Analytical Solution of the Field Integrals of a Cylindrical Grid Element

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Abstract

Field integrals on a cylindrical grid require numerical integrations which can become expensive in terms of computing time and require to compromise the accuracy, respectively. Here the analytical solution of the field integrals over a cylindrical volume element are presented and a very efficient implementation is discussed. The numerical implementation is based on modified relations originally derived by Carlson. A new limiting value and a new transformation relation of the complex elliptic integral of third kind are presented.

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I. INTRODUCTION

Many physical and technological problems are described by a Poisson potential problem – or the associated field problem – in cylindrical coordinates. Examples range from the calculation of the gravitational fields in astronomy (cosmological clouds, stars, galaxies) [1–4], to the calculation of electromagnetic fields in plasma physics [5–7], coil design [8] or charged particle beam optics (ion and electron accelerators, electron microscopy, spectroscopy). In numerical approaches, often, a cylindrical grid consisting of concentric rings with inner radius r_i, outer radius r_o and longitudinal coordinates z_1 and z_2 is employed. With sufficient grid resolution the source term, i.e. the mass, charge or current density, within a ring can be approximated as a constant, so that the problem is essentially reduced to the integration of the potential or field equations over a ring of constant density and rectangular cross-section. The standard approach for this integration is based on an analytical integration over the angular coordinate, which leads to equations containing elliptic integrals. The remaining integrations over the radial and longitudinal coordinates are then performed numerically, e.g., with a Gauss quadrature [9].

Widespread tools for the design of charged particle beams are numerical tracking codes, which calculate the trajectories of the particles self-consistently under the influence of the space charge field generated by the bunch of all particles and additional external fields of magnets and cavities [10, 11]. In order to take the dynamical development of the bunch parameters (transverse and longitudinal size, energy, particle number) into account the space charge field needs to be frequently recalculated. Besides precision the numerical efficiency, i.e. the computing time, is hence of great significance.

The space-charge calculations in the tracking codes are in general performed as electro-static calculations in the average rest system of the bunch. A challenge for an efficient numerical integration is that the aspect ratio of the grid cells $\frac{r_o - r_i}{z_2 - z_1}$ varies drastically when a bunch of particles is emitted from a cathode and accelerated up to relativistic energies. In front of the cathode the particles pile up due to near-zero velocities, making the aspect ratio very large, while it decreases to very small values later, when the bunch length in the rest system increases due to dynamical effects and due to relativistic spatial dilation.

In the following a full analytical solution of the integral over a ring with a rectangular cross-section is presented. In the second section the problem is formulated in a most general way;

in the following section the derivation of the field integrals is discussed. Programs for the symbolic computation of the integrals, mainly Mathematica [12], but also Maple [13], were heavily employed throughout the work. The programs were, however, not able to directly calculate the complete integrals and bring them into a suitable form, so that many interventions were required. Some of the related problems are discussed in section III where also an unknown limiting value and a new transformation formula for the complex elliptic integral of the third kind are presented. Both relations are not documented in the standard literature. While the derivations and the intermediate results tend to be long and intricate, the final results are amazingly simple. The implementation of the analytical solution in a numerical code is presented in section IV. In order to circumvent the poles inherent to the elliptic integrals, modified versions of algorithms, originally proposed by Carlson for the calculation of the complete integrals are presented in section V, which leads to simple and efficient routines for the calculation of the fields.

II. GENERALIZED FORMULATION

The scalar potential at a probe point $\vec{r}_p = r_p, \varphi_p, z_p$ due to a source with coordinates $\vec{r}_r = r_r, \varphi_r, z_r$ inside the ring reads as

$$\phi(r_p, \varphi_p, z_p) = C_\varrho \int\limits_{r_i}^{r_o} \int\limits_{0}^{2\pi} \int\limits_{z_1}^{z_2} \frac{1}{\tilde{R}}\, r_r\, dr_r\, d\varphi_r\, dz_r, \tag{1}$$

where $\tilde{R}$ denotes the distance between source and probe point and where $C_\varrho = \frac{\varrho_e}{4\pi\varepsilon_0}$ with the charge density ϱ_e and the vacuum permeability ε_0 for electro-static problems, or $C_\varrho = \gamma\varrho_g$ with the mass density ϱ_g and the gravitational constant γ for gravitational problems. Boundaries of the integrals are dropped henceforth.

Equivalently the vector potential of a current distribution is given by

$$\vec{A}(r_p, \varphi_p, z_p) = \vec{C}_j \times \iiint \frac{1}{\tilde{R}}\, r_r\, dr_r\, d\varphi_r\, dz_r, \tag{2}$$

where $\vec{C}_j = \frac{\mu_0}{4\pi}(j_r, j_\varphi, j_z)$ with the current densities j_r, j_φ and j_z.

Without loss of generality $\varphi_p = z_p = 0$ is assumed and the index r on φ and z is dropped. Moreover, the dimensionless quantities $\rho = \frac{r_r}{r_p}$ and $\zeta = \frac{z}{r_p}$ are introduced. The on-axis fields, i.e. $r_p = 0$, are well known and can be calculated separately. This case is hence ignored in the following in favor of the description in dimensionless quantities, except for the very end, were the transition to this limiting case is briefly discussed. The distance $R = \frac{\tilde{R}}{r_p}$ is thus given by

$$R^2 = \frac{\tilde{R}^2}{r_p^2} = 1 + \rho^2 + \zeta^2 - 2\rho\cos\varphi \tag{3}$$

and the derivatives of the integral part of Eq. 1 times -1 read as

$$\begin{aligned} G_r &= -\tfrac{1}{C_\varrho}\tfrac{d}{dr_p}\phi(r_p) = r_p \iiint \tfrac{\rho(1-\rho\cos\varphi)}{R^3}\,d\rho\,d\varphi\,d\zeta, \\ G_\varphi &= -\tfrac{1}{C_\varrho r_p}\tfrac{d}{d\varphi}\phi(r_p) = r_p \iiint \tfrac{\rho^2\sin\varphi}{R^3}\,d\rho\,d\varphi\,d\zeta = 0, \\ G_z &= -\tfrac{1}{C_\varrho}\tfrac{d}{dz}\phi(r_p) = r_p \iiint \tfrac{\rho\zeta}{R^3}\,d\rho\,d\varphi\,d\zeta, \end{aligned} \tag{4}$$

The electro-static or gravitational field components are given by

$$\begin{aligned} F_r &= C_\varrho G_r, \\ F_z &= C_\varrho G_z, \end{aligned} \tag{5}$$

while the magneto-static fields are given by

$$\begin{aligned} B_r &= \tfrac{\mu_0}{4\pi} i_\varphi G_z, \\ B_\varphi &= \tfrac{\mu_0}{4\pi}\left(i_z G_r - i_r G_z\right), \\ B_z &= -\tfrac{\mu_0}{4\pi} i_\varphi G_r. \end{aligned} \tag{6}$$

In the following only the field components will be considered, which are all determined by G_r and G_z. The potential is not discussed.

