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THE GENERAL BESSELIAN AND LEGENDRIAN PATH INTEGRAL

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ABSTRACT

The general Besselian and Legendrian path integrals based on the confluent and hypergeometric Natanzon potentials are calculated. These two solutions cover all other path integral representations which are related to the radial harmonic oscillator and the (modified) Pöschl-Teller path integral.

The general structure of potentials usually known nowadays as Natanzon potentials were introduced in [1]. With their six parameter structure they are designed in such a way that a wider range of shapes and potential wells is allowed in comparison with other well-known potential problems in quantum mechanics, e.g., the Morse potential, the radial harmonic oscillator, the Coulomb potential, and the class of hypergeometric potentials as contained in the Pöschl-Teller and modified Pöschl-Teller potentials [2], e.g., the Rosen-Morse-, Manning-Rosen-, Eckart-, Scarf-like-, or the Hultén-potential. They are subject to many applications, e.g., in the study of solvable potentials in quantum mechanics in general, c.f. [3]–[10], in the study of molecular physics for modeling a more realistic single particle electronic shell structure, in atomic physics for quark-antiquark forces, charge densities of nuclei, or in solid state physics, c.f.e.g., [4, 5] and references therein. These two classes of Natanzon potentials cover most known potentials for which an analytic solution to the bound and continuous state problem can be found.

After the publication of the original paper [1], where already the general structure of the solutions, i.e., the energy eigenvalue conditions, the (unnormalized) wave functions, and (in the confluent case) the Green's function, was derived, a considerable amount of work was devoted to the study of the dynamical symmetry of these potentials. It turned out that the spectrum generating algebra is of a SO(2, 1) structure, e.g. Refs. [3, 4, 7, 8]. Also the relation to supersymmetric quantum mechanics was investigated, e.g. [9, 11]–[13]. A subclass of a two-parameter symmetric hypergeometric Natanzon potential was studied in [4, 6, 11]. In general, the Natanzon potentials turn out to be not shape invariant [11].

However, for the two most general classes of Natanzon potentials there seems to exist no systematic path integral discussion. The only exception was [6], where a two-parameter hypergeometric subclass was investigated by path integration. The purpose of this Letter is to fill this gap. The two path integral solutions of the confluent and hypergeometric Natanzon potentials which I find accessible to path integration by means of the path integral representation of the radial harmonic oscillator and the (modified) Pöschl-Teller potential, I want to call the *general Besselian* and *Legendrian path integrals*, respectively. By choosing a path integral approach we succeed in gaining comprehensive information about the bound-stated solutions of these potentials (if they exist), and what is often more important, about the scattering states which eventually allow for the calculation of cross-sections and phase-shifts. As we shall see, in both cases it is only possible to evaluate the corresponding Green's function of the problem, and not the propagator itself. However, I consider this as a pure technical difficulty, and not as a principal conceptual drawback of the path integral calculations.

Let us start with the case of the hypergeometric Natanzon potentials. I use the canonical path integral formulation as developed in, e.g. [14]–[18]. The potentials are defined by (note the different notations used in the literature)

$$V(r) = \frac{\hbar^2}{2m}\frac{fz(z-1)+h_0(1-z)+h_1 z}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \ , \tag{1}$$

where $R(z) = a_0 z^2 + b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$. In order to calculate the path integral representation corresponding to the potential (1) we perform the transformation $r \mapsto z$ together with the time-substitution $dt = ds/z'^2$ according to, e.g., [17]–[19] and references therein, such that the new pseudo-time s'' can be introduced via the constraint $\int_0^{s''} ds/z'^2 = T = t'' - t'$. This space-time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation

$$K(r'',r';T)=\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2-V(r)\right]dt\right\}$$

$$=\left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\int\limits_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot z^2-\frac{\hbar^2}{8m}\left(\frac{f}{z(z-1)}+\frac{h_0}{z^2(1-z)}+\frac{h_1}{z(1-z)^2}\right)+\frac{ER(z)}{4z^2(1-z)^2}\right]ds\right\}\ . \tag{2}$$

