equality

$$\mathfrak{A}(I) = \mathfrak{A}(I^c)'.$$

In the remaining part of this section we give the proof of the lemma. For the sake of clarity we number the consecutive steps.

1. To begin with we define for any $k \in \mathbb{N}$ and complex numbers $w_1, \ldots, w_k$ such that $1 > |w_1| > \ldots > |w_k| > 0$ standard vectors

$$\Phi(w_1,\ldots,w_k) = \sum_{n_1,\ldots,n_k \in \mathbb{N}_0} w_1^{n_1}\left(\frac{w_2}{w_1}\right)^{n_2} \ldots \left(\frac{w_k}{w_{k-1}}\right)^{n_k} \varphi_{n_1-n_2}\varphi_{n_2-n_3}\cdots\varphi_{n_k}\Omega. \tag{3.6}$$

The sum is absolutely convergent because of the temperedness of the field φ. As a consequence each function $w_1, \ldots, w_k \to \Phi(w_1, \ldots, w_k)$ is analytic in the specified domain.

The summation in (3.6) can actually be extended to $n_1, \ldots, n_k \in \mathbb{Z}$, terms involving some negative n_i vanish identically since $T_0 \geq 0$, cp. relation (2.11). Setting $w_i = u_i \cdot z_i$ for $i = 1, \ldots, k$ where $z_i \in S^1$ and $1 > u_1 > \cdots > u_k > 0$, it therefore follows from (3.6) after resummation that in the sense of distributions

$$\text{s-lim}\ \Phi(u_1 z_1, \ldots, u_k z_k) = \varphi(z_1) \ldots \varphi(z_k)\Omega \tag{3.7}$$

if the parameters u_i approach 1 in the given order.

2. Next let I be any open interval and let $A' \in \mathfrak{A}(I)'$. We want to relate the functions

$$\begin{aligned} H_+(w_1, \ldots, w_k) &= \left(A'^*\Omega,\ \Phi(w_1, \ldots, w_k)\right) \\ H_-(w_1, \ldots, w_k) &= \left(\Phi(\bar{w}_1^{-1}, \ldots, \bar{w}_k^{-1}),\ A'\Omega\right) \end{aligned} \tag{3.8}$$

which are defined and are analytic on the domains $1 > |w_1| > \cdots > |w_k| > 0$, respectively $|w_k| > \cdots > |w_1| > 1$. Proceeding as in step 1 we obtain for $1 > u_1 > \cdots u_k > 0$ and $z_1, \ldots, z_k \in S^1$ in the sense of distributions

$$\begin{aligned} \lim H_+(u_1 z_1, \ldots, u_k z_k) &= \left(A'^*\Omega,\ \varphi(z_1) \cdots \varphi(z_k)\Omega\right) \\ \lim H_-(u_1^{-1} z_1, \ldots, u_k^{-1} z_k) &= \left(\varphi(z_1) \cdots \varphi(z_k)\Omega,\ A'\Omega\right). \end{aligned} \tag{3.9}$$

These two distributions coincide on the region $\dot{I}^k = \{z_1, \ldots, z_k : z_i \in I,\ z_i \neq z_j$ for $i \neq j\}$ since φ is a local hermitian field and $A' \in \{\varphi(f)^*, \varphi(f)$: f real, supp f $\subset I\}'$. Hence $H_\pm$ are, by the Edge-of-the-Wedge theorem, determinations of a single function H which is analytic on some extended domain including an open neighbourhood of $\dot{I}^k \subset \mathbb{C}^k$. This neighbourhood does not depend on $A' \in \mathfrak{A}(I)'$.

3. We proceed with four elementary remarks pertaining to the dilations $\delta(\lambda)$ defined in (2.5). First, the function D_s, which appears in the transformation law (2.8b) of the field φ, can be brought into the form

$$D_s(z;\ \delta(\lambda)) = z^s \left(z\,\mathrm{ch}\,2\lambda + \frac{1+z^2}{2}\mathrm{sh}2\lambda\right)^{-s}. \tag{3.10}$$

Hence it can be continued in z and λ to an analytic function with poles at $z = -\,\mathrm{th}\lambda$ and $z = -\,\mathrm{cth}\lambda$. A similar statement holds for

$$\delta(\lambda) \cdot z = \frac{z \cdot \mathrm{ch}\lambda + \mathrm{sh}\lambda}{\mathrm{ch}\lambda + z \cdot \mathrm{sh}\lambda}. \tag{3.11}$$

Second we note that for any pair of complex numbers w_1, w_2 for which $|w_1| > |w_2|$ and $\mathrm{Re}w_1 > \mathrm{Re}w_2 > 0$ one has $|\delta(\lambda)\cdot w_1| > |\delta(\lambda)\cdot w_2| > 0$ if $\lambda \geq 0$. Third, if $0 < \mu < \pi/4$ and if w_1, w_2 are such that $|w_1| > |w_2|$ and $0 > \mathrm{Im}w_1 > \mathrm{Im}w_2$ there holds $|\delta(i\mu)\cdot w_1| > |\delta(i\mu)\cdot w_2|$. Finally, if $z_1, z_2 \in S^1$ and $0 > \mathrm{Im}\,z_1 > \mathrm{Im}\,z_2$ one finds that $1 > |\delta(i\mu)\cdot z_1| > |\delta(i\mu)\cdot z_2| > 0$ for $0 < \mu < \pi/2$.