III. ANALYTICAL SOLUTION

Rather than starting with the integration over the angle the integrations over radius and longitudinal coordinate are performed in a first step by Mathematica. Integration by parts of some terms to remove logarithmic terms brings the results into a form suitable for the final integration as

$$G_r = r_p \int_0^{2\pi} d\varphi \left. \left[\begin{array}{l} -\dfrac{1}{4}\dfrac{\rho\left(\cos(\varphi) - \cos(3\varphi)\right)}{R^2 + \zeta R} \\ +\zeta\dfrac{(1+\zeta^2 - \rho\cos\varphi)\sin^2\varphi}{R\left(1+\zeta^2-\cos^2\varphi\right)} \\ -\operatorname{Arctan}\left[\dfrac{R\sin\varphi}{\zeta\left(\rho - \cos\varphi\right)}\right]\sin(2\varphi) \end{array} \right] \right|_{\rho_o,\rho_i,\zeta_2,\zeta_1} , \tag{7}$$

$$G_z = r_p \int_0^{2\pi} d\varphi \left[\frac{(1+\zeta^2 - \rho\cos\varphi)\sin^2\varphi}{R\,(1+\zeta^2-\cos^2\varphi)} - R \right] \Bigg|_{\rho_o,\,\rho_i,\,\zeta_2,\,\zeta_1}, \tag{8}$$

with $\rho_o = \frac{r_o}{r_p}$, $\rho_i = \frac{r_i}{r_p}$, $\zeta_1 = \frac{z_1}{r_p}$ and $\zeta_2 = \frac{z_2}{r_p}$.

The final equations need to be evaluated at the four corner points of the ring and summed up as $G(\rho_o, \zeta_2) - G(\rho_o, \zeta_1) - G(\rho_i, \zeta_2) + G(\rho_i, \zeta_1)$. Integration constants and all terms which depend only on ρ or on ζ alone cancel therefore in this sum and can in general be ignored. Eqs. 7 and 8 can now be rewritten as

$$G_r = r_p \int T_1 + \zeta T_2 - T_3 \, d\varphi, \tag{9}$$

$$G_z = r_p \int T_2 - T_4 \, d\varphi \tag{10}$$

with

$$\begin{aligned}
T_1 &= -\frac{1}{4}\frac{\rho\,(\cos(\varphi) - \cos(3\varphi))}{R^2 + \zeta R}, \\
T_2 &= \frac{(1+\zeta^2 - \rho\cos\varphi)\sin^2\varphi}{R\,(1+\zeta^2 - \cos^2\varphi)}, \\
T_3 &= \mathrm{Arctan}\left[\frac{R\sin\varphi}{\zeta\,(\rho - \cos\varphi)}\right]\sin(2\varphi), \\
T_4 &= R.
\end{aligned} \tag{11}$$

The final integration over the angular coordinate leads to expressions containing elliptic integrals of first, second and third kind. In the following a notation is adapted where the square of the classical modulus k is replaced by m, i.e. $k^2 = m$, thus the three incomplete integrals read as:

$$\begin{aligned}
F(\phi, m) &= \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2\varphi}}\, d\varphi, \\
E(\phi, m) &= \int_0^{\phi} \sqrt{1 - m\sin^2\varphi}\, d\varphi, \\
\Pi(n, \phi, m) &= \int_0^{\phi} \frac{1}{(1 - n\sin^2\varphi)\sqrt{1 - m\sin^2\varphi}}\, d\varphi.
\end{aligned} \tag{12}$$

For $\phi = \frac{\pi}{2}$ the incomplete integrals turn into complete integrals

$$\begin{aligned} F\left(\frac{\pi}{2}, m\right) &= K(m), \\ E\left(\frac{\pi}{2}, m\right) &= E(m), \\ \Pi\left(n, \frac{\pi}{2}, m\right) &= \Pi(n, m). \end{aligned} \tag{13}$$

The integral of first kind has a pole at $m = 1$, the integral of third kind has a pole at $m = 1$ and at $n = 1$, thus for $m > 1$ and also for $n > 1$ the Cauchy principal value has to be calculated.

Mathematica cannot calculate the complete integrals over the angle coordinate directly for all terms listed in Eq. 11, but gives in general only the incomplete integral in a lengthy and unsorted form. Besides reordering and summarizing the terms, limiting values at the upper and lower boundaries of the integration interval have to be determined to solve the complete integral.

The elliptic integrals of the Mathematica solutions are of incomplete form, have a complex phase and a preceding imaginary factor i. The final result is of course real valued; hence all imaginary terms add up to zero and can be ignored in the derivation. At the lower boundary $\varphi = 0$ elliptic integrals of the first and the second kind yield zero, but elliptic integrals of the third kind have to be replaced by correct limiting values.

The integrals have a form like $i\Pi\left(n, -\operatorname{Arcsin}\sqrt{q}, m\right)$, where parameter $q \geq 1$. q approaches 1 from above when the phase approaches zero. The phase is thus of the form $\phi = -\operatorname{Arcsin}\sqrt{q} = -\frac{\pi}{2} + ix$, where x approaches 0^+. At $x = 0$ the integral has a jump discontinuity; the limit reads as:

$$\xrightarrow[x \to 0^+]{\lim} \left[\operatorname{Re}\left(i\Pi\left(n, -\frac{\pi}{2} + ix, m\right)\right)\right] = \begin{cases} -\frac{\pi}{2}\sqrt{\frac{n}{(n-1)(n-m)}} \text{ for } n > 1 \\ -\infty \text{ for } n = 1 \\ \quad 0 \quad \text{for } n < 1 \end{cases} . \tag{14}$$

For $x \to 0^-$ signs have to be reversed. This limit is not documented in the standard literature for elliptic integrals [14, 15].