We perform a further space-time transformation $z=\tanh^2x$, $x>0$, together with the time substitution $dt=4\tanh^2 x ds/\cosh^4 x$. The quantum potential emerging from the Schwarz derivative of z with respect to x is given by $\Delta V=\hbar^2(4+3/\sinh^2x-3/\cosh^2x)/8m$, and we obtain the path integral representation for the modified Pöschl-Teller potential with the solution according to [18, 20]–[22]

$$K(r'',r';T)=\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{dE}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty ds''\,\mathrm{e}^{\mathrm{i}[(a_0+b_0+c_0)E-\hbar^2(h_1+1)/2m]s''/\hbar}$$
$$\times\int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot x^2-\frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\sinh^2x}-\frac{\nu^2-1/4}{\cosh^2x}\right)\right]ds\right\}$$
$$=\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}G(r'',r';E)\ , \tag{3}$$

with the Green's function $G(E)$ of the hypergeometric Natanzon potential given by

$$G(r'',r';E)=\Big(R(r')R(r'')\Big)^{1/4}\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\Big((1-z(r'))(1-z(r''))\Big)^{(m_1-m_2)/2}\Big(z(r')z(r'')\Big)^{(m_1+m_2)/2}$$
$$\times{}_2F_1\Big(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;1-z_<^2\Big)$$
$$\times{}_2F_1\Big(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;z_>^2\Big)\ . \tag{4}$$

${}_2F_1(a,b;c;z)$ is the hypergeometric function. Here denote $m_{1,2}=\frac{1}{2}(\eta\pm\sqrt{-2mE'}/\hbar)$, $L_\nu=\frac{1}{2}(\nu-1)$, and I have abbreviated

$$\eta^2=h_0+1-\frac{2mc_0E}{\hbar^2}\ ,\qquad \nu^2=f+1-\frac{2ma_0E}{\hbar^2}\ , \tag{5}$$

and $E'=(a_0+b_0+c_0)E-\hbar^2(h_1+1)/2m$. Note that the numbers η and ν are square roots and the specific sign they take may vary in different examples. The poles of the Green's function determine the bound state energy eigenvalues E_n ($n\in\mathbb{N}_0$)

$$\sqrt{h_0+1-\frac{2mc_0E_n}{\hbar^2}}-\sqrt{f+1-\frac{2ma_0E_n}{\hbar^2}}-\sqrt{h_1+1-\frac{2m}{\hbar^2}(a_0+b_0+c_0)E_n}=2n+1\ . \tag{6}$$

(6) is an equation of fourth degree in E_n and can be cast into the canonical form $\Big(B_1 = \frac{\hbar^2}{2m}(h_0+1)$, $B_2 = \frac{\hbar^2}{2m}(f+1)$, $C_1 = \frac{\hbar^2}{2m}(h_1+1)$, $\alpha = a_0+b_0+c_0$, $C_2 = C_1 - B_1 - B_2 + 4\tilde{n}^2$, $\tilde{n} = \hbar(n+\frac{1}{2})/\sqrt{2m}\Big)$

$$E_n^4 + bE_n^3 + cE_n^2 + dE_n + e = 0 \ , \tag{7}$$

$$\left.\begin{aligned}
a &= (b_0^2 - 4a_0c_0)^2 \ , \\
b &= \frac{4}{a}\Big[(b_0^2 - 4a_0c_0)(2a_0B_1 + 2B_2c_0 - b_0C_2 + 8\alpha\tilde{n}^2) + 64\alpha a_0c_0\tilde{n}^2\Big] \ , \\
c &= \frac{8}{a}\Big[2(4\alpha\tilde{n}^2 - a_0B_1)^2 + \tfrac{1}{2}(2B_2c_0 - b_0C_2)^2 + \tfrac{1}{4}b_0^2C_2^2 + B_1B_2(8a_0c_0 - b_0^2) \\
&\qquad -a_0C_2(c_0C_2 + 2b_0B_1) - 4\tilde{n}^2(2\alpha b_0C_2 + b_0^2C_1 + 4\alpha c_0B_2 + 4a_0c_0C_1)\Big] \ , \\
d &= \frac{8}{a}\Big[(B_2c_0 - \tfrac{1}{2}b_0C_2 - 4\alpha\tilde{n}^2 + a_0B_1)(C_2^2 - 4B_1B_2 + 16C_1\tilde{n}^2) \\
&\qquad +8C_2\tilde{n}^2(\alpha C_2 + 2b_0C_1)\Big] \ , \\
e &= \frac{1}{a}\Big[(C_2^2 - 4B_1B_2 + 16C_1\tilde{n}^2)^2 - 64C_1C_2^2\tilde{n}^2\Big] \ .
\end{aligned}\right\} \tag{8}$$

Equation (7) can be solved by considering the solutions of the quadratic equation [26]

$$E_n^2 + (b+A)\frac{E_n}{2} + y + \frac{by-d}{A} = 0 \ , \tag{9}$$

where $A = \pm\sqrt{8y + b^2 - 4c}$, and y is any of the real roots of the cubic equation

$$8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 = 0 \ . \tag{10}$$

The residua at the poles of the Green's function give the correctly normalized bound state wave functions in terms of Jacobi polynomials in $z(r)$, and the analysis of the behaviour of $G(E)$ on the cuts yields the scattering states. In the special case of Refs. [4, 6, 11] (7) reduces to a quadratic equation in E_n which is easy to solve. The results concerning the bound states coincide with Refs. [1, 8, 11]–[13].