4. After these preparations we can determine the action of the dilation operators $V(\lambda)$ on the standard vectors. To this end we pick any operator $A' \in \mathfrak{A}(I_-)'$ and consider for fixed $\lambda \in \mathbb{R}$ the distribution

$$\left(V(\lambda)^{-1} \cdot A'\Omega,\ \varphi(z_1) \cdots \varphi(z_k)\Omega\right). \tag{3.12}$$

Now $V(\lambda)\Omega = \Omega$, and since $\delta(\lambda) \cdot I_- = I_-$ we also have $V(\lambda)^{-1}A'V(\lambda) \in \mathfrak{A}(I_-)'$. Hence we can apply the result of step 2, saying that the distribution (3.12) can be represented at non-coinciding points $z_1, \dots, z_k \in I_-$ by some analytic function $H_\lambda(z_1, \dots, z_k)$. On the other hand we can rewrite the distribution (3.12) in the form

$$\prod_{i=1}^{k} D_s(z_i;\ \delta(\lambda)) \cdot \left(A'\Omega,\ \varphi(\delta(\lambda) \cdot z_1) \cdots \varphi(\delta(\lambda) \cdot z_k)\Omega\right) \tag{3.13}$$

because of the transformation properties (2.8b) of the field φ. Appealing once more to step 2 we can represent the matrix element in (3.13) at non-coinciding points $z_1, \dots, z_k \in I_-$ by the analytic function $H_o\left(\delta(\lambda) \cdot z_1, \dots, \delta(\lambda) \cdot z_k\right)$. Hence we have on that region the equality

$$H_\lambda(z_1, \dots, z_k) = \prod_{i=1}^{k} D_s\left(z_i;\ \delta(\lambda)\right) \cdot H_o\left(\delta(\lambda) \cdot z_1, \dots, \delta(\lambda) \cdot z_k\right). \tag{3.14}$$

Because of the analyticity properties of the functions H_λ and the first and second remark in the preceding step we see that equality (3.14) extends, for $\lambda \geq 0$, to the domain $1 > |w_1| > \cdots > |w_k| > 0$, $\mathrm{Re}w_1 > \mathrm{Re}w_2 > \cdots > \mathrm{Re}w_k > 0$. Thus we have on that domain, cp. (3.8),

$$\left(V(\lambda)^{-1}A'\Omega,\ \Phi(w_1, \dots, w_k)\right) = \prod_{i=1}^{k} D_s(w_i;\ \delta(\lambda))\left(A'\Omega,\ \Phi(\delta(\lambda) \cdot w_1, \dots, \delta(\lambda) \cdot w_k)\right). \tag{3.15}$$

Since the set of vectors $A'\Omega$, $A' \in \mathfrak{A}(I_-)' \supset \mathfrak{A}(I_+)$ is dense in $\mathfrak{H}$ we conclude that

$$V(\lambda) \cdot \Phi(w_1, \dots, w_k) = \prod_{i=1}^{k} D_s\left(w_i; \delta(\lambda)\right) \cdot \Phi\left(\delta(\lambda) \cdot w_1, \dots, \delta(\lambda) \cdot w_k\right). \tag{3.16}$$

The expression on the right hand side of this equality is, for fixed $w_1, \dots, w_k$, analytic in λ in a neighbourhood of $\lambda = 0$. Hence $\Phi(w_1, \dots, w_k)$ is an analytic vector for the group V. If we further restrict $w_1, \dots, w_k$ by demanding that also $0 > \mathrm{Im}w_1 > \cdots > \mathrm{Im}w_k$ it follows from the analyticity properties of the standard vectors and from the third remark in the preceding step that equation (3.16) holds if λ is extended to the positive imaginary axis up to $i\pi/4$.

5. Let $A \in \mathfrak{A}(I_-)$ and let h be an entire analytic function such that $\int d\lambda |h(\lambda + i\mu)| < \infty$ for $\mu \in \mathbb{R}$. We consider the regularized operator

$$A_h = \int d\lambda\, h(\lambda) V(\lambda) A V(\lambda)^{-1} \tag{3.17}$$

which is an element of $\mathfrak{A}(I_-)$ since $V(\lambda)\mathfrak{A}(I_-)V(\lambda)^{-1} \subset \mathfrak{A}(I_-)$ and $\mathfrak{A}(I_-)$ is weakly closed. Due to the regularization it is possible to continue the function $\lambda \to V(\lambda)A_h\Omega$ to the complex plane.