At the upper boundary $\varphi = \pi$ the elliptic integrals of the Mathematica results take a form as $iF\left(i \operatorname{Arcsinh}\sqrt{-m^{-1}}, m\right) = iF\left(-\operatorname{Arcsin}\sqrt{m^{-1}}, m\right)$, with equivalent relations for the integrals of second and third kind. It is known, that in such a case the imaginary part of the elliptic integral (which leads to the real part of the complete term) can be extracted and that it leads to complete integrals. The transformation equations for the integrals of first and second kind are found in the literature [14] (Eq. 111.09) as

$$\begin{aligned} \mathrm{Re}\left[iF\left(-\,\mathrm{Arcsin}\,\sqrt{m^{-1}},m\right)\right] &= -K(1-m), \\ \mathrm{Re}\left[iE\left(-\,\mathrm{Arcsin}\,\sqrt{m^{-1}},m\right)\right] &= E(1-m)-K(1-m). \end{aligned} \tag{15}$$

However, a corresponding equation for the integral of third kind is not published. Note, that since $m<1$, i.e. $m^{-1}>1$, $-\,\mathrm{Arcsin}\,\sqrt{m^{-1}}$ is complex and of the form $-\frac{\pi}{2}+ix$. A closer look at the properties of the complex elliptic integral of third kind along the axis $\phi=-\frac{\pi}{2}+ix$ reveals that the pole which is located along the real axis at $n=1$ appears along this line at the position $n=m$. Thus, the Cauchy principal value has to be calculated already for $n>m$ in this case. The general equation for the transformation of a complex phase argument [14] (Eq. 161.02) leads for $0\le m\le 1$ to:

$$\mathrm{Re}\left[i\Pi\left(n,-\,\mathrm{Arcsin}\,\sqrt{m^{-1}},m\right)\right] = \begin{cases} -\frac{1}{1-n}\left(K\left(1-m\right)-n\Pi\left(1-n,1-m\right)\right) & \text{for } n<m \\ -\frac{1}{1-n}\left(K\left(1-m\right)-n\Pi_{PV}\left(1-n,1-m\right)\right) & \text{for } n>m \end{cases}, \tag{16}$$

where Π_{PV} indicates the Cauchy principal value.
The relation $\Pi_{PV}(1-n,1-m)=K(1-m)-\Pi\left(\frac{1-m}{1-n},1-m\right)$ [15] (Eq. 19.6.5) leads finally to

$$\mathrm{Re}\left[i\Pi\left(n,-\,\mathrm{Arcsin}\,\sqrt{m^{-1}},m\right)\right] = \begin{cases} -\frac{1}{n-1}\left(K\left(1-m\right)-n\Pi\left(1-n,1-m\right)\right) & \text{for } n<m \\ \frac{n}{n-1}\Pi\left(\frac{1-m}{1-n},1-m\right)-K\left(1-m\right) & \text{for } n>m \end{cases}. \tag{17}$$

Additional useful relations not documented in the standard literature are the transformation relations for negative parameters m [16]. They are here reproduced for completeness:

$$\begin{aligned} K(-m) &= \frac{1}{\sqrt{m+1}}K\left(\frac{m}{m+1}\right), \\ E(-m) &= \sqrt{m+1}\,E\left(\frac{m}{m+1}\right), \\ \Pi(n,-m) &= \frac{1}{(1-n)\sqrt{m+1}}\Pi\left(\frac{n}{n-1},\frac{m}{m+1}\right). \end{aligned} \tag{18}$$

With these relations the integrations of the terms T_1-T_4 (Eq. 11) can be executed and the results can be brought into a suitable form. Challenging is the integration of T_3 for which Mathematica finds a complete form for $\rho>1$, but only an incomplete form for $\rho<1$. For $\rho<1$ the integration interval has to be split, because the Arcus Tangent term exhibits a jump discontinuity at $\cos\varphi = \rho$. An additional jump discontinuity, which needs to be removed, shows up for $\rho < 1$ at $\zeta = 0$. It turns out, that after removal of this jump discontinuity, the solution found for $\rho > 1$ describes the case $\rho < 1$ without the jump discontinuity as well. This finding simplifies the results significantly.

More details on the integration and the final integral results of the individual terms $T_1 - T_4$ are summarized in the appendix, as they might be useful for other cases. Here the combined functions G_r and G_z are presented:

$$G_r = \begin{cases} 0 \quad \text{for } \rho = 1 \ \wedge \ \zeta = 0 \\ r_p \zeta \sqrt{4+\zeta^2} \left[\mathrm{K}(m) - \mathrm{E}(m)\right] \quad \text{for } \rho = 1 \wedge \zeta \neq 0 \\ r_p \frac{\zeta}{\sqrt{(\rho+1)^2+\zeta^2}} \begin{bmatrix} (2+2\rho^2+\zeta^2)\,\mathrm{K}(m) \\ -\left((\rho+1)^2+\zeta^2\right)\mathrm{E}(m) \\ -(\rho-1)^2 \Pi(n_0, m) \end{bmatrix} \quad \text{else} \end{cases}, \tag{19}$$

$$G_z = \begin{cases} -4r_p \quad \text{for} \quad \rho = 1 \ \wedge \zeta = 0 \\ 2r_p \left[(1-\rho)\,\mathrm{K}(m) - (\rho+1)\,\mathrm{E}(m)\right] \quad \text{for} \quad \rho \neq 1 \ \wedge \zeta = 0 \\ r_p \frac{2}{\sqrt{(\rho+1)^2+\zeta^2}} \begin{bmatrix} (1-\rho^2+\zeta^2)\,\mathrm{K}(m) - \left((\rho+1)^2+\zeta^2\right)\mathrm{E}(m) \\ -\left((\rho+1)\left(1-\sqrt{1+\zeta^2}\right)+\zeta^2\right)\Pi(n_1, m) \\ -\left((\rho+1)\left(1+\sqrt{1+\zeta^2}\right)+\zeta^2\right)\Pi(\bar{n}_1, m) \end{bmatrix} \quad \text{else} \end{cases}, \tag{20}$$

with

$$\begin{aligned} m &= \tfrac{4\rho}{(\rho+1)^2+\zeta^2}, \\ n_0 &= \tfrac{4\rho}{(\rho+1)^2}, \\ n_1 &= \tfrac{2}{1+\sqrt{1+\zeta^2}}, \\ \bar{n}_1 &= \tfrac{2}{1-\sqrt{1+\zeta^2}}. \end{aligned} \tag{21}$$

IV. IMPLEMENTATION

Eqs.19 to 21 are analytic, but are still impaired by the poles inherent to the elliptic integrals. Carlson [17] and [15] (Eqs. 19.22.8-19.22.13) developed simple and efficient procedures for the calculation of the complete elliptic integrals which can be used to remove the poles and simplify the results further. In the following the complementary parameters $m' = 1 - m$, $n_0' = 1 - n_0$, $n_1' = 1 - n_1$ and $\bar{n}_1' = 1 - \bar{n}_1$ are employed.