The class of the confluent Natanzon potentials can be obtained by the substitution [1] $a_0 = \sigma_2/\tau^2$, $b_0 = \sigma_1/\tau$, $f = g_2/\tau^2$, $h_1 - h_0 - f = g_1/\tau$, $z = h/\tau$, and taking into account the limit $\tau \to 0$. This yields

$$V(r) = \frac{\hbar^2}{2m}\frac{g_2h^2 + g_1h + \eta}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \ , \tag{11}$$

where $R(r) = \sigma_2h^2 + \sigma_1h + c_0$, and $h = h(r)$ is implicitly defined by the differential equation $h'/2h = 1/\sqrt{R(r)}$. The variable r and the function $h = h(r)$ are assumed to be positive. In order to make the corresponding representation of the propagator of the potential (11) accessible to path integration we perform the transformation from the coordinate $r \mapsto h$ accompanied by the time-substitution $dt = R(h)ds/4h^2$. This space-time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation of the Coulomb potential in the polar coordinate h. This in turn can be evaluated by a further space-time transformation by means of the path integral solution of the radial harmonic oscillator [23], e.g. [16, 18, 19, 24, 25] and references therein. Therefore we obtain

$$K(r'', r'; T) = \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V(r)\right]dt\right\}$$

$$
= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')} \right)^{1/2} \int_{\mathrm{I\!R}} \frac{dE}{2\pi\hbar} \, \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds'' \, \mathrm{e}^{\mathrm{i}(\sigma_2 E - \hbar^2 g_2/2m)s''/4\hbar}
$$

$$
\times \int\limits_{h(0)=h'}^{h(s'')=h''} \mathcal{D}h(s) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_0^{s''} \left[\frac{m}{2}\dot{h}^2 + \frac{E\sigma_1 - \hbar^2 g_1/2m}{4h} - \frac{\hbar^2}{2m} \frac{\eta - 2mc_0 E/\hbar^2}{4h^2} \right] ds \right\}
$$

$$
= \int_{\mathrm{I\!R}} \frac{dE}{2\pi\mathrm{i}} \, \mathrm{e}^{-\mathrm{i}ET/\hbar} G(r'', r'; E) \tag{12}
$$

with the Green's function $G(E)$ of the confluent Natanzon potential given by

$$
G(r'', r'; E) = \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')} \right)^{1/2} \frac{1}{\hbar} \sqrt{-\frac{m}{2E'}}
$$

$$
\times \frac{\Gamma(\frac{1}{2} + \lambda - \kappa)}{\Gamma(2\lambda + 1)} W_{\kappa,\lambda}\left(\sqrt{-8mE'} \frac{h_>}{\hbar} \right) M_{\kappa,\lambda}\left(\sqrt{-8mE'} \frac{h_<}{\hbar} \right) \quad . \tag{13}
$$

$W_{\kappa,\lambda}(z)$ and $M_{\kappa,\lambda(z)}$ are Whittaker functions. Here denote

$$
\lambda^2 = \frac{1}{4}\left(\eta + 1 - \frac{2mc_0 E}{\hbar^2} \right) \quad , \qquad \kappa = \frac{E\sigma_1 - \hbar^2 g_1/2m}{4\hbar} \sqrt{-\frac{m}{2E'}} \quad , \tag{14}
$$

and $E' = (\sigma_2 E - \hbar^2 g_2/2m)/4$. From the poles in the Green's function the bound state energy eigenvalues E_n can be determined by the equation ($n \in \mathrm{I\!N}_0$)

$$
\frac{E_n\sigma_1 - \hbar^2 g_1/2m}{2\hbar} \sqrt{\frac{m}{\hbar^2 g_2/m - 2\sigma_2 E_n}} - \sqrt{\eta + 1 - \frac{2mc_0 E_n}{\hbar^2}} = 2n + 1 \quad , \tag{15}
$$

which is again an equation of fourth degree in E_n and can be cast into the canonical form ($\tilde{\eta} = (2n+1)^2 + \eta + 1, \hat{\eta} = (2n+1)^2 - \eta - 1$)