Next, let $0 < \mu < \pi/4$ and let $z_1, \dots, z_k \in S^1$ be such that $\mathrm{Re}z_1 > \cdots > \mathrm{Re}z_k > 0$ and $0 > \mathrm{Im}z_1 > \cdots > \mathrm{Im}z_k$. Because of the result in step 4 and the fact that $V(i\mu)$ is positive and selfadjoint (by Stone's theorem) we obtain in the sense of distributions

$$\begin{aligned}
&\left(V(i\mu)A_h\Omega,\ \varphi(z_1) \cdots \varphi(z_k)\Omega\right) = \\
&= \lim \left(V(i\mu)A_h\Omega,\ \Phi(u_1 z_1, \dots, u_k z_k)\right) = \\
&= \lim \left(A_h\Omega,\ V(i\mu)\Phi(u_1 z_1, \dots, u_k z_k)\right) = \\
&= \prod_{j=1}^{k} D_s\left(z_j;\ \delta(i\mu)\right) \cdot \left(A_h\Omega,\ \Phi(\delta(i\mu) \cdot z_1, \dots, \delta(i\mu) \cdot z_k)\right),
\end{aligned} \tag{3.18}$$

provided the parameters $u_1, \dots, u_k$ approach 1 in the proper order $1 > u_1 > \cdots > u_k$. Making use of the analyticity properties of the standard vectors and the last remark in step 3 the equality of the first and last member in (3.18) can in fact be extended to $0 < \mu < \pi/2$.

We want to proceed to the boundary $\mu = \pi/2$. This is possible since $\delta(i\pi/2) \cdot z = z^{-1}$ and since, due to the restrictions on $z_1, \dots, z_k$, the points $z_1^{-1}, \dots, z_k^{-1} \in I_+$ belong to the region of analyticity of the distribution $\left(A_h\Omega,\ \varphi(z'_1) \dots \varphi(z'_k)\Omega\right)$. (Since $A_h \in \mathfrak{A}(I_-) \subset \mathfrak{A}(I_+)'$ we can apply once again the result of step 2.) Hence, taking into account that $D_s\left(z;\ \delta(i\pi/2)\right) = (-1)^s$, we derive from relation (3.18) the equality of distributions

$$\left(V(i\pi/2)A_h\Omega,\ \varphi(z_1) \cdots \varphi(z_k)\Omega\right) = (-1)^{k \cdot s}\left(A_h\Omega, \varphi(z_1^{-1}) \cdots \varphi(z_k^{-1})\Omega\right) \tag{3.19}$$

at the points $z_1, \dots, z_k$ specified above.

6. We note in passing that the PCT-theorem is a simple consequence of relation (3.19): putting $A = 1$ and making use of the fact that φ is local we infer from (3.19) that

$$\left(\Omega,\ \varphi(z_1) \cdots \varphi(z_k)\Omega\right) = (-1)^{k \cdot s} \cdot \left(\Omega,\ \varphi(z_k^{-1}) \cdots \varphi(z_1^{-1})\Omega\right) \tag{3.20}$$

on certain open sets of points $z_1, \dots, z_k \in S^1$. The distribution on the left hand side of this equation can analytically be continued in $z_1, \dots, z_k$ to the domain $|w_1| > \cdots > |w_k| > 0$, cp. step. 1. Similarly, the distribution on the right hand side can be continued to the domain $|w_k|^{-1} > \cdots > |w_1|^{-1} > 0$. Since the two domains coincide equality (3.20) implies by the Edge-of-the-Wedge theorem that the two analytic functions are equal. Thus in particular their boundary values coincide, proving that equation (3.20) holds for arbitrary $z_1, \dots, z_k \in S^1$. It then follows by direct computation that the operator Θ defined in (3.5) is antiunitary.

7. Since φ is hermitian and local and since $A_h \in \mathfrak{A}(I_+)'$ we can rewrite equation (3.19) in the form

$$\left(V(i\pi/2)A_h\Omega,\ \varphi(z_1)\cdots\varphi(z_k)\Omega\right) = \left((-1)^{ks}\cdot(\varphi(z_k^{-1})\cdots\varphi(z_1^{-1}))^*\Omega,\ A_h^*\Omega\right) \quad (3.21)$$

for the restricted set of points $z_1,\dots,z_k$ specified in step 5. Employing on the right hand side of (3.21) the definition of the PCT-operator Θ and making use of the fact that Θ is antiunitary and $\Theta^2 = 1$ we can proceed further to

$$\left(V(i\pi/2)A_h\Omega,\ \varphi(z_1)\cdots\varphi(z_k)\Omega\right) = \left(\Theta A_h^*\Omega,\ \varphi(z_1)\cdots\varphi(z_k)\Omega\right). \quad (3.22)$$