It is well-known, that the first elliptic integral can be expressed in terms of Gauss'

arithmetic-geometric mean $M(x,y)$ as $\mathrm{K}(m) = \frac{\pi}{2}\frac{1}{M(m',1)} = \frac{\pi}{2}\frac{1}{M(m')}$. Carlson's relations for the second and third integral read as: $\mathrm{E}(m) = \frac{\pi}{2}\frac{S_E(m')}{M(m')}$ and $\Pi(n,m) = \frac{\pi}{2}\frac{1}{M(m')}\left[1 + \frac{1-n'}{2n'}\left(1 + S_{Pi}(n',m')\right)\right]$.
$M(m')$, $S_E(m')$ and $S_{Pi}(n',m')$ are simple, fast converging sums which take values between +1 and -1, see the following figures.

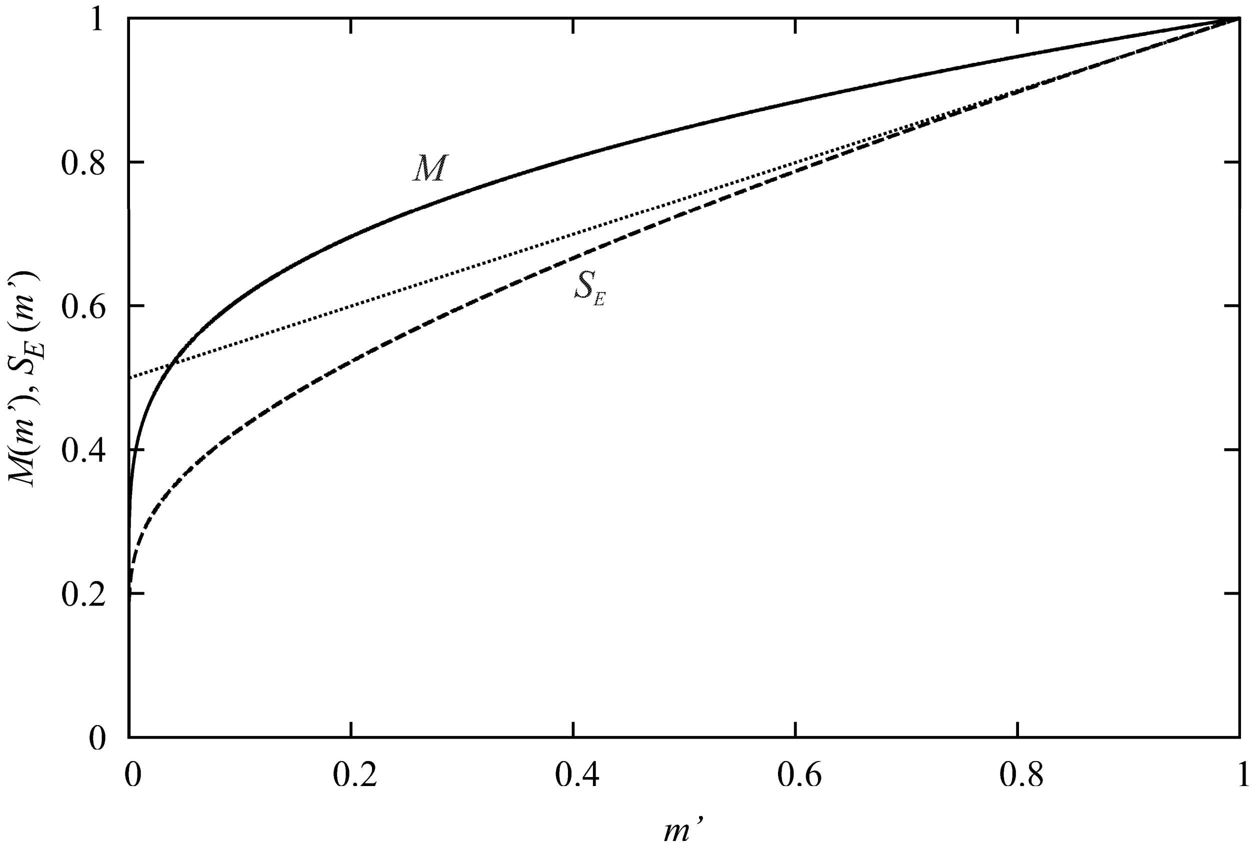

FIG. 1: Gauss arithmetic geometric mean M (solid) and sum S_E (broken) vs. m'. Both functions start at 0 for $m' = 0$ and reach 1 for $m' = 1$. S_E converges to $\frac{1}{2}(m'+1)$ toward $m' = 1$, as indicated by the dotted line.