$$
E_n^4 + bE_n^3 + cE_n^2 + dE_n + e = 0 \quad , \tag{16}
$$

$$
\left.
\begin{aligned}
a &= 16m^4\left(\sigma_1^2 - 4c_0\sigma_2\right)^2 \quad , \\
b &= \frac{32m^3\hbar^2}{a}\left[2\sigma_1^2\sigma_2\tilde{\eta} + 8c_0\sigma_2^2\hat{\eta} - \sigma_1^3 g_1 - 8c_0^2 g_2\sigma_2 + 2c_0 g_2\sigma_1^2 + 4c_0 g_1\sigma_1\sigma_2\right] \quad , \\
c &= \frac{8m^2\hbar^4}{a}\Big[3g_1^2\sigma_1^2 + 8\sigma_2\hat{\eta}(\sigma_2\hat{\eta} - 4c_0 g_2) + 8c_0^2 g_2^2 \\
&\qquad\qquad -4\sigma_1\tilde{\eta}(g_2\sigma_1 + 2g_1\sigma_2) - 8c_0 g_1 g_2\sigma_1 - 4c_0 g_1^2\sigma_2\Big] \quad , \\
d &= \frac{16m\hbar^6}{a}\left[4g_2\hat{\eta}(c_0 g_2 - \sigma_2\hat{\eta}) + \tilde{\eta}(2g_1 g_2\sigma_1 + g_1^2\sigma_2) + c_0 g_1^2 g_2 - \tfrac{1}{2}g_1^3\sigma_1\right] \quad , \\
e &= \frac{\hbar^8}{a}\left[g_1^4 + 8g_2(g_2\hat{\eta}^2 - g_1^2\tilde{\eta})\right] \quad .
\end{aligned}
\right\} \tag{17}
$$

Equation (16) can be solved in the same way as (7). The results coincide with, e.g., Refs. [1, 7, 11, 13]. Taking the residua in $G(E)$ yields the correctly normalized bound-state wave functions which are proportional to Laguerre polynomials in $h(r)$, and the scattering states are determined by analyzing the cuts in $G(E)$, and are proportional to Whittaker functions. In particular, the bound-state solutions stated in the literature lack the proper normalization, and the continuous spectrum is completely neglected in most treatments. Due to the wide range of classes covered by the confluent Natanzon potential it can not be expected that a simple formula for the scattering states can be derived for all cases. Because the roots of the equations (7,16) are analytically known the explicit form of the bound state wave functions, if they exist, can be derived in principle. However, due to the very complicated structure of the relevant equations, this is omitted here.

In this Letter I have presented path integral treatment for the hypergeometric and confluent Natanzon potentials, which I have called the general Legendrian and Besselian path integrals, respectively. In both cases a space-time transformation was essential in order to solve the problem. My approach show that the path integral is superior to other methods. The path integral provides the natural way in which the structure of the solutions is manifest. In spite of the fact that the bound state energy-level conditions are rather complicated, closed form solutions in terms of the Green's functions are still possible. This is quite surprising because the exact analytic form of a particular Natanzon potential is only implicitly defined and may even not be known analytically. The poles of the Green's functions gave the bound state energy-levels, and the cuts provided the scattering states. The fact that the eigenvalue equations for E_n are actually fourth degree equations does not seem to have been noticed previously. In addition, the Green's function representation of the hypergeometric Natanzon potentials wasn't been yet available in the literature. Therefore the two path integral representations (4) and (13) now contain *all former* path integral solutions as special cases, and at the same time generalizing them, except where in addition particular boundary conditions must be taken into account, c.f.e.g. [5].

The results are very satisfactory. The Schrödinger approach, may it be the usual study in non-relativistic quantum mechanics or a super symmetric investigation, fails to see the potential problem as a whole. In comparison, the path integral provides comprehensive information, about the propagator, in case it can be explicitly computed, the Green's function with its poles and cuts, the bound-state wave functions, the continuous spectrum, and the necessary boundary conditions. The latter could be extracted by a detailed Green's function analysis.

A generalization of the one-dimensional Natanzon potentials to a three-dimensional investigation is straightforward as long as one is allowed to choose the angular momentum dependence freely. The examples of Ref. [4] show that this is not always the case, or the question of physical reasonability may arise.

Let us finally note that by some appropriate changes the notion "conditionally solvable" can be given to a modification of the potential (11). The choices $c_0 = \eta = 0$, $g_1 h \mapsto g_1 h^3$, and $\sigma_1 h \mapsto \sigma_1 h^4$, respectively $c_0 = \eta = 0$, $g_1 h \mapsto g_1 h^4$, $\sigma_1 h \mapsto \sigma_1 h^3$ produce generalizations of the two kinds of conditionally solvable potentials as discussed in [5]. They have been called a modified Coulomb potential and a radial confinement potential, respectively. These new "conditionally solvable Natanzon potentials" have four free parameters. Of course, a similar consideration can be made for (1), where four new classes can be found. A detailed analysis of these new classes of potentials will be presented in a future publication.

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