This equality extends, by the Edge-of-the-Wedge theorem, to all points $z_1,\dots,z_k \in S^1$. Thus, bearing in mind that the field φ generates from the vacuum a dense set of vectors, we find that

$$V(i\pi/2)A_h\Omega = \Theta\cdot A_h^*\Omega. \quad (3.23)$$

We plug now into the definition (3.17) of A_h a sequence af admissible functions h_n which approximate the δ-function, e.g. $h_n(\lambda) = n\cdot h_1(n\lambda)$ for suitable h_1. Because of the continuity properties of the dilations V we then have in the sense of strong convergence $A_{h_n}\Omega \to A\Omega$ and $A^*_{h_n}\Omega \to A^*\Omega$. Replacing in relation (3.23) the function h by the sequence h_n and taking into account that the operator $V(i\pi/2)$ is closed, Θ is bounded, and $\Theta^2 = 1$ we obtain in the limit

$$\Theta\cdot V(i\pi/2)A\Omega = A^*\Omega \text{ for } A \in \mathfrak{A}(I_-). \quad (3.24)$$

As already mentioned, the set $\mathfrak{A}(I_-)\Omega$ is, for real λ, stable under the action of $V(\lambda)$. Hence it is a core for $V(i\pi/2)$. Moreover, $V(i\pi/2)$ is positive and selfadjoint and Θ is antiunitary. By comparing relation (3.24) with the definition (3.3) of the operator S we conclude that S has the polar decomposition $S = \Theta\cdot V(i\pi/2)$. This completes the proof of the lemma.

4. Haag duality and the point at infinity

It is an interesting fact that Haag duality can fail for the restriction of a conformal field to the punctured circle (light ray). The simplest example of this kind, discussed in this section, is the energy density with central charge $c > 1$. We are indebted to Martin Lüscher for pointing out to us the relevant features of this field [L].

We will exploit the fact that the restriction of the energy density to the punctured circle can be expressed in terms of a U(1)-current J by an appropriate modification of the Sugawara formula [FST]. We recall that the current J is fixed by the general conditions in Sect. 2 and the commutation relations

$$\left[J(z_1),\ J(z_2)\right] = i\ \delta'(z_{12}). \quad (4.1)$$

Here we made use of the convenient notation

$$F'(z) := i\ z\ \frac{d}{dz}F(z). \quad (4.2)$$

Note that $F'(z)$ is real if $F(z)$ is real. The Dirac δ-function on the circle is given by

$$\delta(z_{12}) = \sum_n \left(\frac{z_1}{z_2}\right)^n, \quad (4.3)$$

in accord with our choice of a measure on S^1. It is apparent from the c-number commutation relations of the current that we are dealing with the Fock-space representation of a free field.

Let $\mathfrak{D} \subset \mathfrak{H}$ be the dense set of vectors which are in the domain of all positive powers of the conformal Hamiltonian T_0. As is well known, one can represent on $\mathfrak{D}$ the conformal energy density T associated with the current J by the Sugawara formula [FST], which formally reads

$$T(z) = \frac{1}{2} : J^2 : (z). \quad (4.4)$$

Here the dots denote Wick ordering with respect to the underlying Fock-vacuum.

We remove now from S^1 the "point at infinity" $z = -1$ and define on the punctured circle $R = S^1\backslash\{-1\}$ the field [5)]

$$P(z) = J'(z) + i\,\frac{z-1}{z+1}J(z). \quad (4.5)$$

This field is manifestly hermitian and local. It requires some elementary calculations to see that it transforms like a spin 2 field under conformal transformations $W(\gamma)$ induced by the elements $\gamma \in SU(1,1)$ which leave the point -1 fixed. With the help of $P(z)$ one can define perturbed energy densities [5)]

$$T_\varkappa(z) = T(z) + \varkappa\cdot P(z),\ \varkappa \in \mathbb{R}. \quad (4.6)$$

Making use of the c-number commutation relations (4.1) and the Sugawara formula (4.4) one finds by a straight-forward computation that

$$\left[T_\varkappa(z_1),\ T_\varkappa(z_2)\right] = i\left(T_\varkappa(z_1)+T_\varkappa(z_2)\right)\delta'(z_{12}) - i\left(\frac{1}{12}+\varkappa^2\right)\left(\delta'''(z_{12})+\delta'(z_{12})\right). \quad (4.7)$$

Apart from the restriction $z_1, z_2 \in R$, these are the commutation relations of the energy density for central charge $c = 1+12\varkappa^2$ [FST].