Details on the summation follow below. Note, that the relation for the third integral is slightly modified in comparison to Carlson's version. Splitting off the zeroth term of Carlson sum and taking the factor $\frac{1-n'}{2n'}$ in front of the sum allows to combine these terms with the

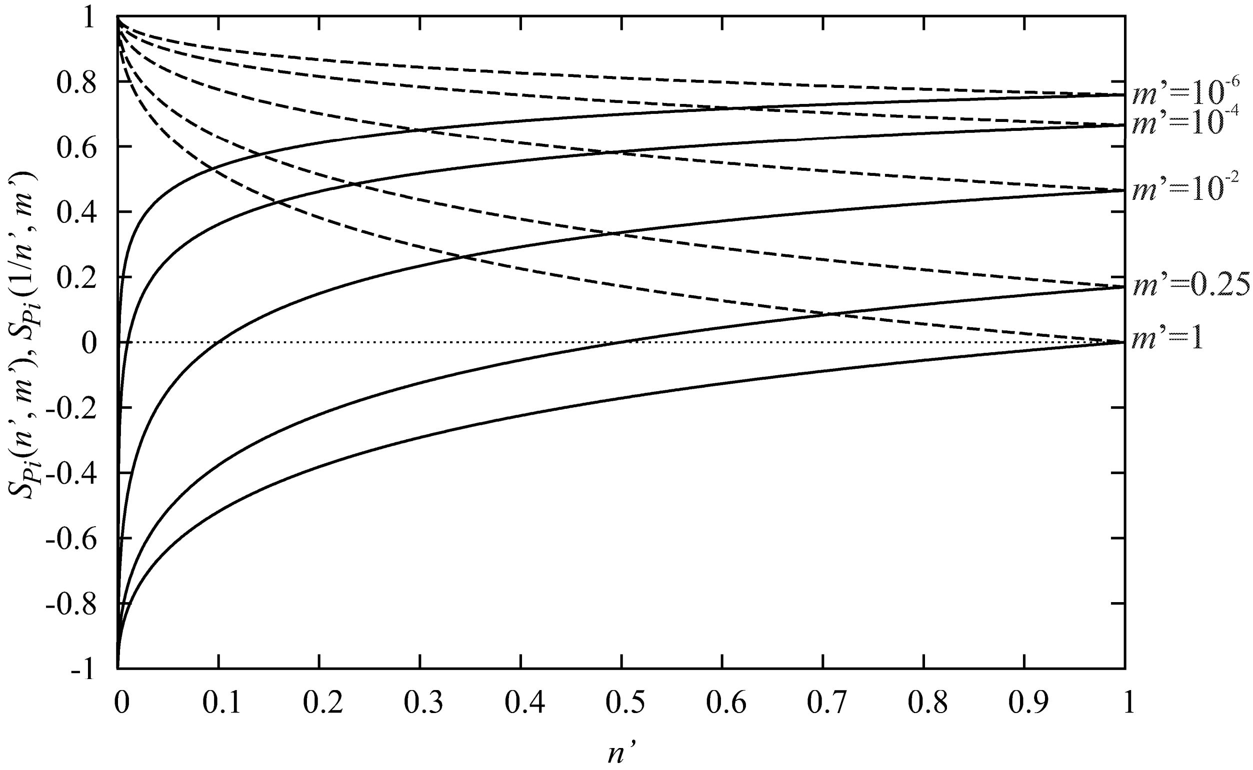

FIG. 2: Sum $S_{Pi}(n',\ m')$ (solid) and $S_{Pi}(n'^{-1},\ m')$ (broken) vs. n' for various values of m'. Some special values: $S_{Pi}(1,\ 1) = 0$; $S_{Pi}(n',\ 0) = S_{Pi}(n'^{-1},\ 0) = 1$ for $0 < n' \leq 1$; $S_{Pi}(0,\ m') = -1$; $S_{Pi}(\infty,\ m') = 1$.

prefactors of the terms in Eq. 19 and 20. The modified field equations read now:

$$G_r = \begin{cases} 0 \quad \text{for } \rho = 1 \wedge \zeta = 0 \\ \frac{\pi}{2} r_p \frac{\zeta}{M(m')\sqrt{4+\zeta^2}} \left[4 + \zeta^2 - (4+\zeta^2)\, S_E(m')\right] \quad \text{for } \rho = 1 \wedge \zeta \neq 0 \\ \frac{\pi}{2} r_p \frac{\zeta}{M(m')\sqrt{(\rho+1)^2+\zeta^2}} \begin{bmatrix} \rho^2 + 1 + \zeta^2 - \left((\rho+1)^2 + \zeta^2\right) S_E(m') \\ -2\rho S_{Pi}(n'_0, m') \end{bmatrix} \text{else} \end{cases}, \tag{22}$$

$$G_z = \begin{cases} -4 r_p \quad \text{for } \rho = 1 \wedge \zeta = 0 \\ -\pi r_p \frac{1}{M(m')} \left[\rho - 1 + (\rho+1)\, S_E(m')\right] \quad \text{for } \rho \neq 1 \wedge \zeta = 0 \\ -\pi r_p \frac{1}{M(m')\sqrt{(\rho+1)^2+\zeta^2}} \begin{bmatrix} \rho^2 + 1 + \zeta^2 + \left((\rho+1)^2 + \zeta^2\right) S_E(m') \\ -\left(\rho - \sqrt{1+\zeta^2}\right) S_{Pi}(n'_1, m') \\ -\left(\rho + \sqrt{1+\zeta^2}\right) S_{Pi}(\bar{n}'_1, m') \end{bmatrix} \text{else} \end{cases}, \tag{23}$$

$$
\begin{aligned}
m' &= \tfrac{(\rho-1)^2+\zeta^2}{(\rho+1)^2+\zeta^2},\\
n'_0 &= \left(\tfrac{\rho-1}{\rho+1}\right)^2,\\
n'_1 &= \tfrac{\sqrt{1+\zeta^2}-1}{\sqrt{1+\zeta^2}+1},\\
\bar{n}'_1 &= \tfrac{1}{n'_1}.
\end{aligned}
\tag{24}
$$

Close to the symmetry axis ρ and ζ get large and absolute parameters might be appropriate. Transformation of Eqs. 22 and 23 is straight forward; parameters m', n'_0, n'_1 and $\bar{n}'_1$ do not change. In the limit $r_p \to 0$ $m' = n'_0 = n'_1 = \bar{n}'_1 = 1$ and thus $M(m') = 1$, $S_{Pi}(n'_0, m') = S_{Pi}(n'_1, m') = S_{Pi}(\bar{n}'_1, m') = 0$. S_E converges to $\frac{1}{2}(m'+1)$ toward $m' = 1$, which brings Eq. 22 to $G_r = 0$ and Eq. 23 to $G_z = -2\pi\sqrt{r_r^2 + z^2}$, i.e. the known on-axis result.