The field $T_\varkappa$ transforms covariantly under $W(\gamma)$ for γ in the little group of -1, and the Fock-vacuum Ω is invariant under $W(\gamma)$. It therefore follows from the Lüscher-Mack theorem (or by explicit computations) that the Wightman functions

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5) One may regard this field as section in a field bundle over S^1.

$$\left(\Omega,\ T_x(z_1)\ \dots\ T_x(z_n)\Omega\right) \tag{4.8}$$

can be extended in the variables $z_1,\dots,z_n$ to the entire circle S^1. Moreover, these extensions are invariant under the action of the full conformal group SU(1,1) [FST]. By the reconstruction theorem we thus arrive at the vacuum representation of the conformal energy density for central charge $c = 1+12\cdot x^2$. We denote the corresponding Hilbert space by $\mathfrak{H}_x$ and the field by $\overline{T}_x(z)$, $z \in S^1$. In fact, $\mathfrak{H}_x$ can be identified with a subspace of the original Fock-space $\mathfrak{H}$, and $\overline{T}_x(z)$, $z \in R$ then coincides with the restriction of $T_x(z)$ to that subspace. This identification is possible since the polynomials in fields, smoothed-out with testfunctions from a fixed region, generate a core from the vacuum for all field operators [BZ]. Thus if $\mathfrak{D}_x$ is the subspace of $\mathfrak{H}$ obtained by applying all polynomials in the field operators $T_x(g)$, $\mathrm{supp}\, g \subset R$ to the Fock vacuum Ω, we can identify $\mathfrak{H}_x$ with the closure of $\mathfrak{D}_x$, and we then have

$$\overline{T}_x(f) = \left(T_x(f) \restriction \mathfrak{D}_x\right)^{**} \text{ if } \mathrm{supp} f \subset R. \tag{4.9}$$

Relation (4.9) is the precise statement of the assertion that T_x is the restriction of the conformal energy density $\overline{T}_x$ to the punctured circle.

In analogy to the discussion in Sect. 2 we proceed now to bounded functions of the field operators $T_x(f)$, $\mathrm{supp} f \subset R$ on $\mathfrak{H}$. This step causes no problems since each $T_x(f)$ is a sum of the smoothed-out fields T and J. For the latter fields there hold energy bounds, c.p. relations (2.21) and (2.23), which therefore also apply to $T_x(f)$.

For each interval $I \subset R$ we define with the help of the field T_x a von Neumann algebra $\mathfrak{A}_x(I)$ as in relation (2.13). Since the field operators $T_x(f)$ are selfadjoint for real f it amounts to the same if we set

$$\mathfrak{A}_x(I) = \left\{e^{iT_x(f)} : f \text{ real}, \mathrm{supp} f \subset I\right\}''. \tag{4.10}$$

The system of algebras so defined is a local and covariant net on R. By local we mean here that operators assigned to disjoint intervals in R commute.

We also have to consider the algebras associated with the complement of intervals. If $I \subset R$ is any interval then its complement $I' = R\backslash I$ consists in general of two disjoint pieces $I_<$, $I_>$. Hence if f is any test function with support in I' we can decompose the corresponding field operator $T_x(f)$ into a sum $T_x(f_<) + T_x(f_>)$, where $f_<$, $f_>$ are test functions with support in $I_<$, $I_>$, respectively. The operators $T_x(f_<)$ and $T_x(f_>)$ commute. Thus we conclude that the algebra $\mathfrak{A}_x(I')$, which is generated by the field T_x according to relation (4.10), coincides with the von Neumann algebra generated by $\mathfrak{A}_x(I_<)$, $\mathfrak{A}_x(I_>)$. In formulae

$$\mathfrak{A}_x(I') = \mathfrak{A}_x(I_<) \vee \mathfrak{A}_x(I_>). \tag{4.11}$$

We proceed now to the vacuum representation of this net by restricting the respective algebras to $\mathfrak{H}_x$. In accord with our notation for the conformal field we denote the reduced algebras $\mathfrak{A}_x(I) \restriction \mathfrak{H}_x$ by $\overline{\mathfrak{A}}_x(I)$. It follows from relation (4.9) that these reduced algebras are generated by $\overline{T}_x$.

Let us pick now any closed interval $I \subset R$. Because of locality we have the inclusion

$$\overline{\mathfrak{A}}_x(I) \subset \overline{\mathfrak{A}}_x(I')', \tag{4.12}$$

where the commutant refers to $\mathfrak{H}_x$. It will turn out that this inclusion is proper if $x \neq 0$, i.e. Haag duality fails. In the proof it will pay that we have embedded the field T_x into the current algebra.