V. ALGORITHM FOR THE CALCULATION OF M, S_E AND S_{Pi}

Carlson presented individual routines for the calculation of the complete elliptic integrals. Here a combined routine, which is adapted to the definitions given above is presented. At entry calculate the zeros and the first step as:

I=0

$$xt = \sqrt{m'}; \quad M_0 = 0.25(xt+1)^2; \quad xy = xt; \quad p2 = n'; \quad p = \sqrt{p2}$$

I=1

$$
\begin{aligned}
&xt = 0.5\,(xt+1)\,; \quad yt = \sqrt{xy}; \quad Q = 0.5\left(\tfrac{p2-xy}{p2+xy}\right); \quad p = 0.5\left(p + \tfrac{xy}{p}\right)\\
&S_{Pi} = Q; \quad Fac = 0.5; \quad S_E = M_0 - Fac(xt-yt)^2;\\
&xy = xt \times yt; \quad p2 = p^2; \quad Del = 1
\end{aligned}
$$

Do While ($Del > eps$)

I=I+1

$$
\begin{aligned}
&xt = 0.5\,(xt+yt)\,; \quad yt = \sqrt{xy}; \quad Q = 0.5\,Q\left(\tfrac{p2-xy}{p2+xy}\right); \quad p = 0.5\left(p + \tfrac{xy}{p}\right)\\
&S_{Pi} = S_{Pi} + Q; \quad Fac = 2 \times Fac; \quad S_E = S_E - Fac(xt-yt)^2;\\
&xy = xt \times yt; \quad p2 = p^2; \quad Del = \left|\tfrac{Q}{S_{Pi}}\right|
\end{aligned}
$$

On exit set $M = xt$.

Parameter *eps* controls the accuracy of the calculation. The sums converge very fast; the following figures show some example cases. Convergence is fastest for $m' = n' \to 1$. It is slower for smaller m' and when n' differs significantly from m'.

Figure 3 shows the case for $m' = n' = 10^{-2}$. Here $\frac{\Delta M}{M} = \left|\frac{M(I)-M(I=30)}{M(I=30)}\right|$; $\frac{\Delta S_E}{S_E} = \left|\frac{S_E(I)-S_E(I=30)}{S_E(I=30)}\right|$; $\frac{\Delta S_{Pi}}{S_{Pi}} = \left|\frac{S_{Pi}(I)-S_{Pi}(I=30)}{S_{Pi}(I=30)}\right|$ and $\left|\frac{Q}{S_{Pi}}\right| = \left|\frac{Q(I)}{S_{Pi}(I)}\right| = Del$. The lower boundary of the vertical axis is limited at 10^{-15}; $I = 30$ is arbitrary, but large enough to reach full convergence for all tested parameters. In Figure 3 $\left|\frac{Q}{S_{Pi}}\right| > \frac{\Delta M}{M} > \frac{\Delta S_{Pi}}{S_{Pi}} > \frac{\Delta S_E}{S_E}$.

Figure 4 shows the case $m' = 10^{-6}$ and $n' = 10^{6}$. S_{Pi} converges much slower, still $\left|\frac{Q}{S_{Pi}}\right| > \frac{\Delta S_{Pi}}{S_{Pi}} > \frac{\Delta M}{M} > \frac{\Delta S_E}{S_E}$ holds. Thus $\left|\frac{Q}{S_{Pi}}\right|$ is a suitable parameter to control the convergence for all values of m' and n'.

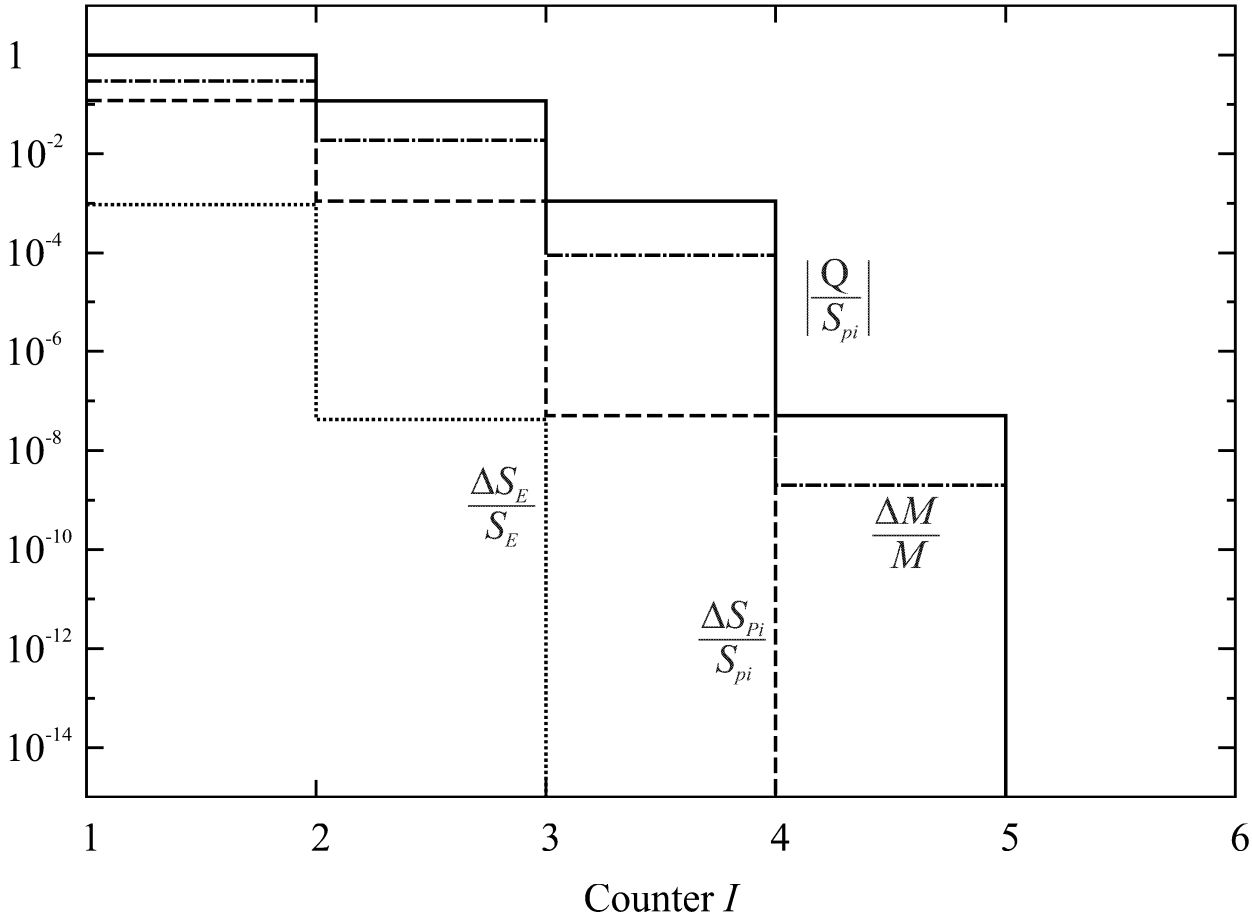

FIG. 3: Convergence of M, S_E and S_{Pi} vs. step number for $m' = n' = 10^{-2}$. For details see text.

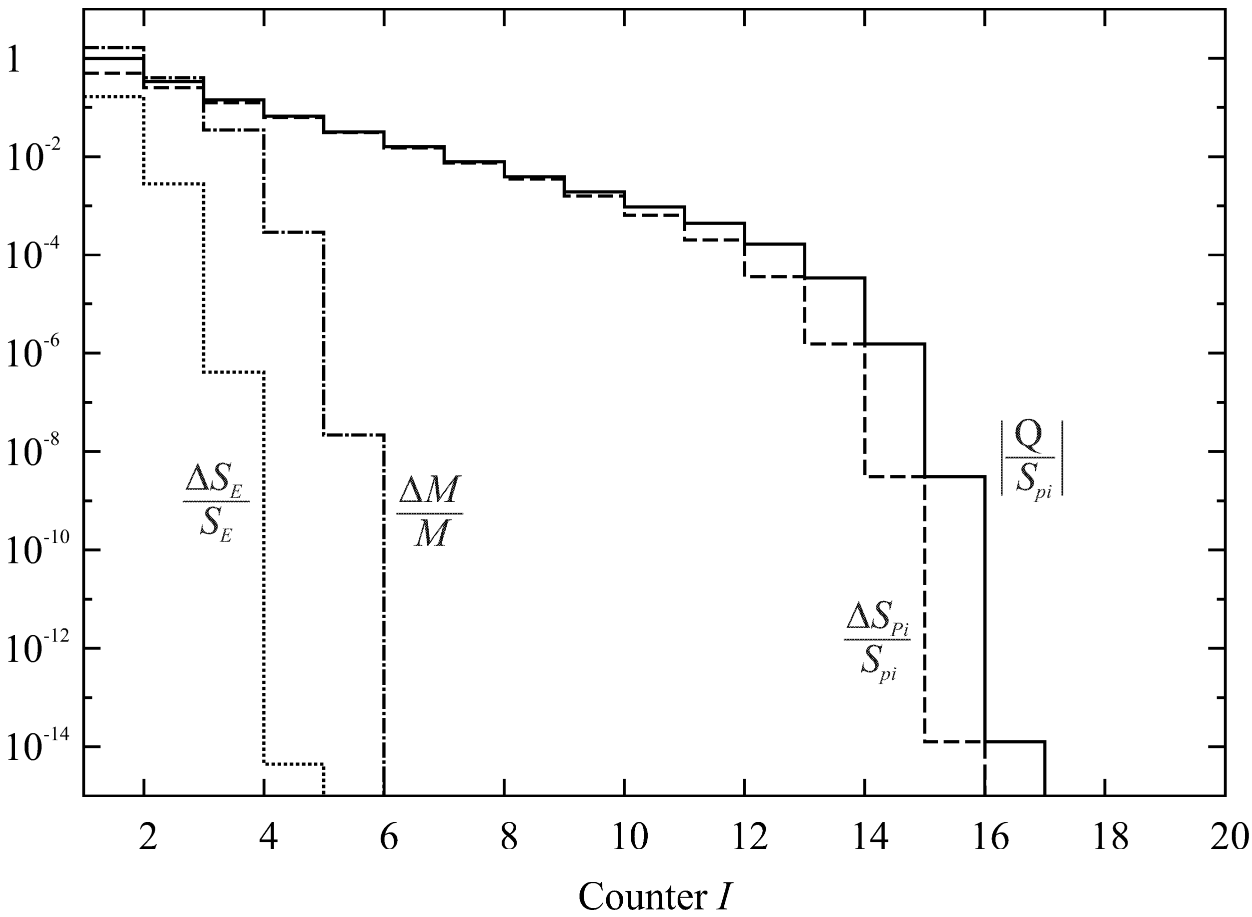

FIG. 4: Convergence of M, S_E and S_{Pi} vs. step number for $m' = 10^{-6}$ and $n' = 10^6$. For details see text.

VI. SUMMARY

Field integrals on a cylindrical grid can be used to calculate electro-magnetic or gravitational fields with cylindrical symmetry. Eqs. 19 to 21 present the analytical solution for these field integrals. The derivation of the solution is based on intensive usage of symbolic computational programs, which are however not able to perform the calculations directly. A new limit for the complex elliptic integral of the third kind, which is required for the solution of the problem, and a new transformation relation for the same integral are presented in Eqs. 14 and 18.

While Eqs. 19 to 21 are analytical, they are still impaired by poles inherent to the elliptic integrals. The proposed implementation in a numerical algorithm is based on relations derived by Carlson for the calculation of complete elliptic integrals. A small modification of Carlsons relations allows to remove the poles and formulate a very efficient and precise algorithm for the calculation of the field integrals.

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VII. APPENDIX

Here the integrals of the terms T_1 to T_4 (Eq. 11) are summarized. The arguments of the elliptic integrals are defined in Eq. 21.