Let E_x be the orthogonal projection onto the subspace $\mathfrak{H}_x \subset \mathfrak{H}$. Since $\mathfrak{H}_x$ reduces the von Neumann algebras $\mathfrak{A}_x(I)$, $I \subset R$ it follows that E_x commutes with the corresponding operators. Hence, making use of locality, we get

$$E_x\, e^{iJ(f)} E_x \in \overline{\mathfrak{A}}_x(I')' \text{ if } \mathrm{supp} f \subset I. \tag{4.13}$$

We will show that the operators $E_x\, e^{iJ(f)} E_x$ are in general not contained in $\overline{\mathfrak{A}}_x(I)$. For that purpose we need two simple relations between exponentials of the current and the energy density: let $I_1 \subset R$ be any closed interval containing the given interval I in its interior, and let h be a real test function with support in R which is equal to 1 on I_1. Furthermore, let f be any real testfunction with support in I and let $t \in \mathbb{R}$ be sufficiently small such that the support of $f_t(z) := f(e^{-it}z)$ is contained in I_1. We then have the equality

$$e^{itT_x(h)}\, e^{iT_x(f)}\, e^{-itT_x(h)} = e^{iT_x(f_t)}. \tag{4.14}$$

Even though this relation seems plausible, let us indicate its proof. We first note that is suffices to establish relation (4.14) on $\mathfrak{H}_x$ since $\Omega \in \mathfrak{H}_x$ is separating for the local algebras. On $\mathfrak{H}_x$ we can replace T_x by the conformal field $\overline{T}_x$ and write $\overline{T}_x(h) = \overline{T}_x(e_o) + \overline{T}_x(h-e_o)$, where e_o is the constant function $e_o(z) = 1$, $z \in S^1$. The operator $\overline{T}_x(e_o)$ is the conformal Hamiltonian, generating rotations of S^1, and the operator $\overline{T}_x(h-e_o)$ commutes with all operators in $\overline{\mathfrak{A}}_x(I_1)$ since $h-e_o$ has support in $S^1\backslash I_1$. Moreover, all operators involved are essentially self-adjoint on a common dense domain in $\mathfrak{H}_x$. Relation (4.14) thus follows from the Trotter product formula [RS].

Let us determine next the action of the conformal transformations on the exponential function of the current J. There we obtain (with the same notation as before)

$$e^{itT_x(h)}\, e^{iJ(f)}\, e^{-itT_x(h)} = \eta \cdot e^{iJ(f_t)}, \tag{4.15a}$$

where η is a phase factor given by

$$\eta = \exp\left(2x \oint \frac{dz}{2\pi i z}\left[\frac{1}{1+e^{it}z} - \frac{1}{1+z}\right] f(z)\right). \tag{4.15b}$$

The proof of this statement is accomplished in a similar fashion as that of relation (4.14): making use of relation (4.6) one splits now the operator $T_\chi(h)$ into three pieces,

$$T_\chi(h) = T(e_o) + \chi P(h) + T(h-e_o'). \tag{4.16}$$

The first term in this sum is the conformal Hamiltonian associated with the current J. The second term is linear in J and hence induces the automorphism $J(g) \rightarrow J(g) + i\chi\left[P(h),\, J(g)\right]\cdot 1$ of the current algebra. The third term, finally, commutes with all operators $e^{iJ(g)}$, $\mathrm{supp}\, g \subset I_1$ since $h-e_o$ has support in $S^1\backslash I_1$. Hence, applying the Trotter product formula twice, one arrives at relation (4.15) in a straight-forward manner.

Relations (4.14) and (4.10) tell us that the operators $e^{it\bar{T}_\chi(h)}$ induce, for sufficiently small t, special conformal transformations of the algebra $\bar{\mathfrak{A}}_\chi(I)$. Hence in view of the invariance of the vacuum state under these transformations (by the Lüscher-Mack theorem) we get

$$\left(\Omega,\, e^{it\bar{T}_\chi(h)}\bar{A}e^{-it\bar{T}_\chi(h)}\Omega\right) = (\Omega,\, \bar{A}\Omega) \text{ if } \bar{A}\in\bar{\mathfrak{A}}_\chi(I). \tag{4.17}$$

On the other hand we obtain with the help of relation (4.15a) the equality

$$\left(\Omega,\, e^{it\bar{T}_\chi(h)}E_\chi e^{iJ(f)}E_\chi e^{-it\bar{T}_\chi(h)}\Omega\right) = \eta\cdot\left(\Omega,\, E_\chi e^{iJ(f)}E_\chi\Omega\right). \tag{4.18}$$

Moreover, from (4.15b) we see that for $\chi \neq 0$, $f \neq 0$, and almost all t there holds $\eta \neq 1$. By comparing relations (4.17) and (4.18) we conclude that $E_\chi e^{iJ(f)}E_\chi$ is not an element of the algebra $\bar{\mathfrak{A}}_\chi(I)$. Hence the inclusion (4.12) is proper, and Haag duality fails.

As already pointd out, one can enforce Haag duality von R by extending the local algebras: the net

$$\bar{\mathfrak{C}}_\chi(I) = \bar{\mathfrak{A}}_\chi(I')',\ I\subset R \tag{4.19}$$

satisfies duality if one defines the algebras associated with I' in analogy to (4.11) by

$$\bar{\mathfrak{C}}_\chi(I') = \bar{\mathfrak{C}}_\chi(I_<) \vee \bar{\mathfrak{C}}_\chi(I_>) = \bar{\mathfrak{A}}_\chi(I'). \tag{4.20}$$

The second equality follows from the fact that the original (conformal) net satisfies duality on S^1. The extended net transforms covariantly under the conformal transformations $W_\chi(\gamma)$ on $\mathfrak{H}_\chi$ if $\gamma\in SU(1,1)$ is element of the little group of -1. But it does not transform properly under (small) rotations $U_\chi(t)$.

To see this, let $I_o \subset R$ be any closed semi-circle. As we have shown, the algebra $\bar{\mathfrak{C}}_\chi(I_o)$ is a proper extension of $\bar{\mathfrak{A}}_\chi(I_o)$. In particular it contains some operator which does not commute with all operators in $\bar{\mathfrak{A}}_\chi(I_o)' = \bar{\mathfrak{A}}_\chi(I_o{}^c)$. By a rotation about an angle t_o we can turn I_o onto the upper semicircle I_+. It then follows from covariance of the original net that the algebra

$U_\chi(t_o)\bar{\mathfrak{C}}_\chi(I_o)U_\chi(t_o)^{-1}$ contains some operator which does not commute with $\bar{\mathfrak{A}}_\chi(I_-)$. But $\bar{\mathfrak{A}}_\chi(I_-) = \bar{\mathfrak{C}}_\chi(I_-)$, hence the algebra $U_\chi(t_o)\bar{\mathfrak{C}}_\chi(I_o)U_\chi(t_o)^{-1}$ does not coincide with $\bar{\mathfrak{C}}_\chi(I_+)$, as it would be required by covariance.

It should be noticed that Haag duality may hold for the restriction of some conformal fields to R. The simplest example of this kind is the current J, as can be deduced from the discussion of free fields in [HL]. It is instructive to see explicitly how the equality of the algebras $\mathfrak{A}(I)' = \mathfrak{A}(I^c)$ and $\mathfrak{A}(I')$ comes about in this case: let f be any real test function with support in I^c. We decompose f into the sum of three testfunctions $f_<$, $f_>$, and f_δ, which have support in $I_<$, $I_>$, and a small interval I_δ of length $\delta > 0$ about the point $z = -1$, respectively. Making use of the c-number commutation relations of J we obtain

$$e^{iJ(f_< + f_>)}\, e^{-iJ(f)} = \zeta\cdot e^{-iJ(f_\delta)}, \tag{4.21}$$

where ζ is a phase factor. The unitary operator on the right hand side is localized in I_δ. Hence, as δ approaches 0, all weak limit points of this operator are multiples of the identity since there do not exist non-trivial bounded operators which are localized at a point [Ro2]. It is crucial now that, because of the tame ultraviolet behaviour of J, these multiples are different from 0 if f_δ is properly chosen. (This can be verified by calculating the vacuum expectation value of $e^{-iJ(f_\delta)}$.) Thus the operators $e^{iJ(f_< + f_>)}\in\mathfrak{A}(I')$ have the operator $c\cdot e^{iJ(f)}$ for some $c \neq 0$ as a weak limit point. This shows that $\mathfrak{A}(I') = \mathfrak{A}(I^c)$. In view of this example one may interpret the lack of duality in the case of the energy density as an ultraviolet problem.

We conclude this paper with the remark that the explicit representation (4.6) of the fields T_χ can also be used to construct representations of the energy density for lowest weight $h > 0$. These representations are induced by automorphisms $J(z) \rightarrow J(z) + \rho(z)\cdot 1$ of the current algebra on R, where ρ is a real function on R. Applying these automorphisms to T_χ we obtain fields

$$T_\chi(z) + \rho(z)\cdot J(z) + \left(\frac{1}{2}\rho^2(z) + \chi\rho'(z) + i\chi\,\rho(z)\frac{z-1}{z+1}\right) \tag{4.22}$$

which also satisfy the commutation relations (4.7). As in the case of T_χ we consider the subrepresentation of these fields which is induced by the Fock-vacuum Ω. If Ω is to represent a state of lowest weight, then ρ must be chosen in such a way that the c-number term in (4.22) is constant. The ensuing differential equation for ρ is of Riccati type and can be solved explicitly. Its meromorphic solutions are of the form

$$\rho(z) = -\, i\chi\,\frac{z-1}{z+1} + \eta, \tag{4.23}$$

where η is an arbitrary real constant. Inserting these solutions into (4.22) and recalling the definition of T_χ we arrive at the fields

$$T_{\chi,\eta}(z) = T(z) + \chi J'(z) + \eta J(z) + \frac{\chi^2+\eta^2}{2}. \tag{4.24}$$

It is readily checked that these fields can be extended to conformal fields on S^1 in the representation induced by Ω, and Ω then represents a state of lowest weight $h = (x^2+\eta^2)/2$. Hence we obtain this way all representations of central charge $c \geq 1$ and lowest weight $h \geq (c-1)/24$. These representations were also found by Carey and Ruijsenaars within a field-theoretic setting [R] [6].