$\mathbf{T_1}$:

$$\int T_1 d\varphi = \int_0^{2\pi} -\frac{1}{4}\frac{\rho\left(\cos(\varphi)-\cos(3\varphi)\right)}{R^2+\zeta R}d\varphi + \frac{\pi}{2}\rho^2 \tag{25}$$

The integral of the first part ($\cos\varphi$) is based on the incomplete solution of Mathematica. The Mathematica result of the second part ($\cos(3\varphi)$) is not correct, instead the incomplete result of Maple has been used. The integration constant $\frac{\pi}{2}\rho^2$ is added to correct for an offset. The combined result (addition of $\cos\varphi$ and $\cos(3\varphi)$ part) reads as:

$$\int T_1 d\varphi = \begin{cases} \frac{1}{3}\begin{pmatrix} \frac{\zeta}{\sqrt{4+\zeta^2}}\left(12+7\zeta^2+\zeta^4\right)\mathrm{K}(m) \\ -\frac{\sqrt{4+\zeta^2}}{\zeta}\left(5\zeta^2+\zeta^4\right)\mathrm{E}(m) \end{pmatrix} & \text{for } \rho=1 \\ \frac{\zeta}{6\rho^2\sqrt{(1+\rho)^2+\zeta^2}}\begin{pmatrix} \left(8(1+\rho^2+\rho^4)+7\zeta^2\left(1+\rho^2\right)+2\zeta^4\right)\mathrm{K}(m) \\ -\left((1+\rho)^2+\zeta^2\right)\left(5(1+\rho^2)+2\zeta^2\right)\mathrm{E}(m) \\ -3(\rho-1)^2\left(\rho^2+1\right)\Pi\left(n_0,m\right) \end{pmatrix} & \text{else} \end{cases} \tag{26}$$

$\mathbf{T_2}$:

$$\int T_2 d\varphi = \int_0^{2\pi} \frac{\left(1+\zeta^2-\rho\cos\varphi\right)\sin^2\varphi}{R\left(1+\zeta^2-\cos^2\varphi\right)}d\varphi \tag{27}$$

The integral is based on the incomplete result of Mathematica

$$\int T_2 d\varphi = \frac{2}{\sqrt{(1+\rho)^2+\zeta^2}}\begin{pmatrix} \left(1-\rho^2+\zeta^2\right)\mathrm{K}(m)+\left((1+\rho)^2+\zeta^2\right)\mathrm{E}(m) \\ -\left(1+\rho+\zeta^2-(1+\rho)\sqrt{1+\zeta^2}\right)\Pi\left(n_1,m\right) \\ -\left(1+\rho+\zeta^2+(1+\rho)\sqrt{1+\zeta^2}\right)\Pi\left(\bar{n}_1,m\right) \end{pmatrix} \tag{28}$$

$\mathbf{T_3}$:

$$\int T_3 d\varphi = \int_0^{2\pi} \mathrm{Arctan}\left[\frac{R\sin\varphi}{\zeta\left(\rho-\cos\varphi\right)}\right]\sin(2\varphi)d\varphi \tag{29}$$

Mathematica finds a complete form for the integral of T_3 for $\rho > 1$, but only an incomplete form for $\rho < 1$.

For $\rho < 1$ the integration interval has to be split, because the Arcus Tangent term exhibits a jump discontinuity at $\cos\varphi = \rho$. An additional (unphysical) jump discontinuity, which needs to be removed, shows up for $\rho < 1$ at $\zeta = 0$. The integral reads hence for $\rho < 1$

$$\begin{aligned}\int T_3 d\varphi = & \int_0^{\xrightarrow[\varphi\to\mathrm{Arccos}\,(\rho)^-]{\lim}} 2T_3 d\varphi + \int_{\xrightarrow[\varphi\to\mathrm{Arccos}\,(\rho)^+]{\lim}}^{\pi} 2T_3 d\varphi \\ & - \xrightarrow[\zeta\to 0^\pm]{\lim}\left[\int_0^{\xrightarrow[\varphi\to\mathrm{Arccos}\,(\rho)^-]{\lim}} 2T_3 d\varphi + \int_{\xrightarrow[\varphi\to\mathrm{Arccos}\,(\rho)^+]{\lim}}^{\pi} 2T_3 d\varphi\right]\end{aligned}$$

It turns out the solution found for $\rho > 1$ describes the case $\rho < 1$ without jump discontinuity as well. This finding simplifies the results significantly.

$$\int T_3 d\varphi = \begin{cases} -\pi\frac{\zeta^3}{|\zeta|} - \\ \frac{1}{3\zeta\sqrt{4+\zeta^2}}\begin{pmatrix}\left(6\sqrt{1+\zeta^2}\left(1+2\zeta^2\right)-\left(6+12\zeta^2+4\zeta^4+\zeta^6\right)\right)\mathrm{K}(m) \\ -\zeta^2\left(16-\zeta^4\right)\mathrm{E}(m) \\ +3\left(1+2\zeta^2\right)\begin{pmatrix}\left(1-\sqrt{1+\zeta^2}\right)^2\Pi\left(n_2,m\right) \\ -\left(1+\sqrt{1+\zeta^2}\right)^2\Pi\left(\bar{n}_1,m\right)\end{pmatrix}\end{pmatrix} & \text{for } \rho = 1 \\ \frac{\zeta}{6\rho^2\sqrt{(1+\rho)^2+\zeta^2}} \\ \begin{pmatrix}\left(8\left(1+\rho^2-2\rho^4\right)+\left(7+13\rho^2\right)\zeta^2+2\zeta^4\right)\mathrm{K}(m) \\ -\left(\left(1+\rho\right)^2+\zeta^2\right)\left(5-13\rho^2+2\zeta^2\right)\mathrm{E}(m) \\ -3(1-\rho)^3\left(1+\rho\right)\Pi\left(n_0,m\right) \\ -12\rho^2\left[\begin{matrix}\left(1+\rho+\zeta^2-\frac{1+\rho\left(1+\zeta^2\right)+\zeta^2}{\sqrt{1+\zeta^2}}\right)\Pi\left(n_1,m\right) \\ +\left(1+\rho+\zeta^2+\frac{1+\rho\left(1+\zeta^2\right)+\zeta^2}{\sqrt{1+\zeta^2}}\right)\Pi\left(\bar{n}_1,m\right)\end{matrix}\right]\end{pmatrix} & \text{else}\end{cases} \tag{30}$$

with

$$n_2 = \frac{2\left(1 + \sqrt{1+\zeta^2}\right)}{4+\zeta^2}$$

$\mathbf{T_4}$:

$$\int T_4 d\varphi = \int_0^{2\pi} R d\varphi \tag{31}$$

The integral T_4 corresponds to the standard integral for an infinitely thin charged ring

$$\int T_4 d\varphi = 4\sqrt{(1+\rho)^2 + \zeta^2}\mathrm{E}(m). \tag{32}$$

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