Since the above automorphisms of the current algebra on R are locally unitarily implementable, one can show that the representations of the energy density for fixed $c \geq 1$ and the given range of h are locally normal with respect to each other. This result is the mathematical expression of the fact, mentioned in the Introduction, that h is a localizable charge. Making use of Haag duality it then follows by a standard argument [DHR] that all these representations arise from endomorphisms of the net defined in (4.10).

This argument merely establishes the existence of such endomorphisms. In view of the concrete and simple form (4.24) of the underlying representations one may, however, hope to construct them also explicitly. Such a result would be a vital step [BMT] towards the classification of conformal fields associated with an energy density with central charge $c > 1$.

6) We note that one may proceed further and consider the fields $\left(T_x(z) \otimes 1 + 1 \otimes T_{0,\eta}(z)\right)$ which carry central charge $c = 2+12x^2$. The vector $\Omega \otimes \Omega$ then represents a state of lowest weight $h = \eta^2/2$.

References

[BMT] D. Buchholz, G. Mack, I.T. Todorov: The current algebra on the circle as a germ of local field theories. Nucl. Phys. B (Proc. suppl.) 5B (1988) 20-56

[FRS] K. Fredenhagen, K.H. Rehren, B. Schroer: Superselction sectors with braid group statistics and exchange algebras. I: General theory. Commun. Math. Phys. 125 (1989) 201-226

[Sch] B. Schroer: A trip to scaling land. In: V Brazilean Symposium on Theoretical Physics. E. Ferreira (ed.) Livros Tècnicos e Scientificos Edit. Rio de Janeiro 1974

[ML] G. Mack, M. Lüscher: The energy momentum tensor of a critical quantum field theory in 1+1 dimensions. Unpublished manuscript (1976)

[FST] P. Furlan, G.M. Sotkov, I.T. Todorov: Two dimensional conformal quantum field theory. Riv. Nuovo Cimento 12, N.6 (1989) 1-202

[DHR] S. Doplicher, R. Haag, J.E. Roberts: Local observables and particle statistics. I. Commun. Math. Phys. 23 (1971) 199-230 II. Commun. Math. Phys. 35 (1974) 49-85

[H] R. Haag: Les problèmes mathématiques de la théorie quantique des champs. In: Lille Conference 1957. Paris: Èditions du CNRS 1959

[Bo] H.J. Borchers: Über die Mannigfaltigkeiten der interpolierenden Felder zu einer kausalen S-Matrix. Il Nuovo Cimento 15 (1960) 784-794

[La] L. Landau: On local functions of fields. Commun. Math. Phys. 39 (1974) 49-62

[BW] J.J. Bisognano, E.H. Wichmann: On the duality condition for a hermitian scalar field. Journ. Math. Physics 16 (1975) 985-1007

[Ro1] J.E. Roberts:
Spontaneously broken gauge symmetries and superselection rules.
In: International school of Mathematical Physics.
Universita' di Camerino 1974

[Ro2] J.E. Roberts:
Some applications of dilation invariance to structural questions in the theory of local observables.
Commun. Math. Phys. 37 (1974) 273-286

[Sa] S. Sakai:
C^*-algebras and W^*-algebras.
Berlin, Heidelberg, New York: Springer 1971

[DSW] W. Driessler, S.J. Summers, E.H. Wichmann:
On the connection between quantum fields and von Neumann algebras of local operators.
Commun. Math. Phys. 105 (1986) 49-84

[LRT] P. Leyland, J.E. Roberts, D. Testard:
Duality for quantum free fields.
Preprint, Marseille 1978

[DF] W. Driessler, J. Fröhlich:
The reconstruction of local observable algebras from the Euclidean Greens functions of a relativistic quantum field theory.
Ann. Inst. H. Poincaré 27 (1977) 221-236

[Bu] D. Buchholz:
Harmonic analysis of local operators.
Commun. Math. Phys. (to appear)

[BR] O. Bratelli, D.W. Robinson:
Operator algebras and quantum statistical mechanics I.
Berlin, Heidelberg, New York: Springer 1979

[L] M. Lüscher:
How to derive the Virasoro algebra from dilation invariance.
Talk given at Max Planck Institut für Physik und Astrophysik, Munich 1988

[BZ] H.J. Borchers, W. Zimmermann:
On the self-adjointness of field operators.
Il Nuovo Cimento 31 (1963) 1047-1059

[RS] M. Reed, B. Simon:
Methods of modern mathematical physics. I: Functional analysis.
New York, San Francisco, London: Academic Press 1972

[HL] P.D. Hislop, R. Longo:
Modular structure of the local algebras associated with the massless scalar field
Math. Phys. 84 (1982) 71-85

[R] S. Ruijsenaars:
Dirac quantization, Virasoro and Kac-Moody algebras and boson-fermion correspondence in 2N dimensions:
In: XIth International Congress on Mathematical Physics, Swansea 1988.
Bristol, New York: Adam Hilger 1989